Advanced Studies in Pure Mathematics 64, 2015
Nonlinear Dynamics in Partial Differential Equations
pp. 221–228

Minimax approach to the n-body problem

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Abstract.

Using the variational method Chenciner and Montgomery proved the existence of a new periodic solution of figure-eight shape to the planar three-body problem. Since then, a number of periodic solutions have been discovered as minimizers. We present a minimax approach to the n-body problem and prove the existence of some periodic solutions as minimax points of the action functional.

§1. Introduction

We consider the classical n-body problem for which the equation of motion is given by

$$m_\ell \ddot{q}_\ell = \frac{\partial V}{\partial q_\ell}, \quad q_\ell \in \mathbb{R}^3, \quad \ell = 1, 2, \ldots, n$$

where an overdot denotes differentiation with respect to the time variable, m_ℓ (> 0) is the ℓ-th mass and

$$V(q_1, \ldots, q_n) = \sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|}$$

represents the (negative-)potential energy with the unit gravitational constant.

Using the variational method Chenciner and Montgomery [1] proved the existence of a new periodic solution of figure-eight shape to the planar three-body problem. Since then, a number of periodic solutions have been found as minimizers of variational formulation of the n-body problem in various different settings.

Received February 29, 2012.
Revised October 23, 2012.
2010 *Mathematics Subject Classification.* 70F10.
Key words and phrases. Choreography, variational method.

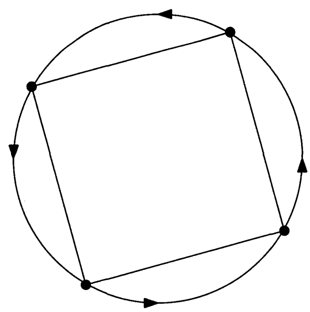

Fig. 1. The rotating regular n-gon solution: this figure represents the case of $n = 4$.

We present a minimax approach to the n-body problem. We consider a variational structure under a certain symmetric constraint in the spatial n-body problem. In the setting, the minimizers are the rotating regular n-gons which is trivial solutions (Fig. 1). The set of the minimizers consists of two connected components and each component is homeomorphically equivalent to the circle. We prove the existence of a mountain pass solution between these two components (Theorem 3). We also obtain other solutions as the top of the hill whose foot is one of these components (Theorem 4). Morse index plays an important role to estimate the number of collisions and to show that the obtained solutions are different from the classical solutions.

Section 2 collects some known results about variational structure for symmetric curves. In Section 3 we show the existence of periodic solutions which attain mountain pass points. In Section 4 we show the existence of other periodic solutions which attain a different type of minimax point.

§2. Variational formulation and symmetric constraint

The n-body problem is equivalent to the variational problem with respect to the action functional

$$\mathcal{A}(q) = \int_0^T L(q, \dot{q})dt$$

where the function L is the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}\sum m_k|\dot{q}_k|^2 + \sum_{i<j} \frac{m_i m_j}{|q_i - q_j|}.$$

Denote the configuration space by $\hat{\mathcal{X}}$ where

$$\mathcal{X} = \{q = (q_1, \dots, q_n) \in (\mathbb{R}^3)^n \mid \sum_{k=1}^{n} m_k q_k = 0\}$$
$$\Delta_{ij} = \{q \in \mathcal{X} \mid q_i = q_j\}, \quad \Delta = \bigcup_{i<j} \Delta_{ij}, \quad \hat{\mathcal{X}} = \mathcal{X} - \Delta$$

and let

$$\Lambda = H^1(\mathbb{R}/T\mathbb{Z}, \hat{\mathcal{X}})$$

where H^1 stands for the Sobolev space. Let G be a group and let

$$\tau : G \to O(2), \quad \rho : G \to O(3), \quad \sigma : G \to \mathfrak{S}_n,$$

be homomorphisms. Here $\mathbb{R}/T\mathbb{Z}$ is naturally embedded into $\mathbb{R}^2$ so that the image is the unit circle $\mathbb{S}^1$. We regard time t as belonging $\mathbb{S} \subset \mathbb{R}^2$ and the orthogonal group $O(2)$ as acting $\mathbb{S}^1$.

We define the action of G to Λ by

$$g \cdot ((q_1, \dots, q_n)(t)) = (\rho(g) q_{\sigma(g^{-1})(1)}, \dots, \rho(g) q_{\sigma(g^{-1})(n)})(\tau(g^{-1})t)$$

for $g \in G$ and $q(t) = (q_1, \dots, q_n)(t) \in \Lambda$. Let

$$\Lambda^G = \{q \in \Lambda \mid g \cdot q = q\}, \quad \mathcal{A}^G = \mathcal{A}|_{\Lambda^G}.$$

Palais' theorem is important in this setting.

Theorem 1 ([3]). *If $\mathcal{A}$ is invariant under the group action of G, then a critical point of A^G in Λ^G is a critical point of $\mathcal{A}$ in Λ.*

As a special case, we assume the all masses are equal and take G as the cyclic group $C_n = \langle g \mid g^n = 1 \rangle$ of order n. The homomorphisms are defined by

$$\tau(g) = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} \qquad \rho(g) = E_3 \qquad \sigma(g) = (1, 2, \dots, n).$$

We can obtain periodic solutions as minimizers of $\mathcal{A}^{C_n}$, but these are trivial as follows.

Theorem 2 ([2]). *The minimizers of $\mathcal{A}^{C_n}$ are just rotating regular n-gons.*

Next we take G as the dihedral group $D_n = \langle g_1, g_2 \mid g_1^2 = g_2^n = (g_1 g_2)^2 = 1 \rangle$. The homeomorphisms ρ, σ, τ are defined by

$$\tau(g_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho(g_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\sigma(g_1) = (2, n)(3, n-1) \dots ([\frac{n-1}{2}] + 1, n - [\frac{n-1}{2}] + 1)$$

$$\tau(g_2) = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix}, \qquad \rho(g_2) = E_3, \qquad \sigma(g_2) = (1, 2, \dots, n).$$

The D_n-equivalent set Λ^{D_n} is a subset of Λ^{C_n} and Λ^{D_n} includes the rotating regular n-gons. Therefor the minimizers of $\mathcal{A}^{D_n}$ are also the rotating regular n-gons. The minimizing rotating regular n-gon solutions are denoted by

$$q = (\zeta_+^\theta(t), \zeta_+^\theta(t - 2\pi/n), \dots, \zeta_+^\theta(t - 2\pi(n-1)/n))$$

and

$$q = (\zeta_-^\theta(t), \zeta_-^\theta(t - 2\pi/n), \dots, \zeta_-^\theta(t - 2\pi(n-1)/n))$$

where

$$\zeta_+^\theta(t) = \sum_{j=1}^{n-1} \frac{1}{4|\sin \frac{\pi j}{n}|} \begin{pmatrix} \cos\theta \cos t \\ \sin\theta \cos t \\ \sin t \end{pmatrix}$$

$$\zeta_-^\theta(t) = \sum_{j=1}^{n-1} \frac{1}{4|\sin \frac{\pi j}{n}|} \begin{pmatrix} \cos\theta \cos t \\ \sin\theta \cos t \\ -\sin t \end{pmatrix}.$$

The set of minimizers is the disjoint union of two sets R_+ and R_- where

$$R_+ = \left\{ (\zeta_+^\theta(t), \zeta_+^\theta(t - 2\pi/n), \dots, \zeta_+^\theta(t - 2\pi(n-1)/n)) \in \Lambda^{D_n} \mid \theta \in \mathbb{R} \right\}$$

$$R_- = \left\{ (\zeta_-^\theta(t), \zeta_-^\theta(t - 2\pi/n), \dots, \zeta_-^\theta(t - 2\pi(n-1)/n)) \in \Lambda^{D_n} \mid \theta \in \mathbb{R} \right\}.$$

Each of R_+ and R_- is topologically equivalent to S^1 by mappings

$$\theta \mapsto (\zeta_+^\theta(t), \zeta_+^\theta(t - 2\pi/n), \dots, \zeta_+^\theta(t - 2\pi(n-1)/n))$$

$$\theta \mapsto (\zeta_-^\theta(t), \zeta_-^\theta(t - 2\pi/n), \dots, \zeta_-^\theta(t - 2\pi(n-1)/n)).$$

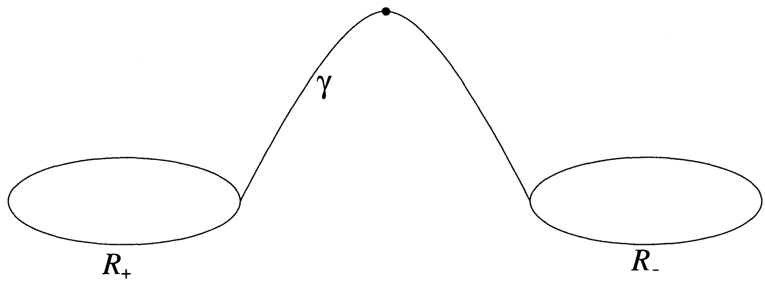

Fig. 2. Mountain pass point

§3. Mountain pass solution

Let

$$\Gamma = \{\gamma \in C([0,1], \Lambda^{D_n}) \mid \gamma(0) \in R_-, \gamma(1) \in R_+\}$$

and let

$$d = \inf_{\gamma \in \Gamma} \max_{q \in \gamma([0,1])} \mathcal{A}(q).$$

where $C(X, Y)$ stands for the set of continuous maps from X to Y. See Fig 2.

Theorem 3. *In the n-body problem with equal masses, there is a periodic solution which attains d. The solution has binary collision at most once. If $n \geq 4$ is even, this solution is not circular solution.*

Outline of the proof. Since the action functional does not satisfy Palais–Smale condition, we first consider an action functional added a strong force part:

$$\mathcal{A}^{\varepsilon}(q) = \mathcal{A}(q) + \varepsilon \int_0^T \sum_{i<j} \frac{m_i m_j}{|q_i - q_j|^2}$$

where $\varepsilon > 0$. The modified action functional $\mathcal{A}^{\varepsilon}$ satisfies Palais–Smale condition.

Applying the mountain pass theorem on the set of curves connecting these two components, it turns out that there is a mountain pass solution q^{ε}. The Morse index of the mountain pass solution q^{ε} is no more than 1. We obtain a periodic solution q^0 as a limit of a convergence subsequence q^{ε_k} of q^{ε} as ε converses to $+0$.

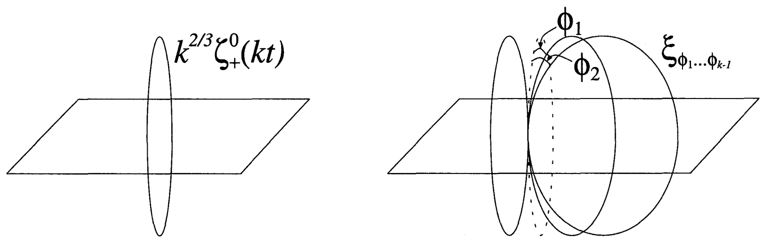

Fig. 3. $k^{2/3}\zeta_+^0(kt)$ and $\xi_{\phi_1,\dots,\phi_{k-1}}(t)$

Tanaka [4], [5] studied the behavior of a solution with a collision in the Kepler-like problem and proved that the limit of the Morse index as k diverges to ∞ is no less than the number of collisions. The theory can be applied to our setting. Therefor the number of collision is at most one.

Moreover we need prove that the obtained solution is not trivial. The other trivial solutions are the rotating regular n-gon which rotates k times per the period T:

$$k^{2/3}\zeta_+^0(kt)$$

where n and k must be relatively prime. Let us consider the modification as follows:

$$\xi_{\phi_1,\dots,\phi_{k-1}}(t) = \begin{cases} k^{2/3}P_{\phi_1}(\zeta_+^0(kt) - \zeta_+^0(0)) + k^{2/3}\zeta_+^0(0) & (0 \le t \le \frac{2\pi}{k}) \\ k^{2/3}P_{\phi_2}(\zeta_+^0(kt) - \zeta_+^0(0)) + k^{2/3}\zeta_+^0(0) & (\frac{2\pi}{k} < t \le \frac{2\pi}{k}) \\ \vdots & \\ k^{2/3}P_{\phi_{k-1}}(\zeta_+^0(kt) - \zeta_+^0(0)) + k^{2/3}\zeta_+^0(0) & \\ \qquad\qquad (\frac{2\pi(k-2)}{k} < t \le \frac{2\pi(k-1)}{k}). & \end{cases}$$

See Fig. 3. Here

$$P_\phi = \begin{pmatrix} cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\xi_{\phi_1,\dots,\phi_{k-1}}(t)$ for $t \in [\frac{2\pi(k-1)}{k}, 2\pi]$ is defined such that the center of masses is at zero. There is $k-1$ dimension space on which the second variation is negative definite. This means that the Morse index is greater than $k-1$. If n is even, k is greater than 2. Hence the mountain pass solution is non-trivial solution. Q.E.D.

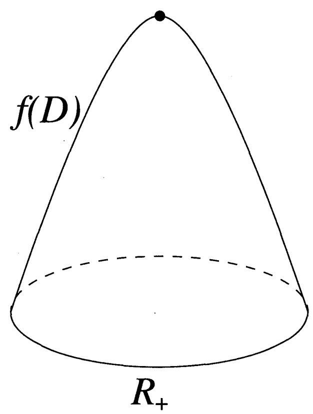

Fig. 4. Minimax point

§4. Another minimax solution

Let

$$\Omega = \{f \in C(D, \Lambda^{D_n}) \mid f|_{\partial D} \in \mathrm{Homeo}(\partial D, R_+)\}$$

and let

$$c = \inf_{f \in \Omega} \max_{q \in f(D)} \mathcal{A}(q).$$

Here D is the 2-dimensional disc (Fig. 4).

Theorem 4. *In the n-body problem with equal masses there is a periodic solution which attains c. The solution has a collision at most twice. If n is a multiple of 6, this solution is not circular solution.*

Outline of the proof. The proof is similar as one of the previous theorem. The existence follows from the minimax theorem for the action functional added a strong force. The limit of the Morse index is no more than 2. Hence the number of collisions is at most two. If n is a multiple of 6, the Morse index of the other circular solution is greater than 2. Hence the obtained solution is non-trivial solution.

Q.E.D.

Acknowledgements. The author is supported in part by the Sumitomo Foundation, Grant for Basic Science Research Projects No. 111153 and by the Japan Society for the Promotion of Science (JSPS), Grant-in-Aid for Young Scientists (B) No. 40467444.

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