Advanced Studies in Pure Mathematics 64, 2015
Nonlinear Dynamics in Partial Differential Equations
pp. 165–173

Stable patterns and Morse index one solutions

Yasuhito Miyamoto

Abstract.

We survey results on shapes of the stable steady states of two nonlinear problems: a variational problem with a mass constraint and the shadow system of activator-inhibitor type. We see that the stable steady states of the two problems are the Morse index one solutions of a scalar reaction-diffusion equation. We study shapes of the Morse index one solutions and see that the shapes of the Morse index one solutions are deeply related to the "hot spots" conjecture of J. Rauch. We also survey results on the "hot spots" conjecture and related problems.

§1. Introduction

In this article we survey results on

(I) shapes of the stable patterns of two nonlinear problems, which are (1) and (2) below, in a convex domain and

(II) shapes of the second eigenfunctions of the Neumann Laplacian.

In Sections 2, 3, and 4 we study (I). In Section 2 we study shapes of the local minimizers of the variational problem with a mass constraint

$$I(u) := \int_\Omega \left(\frac{|\nabla u|^2}{2} - F(u) \right) dx, \qquad m = \int_\Omega u dx, \tag{1}$$

where Ω is a convex domain. In Section 3 we study shapes of the stable steady states of the shadow system of activator-inhibitor type

$$\begin{gathered} u_t = \Delta u + f(u,\xi) \ \text{ in } \ \Omega, \qquad \tau\xi_t = \frac{1}{|\Omega|}\int_\Omega g(u,\xi)dx \ \text{ in } \ \Omega, \\ \partial_\nu u = 0 \ \text{ on } \ \partial\Omega, \end{gathered} \tag{2}$$

Received January 11, 2012.
Revised January 28, 2013.
2010 *Mathematics Subject Classification.* 35K50, 35J20, 35B35.
Key words and phrases. Shadow system, stability analysis, hot spots, second eigenvalue.

where $u = u(x,t)$, $\xi = \xi(t)$, and $\tau > 0$. We show that if a nonconstant critical point of (1) is a local minimizer, then u is a Morse index one solution of the Euler–Lagrange equation

$$\Delta u + f(u) = c \ \text{ in } \ \Omega, \qquad \partial_\nu u = 0 \ \text{ on } \ \partial\Omega, \tag{3}$$

where $f(u) := F'(u)$ and c is the Lagrange multiplier. Since c is constant and f is a general homogeneous nonlinear term, (3) is essentially equivalent to

$$\Delta u + f(u) = 0 \ \text{ in } \ \Omega, \qquad \partial_\nu u = 0 \ \text{ on } \ \partial\Omega. \tag{4}$$

Here the Morse index is the number of the strictly positive eigenvalues. We also show that if a nonconstant steady state (u, ξ) of (2) is stable, then u is a Morse index one solution of the stationary problem of the first equation of (2) with fixed ξ. Thus, it is important to study shapes of the Morse index one solutions of the scalar reaction-diffusion equation (4). In Section 4 we study those shapes in the case where the domain is a disk or rectangle. In Section 5 we study (II). Since the study of those shapes in a general convex domain is difficult, we study shapes of the second eigenfunctions of the Neumann Laplacian which is a Morse index one solution of the simplest reaction-diffusion equation $\Delta u + \mu u = 0$. In particular, we study the locations of the local maximum points of the second eigenfunctions which we call "hot spots".

§2. Variational problem

Let $\varepsilon > 0$ be small. The shape of a global minimizer of the variational problem with a mass constraint

$$J(u) = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + \frac{(u^2-1)^2}{4} \right) dx, \qquad m = \int_\Omega u dx \tag{5}$$

is well understood. Let u_ε be a minimizing sequence of (5). Modica [20] and Sternberg [25] have shown that the limit of minimizers u_ε as $\varepsilon \downarrow 0$ is a function with values ± 1 almost everywhere and that the interface minimizes the area under the constraint that the ratio of $|\{u_\varepsilon \approx 1\}|$ and $|\{u_\varepsilon \approx -1\}|$ is a certain value. Luckhaus–Modica [11] have shown that the area of minimizing interface is a hypersurface with constant mean curvature. (Let Ω be a domain in $\mathbb{R}^N$. The theory of minimal surfaces ensures the following: If $N \leq 7$, then the interface is smooth, and if $N \geq 8$, then the interface may have singularities, however the Hausdorff dimension of the set of the singularities is at most $N-8$.) Sternberg–Zumbrun [26] studied (5) with a general double-well potential. They have shown that, if Ω is strictly convex, then, for some $k \geq 1$, the interface $\{a_\varepsilon + \varepsilon^k < u_\varepsilon < b_\varepsilon - \varepsilon^k\}$, the superlevel set $\{u_\varepsilon > a_\varepsilon + \varepsilon^k\}$, and the sublevel set $\{u_\varepsilon < b_\varepsilon - \varepsilon^k\}$ are connected, where a_ε and b_ε $(a_\varepsilon < b_\varepsilon)$ go to two stable zeros of the derivative of the potential term as $\varepsilon \downarrow 0$. In [26] the connectivity of the interface and the boundary was also shown.

When $\varepsilon > 0$ is not necessarily small, Gurtin–Matano [7] studied the shape of the local (and global) minimizers in the case where Ω is a disk, annulus, or cylinder. In [7] they have shown that when Ω is a disk, each global minimizer is monotone in some direction.

We study (1). In particular, we do not assume the smallness of the diffusion coefficient or the double-well potential term.

The next lemma is an abstract instability criterion.

Lemma 1. *Let u be a critical point of* (1). *If the second eigenvalue of the eigenvalue problem*

$$\Delta\phi + f'(u)\phi = \mu\phi \quad in \quad \Omega, \qquad \partial_\nu\phi = 0 \quad on \quad \partial\Omega \tag{6}$$

is positive, then u is not a local minimizer.

Proof. This lemma is well known. See [18, Lemma 2.2] for example. Q.E.D.

The contrapositive of the lemma is a necessary condition for a local minimizer.

Corollary 2. *Let u be a local minimizer of* (1). *Then the Morse index of u with respect to* (6) *is zero or one.*

It is well known that if the domain is convex, then every Morse index zero solution is constant (See [6], [12]). Thus, if a local minimizer is nonconstant, then the Morse index should be one provided that the domain is convex.

§3. Shadow system of the activator-inhibitor type

In this section we consider (2). We impose the following assumptions:

$$\begin{gathered} f_\xi(u,\xi) < 0, \ g_\xi(u,\xi) < 0, \ \text{and} \\ \text{there is } k(\xi) \text{ such that } g_u(u,\xi) = k(\xi) f_\xi(u,\xi). \end{gathered} \tag{7}$$

We call (2) with (7) the shadow activator-inhibitor system. The assumption (7) appeared in [13] and this system was studied in [13], [14], [15], [16]. The first and second assumptions are natural, since they are included in a certain definition of the activator-inhibitor system (See

[14]). The third assumption is technical. However, a special case of the shadow Gierer–Meinhardt system and the shadow system with the FitzHugh–Nagumo type nonlinearity are included in (7).

The next lemma is an abstract instability criterion.

Lemma 3. *Let (u,ξ) be a steady state of* (2) *with* (7)*. If the second eigenvalue of the eigenvalue problem*

$$(8) \qquad \Delta\phi + f_u(u,\xi)\phi = \mu\phi \quad in \quad \Omega, \qquad \partial_\nu \phi = 0 \quad on \quad \partial\Omega$$

is positive, then (u,ξ) is unstable.

Proof. See [16, Lemma 2.3] or [13, Lemma 3,2]. Q.E.D.

The contrapositive of the lemma is a necessary condition for a stable steady state.

Corollary 4. *Let (u,ξ) be a steady state. If (u,ξ) is stable for some $\tau > 0$, then the Morse index of u with respect to the eigenvalue problem (8) is zero or one.*

Since ξ is constant, $f(\,\cdot\,,\xi)$ can be seen as a general homogeneous nonlinear term of one variable, i.e., $f(u,\xi)$ is essentially equivalent to $f(u)$. Hence, (8) is equivalent to (6).

The two problems (1) and (2) have the same structure. Shapes of the Morse index one solutions of (4) become important.

§4. Shapes of the Morse index one solutions

In Sections 2 and 3 we saw that the study of shapes of the local minimizers of (1) and the stable steady states of (2) can be reduced to the study of shapes of the Morse index one solutions of (4). When the domain is an interval, the nonconstant Morse index one solution is monotone. This fact was used in the study of the stable patterns of (2) by [21], [22]. However, in the high-dimensional case it is difficult to characterize shapes of the Morse index one solution. Therefore, we study cases of simple domains, e.g., a disk and rectangle. In [13], [14], [18] the author studied the case where the domain is a disk.

Theorem 5 ([18, Theorem A])**.** *Let Ω be a disk D, and let u be a nonconstant solution of* (4)*. If the Morse index of u is one, then u satisfies the following* (a) *and* (b):

(a) *u has exactly two critical points in $\overline{D}$ and those are on ∂D. In particular, u attains its maximum and minimum at those two points and there is no critical point in D.*

(b) *For every $c \in (\min_{x\in\overline{D}} u(x), \max_{x\in\overline{D}} u(x))$, the c-level set of u is* *a unique C^1-curve whose edges hit ∂D at two different points and it divides D into exactly two simply connected regions.*

In [15] the case of a rectangle is studied.

Theorem 6 ([15, Lemma 3.2]). *Let Ω be a rectangle $R := [0,a] \times [0,b]$, and let u be a nonconstant solution of* (4). *If the Morse index of u is one, then u satisfies the following* (a) *or* (b):
(a) *$u(x,y)$ is strictly monotone in both x and y.*
(b) *$u(x,y)$ is strictly monotone in x and constant in y (or constant in x and monotone in y).*

In both disk and rectangle cases u has no critical point inside the domain. Yanagida [27] posed the following conjecture:

Conjecture 7. *Let Ω be a convex domain, and let u be a nonconstant solution of* (4). *If u has a critical point inside Ω, then the Morse index is two or larger.*

Theorems 5 and 6 are partial positive answers to Conjecture 7. He pointed out that Conjecture 7 is a nonlinear version of the "hot spots" conjecture of J. Rauch:

Conjecture 8 ([24]). *Let Ω be a convex domain. Then every second Neumann eigenfunction on Ω attains its maximum only on the boundary $\partial\Omega$.*

We set $f(u) = \mu_1 u$, where μ_1 is the second Neumann eigenvalue. Then Conjecture 8 immediately follows from the contrapositive of Conjecture 7.

§5. Second eigenfunctions of the Neumann Laplacian

When the domain is not convex, it was believed that a counter-example to Conjecture 8 existed (See [9, p. 56]). Burdzy–Werner [5] gave the first counter-example in 1999. Their domain is a planar domain with three holes. Several counter-examples were later given by [2], [3], [4]. These domains have hole(s). However, a counter-example without hole is not known.

On the other hand, there are partial positive answers. Kawohl [9] proved Conjecture 8 for the domain of the type $\Omega = D \times (0,1)$. Bañuelos–Burdzy [3] and Jerison–Nadirashivili [8] proved Conjecture 8 for planar convex domains with two axes of symmetry. ([3] imposed an additional assumption which was removed by [8].) Pascu [23] proved Conjecture 8 if the domain is a planar convex domain with one axis of symmetry and if the second eigenfunction is anti-symmetric. Using results of [10], the

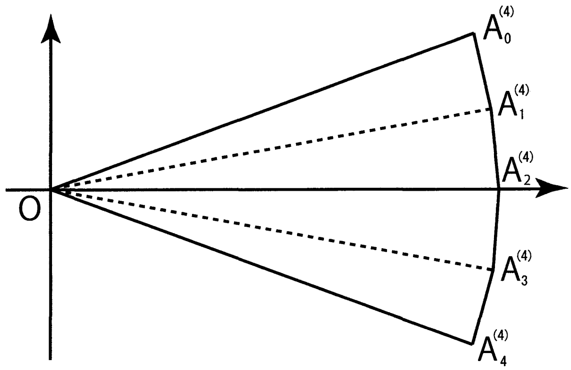

Fig. 1. The shape of $\Omega_{4,\theta}$ given in Theorem 9. The second eigenfunction attains its maximum at five vertices $A_0^{(4)}, \ldots, A_4^{(4)}$.

author [19] showed that the conjecture holds for the isosceles triangles. In [3], the conjecture was proved for the obtuse triangles. See [1], [17] for other domains without symmetry. Conjecture 8 remains open even for a general triangle.

Our ultimate goal is to characterize the shapes of the Morse index one solutions of (4). Even if Conjecture 7 were proved, we do not obtain the information of the shapes of the solutions on the boundary. Hereafter we study shapes of the Morse index one solutions on the boundary. In particular, we consider the number of the critical points on the boundary. Since the nonlinear problem (4) is difficult, we study the Morse index one solution of the linear problem

$$\Delta u + \mu_1 u = 0 \text{ in } \Omega, \qquad \partial_\nu \phi = 0 \text{ on } \partial\Omega$$

which is the second Neumann eigenfunction. When the domain is a polygon, the second Neumann eigenfunction has a critical point at each corner, hence the number of the critical points on the boundary can be large even if the domain is convex. Therefore, we consider the number of the local maximum points on the boundary. When the domain is a disk, Theorem 5 indicates that only a single boundary peak solution can be stable. However, the next theorem says that many hot spots can exist on the boundary even if the domain is convex.

Theorem 9 ([19, Theorem A]). *Let $\theta > 0$ be small. Let O be the origin of $\mathbb{R}^2$, and let $A_k^{(n)} = (\cos(\frac{n-2k}{2}\theta), \sin(\frac{n-2k}{2}\theta))$. Let $\Omega_{n,\theta}$ denote* *the convex polygon* $OA_0^{(n)}A_1^{(n)}\cdots A_n^{(n)}$. *For each integer* $n \geq 1$, *there is a small* $\theta > 0$ *such that* $\mu_1(\Omega_{n,\theta})$ *is simple, the associated eigenfunction attains its local and global maximum at* $A_0^{(n)}, \ldots, A_n^{(n)}$, *and it does not have an interior critical point. In particular, the eigenfunction has exactly* $n+1$ *isolated local and global maximum points on the boundary. See Fig.* 1 *for the case* $n = 4$.

This theorem indicates that the number of the isolated local maximum points on the boundary can be arbitrary large, hence it is impossible to characterize the shapes of the Morse index one solutions with the number of the local maximum points on the boundary. We have to find other properties in order to characterize the shapes of the Morse index one solutions.

The relation between the location of each hot spot and the curvature of the boundary is not clear from our study. This relation is a key to further research.

References

[1] R. Atar and K. Burdzy, On Neumann eigenfunctions in lip domains, J. Amer. Math. Soc., **17** (2004), 243–265.

[2] K. Burdzy, The hot spots problem in planar domains with one hole, Duke Math. J., **129** (2005), 481–502.

[3] R. Bañuelos and K. Burdzy, On the "hot spots" conjecture of J. Rauch, J. Funct. Anal., **164** (1999), 1–33.

[4] R. Bass and K. Burdzy, Fiber Brownian motion and the "hot spots" problem, Duke Math. J., **105** (2000), 25–58.

[5] K. Burdzy and W. Werner, A counterexample to the "hot spots" conjecture, Ann. of Math. (2), **149** (1999), 309–317.

[6] R. Casten and C. Holland, Instability results for reaction diffusion equations with Neumann boundary conditions, J. Differential Equations, **27** (1978), 266–273.

[7] M. Gurtin and H. Matano, On the structure of equilibrium phase transitions within the gradient theory of fluids, Quart. Appl. Math., **46** (1988), 301–317.

[8] D. Jerison and N. Nadirashvili, The "hot spots" conjecture for domains with two axes of symmetry, J. Amer. Math. Soc., **13** (2000), 741–772.

[9] B. Kawohl, Rearrangements and convexity of level sets in PDE, Lecture Notes in Math., **1150**, Springer-Verlag, 1985.

[10] R. Laugesen and B. Siudeja, Minimizing Neumann fundamental tones of triangles: an optimal Poincaré inequality, J. Differential Equations, **249** (2010), 118–135.

[11] S. Luckhaus and L. Modica, The Gibbs–Thompson relation within the gradient theory of phase transitions, Arch. Rational Mech. Anal., **107** (1989), 71–83.

[12] H. Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations, Publ. Res. Inst. Math. Sci., **15** (1979), 401–454.

[13] Y. Miyamoto, An instability criterion for activator-inhibitor systems in a two-dimensional ball, J. Differential Equations, **229** (2006), 494–508.

[14] Y. Miyamoto, An instability criterion for activator-inhibitor systems in a two-dimensional ball, II, J. Differential Equations, **239** (2007), 61–71.

[15] Y. Miyamoto, On the shape of the stable patterns for activator-inhibitor systems in two-dimensional domains, Quart. Appl. Math., **65** (2007), 357–374.

[16] Y. Miyamoto, On stable patterns for reaction-diffusion equations and systems, Workshops on Pattern Formation Problems in Dissipative Systems and Mathematical Modeling and Analysis for Nonlinear Phenomena, RIMS Kôkyûroku Bessatsu, **B3**, Res. Inst. Math. Sci. (RIMS), Kyoto, 2007, pp. 59–82.

[17] Y. Miyamoto, The "hot spots" conjecture for a certain class of planar convex domains, J. Math. Phys., **50** (2009), 103530.

[18] Y. Miyamoto, Global bifurcation and stable two-phase separation for a phase field model in a disk, Discrete Contin. Dyn. Syst., **30** (2011), 791–806.

[19] Y. Miyamoto, A planar convex domain with many isolated "hot spots" on the boundary, Jpn. J. Ind. Appl. Math., **30** (2013), 145–164.

[20] L. Modica, The gradient theory of phase transitions and the minimal interface criterion, Arch. Rational Mech. Anal., **98** (1987), 123–142.

[21] Y. Nishiura, Coexistence of infinitely many stable solutions to reaction diffusion systems in the singular limit, Dynamics Reported, **3** (1994), 25–103.

[22] W. M. Ni, P. Poláčik, and E. Yanagida, Monotonicity of stable solutions in shadow systems, Trans. Amer. Math. Soc., **353** (2001), 5057–5069.

[23] M. Pascu, Scaling coupling of reflecting Brownian motions and the hot spots problem, Trans. Amer. Math. Soc., **354** (2002), 4681–4702.

[24] J. Rauch, Five problems: an introduction to the qualitative theory of partial differential equations, In: Partial Differential Equations and Related Topics, Tulane Univ., New Orleans, La., 1974, Lecture Notes in Math., **446**, Springer-Verlag, 1975, pp. 355–369.

[25] P. Sternberg, The effect of a singular perturbation on nonconvex variational problems, Arch. Rational Mech. Anal., **101** (1988), 209–260.

[26] P. Sternberg and K. Zumbrun, Connectivity of phase boundaries in strictly convex domains, Arch. Rational Mech. Anal., **141** (1998), 375–400.

[27] E. Yanagida, Private communation, 2006.

Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1 Hiyoshi, kohoku-ku, Yokohama
223-8522, Japan
E-mail address: `miyamoto@math.titech.ac.jp`