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A shallow water approximation for water waves over a moving bottom

Hiroyasu Fujiwara and Tatsuo Iguchi

Abstract.

It is known that the Green–Naghdi equations are higher order approximations for the water wave problem in a shallow water regime. We derive corresponding equations in the case of moving bottom as a mode of tsunamis caused by the slow deformation of the bottom, and give a mathematically rigorous justification of the model.

§1. Introduction

In this paper we are concerned with model equations for generation and propagation of tsunamis. In a standard tsunami model, the shallow water equations

$$\begin{cases} \eta_t + \nabla \cdot \big((h + \eta - b_1)u\big) = 0, \\ u_t + (u \cdot \nabla)u + g\nabla\eta = 0 \end{cases}$$

are used to simulate the propagation of tsunami under the assumption that the initial profile of tsunami is equal to the permanent shift of the seabed and the initial velocity field is zero

$$\eta = b_1 - b_0, \quad u = 0 \quad \text{at} \quad t = 0,$$

where η is the elevation of the water surface, u is the velocity field in the horizontal direction on the water surface, h is the mean depth of the water, g is the gravitational constant, b_0 is the bottom topography before the submarine earthquake, and b_1 is that after the earthquake. In fact, in [7] it was shown that the solution of the full water wave problem can be approximated by the solution of this tsunami model in the scaling regime $\delta^2 \ll \varepsilon \ll 1$ under appropriate assumptions on the initial data

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and the bottom topography. Here the nondimensional parameters δ and ε are defined by $\delta = \frac{h}{\lambda}$ and $\varepsilon = \frac{t_0}{\lambda/\sqrt{gh}}$, where λ is a typical wave length and t_0 is the time when the submarine earthquake takes place.

However, very rarely, the condition $\delta^2 \ll \varepsilon \ll 1$ is not satisfied, particularly, the condition on ε. One of such events is the Meiji-Sanriku earthquake, which occurred at June 15 in 1896. The seismic scale of this earthquake was small, but it continued for several minutes. As a result, a huge tsunami attacked the Sanriku coast line. To simulate such a tsunami, it might be better to consider the limit $\delta \to 0$ keeping ε is of order one. In this paper we will consider this kind of tsunamis, so that in the following we always assume that $\varepsilon = 1$. In this case, the standard tsunami model should be replaced by

$$\begin{cases} \eta_t + \nabla \cdot \big((h + \eta - b)u\big) = b_t, \\ u_t + (u \cdot \nabla)u + g\nabla\eta = 0 \end{cases}$$

with zero initial conditions, where b is the bottom topography. In fact, using the techniques in [7] we can easily show that the solution of the full water wave problem can be approximated by the solution of the above tsunami model with a source term in the scaling regime $\delta \ll 1$ and $\varepsilon = 1$. Therefore, in this paper we will consider a higher order approximation.

It was shown by Li [10] that the solution of the two-dimensional water waves over a flat bottom can be approximated by the solution of the so-called Green–Naghdi equations up to order $O(\delta^4)$. Alvarez-Samaniego and Lannes [2] extended her result to the three-dimensional water waves over a nonflat bottom by using the Nash–Moser technique to show the existence of solution, so that they imposed much regularity of the initial data. In this paper, we extend the result to the case of moving bottom without using the Nash–Moser technique. Therefore, in our result the regularity assumption on the initial data is much weaker than those in [2].

§2. Formulation of the problem

Under the assumption $\varepsilon = 1$, the basic equations for water waves in the nondimensional form have the form

$$\delta^2 \Delta\Phi + \partial_3^2 \Phi = 0 \quad \text{in} \quad \Omega(t),\ t > 0, \tag{1}$$

$$\begin{cases} \delta^2\big(\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + \eta\big) + \frac{1}{2}(\partial_3\Phi)^2 = 0, \\ \delta^2(\eta_t + \nabla\Phi \cdot \nabla\eta) - \partial_3\Phi = 0 \quad \text{on} \quad \Gamma(t),\ t > 0, \end{cases} \tag{2}$$

$$\delta^2(b_t + \nabla\Phi\cdot\nabla b) - \partial_3\Phi = 0 \quad \text{on} \quad \Sigma(t),\ t>0, \tag{3}$$

$$\eta(x,0) = \eta_0(x), \quad \Phi(X,0) = \Phi_0(X), \tag{4}$$

where

$$\begin{aligned}
\Omega(t) &= \{X = (x, x_3) \in \mathbf{R}^3\,;\, b(x,t) < x_3 < 1 + \eta(x,t)\},\\
\Gamma(t) &= \{X = (x, x_3) \in \mathbf{R}^3\,;\, x_3 = 1 + \eta(x,t)\},\\
\Sigma(t) &= \{X = (x, x_3) \in \mathbf{R}^3\,;\, x_3 = b(x,t)\}.
\end{aligned}$$

Here, Φ is the velocity potential and η is the surface elevation. Both of them are unknown functions, while b is a given function in this paper. $x = (x_1, x_2)$ and x_3 are the horizontal and vertical spatial variables, respectively, $\nabla = (\partial_1, \partial_2)$, and $\Delta = \partial_1^2 + \partial_2^2$.

We reformulate the initial value problem (1)–(4) to a problem on the water surface. To this end, we introduce a new unknown function ϕ as the trace of the velocity on the water surface, that is,

$$\phi(x,t) = \Phi(x, 1 + \eta(x,t), t).$$

Then, we have

$$\begin{cases}
\eta_t - \Lambda^{\mathrm{DN}}(\eta, b, \delta)\phi - \Lambda^{\mathrm{NN}}(\eta, b, \delta) b_t = 0,\\
\phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}\\
\times\big(\Lambda^{\mathrm{DN}}(\eta, b, \delta)\phi + \Lambda^{\mathrm{NN}}(\eta, b, \delta) b_t + \nabla\phi\cdot\nabla\eta\big)^2 = 0 \ \text{for} \ t>0,
\end{cases} \tag{5}$$

$$\eta = \eta_0, \quad \phi = \phi_0 \quad \text{at} \quad t = 0, \tag{6}$$

where Λ^{DN} and Λ^{NN} are Dirichlet–Neumann and Neumann–Neumann maps, respectively, and $\phi_0 = \Phi_0(\cdot, 1 + \eta_0(\cdot))$. We refer to [7] for the derivation of these equations. The following theorem is one of the main results in this paper and asserts the existence of the solution of (5) and (6) with uniform bounds of the solution on a time interval independent of small $\delta > 0$.

Theorem 1. *Let $s > 3$ and $M_0, c_0 > 0$. Then, there exist a time $T > 0$ and constants $C_0, \delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, $\eta_0 \in H^{s+7/2}$, $\nabla\phi_0 \in H^{s+3}$, and $b \in C([0,T]; H^{s+4})$ satisfying*

$$\begin{cases}
\|b(t)\|_{s+4} + \|b_t(t)\|_{s+3} + \|b_{tt}(t)\|_{s+1} + \|b_{ttt}(t)\|_s \le M_0,\\
\|\eta_0\|_{s+7/2} + \|\nabla\phi_0\|_{s+3} \le M_0,\\
1 + \eta_0(x) - b_0(x) \ge c_0 \quad \text{for} \quad (x,t) \in \mathbf{R}^2 \times [0,T],
\end{cases}$$

the initial value problem (5) *and* (6) *has a unique solution* $(\eta, \phi) = (\eta^\delta, \phi^\delta)$ *on the time interval* [0,T] *satisfying*

$$\begin{cases} \|\eta^\delta(t)\|_{s+3} + \|\nabla\phi^\delta(t)\|_{s+2} + \|(\eta_t^\delta(t), \phi_t^\delta(t))\|_{s+2} \le C_0, \\ 1 + \eta^\delta(x,t) - b(x,t) \ge c_0/2 \quad \text{for} \quad (x,t) \in \mathbf{R}^2 \times [0,T], \quad \delta \in (0, \delta_0]. \end{cases}$$

§3. Shallow water approximations

The following proposition was obtained by Alvarez-Samaniego and Lannes [2].

Proposition 1. *Let* $s > 1$ *and* $M, c_1 > 0$. *Suppose that*

$$\begin{cases} \|\eta\|_{s+9/2} + \|b\|_{s+11/2} \le M, \\ 1 + \eta(x) - b(x) \ge c_1 \quad \text{for} \quad x \in \mathbf{R}^2. \end{cases}$$

Then, there exists a constant $C = C(M, c_1, s) > 0$ *independent of* δ *such that for any* $\delta \in (0,1]$ *we have*

$$\begin{aligned} &\big\|\Lambda^{\mathrm{DN}}(\eta, b, \delta)\phi + \nabla\cdot\big((1+\eta-b)\nabla\phi\big) + \delta^2\Delta\big(\tfrac{1}{3}(1+\eta-b)^3\Delta\phi\big) \\ &\quad - \delta^2\Delta\big(\tfrac{1}{2}(1+\eta-b)^2\nabla b\cdot\nabla\phi\big) + \delta^2\nabla\cdot\big(\tfrac{1}{2}(1+\eta-b)^2(\nabla b)\Delta\phi\big) \\ &\quad - \delta^2\nabla\cdot\big((1+\eta-b)(\nabla b)\nabla b\cdot\nabla\phi\big)\big\|_s \le C\delta^4\|\nabla\phi\|_{s+11/2}. \end{aligned}$$

The following proposition was obtained in [7].

Proposition 2. *Let* $s > 1$ *and* $M, c_1 > 0$. *Suppose that*

$$\begin{cases} \|\eta\|_{s+9/2} + \|b\|_{s+11/2} \le M, \\ 1 + \eta(x) - b(x) \ge c_1 \quad \text{for} \quad x \in \mathbf{R}^2. \end{cases}$$

Then, there exist constants $C = C(M, c_1, s) > 0$ *and* $\delta_0 = \delta_0(M, c_1, s) > 0$ *such that for any* $\delta \in (0, \delta_0]$ *we have*

$$\begin{aligned} &\big\|\Lambda^{\mathrm{NN}}(\eta, b, \delta)b_t - b_t \\ &\quad - \delta^2\nabla\cdot\big\{(1+\eta-b)\big(b_t\nabla\eta + \tfrac{1}{2}(1+\eta-b)\nabla b_t\big)\big\}\big\|_s \le C\delta^4\|b_t\|_{s+4}. \end{aligned}$$

Using these two propositions, we can approximate the equations (5) by the following partial differential equations up to order $O(\delta^4)$.

$$(7)\quad \begin{cases} \eta_t - b_t + \nabla\cdot\big((1+\eta-b)\nabla\phi\big) + \delta^2\Delta\big(\tfrac{1}{3}(1+\eta-b)^3\Delta\phi\big) \\ \quad -\delta^2\Delta\big(\tfrac{1}{2}(1+\eta-b)^2\nabla b\cdot\nabla\phi\big) + \delta^2\nabla\cdot\big(\tfrac{1}{2}(1+\eta-b)^2\nabla b\Delta\phi\big) \\ \quad -\delta^2\nabla\cdot\big((1+\eta-b)\nabla b(\nabla b\cdot\nabla\phi)\big) \\ \quad -\delta^2\nabla\cdot\big\{(1+\eta-b)\big(b_t\nabla\eta + \tfrac{1}{2}(1+\eta-b)\nabla b_t\big)\big\} = O(\delta^4), \\ \phi_t + \eta + \tfrac{1}{2}|\nabla\phi|^2 \\ \quad -\tfrac{1}{2}\delta^2\big(\nabla b\cdot\nabla\phi - (1+\eta-b)\Delta\phi + b_t\big)^2 = O(\delta^4). \end{cases}$$

Here, we define a second order differential operator $T(\eta, b)$ depending on η and b and acting on vector fields by

$$\begin{aligned} T(\eta,b)u := &-\nabla\big(\tfrac{1}{3}(1+\eta-b)^3(\nabla\cdot u)\big)+\nabla\big(\tfrac{1}{2}(1+\eta-b)^2(\nabla b\cdot u)\big)\\ &-\tfrac{1}{2}(1+\eta-b)^2\nabla b(\nabla\cdot u)+(1+\eta-b)\nabla b(\nabla b\cdot u) \end{aligned}$$

and introduce a new variable u by

$$(8)\quad \nabla\phi = u+\delta^2(1+\eta-b)^{-1}T(\eta,b)u+\delta^2\big(b_t\nabla\eta+\tfrac{1}{2}(1+\eta-b)\nabla b_t\big).$$

Putting this into (7) and neglecting the terms of order $O(\delta^4)$, we obtain the Green–Naghdi equations

$$(9)\qquad \begin{cases} \eta_t+\nabla\cdot\big((1+\eta-b)u\big)=b_t,\\ \big((1+\eta-b)+\delta^2T(\eta,b)\big)u_t+(1+\eta-b)\big(\nabla\eta+(u\cdot\nabla)u\big)\\ \qquad +\delta^2\big\{\tfrac{1}{3}\nabla\big((1+\eta-b)^3P_u(\nabla\cdot u)\big)+Q(\eta,u,b)\\ \qquad\qquad +R_1(\eta,u,b)b_t+R_2(\eta,b)b_{tt}\big\}=0 \quad\text{for}\quad t>0, \end{cases}$$

$$(10)\qquad \eta=\eta_0,\quad u=u_0\quad\text{at}\quad t=0,$$

where

$$\begin{aligned} &P_u=\nabla\cdot u-u\cdot\nabla,\\ &Q(\eta,u,b)=\tfrac{1}{2}\nabla\big((1+\eta-b)^2(u\cdot\nabla)^2b\big)+\tfrac{1}{2}\big((1+\eta-b)^2P_u(\nabla\cdot u)\big)\nabla b\\ &\qquad\qquad +(1+\eta-b)\big((u\cdot\nabla)^2b\big)\nabla b,\\ &R_1(\eta,u,b)b_t=(1+\eta-b)^2\nabla(u\cdot\nabla b_t)+2(1+\eta-b)(u\cdot\nabla b_t)\nabla\eta,\\ &R_2(\eta,b)b_{tt}=\tfrac{1}{2}(1+\eta-b)^2\nabla b_{tt}+(1+\eta-b)b_{tt}\nabla\eta, \end{aligned}$$

and u_0 is determined by (8) from (η_0, b_0).

Now, we are ready to give the main result in this paper, which asserts the rigorous justification of the Green–Naghdi approximation.

Theorem 2. *Let $s > 3$ and $M_0, c_0 > 0$. Then, there exist a time $T > 0$ and constants $C, \delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, $\eta_0 \in H^{s+15/2}$, $\nabla\phi_0 \in H^{s+7}$, and $b \in C([0,T]; H^{s+8})$ satisfying*

$$\begin{cases} \|b(t)\|_{s+8}+\|b_t(t)\|_{s+7}+\|b_{tt}(t)\|_{s+5}+\|b_{ttt}(t)\|_{s+4}\le M_0,\\ \|\eta_0\|_{s+15/2}+\|\nabla\phi_0\|_{s+7}\le M_0,\\ 1+\eta_0(x)-b_0(x)\ge c_0 \quad\text{for}\quad (x,t)\in\mathbf{R}^2\times[0,T], \end{cases}$$

the solution $(\eta,\phi)=(\eta^\delta,\phi^\delta)$ obtained in Theorem 1 and the function u^δ determined by (8) from $(\eta^\delta,\phi^\delta)$ and b satisfy

$$\|\eta^\delta(t)-\tilde\eta^\delta(t)\|_s+\|u^\delta(t)-\tilde u^\delta(t)\|_s+\delta\|\nabla\cdot(u^\delta(t)-\tilde u^\delta(t))\|_s\le C\delta^4$$

for $0 \le t \le T$, *where* $(\eta, u) = (\tilde{\eta}^\delta, \tilde{u}^\delta)$ *is a unique solution of the initial value problem for the Green–Naghdi equations* (9) *and* (10).

§4. The Green–Naghdi equations

We define a second order differential operator $L(\eta, b, \delta)$ by

$$L(\eta, b, \delta)u := \big((1+\eta-b) + \delta^2 T(\eta, b)\big)u$$

and consider the partial differential equation

$$L(\eta, b, \delta)u = F + \delta a \nabla f. \tag{11}$$

Lemma 1. *Let* $s > 2$ *and* $M, c_1 > 0$. *Suppose that*

$$\begin{cases} \|\eta\|_s + \|b\|_{s+1} + \|a\|_s \le M, \\ 1 + \eta(x) - b(x) \ge c_1 \quad \text{for} \quad x \in \mathbf{R}^2. \end{cases}$$

Then, for any $F, f \in H^s$ *and* $\delta \in (0, 1]$, *equation* (11) *has a unique solution* $u \in H^s$ *satisfying* $\nabla \cdot u \in H^s$. *Moreover, we have*

$$\|u\|_s + \delta \|\nabla \cdot u\|_s \le C(\|F\|_s + \|f\|_s),$$

where $C = C(M, c_1, s) > 0$ *is independent of* δ.

Proof. For any $u, \phi \in H := \{u \in L^2; \nabla \cdot u \in L^2\}$ we have

$$\begin{aligned} &(Lu, \phi) \\ &= ((1+\eta-b)u, \phi) + \tfrac{\delta^2}{3}((1+\eta-b)^3(\nabla \cdot u), \nabla \cdot \phi) \\ &\quad - \tfrac{\delta^2}{2}((1+\eta-b)^2(\nabla b \cdot u), \nabla \cdot \phi) - \tfrac{\delta^2}{2}((1+\eta-b)^2(\nabla \cdot u), \nabla b \cdot \phi) \\ &\quad + \delta^2((1+\eta-b)(\nabla b \cdot u), \nabla b \cdot \phi) \\ &=: (u, \phi)_H. \end{aligned}$$

By the hypothesis we have

$$C^{-1}(\|u\|^2 + \delta^2\|\nabla \cdot u\|^2) \le (Lu, u) \le C(\|u\|^2 + \delta^2\|\nabla \cdot u\|^2). \tag{12}$$

Therefore, H is the Hilbert space with the inner product $(\cdot, \cdot)_H$, so that Riesz's representation theorem implies the existence of a unique solution $u \in H$ to the equation

$$(u, \phi)_H = (F + \delta a \nabla f, \phi).$$

This solution u is a unique weak solution of (11). Moreover, it is easy to see that u satisfies $\|u\| + \delta\|\nabla \cdot u\| \le C(\|F\| + \|f\|)$. A higher order estimate is obtained in a standard manner. Q.E.D.

The following proposition asserts the existence of the solution to the initial value problem (9) and (10) with a uniform bound of the solution on a time interval independent of $\delta \in (0,1]$.

Proposition 3. *Let $s > 3$ and $M, c_1 > 0$. Then, there exist a time $T > 0$ and a constant $C_0 > 0$ such that for any $\delta \in (0,1]$, $\eta_0 \in H^s$, $u_0 \in H^s$, and $b \in C([0,T]; H^{s+2})$ satisfying*

$$\begin{cases} \|\eta_0\|_s + \|u_0\|_s + \delta\|\nabla \cdot u_0\|_s \le M, \\ \|b(t)\|_{s+2} + \|b_t(t)\|_{s+2} + \|b_{tt}(t)\|_{s+1} \le M, \\ 1 + \eta_0(x) - b_0(x) \ge c_1 \quad \text{for} \quad (x,t) \in \mathbf{R}^2 \times [0,T], \end{cases}$$

the initial value problem for the Green–Naghdi equations (9) *and* (10) *has a unique solution* (η, u) *on the time interval* $[0,T]$ *satisfying*

$$\begin{cases} \|\eta(t)\|_s + \|u(t)\|_s + \delta\|\nabla \cdot u(t)\|_s \le C_0, \\ 1 + \eta(x,t) - b(x,t) \ge c_0/2 \quad \text{for} \quad (x,t) \in \mathbf{R}^2 \times [0,T]. \end{cases}$$

Proof. We first give an a priori estimate of the solution by the energy method. Let (η, u) be a solution of (9) and (10) satisfying

$$\begin{cases} \|\eta(t)\|_{s-1} \le N_1, \\ E_s(t) := \|\eta(t)\|_s^2 + \|u(t)\|_s^2 + \delta^2\|\nabla \cdot u(t)\|_s^2 \le N_2, \\ 1 + \eta(x,t) - b(x,t) \ge c_0/2 \quad \text{for} \quad (x,t) \in \mathbf{R}^2 \times [0,T], \end{cases} \tag{13}$$

where positive constants N_1, N_2, and T will be determined later. In the following, C_1 and C_2 denote positive constants depending only on (M_0, N_1, c_0, s) and (M_0, N_2, c_0, s), respectively. The energy function for (9) is defined by

$$\mathcal{E}_s(t) := \|\eta(t)\|_s^2 + (L(1+|D|)^s u, (1+|D|)^s u),$$

which is equivalent to $E_s(t)$ thanks of (12). Then, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(t) \le C_2\bigl(E_s(t) + 1\bigr).$$

This and Gronwall's inequality yield that

$$E_s(t) \le C_1 \mathrm{e}^{C_2 t}\bigl(E_s(0) + C_2 t\bigr).$$

We also have

$$1 + \eta(x,t) - b(x,t) \ge c_0 - C_2 t.$$

Thus, if we set $N_1 = \|\eta_0\|_{s-1} + 1$, $N_2 = 2C_1(\|\eta_0\|_s^2 + \|u_0\|_s^2 + \delta^2\|\nabla \cdot u_0\|_s^2 + 1)$ and $T = \min\{C_2^{-1}\log 2, (2C_2)^{-1}c_0\}$, then the estimates in (13) hold for $0 \le t \le T$.

To construct the solution, we use, for example, a parabolic regularization of the equations by

$$
(14)\quad \begin{cases} \eta_t - \varepsilon\Delta\eta + \nabla\cdot\big((1+\eta-b)u\big) = b_t, \\ \big((1+\eta-b)+\delta^2 T(\eta,b)\big)(u_t - \varepsilon\Delta u) \\ \qquad +(1+\eta-b)\big(\nabla\eta + (u\cdot\nabla)u\big) \\ \qquad +\delta^2\big\{\frac{1}{3}\nabla\big((1+\eta-b)^3 P_u(\nabla\cdot u)\big) + Q(\eta,u,b) \\ \qquad\qquad +R_1(\eta,u,b)b_t + R_2(\eta,b)b_{tt}\big\} = 0 \quad \text{for} \quad t>0. \end{cases}
$$

For each $\varepsilon \in (0,1]$ the initial value problem for the regularized Green–Naghdi equations (14) and (10) has a unique solution $(\eta^\varepsilon, u^\varepsilon)$, which satisfies a uniform bound on a time interval independent of ε. Moreover, the solution $(\eta^\varepsilon, u^\varepsilon)$ converges as $\varepsilon \to +0$. The limiting function is the desired solution. Q.E.D.

§5. Proof of the main theorem

Let $(\eta^\delta, \phi^\delta)$ be the solution of (5) and (6) obtained in Theorem 2 and define h^δ, W^δ, and u^δ by

$$
\begin{cases} h^\delta = 1+\eta^\delta - b, \quad W^\delta = b_t\nabla\eta^\delta + \frac{1}{2}h^\delta\nabla b_t, \\ L(\eta^\delta, b, \delta)u^\delta = \nabla\phi^\delta - \delta^2 h^\delta W^\delta. \end{cases}
$$

Then, we have

$$
(15)\quad \begin{cases} \eta_t^\delta + \nabla\cdot\big((1+\eta^\delta-b)u^\delta\big) = b_t + \delta^4 g_1^\delta, \\ L(\eta^\delta,b,\delta)u_t^\delta + (1+\eta^\delta-b)\big(\nabla\eta^\delta + (1+\eta^\delta-b)(u^\delta\cdot\nabla)u^\delta\big) \\ \qquad +\delta^2\big\{\frac{1}{3}\nabla\big((1+\eta^\delta-b)^3 P_{u^\delta}(\nabla\cdot u^\delta)\big) + Q(\eta^\delta,u^\delta,b) \\ \qquad\qquad +R_1(\eta^\delta,u^\delta,b)b_t + R_2(\eta^\delta,b)b_{tt}\big\} = \delta^4 g_2^\delta, \end{cases}
$$

where $g_1^\delta = \delta^{-4} g_{11}^\delta + g_{12}^\delta$,

$$
\begin{aligned} g_{11}^\delta = {} & \Lambda^{\mathrm{DN}}(\eta^\delta,b,\delta)\phi^\delta + \nabla\cdot(h^\delta\nabla\phi^\delta) + \delta^2\Delta\big(\tfrac{1}{3}(h^\delta)^3\Delta\phi^\delta\big) \\ & - \delta^2\Delta\big(\tfrac{1}{2}(h^\delta)^2\nabla b\cdot\nabla\phi^\delta\big) + \delta^2\nabla\cdot\big(\tfrac{1}{2}(h^\delta)^2(\nabla b)\Delta\phi^\delta\big) \\ & - \delta^2\nabla\cdot\big(h^\delta(\nabla b)\nabla b\cdot\nabla\phi^\delta\big) \\ & + \Lambda^{\mathrm{NN}}(\eta^\delta,b,\delta)b_t - b_t - \delta^2\nabla\cdot(h^\delta W^\delta), \end{aligned}
$$

$$
g_{12}^\delta = \nabla\cdot T^\delta(\eta^\delta,b)\big((h^\delta)^{-1}T^\delta(\eta^\delta,b)u^\delta + W^\delta\big),
$$

$$
\begin{aligned}
g_2^\delta &= g_{21}^\delta + g_{22}^\delta + g_{23}^\delta + g_{24}^\delta + g_{25}^\delta,\\
g_{21}^\delta &= h^\delta \nabla\big\{\big(-\nabla\cdot(h^\delta\nabla\phi^\delta) + b_t + \nabla\phi^\delta\cdot\nabla\eta^\delta\big)g_{211}^\delta + \tfrac{1}{2}\delta^2(g_{211}^\delta)^2\\
&\quad - \tfrac{1}{2}|\nabla\eta^\delta|^2(1+\delta^2|\nabla\eta^\delta|)^{-2}\\
&\quad \times\big(\Lambda^{\mathrm{DN}}(\eta^\delta,b,\delta)\phi^\delta + \Lambda^{\mathrm{NN}}(\eta^\delta,b,\delta)b_t + \nabla\phi^\delta\cdot\nabla\eta^\delta\big)^2\big\},\\
g_{211}^\delta &= \delta^{-2}\big\{\Lambda^{\mathrm{DN}}(\eta^\delta,b,\delta)\phi^\delta + \nabla\cdot(h^\delta\nabla\phi^\delta) + \Lambda^{\mathrm{NN}}(\eta^\delta,b,\delta)b_t - b_t\big\},\\
g_{22}^\delta &= \delta^2\big\{\nabla\big((h^\delta)^2 g_1^\delta(\nabla\cdot u^\delta) - h^\delta g_1^\delta(\nabla b\cdot u^\delta)\big) + h^\delta g_1^\delta(\nabla b)\nabla\cdot u^\delta\\
&\quad - g_1^\delta(\nabla b)\nabla b\cdot u^\delta - h^\delta b_t\nabla g_1^\delta - \tfrac{1}{2}h^\delta g_1^\delta\nabla b_t\big\},\\
g_{23}^\delta &= -\big((T^\delta(\eta^\delta,b)u^\delta + h^\delta W^\delta)\cdot\nabla\big)\big((h^\delta)^{-1}T^\delta(\eta^\delta,b)u^\delta + W^\delta\big),\\
g_{24}^\delta &= h^\delta\nabla\big\{\big(\nabla b\cdot u^\delta - h^\delta(\nabla\cdot u^\delta)\big)\big(\nabla b\cdot\big((h^\delta)^{-1}T^\delta(\eta^\delta,b)u^\delta + W^\delta\big)\\
&\quad - h^\delta\nabla\cdot\big((h^\delta)^{-1}T^\delta(\eta^\delta,b)u^\delta + W^\delta\big)\big)\big\}\\
&\quad + \tfrac{1}{2}h^\delta\delta^2\nabla\big\{\big(\nabla b\cdot\big((h^\delta)^{-1}T^\delta(\eta^\delta,b)u^\delta + W^\delta\big)\\
&\quad - h^\delta\nabla\cdot\big((h^\delta)^{-1}T^\delta(\eta^\delta,b)u^\delta + W^\delta\big)\big)^2\big\}\\
&\quad + h^\delta\nabla\big\{\big(\nabla b\cdot\big((h^\delta)^{-1}T^\delta(\eta^\delta,b)u^\delta + W^\delta\big)\\
&\quad - h^\delta\nabla\cdot\big((h^\delta)^{-1}T^\delta(\eta^\delta,b)u^\delta + W^\delta\big)\big)b_t\big\},\\
g_{25}^\delta &= \tfrac{1}{3}\delta^{-2}\big\{(u^\delta\cdot\nabla)\nabla\big((h^\delta)^3(\nabla\cdot u^\delta)\big) + \big(\nabla\big((h^\delta)^3(\nabla\cdot u^\delta)\big)\cdot\nabla\big)u^\delta\\
&\quad - \nabla\big(\nabla\big((h^\delta)^3(\nabla\cdot u^\delta)\big)\cdot u^\delta\big)\big\}\\
&\quad - \delta^{-2}h^\delta b_t\big\{(u^\delta\cdot\nabla)\nabla\eta^\delta + (\nabla\eta^\delta\cdot\nabla)u^\delta - \nabla(\nabla\eta^\delta\cdot u^\delta)\big\}\\
&\quad - \tfrac{1}{2}\delta^{-2}(h^\delta)^2\big\{(u^\delta\cdot\nabla)\nabla b_t + (\nabla b_t\cdot\nabla)u^\delta - \nabla(\nabla b_t\cdot u^\delta)\big\}\\
&\quad - \delta^{-2}h^\delta(\nabla b\cdot u^\delta)\big\{(u^\delta\cdot\nabla)\nabla b + (\nabla b\cdot\nabla)u^\delta - \nabla(\nabla b\cdot u^\delta)\big\}\\
&\quad + \tfrac{1}{2}\delta^{-2}(h^\delta)^2(\nabla\cdot u^\delta)\big\{(u^\delta\cdot\nabla)\nabla b + (\nabla b\cdot\nabla)u^\delta - \nabla(\nabla b\cdot u^\delta)\big\}\\
&\quad - \tfrac{1}{2}\delta^{-2}(\nabla b\cdot u^\delta)\big\{(u^\delta\cdot\nabla)\nabla\big((h^\delta)^2\big) + \big(\nabla\big((h^\delta)^2\big)\cdot\nabla\big)u^\delta\\
&\quad - \nabla\big(\nabla\big((h^\delta)^2\big)\cdot u^\delta\big)\big\}\\
&\quad - \tfrac{1}{2}\delta^{-2}(h^\delta)^2\big\{(u^\delta\cdot\nabla)\nabla(\nabla b\cdot u^\delta) + (\nabla(\nabla b\cdot u^\delta)\cdot\nabla)u^\delta\\
&\quad - \nabla(\nabla(\nabla b\cdot u^\delta)\cdot u^\delta)\big\}.
\end{aligned}
$$

Lemma 2. *Under the same hypothesis of Theorem 2, there exists a constant $C = C(M_0, c_0, s) > 0$ such that we have*

$$\|(g_1^\delta(t), g_2^\delta(t))\|_s \le C \quad \text{for} \quad t\in[0,T],\ \ \delta\in(0,\delta_0],$$

where T and δ_0 are the constants in Theorem 1.

Proof. By Theorem 1 we have

$$\|\eta^\delta(t)\|_{s+7} + \|\nabla\phi^\delta(t)\|_{s+6} \le C \quad \text{for} \quad t \in [0,T], \;\; \delta \in (0,\delta_0],$$

which together with Lemma 1 implies that $\|u^\delta(t)\|_{s+6} \le C$. We also note that $\|g_{25}^\delta(t)\|_s \le C\delta^{-2}\|\mathrm{rot}\, u^\delta(t)\|_s \le C$. Now, the desired estimate is obtained by Propositions 1 and 2. Q.E.D.

Let $(\tilde\eta^\delta, \tilde u^\delta)$ be the solution of (9) and (10) obtained in Proposition 3 and put

$$h^\delta = 1 + \eta^\delta - b, \quad \tilde h^\delta = 1 + \tilde\eta^\delta - b, \quad \zeta = \eta^\delta - \tilde\eta^\delta, \quad w = u^\delta - \tilde u^\delta.$$

It follows from (15) that

$$\left\{ \begin{array}{l} \zeta_t + \nabla\cdot(h^\delta w) + \nabla\cdot(\zeta \tilde u^\delta) = \delta^4 g_1, \\ \tilde L^\delta w_t + A_1 w + \nabla(A_2 w) + B\zeta = a + \delta^4 g_2, \end{array} \right. \tag{16}$$

where

$$\begin{aligned} \tilde L^\delta w_t &= (\tilde h^\delta + \delta^2 \tilde T^\delta) w_t \\ &= \tilde h^\delta w_t - \tfrac{\delta^2}{3}\nabla\big((\tilde h^\delta)^3(\nabla\cdot w_t)\big) + \tfrac{\delta^2}{2}\nabla\big((\tilde h^\delta)^2(\nabla b\cdot w_t)\big) \\ &\quad - \tfrac{\delta^2}{2}(\tilde h^\delta)^2\nabla b(\nabla\cdot w_t) + \delta^2\tilde h^\delta\nabla b(\nabla b\cdot w_t), \end{aligned}$$

$$\begin{aligned} A_1 w &= h^\delta(u^\delta\cdot\nabla)w + \tfrac{\delta^2}{3}\nabla\big((h^\delta)^3(\nabla\cdot u^\delta)(\nabla\cdot w)\big) \\ &\quad + \tfrac{\delta^2}{3}\nabla\big((h^\delta)^3(\nabla\cdot\tilde u^\delta)(\nabla\cdot w)\big) - \tfrac{\delta^2}{3}\nabla\big((h^\delta)^3(w\cdot\nabla)(\nabla\cdot\tilde u^\delta)\big) \\ &\quad + \tfrac{\delta^2}{2}\nabla\big((h^\delta)^2(w\cdot\nabla)(\tilde u^\delta\cdot\nabla b)\big) - \tfrac{\delta^2}{2}\big((h^\delta)^2(u^\delta\cdot\nabla)(\nabla\cdot w)\big)\nabla b \\ &\quad + \delta^2\big(h^\delta(u^\delta\cdot\nabla)(w\cdot\nabla b)\big)\nabla b + \delta^2(\tilde h^\delta)^2\nabla(w\cdot\nabla b_t), \end{aligned}$$

$$A_2 w = -\tfrac{\delta^2}{3}\big((h^\delta)^3(u^\delta\cdot\nabla)(\nabla\cdot w)\big) + \tfrac{\delta^2}{2}\big((h^\delta)^2(u^\delta\cdot\nabla)(w\cdot\nabla b)\big),$$

$$\begin{aligned} B\zeta &= -\tfrac{\delta^2}{3}\nabla\big(((h^\delta)^2 + h^\delta\tilde h^\delta + (\tilde h^\delta)^2)(\nabla\cdot u_t^\delta)\zeta\big) \\ &\quad + \tfrac{\delta^2}{2}\nabla\big((h^\delta + \tilde h^\delta)(\nabla b\cdot u_t^\delta)\zeta\big) + h^\delta\nabla\zeta \\ &\quad + \tfrac{\delta^2}{3}\nabla\big(((h^\delta)^2 + h^\delta\tilde h^\delta + (\tilde h^\delta)^2)(\nabla\cdot\tilde u^\delta)^2\zeta\big) \\ &\quad - \tfrac{\delta^2}{3}\nabla\big(((h^\delta)^2 + h^\delta\tilde h^\delta + (\tilde h^\delta)^2)\zeta(\tilde u^\delta\cdot\nabla)(\nabla\cdot\tilde u^\delta)\big) \\ &\quad + \tfrac{\delta^2}{2}\nabla\big((h^\delta + \tilde h^\delta)\zeta(\tilde u^\delta\cdot\nabla)(\tilde u^\delta\cdot\nabla b)\big) \\ &\quad + 2\delta^2 h^\delta(u^\delta\cdot\nabla b_t)\nabla\zeta + \delta^2 h^\delta b_{tt}\nabla\zeta, \end{aligned}$$

$$\begin{aligned}
a = &-\zeta u_t^\delta + \tfrac{\delta^2}{2}\big((h^\delta + \tilde{h}^\delta)(\nabla\cdot u_t^\delta)\zeta\big)\nabla b - \delta^2(\nabla b\cdot u_t^\delta)\zeta\nabla b - \zeta\nabla\tilde{h}^\delta - \zeta\nabla b\\
&- h^\delta(w\cdot\nabla)\tilde{u}^\delta - \zeta(\tilde{u}^\delta\cdot\nabla)\tilde{u}^\delta - \tfrac{\delta^2}{2}\big((h^\delta)^2(\nabla\cdot u^\delta)(\nabla\cdot w)\big)\nabla b\\
&- \tfrac{\delta^2}{2}\big((h^\delta)^2(\nabla\cdot\tilde{u}^\delta)(\nabla\cdot w)\big)\nabla b - \tfrac{\delta^2}{2}\big((h^\delta+\tilde{h}^\delta)(\nabla\cdot\tilde{u}^\delta)^2\zeta\big)\nabla b\\
&+ \tfrac{\delta^2}{2}\big((h^\delta)^2(w\cdot\nabla)(\nabla\cdot\tilde{u}^\delta)\big)\nabla b + \tfrac{\delta^2}{2}\big((h^\delta+\tilde{h})\zeta(\tilde{u}^\delta\cdot\nabla)(\nabla\cdot\tilde{u}^\delta)\big)\nabla b\\
&- \delta^2\big(h^\delta(w\cdot\nabla)(\tilde{u}^\delta\cdot\nabla b)\big)\nabla b - \delta^2\big(\zeta(\tilde{u}^\delta\cdot\nabla)(\tilde{u}^\delta\cdot\nabla b)\big)\nabla b\\
&- \delta^2\zeta(h^\delta+\tilde{h}^\delta)\nabla(u^\delta\cdot\nabla b_t) - \tfrac{\delta^2}{2}\zeta(h^\delta+\tilde{h}^\delta)\nabla b_{tt}\\
&- 2\delta^2 h^\delta(w\cdot\nabla b_t)\nabla\tilde{h}^\delta - 2\delta^2\zeta(\tilde{u}^\delta\cdot\nabla b_t)\nabla\tilde{h}^\delta - 2\delta^2 h^\delta(w\cdot\nabla b_t)\nabla b\\
&- 2\delta^2\zeta(\tilde{u}^\delta\cdot\nabla b_t)\nabla b - \delta^2\zeta b_{tt}\nabla\tilde{h}^\delta - \delta^2\zeta b_{tt}\nabla b,
\end{aligned}$$

where a is a correction of lower order terms in (ζ, w). We note that the equations in (16) are linearized Green–Naghdi equations. Here, we also have

$$C^{-1}(\|w\|^2 + \delta^2\|\nabla\cdot w\|^2) \le (\tilde{L}^\delta w, w) \le C(\|w\|^2 + \delta^2\|\nabla\cdot w\|^2).$$

Therefore, in the same way as the proof of Proposition 3 we obtain

$$E_s(t) \le C\mathrm{e}^{Ct}\int_0^t \mathrm{e}^{C(t-\tau)}(\delta^8(\|g_1(\tau)\|_s^2 + \|g_2(\tau)\|_s^2) + E_s(\tau))\mathrm{d}\tau,$$

where

$$E_s(t) = \|\zeta(t)\|_s^2 + \|w(t)\|_s^2 + \delta^2\|\nabla\cdot w(t)\|_s^2.$$

In the above calculation we used the fact that $E_s(0) = 0$. Therefore, by Gronwall's inequality we obtain

$$E_s(t) \le C\mathrm{e}^{Ct}\int_0^t \delta^8(\|g_1(\tau)\|_s^2 + \|g_2(\tau)\|_s^2)\mathrm{d}\tau,$$

which together with Lemma 2 yields the desired estimate. Q.E.D.

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Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1 Hiyoshi, Kohoku-ku
Yokohama 223-8522
Japan
E-mail address: iguchi@math.keio.ac.jp