Advanced Studies in Pure Mathematics 64, 2015
Nonlinear Dynamics in Partial Differential Equations
pp. 1–33

Asymptotic analysis of compressible, viscous and heat conducting fluids

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Abstract.

This is a survey of recent results concerning the mathematical theory of compressible, viscous, and heat conducting fluids. Starting from the basic physical principles, notably the First and Second laws of thermodynamics, we introduce a concept of weak solutions to complete fluid systems and analyze their asymptotic behavior. In particular, the long time behavior and scale analysis will be performed. We also introduce a new concept of relative entropy for the system and show how it can be used in the problem of weak-strong uniqueness and the inviscid limits.

§1. Introduction

In this survey, we discuss certain general ideas and recent results of the mathematical theory of viscous, compressible, and heat conducting fluids. The fundamental principles are the *First and Second laws of thermodynamics* that play a crucial role in the study of qualitative properties of solutions of such systems. In particualr, the following issues will be addressed:

(1) global-in-time *existence* of (weak) solutions for any physically admissible data, without assuming unnecessary restrictions on their size;
(2) the problem of stability and weak-strong uniqueness;
(3) long-time behavior of the weak solutions;
(4) scale analysis and resulting simplified (target) systems, including some inviscid limits.

Received April 18, 2012.
Revised December 7, 2012.
2010 *Mathematics Subject Classification.* 35Q30, 35Q35, 35E15.
Key words and phrases. Navier–Stokes–Fourier system, long-time behavior, scale analysis.

§2. Mathematical theory of fluid dynamics

We develop a mathematical theory of simple but still physically complete fluid systems, to which all basic thermodynamic principles may be applied. We focus on *energetically closed* systems, where both the total mass of the fluid and its total energy are preserved in time. We choose the *mass density* ϱ and the *absolute temperature* ϑ as fundamental *state variables*, characterizing completely the fluid in thermodynamic equilibrium, while the *velocity field* $\mathbf{u}$ will describe the mass transfer for fluids out of equilibrium states.

2.1. Thermal systems in equilibrium

A simple thermal system (in equilibrium) is fully described by the state variables ϱ, ϑ and the associated *thermodynamic functions*: the internal energy $e = e(\varrho, \vartheta)$, the pressure $p = p(\varrho, \vartheta)$, and the entropy $s = s(\varrho, \vartheta)$, see Callen [3].

The thermodynamic functions e, s, and p are interrelated through GIBBS' EQUATION:

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right). \tag{1}$$

In addition to (1), it is customary to impose the so-called HYPOTHESIS OF THERMODYNAMIC STABILITY:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0 \tag{2}$$

for any ϱ, $\vartheta > 0$.

The former condition in (2) means that *compressibility* of the fluid is always positive, while the latter is equivalent to positivity of the *specific heat at constant volume.* Hypothesis of thermodynamic stability plays a crucial role in the asymptotic analysis of the underlying fluid system as well as in the issues related to stability, in particular, to the problem of *weak-strong uniqueness*, see Section 4.

2.2. Description of motion, velocity

The motion of a fluid is characterized by a *velocity field* $\mathbf{u}$. Velocity describes the *transport of mass* in the fluid and the related balance law is usually termed EQUATION OF CONTINUITY. Its classical formulation reads

$$\partial_t \varrho + \operatorname{div}_x (\varrho \mathbf{u}) = 0. \tag{3}$$

Here, we shall mostly deal with the weak formulation represented by the integral identity

$$\int_0^T \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0, \cdot) \ \mathrm{d}x. \tag{4}$$

Note that, if satisfied for any test function $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, relation (4) includes implicitly the satisfaction of the initial condition $\varrho(0,\cdot) = \varrho_0$ and the no-flux boundary condition $\varrho \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$ —the *impermeability* of the physical boundary.

By *Newton's second law* the flux associated to the momentum vector $\varrho \mathbf{u}$ reads $\varrho \mathbf{u} \otimes \mathbf{u} - \mathbb{T}$, where $\mathbb{T}$ is the Cauchy stress tensor, yielding the force per unit surface that the part of a fluid in contact with an ideal surface element imposes on the part of the fluid on the other side of the same surface element. Fluids are characterized among other materials through *Stokes' law*

$$\mathbb{T} = \mathbb{S} - p\mathbb{I},$$

where the symbol $\mathbb{S}$ denotes the *viscous stress tensor.*

The *balance of linear momentum* or EQUATION OF MOTION reads

$$\int_0^T \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi + p \mathrm{div}_x \varphi \ \mathbb{I} \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{5}$$

$$= \int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x \varphi - \varrho \mathbf{f} \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega (\varrho \mathbf{u})_0 \cdot \varphi(0, \cdot) \ \mathrm{d}x,$$

or, in the classical form,

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}, \ \ \varrho \mathbf{u}(0, \cdot) = (\varrho \mathbf{u})_0, \tag{6}$$

where $\mathbf{f}$ denotes a driving force.

A proper choice of the test functions in (5) is open to discussion. Of course, the space of test functions should contain $C_c^\infty([0,T) \times \Omega; R^3)$ in order to establish, at least at the level of formal interpretation, equation (6). Moreover, in accordance with the hypothesis of *impermeability of the physical boundary*, we restrict ourselves to the case $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$. In particular, taking

$$\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; R^3), \ \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{7}$$

we end up with the *complete slip* boundary conditions for the velocity field

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0. \tag{8}$$

Note that for viscous fluid, where the tensor $\mathbb{S}$ depends effectively on the velocity gradient, it is more customary to use the *no-slip* boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{9}$$

corresponding to the space of test functions $C_c^\infty([0,T) \times \Omega; R^3)$. The reader may consult Málek and Rajagopal [24] for more details concerning the physical background of the boundary conditions for viscous fluids.

We conclude this part by remarking that the concept of *weak* solution was introduced, in the context of the incompressible fluids, by Leray [19], and later developed by many authors, notably Ladyzhenskaya [17], Lions [20], [21], Temam [27], among many others.

2.3. Energy, entropy, Second law of thermodynamics

To simplify presentation, let us assume that $\mathbf{f} = \nabla_x F$, where $F = F(x)$ is a given potential, defined and differentiable in Ω.

Multiplying, formally, the momentum equation (6) by $\mathbf{u}$ we deduce

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 - \varrho F \right) + \mathrm{div}_x \left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u} - \varrho F\mathbf{u} + p\mathbf{u} \right) - \mathrm{div}_x(\mathbb{S}\mathbf{u}) \tag{10}$$
$$= p\mathrm{div}_x\mathbf{u} - \mathbb{S} : \nabla_x\mathbf{u}.$$

The quantity $1/2\varrho|\mathbf{u}|^2 - \varrho F$ represents the mechanical energy of the system; whence (10) may be viewed as a balance of mechanical energy. Since (10) is not in the form of a conservation law, and, at the same time, the boundary $\partial\Omega$ is impermeable, the *total energy* of the system must be conserved. The "missing" part of the energy is converted to its *internal component* e so that the TOTAL ENERGY BALANCE reads

$$E(t) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right)(t,\cdot)\ \mathrm{d}x = E_0 \text{ for any } t > 0. \tag{11}$$

The bridge between (10)–(11) is provided by *Second law of thermodynamics*, specifically by the ENTROPY BALANCE EQUATION:

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\mathbf{q}_s = \sigma, \tag{12}$$

where $\mathbf{q}_s$ is the entropy flux, and σ is the *entropy production rate.* In view of (1), it is more convenient to set

$$\mathbf{q}_s = \frac{\mathbf{q}}{\vartheta},$$

where $\mathbf{q}$ represents the internal energy (heat) diffusion flux.

Multiplying equation (12) by ϑ, we use Gibbs' equation (1) to deduce the *internal energy balance*

$$(13)\quad \partial_t(\varrho e(\varrho,\vartheta)) + \mathrm{div}_x(\varrho e(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\mathbf{q} = \vartheta\sigma + \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta} - p\mathrm{div}_x\mathbf{u}.$$

Since we want to avoid the flux of energy through the boundary, the relevant boundary condition for $\mathbf{q}$ reads

$$(14)\qquad \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

Consequently, integrating (13) over Ω and comparing the resulting expression with (10), (11), we get

(15)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F\right)\,\mathrm{d}x = \int_\Omega \left(\vartheta\sigma - \mathbb{S}:\nabla_x\mathbf{u} + \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\,\mathrm{d}x.$$

As a matter of fact, we derived (15) under the principal assumption that all quantities in question are regular (smooth). Keeping in mind possible singularities we allow the entropy production σ to be non-negative (measure) satisfying

$$(16)\qquad \sigma \geq \frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right) \geq 0.$$

In particular, comparing (11), (15), (16) we arrive at the classical relation

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

provided all quantities are smooth.

Since any non-negative distribution σ can be viewed as a Radon measure, the weak formulation of the ENTROPY BALANCE EQUATION takes the form

(17)

$$\int_0^T\int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}\cdot\nabla_x\varphi}{\vartheta}\right)\,\mathrm{d}x\,\mathrm{d}x + <\sigma;\varphi>$$
$$= -\int_\Omega (\varrho s(\varrho,\vartheta))_0\varphi(0,\cdot)\,\mathrm{d}x$$

for any test function $\varphi\in C_c^\infty([0,T)\times\overline{\Omega})$, where $\sigma\in\mathcal{M}^+([0,T]\times\overline{\Omega})$ is a measure satisfying (16), see [12, Chapter 3].

2.4. Constitutive equations

Constitutive equations describe the *material properties* of a specific fluid. They are (typically non-linear) relations between the fundamental state variables and their partial derivatives.

2.4.1. *Equations of state* A typical example of a constitutive relation is the THERMAL EQUATION OF STATE relating the pressure p to the thermostatic state variables ϱ, ϑ. A universal equation of state characterizing a *monoatomic gas* reads (see Eliezer et al. [7]):

$$p(\varrho,\vartheta) = \frac{2}{3}\varrho e(\varrho,\vartheta). \tag{18}$$

Combining (18) with Gibbs' equation (1) we obtain

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) \text{ for a certain function } P. \tag{19}$$

The hypothesis of thermodynamics stability formulated through (2) leads to

$$P'(Z) > 0 \text{ for any } Z \geq 0, \tag{20}$$

and

$$\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} > 0 \text{ for all } Z \geq 0, \tag{21}$$

in particular,

$$\frac{P(Z)}{Z^{5/3}} \searrow p_\infty \text{ as } Z \to \infty. \tag{22}$$

Accordingly, the specific entropy is given as

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right), \tag{23}$$

with

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} < 0. \tag{24}$$

Finally, *Third law of thermodynamics* requires that

$$\lim_{Z\to\infty} S(Z) = 0, \tag{25}$$

in particular, it is plausible to require the specific heat at constant volume to be bounded,

$$0 < \frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \leq c \text{ for all } Z \geq 0. \tag{26}$$

It is interesting to note that (25) rules out the standard *Boyle–Marriot law* of perfect gas

$$p(\varrho, \vartheta) = \mathrm{R}\varrho\vartheta$$

that is not suitable for describing *real* gases for large values of the degeneracy parameter $\varrho/\vartheta^{3/2}$. Accordingly, we make a realistic assumption that the gas or at least one of its components (electron gas) behaves like a *Fermi gas* in the degenerate area $\varrho/\vartheta^{3/2} >> 1$, specifically, we set $p_\infty > 0$ in (22) (see Eliezer at al. [7]).

In models describing gases under large temperature regime, it is convenient to consider also the effect of thermal radiation. The simplest, but certainly not optimal approach consists in adding the so-called thermal pressure $p_R = a/3\vartheta^4$, with $a > 0$. A prototype example of the pressure in a real gas then reads

(27)

$$p(\varrho, \vartheta) = p_M(\varrho, \vartheta) + p_R(\varrho, \vartheta), \text{ with } p_M(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right), \ p_R = \frac{a}{3}\vartheta^4,$$

where $a > 0$ is a (very small) positive constant.

2.4.2. *Diffusion flux, transport coefficients* We suppose a simple linear dependence of the fluxes $\mathbb{S}$, $\mathbf{q}$ on the affinities $\nabla_x \mathbf{u}$, $\nabla_x \vartheta$. Specifically, the viscous stress $\mathbb{S}$ is given by NEWTON'S RHEOLOGICAL LAW:

$$\mathbb{S} = \mu\Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{28}$$

with the *shear viscosity coefficient* μ and the *bulk viscosity coefficient* η, while the heat flux $\mathbf{q}$ obeys FOURIER'S LAW:

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \tag{29}$$

where κ is called the *heat conductivity coefficient.*

In accordance with *Second law of thermodynamics*, the *transport coefficients* μ, η, and κ must be non-negative. In addition, we focus on (real) viscous and heat conducting fluids therefore we always assume that both μ and κ are strictly positive.

2.5. Navier–Stokes–Fourier system describing a general compressible viscous fluid

We introduce a model problem of an energetically isolated fluid system based on the physical principles and constitutive assumptions discussed in the preceding text.

2.5.1. *Mathematical formulation* We are given a family of thermodynamic function: the pressure $p = p(\varrho, \vartheta)$, the specific internal energy $e = e(\varrho, \vartheta)$, and the specific entropy $s = s(\varrho, \vartheta)$ satisfying Gibbs' equation (1), together with hypothesis of thermodynamic stability (2). The fluid occupies a bounded spatial domain $\Omega \subset R^3$ and is mechanically any thermally insulated, in particular, the total mass M and the total energy E of the fluid are constants of motion:

$$\frac{\mathrm{d}}{\mathrm{dt}} M(t) = 0, \ M(t) = \int_\Omega \varrho(t, \cdot) \ dx, \tag{30}$$

$$\frac{\mathrm{d}}{\mathrm{dt}} E(t) = 0, \ E(t) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) (t, \cdot) \ dx. \tag{31}$$

The time evolution of the fluid is governed by the principal field equations, namely, EQUATION OF CONTINUITY:

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0; \tag{32}$$

MOMENTUM EQUATION:

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F; \tag{33}$$

and ENTROPY EQUATION:

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma, \ \sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{34}$$

To comply with (30), (31), the system of equations (32)–(34) is supplemented by the NO-SLIP boundary conditions:

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{35}$$

or, alternatively, the COMPLETE SLIP boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0. \tag{36}$$

The normal component of the heat flux vanishes on the boundary:

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{37}$$

The viscous stress $\mathbb{S}$ is determined through Newton's law (28), while the heat flux $\mathbf{q}$ obeys Fourier's law (29).

2.6. Existence of global-in-time weak solutions

A rigorous proof of global-in-time weak solutions to the Navier–Stokes–Fourier system for given initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ requires several technical hypotheses stated below.

2.6.1. *Hypotheses* The hypotheses listed below are by no means optimal. The interested reader may consult [12, Chapter 3] for the physical background and possible relaxations.

(1) The initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ satisfy:

$$\varrho_0, \vartheta_0 \in L^\infty(\Omega),\ \mathbf{u}_0 \in L^\infty(\Omega; R^3),\ \varrho_0(x) \geq 0, \vartheta_0(x) > 0 \text{ for a.a. } x \in \Omega.$$

(2) The potential of the driving force F belongs to $W^{1,\infty}(\Omega)$.

(3) The pressure $p = p(\varrho, \vartheta)$ is given by

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4,\ a > 0, \tag{38}$$

where

$$P \in C^1[0, \infty),\ P(0) = 0,\ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{39}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} \leq c \text{ for all } Z > 0,\ \lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{40}$$

In accordance with Gibbs' equation (1), the specific internal energy e obeys

$$e(\varrho, \vartheta) = \frac{3}{2}\frac{\vartheta^{5/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \tag{41}$$

and

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho},\ \text{with } S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2}. \tag{42}$$

(4) The transport coefficients μ, η, and κ are continuously differentiable functions of the temperature ϑ satisfying

$$\mu \in W^{1,\infty}[0, \infty),\ 0 < \underline{\mu}(1 + \vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta^\alpha), \tag{43}$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1 + \vartheta^\alpha),\ \text{where } 1/2 \leq \alpha \leq 1; \tag{44}$$

and

$$0 < \underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3). \tag{45}$$

2.6.2. *Principal existence result concerning the weak solutions* The following result was proved in [12, Chapter 3.3, Theorem 3.1]:

Theorem 1. *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}, \nu > 0$. Suppose that the initial data ϱ_0, ϑ_0, $\mathbf{u}_0$, the driving force potential F, the thermodynamic functions p, e, and s, and the transport coefficients μ, η, and κ satisfy the general hypotheses stated in Section 2.6.1.*

Then the initial-boundary value problem for the Navier–Stokes–Fourier system admits a weak solution ϱ, ϑ, and $\mathbf{u}$ belonging to the class:

$$\varrho \in L^\infty(0,T;L^{5/3}(\Omega)),\ \vartheta \in L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)),$$

$$\mathbf{u} \in L^2(0,T;W^{1,q}(\Omega;R^3)),\ q = \frac{8}{5-\alpha}.$$

The main advantage of the class of *weak* solutions is the fact that they exist globally in time and without any essential restrictions on the size of the data. The reader may consult the result of Bresch and Desjardins [2] for an alternative approach to the weak solutions of the complete fluid systems based on some special relations satisfied by the density-dependent viscosity coefficients.

§3. Long-time behavior

A mathematical object called *dynamical system* is completely characterized by its *state* and the rules called *dynamics* that determine the state at a given future time in terms of the present state. The dynamics of energetically insulated fluid systems considered in this text is governed by the Navier–Stokes–Fourier system of equations introduced in the preceding chapter. In order to fix ideas, we impose the no-slip boundary condition for the velocity

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{46}$$

3.1. Stationary states

We study the equilibrium solutions paying attention to the following commonly accepted but otherwise rather vague statements:

(1) equilibrium solutions minimize the entropy production;
(2) equilibrium solutions maximize the total entropy of the system in the class of all admissible states;
(3) all solutions to the evolutionary system driven by a conservative time-independent external force tend to an equilibrium for large time.

The leading physical principles to be used in the forthcoming analysis are Gibbs' equation stated in (1), together with hypothesis of thermodynamic stability specified in (2).

The entropy equation integrated over Ω, and added to the total energy balance gives rise to the TOTAL DISSIPATION BALANCE in the form

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \overline{\vartheta}\varrho s(\varrho,\vartheta) - \varrho F \right)(\tau,\cdot)\ \mathrm{d}x + \overline{\vartheta}\sigma\left[[0,\tau]\times\overline{\Omega}\right] \tag{47}$$

$$= \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0) - \overline{\vartheta}\varrho_0 s(\varrho_0,\vartheta_0) - \varrho_0 F \right)\ \mathrm{d}x$$

for a.a. $\tau \in [0,T]$ and any positive constant $\overline{\vartheta}$. As we shall see in Section 4, the total dissipation balance will play a crucial role in the construction of the relative entropy for the Navier–Stokes–Fourier system.

Relation (47) implies that equilibrium (time independent) solutions minimize trivially the entropy production rate, namely $\sigma \equiv 0$, specifically,

$$\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I} \right) = 0, \ \text{and}\ \nabla_x \vartheta = 0 \tag{48}$$

for any equilibrium state. In particular, as $\mathbf{u}$ vanishes on the boundary (cf. (46)), a direct application of the standard Korn's inequality yields

$$\mathbf{u} \equiv 0 \ \text{for any equilibrium state.} \tag{49}$$

Thus any equilibrium solution $\tilde{\varrho}$, $\tilde{\vartheta}$ satisfies

$$\nabla_x p(\tilde{\varrho},\tilde{\vartheta}) = \tilde{\varrho}\nabla_x F,\ \tilde{\vartheta} = \mathrm{const} > 0 \ \text{in}\ \Omega.$$

The static states can be identified through their total mass M_0,

$$M_0 = \int_\Omega \tilde{\varrho}\ \mathrm{d}x,$$

and through the asymptotic limit

$$D_\infty[\overline{\vartheta}] = \lim_{\tau\to\infty} \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \overline{\vartheta}\varrho s(\varrho,\vartheta) - \varrho F \right)(\tau,\cdot)\ \mathrm{d}x.$$

3.2. Static states

We examine a solution $\tilde{\varrho}$, $\tilde{\vartheta}$ of the STATIC PROBLEM:

$$\nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) = \tilde{\varrho} \nabla_x F, \ \tilde{\varrho} \geq 0, \ \tilde{\vartheta} = \text{const} > 0 \text{ in } \Omega, \tag{50}$$

satisfying the constraints

$$\int_\Omega \tilde{\varrho} \, \mathrm{d}x = M_0, \int_\Omega \left(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \overline{\vartheta} \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \right) \, \mathrm{d}x = D_\infty[\overline{\vartheta}]. \tag{51}$$

In addition to hypothesis of thermodynamic stability (2), we assume that

$$\lim_{\varrho \to 0} \frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0 \text{ for any fixed } \vartheta > 0. \tag{52}$$

Under these circumstances, given a positive constant $\tilde{\vartheta}$, equation (50) admits only strictly positive solutions $\tilde{\varrho}$ on condition that $\nabla_x F$ is bounded and p satisfies (52).

3.2.1. *Ballistic free energy* Given $\overline{\vartheta} > 0$, we introduce the BALLISTIC FREE ENERGY:

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta). \tag{53}$$

It follows from Gibbs' relation and hypothesis of thermodynamics stability that

- $\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta})$ is a strictly convex function;
- $\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta)$ is decreasing if $\vartheta < \overline{\vartheta}$ and increasing whenever $\vartheta > \overline{\vartheta}$ for any fixed ϱ.

Consequently, the ballistic free energy $H_{\overline{\vartheta}}$ enjoys certain *coercivity properties.* More specifically, for any $\tilde{\varrho}$ such that

$$0 < \underline{\varrho} < \tilde{\varrho} < \overline{\varrho}$$

there exists a positive constant $\Lambda = \Lambda(\underline{\varrho}, \overline{\varrho}, \overline{\vartheta})$ such that

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \tilde{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta}) \tag{54}$$

$$\geq \Lambda \left\{ \begin{array}{l} |\varrho - \tilde{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2 \text{ if } \underline{\varrho} < \varrho < \overline{\varrho}, \ \overline{\vartheta}/2 < \vartheta < 2\overline{\vartheta}, \\ \\ \varrho e(\varrho, \vartheta) + \overline{\vartheta} |s(\varrho, \vartheta)| + 1 \text{ otherwise} \end{array} \right.$$

(see [12, Chapter 3, Proposition 3.2]).

It is easy to check that

$$\frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} = F + c_{\tilde{\varrho}, \tilde{\vartheta}} \text{ in } \Omega \tag{55}$$

whenever $\tilde{\varrho} = \tilde{\varrho}(x)$, $\tilde{\vartheta} > 0$ is a solution of static problem (50) and $c_{\tilde{\varrho}, \tilde{\vartheta}}$ is a suitable constant.

3.2.2. *Principle of maximal entropy* As a consequence of (55), the static solutions minimize the entropy among all states of the system having the same mass and total energy. Indeed let $\tilde{\varrho} = \tilde{\varrho}(x) > 0$, $\tilde{\vartheta} = \overline{\vartheta} > 0$ be a solution of problem (50), and let $\varrho = \varrho(x) \geq 0$, $\vartheta = \vartheta(x) > 0$ be a couple of functions such that

(56)

$$\int_\Omega \tilde{\varrho} \, \mathrm{d}x = \int_\Omega \varrho \, \mathrm{d}x, \ \int_\Omega \Big(\varrho e(\varrho, \vartheta) - \varrho F \Big) \, \mathrm{d}x = \int_\Omega \Big(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \Big) \, \mathrm{d}x.$$

It follows from (55), (56) that

$$\overline{\vartheta} \int_\Omega \Big(\tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}) - \varrho s(\varrho, \vartheta) \Big) \, \mathrm{d}x = \int_\Omega \Big(H_{\overline{\vartheta}}(\varrho, \vartheta) - H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta}) \Big) \, \mathrm{d}x + \int_\Omega (\tilde{\varrho} - \varrho) F \, \mathrm{d}x$$

$$= \int_\Omega \left(H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \tilde{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta}) \right) \, \mathrm{d}x.$$

Thus, in view of the coercivity properties of the ballistic free energy $H_{\overline{\vartheta}}$, we may infer that:

(1) the static solution $\tilde{\varrho}$, $\overline{\vartheta}$ maximizes the total entropy functional

$$(\varrho, \vartheta) \mapsto \int_\Omega \varrho s(\varrho, \vartheta) \, \mathrm{d}x$$

among all admissible states of the system with the same mass and total energy;

(2) if

$$\int_\Omega \varrho s(\varrho, \vartheta) \, \mathrm{d}x = \int_\Omega \tilde{\varrho} s(\tilde{\varrho}, \overline{\vartheta}) \, \mathrm{d}x$$

then, necessarily, $\varrho \equiv \tilde{\varrho}$, $\vartheta \equiv \overline{\vartheta}$, in particular, there is at most one static solution with prescribed mass and energy.

3.3. Conservative systems, attractors

The large time behavior of solutions to the energetically isolated Navier–Stokes–Fourier system is completely determined by *Second law of thermodynamics.* We shall see that all *global* trajectories approach an equilibrium state uniquely determined by the total mass and energy that are constants of motion. Moreover, the set of equilibria is an *attractor* for all trajectories emanating from the states of uniformly bounded mass and energy.

As we have already observed, the total mass

$$M_0 = \int_\Omega \varrho(t,\cdot)\ \mathrm{d}x$$

as well as the total energy

$$E_0 = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right)(t,\cdot)\ \mathrm{d}x$$

are constants of motion. Moreover, we may assume that

$$\int_\Omega \varrho s(\varrho,\vartheta)(t,\cdot)\ \mathrm{d}x \geq S_0$$

where S_0 represents the "initial" entropy of the system. Thus

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}$$

and we suppose that

$$S_0 > M_0 s_\infty,\ s_\infty = \lim_{Z\to\infty} S(Z) \geq -\infty. \tag{57}$$

Our goal is to show that the set of equilibria is an attractor for all trajectories emanating from a set of bounded total mass and energy. In a way, such a conclusion can be viewed as the most pessimistic scenario dictated by *Second law of thermodynamics*.

The following result was proved in [13, Chapter 5, Theorem 5.1].

Theorem 2. *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Assume that the hypotheses of Theorem 1 are satisfied. Let $M_0 > 0$, E_0, S_0 be given, with S_0 satisfying* (57).

Then for any $\varepsilon > 0$, there exists a time $T = T(\varepsilon)$ such that

$$\left.\begin{cases} \|(\varrho\mathbf{u})(t,\cdot)\|_{L^{5/4}(\Omega;R^3)} \leq \varepsilon, \\[2ex] \|\varrho(t,\cdot) - \tilde{\varrho}\|_{L^{5/3}(\Omega)} \leq \varepsilon, \\[2ex] \|\vartheta(t,\cdot) - \overline{\vartheta}\|_{L^4(\Omega)} \leq \varepsilon \end{cases}\right\} \textit{for a.a. } t > T(\varepsilon)$$

for any weak solution $\{\varrho, \mathbf{u}, \vartheta\}$ *of the Navier–Stokes–Fourier system defined on* $(0,\infty)\times\Omega$ *and satisfying*

$$\left\{\begin{array}{c} \int_\Omega \varrho(t,\cdot)\ \mathrm{d}x > M_0, \\[2ex] \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F\right)(t,\cdot)\ \mathrm{d}x < E_0, \\[2ex] \operatorname{ess}\liminf_{t\to 0} \int_\Omega \varrho s(\varrho,\vartheta)(t,\cdot)(t,0)\ \mathrm{d}x > S_0, \end{array}\right\} \tag{58}$$

where $\tilde\varrho$, $\overline{\vartheta}$ *is a solution of the static problem* (50) *determined uniquely by the condition*

$$\int_\Omega \tilde\varrho\ \mathrm{d}x = \int_\Omega \varrho\ \mathrm{d}x,$$

$$\int_\Omega \left(\tilde\varrho e(\tilde\varrho,\overline{\vartheta}) - \tilde\varrho F\right)\ \mathrm{d}x = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F\right)\ \mathrm{d}x$$

The main difficulty in Theorem 2 is showing *uniformity* of the convergence with respect to the time that may be spoiled by the (hypothetical) presence of density oscillations.

3.4. Systems driven by a non-conservative force

It is quite natural to ask what happens if the fluid system is driven by a *non-conservative* driving force $\mathbf{f}$ and/or if sources of heat are present. In such a situation, the total energy balance reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t,\cdot)\ \mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x. \tag{59}$$

We report the following result ([13, Chapter 5.2, Theorem 5.2]) for $\mathbf{f} = \mathbf{f}(x)$ independent of t.

Theorem 3. *Let* $\Omega \subset R^3$ *be a bounded Lipschitz domain. Under the hypotheses of Theorem 1, let* $\{\varrho,\vartheta,\mathbf{u}\}$ *be a weak solution of the Navier–Stokes–Fourier system driven by an external force* $\mathbf{f} = \mathbf{f}(x)$ *on the time interval* $[T_0,\infty)$, *where* $\mathbf{f} \not\equiv \nabla_x F$.

Then

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t,\cdot)\ \mathrm{d}x \to \infty \text{ as } t\to\infty. \tag{60}$$

Another result in this direction reads (see [13, Chapter 5.2, Theorem 5.2]):

Theorem 4. *In addition to the hypotheses of Theorem 3, assume that* $\mathbf{f} = \mathbf{f}(t,x)$, $\mathbf{f} \in L^\infty((0,T)\times\Omega;R^3)$.

Then either

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t,\cdot)\ \mathrm{d}x \to \infty \ \textit{as}\ t\to\infty$$

or

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t,\cdot)\ \mathrm{d}x \le E_\infty \ \textit{for a.a.}\ t > T_0$$

for a certain constant E_∞. Moreover, in the latter case, each sequence $\tau_n \to \infty$ contains a subsequence (not relabeled) such that

$$\mathbf{f}(\tau_n + \cdot, \cdot) \to \nabla_x F \ \textit{weakly-}(*)\ \textit{in}\ L^\infty((0,1)\times\Omega; R^3)$$

for a certain $F = F(x)$, $F \in W^{1,\infty}(\Omega)$ that, in general, may depend on the choice of $\{\tau_n\}_{n=1}^\infty$.

3.4.1. *Highly oscillating driving force* In light of the arguments presented in the previous section, it may seem that almost *any* time-dependent driving force imposed on the energetically insulated Navier–Stokes–Fourier system produces a "grow-up" of the total energy for large values of time. The exceptions are rapidly oscillating forces as shown in the following result [13, Chapter 5.3, Theorem 5.3]:

Theorem 5. *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. In addition to the hypotheses of Theorem 1, assume that the driving force takes the form*

$$\mathbf{f}(t,x) = \omega(t^\beta)\mathbf{w}(x),\ t > 0,\ x \in \Omega,$$

where $\mathbf{w} \in W^{1,\infty}(\Omega)$, $\mathbf{w} \neq 0$, and

$$\omega \in L^\infty(R),\ \omega \neq 0,\ \sup_{\tau>0}\left|\int_0^\tau \omega(t)\ \mathrm{d}t\right| < \infty,$$

are given functions.

Then for all $\beta > 2$ any global-in-time weak solution of the Navier–Stokes–Fourier system satisfies

$$\varrho\mathbf{u}(t,\cdot) \to 0 \ \textit{in}\ L^{5/4}(\Omega; R^3)\ \textit{as}\ t\to\infty,$$

$$\vartheta(t,\cdot) \to \overline{\vartheta}\ \textit{in}\ L^4(\Omega)\ \textit{as}\ t\to\infty,$$

and

$$\varrho(t,\cdot) \to \overline{\varrho}\ \textit{in}\ L^{5/3}(\Omega)\ \textit{as}\ t\to\infty,$$

where ϱ_s, ϑ_s are positive constants,

$$\overline{\varrho} = \frac{1}{|\Omega|}\int_\Omega \varrho\ \mathrm{d}x.$$

The preceding result can be extended to oscillatory forces that may even *increase* in time. More specifically, we report the following result, [1, Theorem 1]:

Theorem 6. *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. In addition to the hypotheses of Theorem 1, assume that the driving force takes the form*

$$\mathbf{f}(t,x) = t^{\delta}\omega(t^{\beta})\mathbf{w}(x), \ t > 0, \ x \in \Omega,$$

where $\mathbf{w} \in W^{1,\infty}(\Omega)$, $\mathbf{w} \neq 0$, and

$$\omega \in L^{\infty}(R), \ \omega \neq 0, \ \sup_{\tau>0}\left|\int_0^{\tau} \omega(t) \ \mathrm{d}t\right| < \infty,$$

are given functions. Suppose that

$$\delta > 0, \ \beta - 2\delta > 2 \ \text{or} \ \delta \leq 0, \ \beta - \delta > 2.$$

Then any global-in-time weak solution of the Navier–Stokes–Fourier system satisfies

$$\varrho\mathbf{u}(t,\cdot) \to 0 \ \text{in} \ L^{5/4}(\Omega; R^3) \ \text{as} \ t \to \infty,$$

$$\vartheta(t,\cdot) \to \overline{\vartheta} \ \text{in} \ L^4(\Omega) \ \text{as} \ t \to \infty,$$

and

$$\varrho(t,\cdot) \to \overline{\varrho} \ \text{in} \ L^{5/3}(\Omega) \ \text{as} \ t \to \infty,$$

where ϱ_s, ϑ_s are positive constants,

$$\overline{\varrho} = \frac{1}{|\Omega|}\int_{\Omega} \varrho \ \mathrm{d}x.$$

Unlike the temperature ϑ, the density ϱ as well as the momentum $\varrho\mathbf{u}$ are weakly continuous with respect to the time variable, in particular, the instantaneous values $\varrho\mathbf{u}(t,\cdot)$, $\varrho(t,\cdot)$ make sense. As a matter of fact, it follows from DiPerna–Lions theory [6] that ϱ is even strongly continuous with values in $L^1(\Omega)$. On the other hand, the convergence of the temperature must be interpreted as

$$\operatorname{ess}\lim_{t\to\infty} \|\vartheta(t,\cdot) - \tilde{\vartheta}\|_{L^4(\Omega)} = 0.$$

§4. Relative entropies and the weak-strong uniqueness problem

Motivated by the properties of *ballistic free energy* (53) discussed and used in the preceding part, we introduce the *relative entropy* for the Navier–Stokes–Fourier system in the form:

$$\text{(61)} \quad \mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\middle|r,\Theta,\mathbf{U}\right) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho - r) - H_\Theta(r,\Theta)\right)\ \mathrm{d}x,$$

where $\{\varrho,\vartheta,\mathbf{u}\}$ is a weak solution to the Navier–Stokes–Fourier system in the sense specified in Section 2.5, and $\{r,\mathbf{U},\Theta\}$ is an arbitrary trio of (smooth) functions, with r, $\Theta > 0$, and $\mathbf{U}$ satisfying the relevant boundary conditions, see [10].

As already observed in the preceding section, where $r = \tilde{\varrho}$, $\Theta = \overline{\vartheta}$, and $\mathbf{U} \equiv 0$ was a suitable static solution, the relative entropy represents a distance between a solution $\{\varrho,\vartheta,\mathbf{u}\}$ and $\{r,\Theta,\mathbf{U}\}$. A remarkable feature of the relative entropy functional is that this distance may be evaluated for any weak solution of the Navier–Stokes–Fourier system as long as the quantities $\{r,\Theta,\mathbf{U}\}$ are smooth. The resulting expression reads (see [10] for details):

$$\text{(62)} \quad \left[\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\middle|r,\Theta,\mathbf{U}\right)\right]_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega \frac{\Theta}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq \int_0^\tau\int_\Omega \Big(\varrho(\mathbf{U}-\mathbf{u})\cdot\partial_t\mathbf{U} + \varrho(\mathbf{U}-\mathbf{u})\otimes\mathbf{u} : \nabla_x\mathbf{U} - p(\varrho,\vartheta)\mathrm{div}_x\mathbf{U}\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$+ \int_0^\tau\int_\Omega \Big(\mathbb{S}:\nabla_x\mathbf{U} + \varrho\nabla_x F\cdot(\mathbf{u}-\mathbf{U})\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$- \int_0^\tau\int_\Omega \Big(\varrho\big(s(\varrho,\vartheta) - s(r,\Theta)\big)\partial_t\Theta + \varrho\big(s(\varrho,\vartheta) - s(r,\Theta)\big)\mathbf{u}\cdot\nabla_x\Theta$$
$$+\frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\Theta\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$+ \int_0^\tau\int_\Omega \left(\left(1-\frac{\varrho}{r}\right)\partial_t p(r,\Theta) - \frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta)\right)\ \mathrm{d}x\ \mathrm{d}t.$$

We emphasize that (62) holds for *any* weak solution $\{\varrho,\vartheta,\mathbf{u}\}$ of the Navier–Stokes–Fourier system and *any* trio of functions $\{r,\Theta,\mathbf{U}\}$ that are continuously differentiable in $[0,T]\times\overline{\Omega}$ satisfying

$$r > 0,\ \Theta > 0,\ \mathbf{U}|_{\partial\Omega} = 0.$$

The concept of *relative entropy* is central in the proof of the property of *weak-strong uniqueness* for the full Navier–Stokes–Fourier system. The leading and apparently simple idea is to take a (hypothetical) strong solution $\{\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\}$ as "test functions" in (62) and to use a Gronwall type argument. A rigorous proof is, however, rather involved yielding the following result, see [8, Theorem 6.2], [10]:

Theorem 7. *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Under the hypotheses of Theorem 1, let $\{\varrho, \vartheta, \mathbf{u}\}$ be a weak solution of the Navier–Stokes–Fourier system defined on the time interval $[0,T]$. Suppose that $\{\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\}$ is a smooth solution of the same problem emanating from the same initial data and defined on $[0,T]$.*

Then

$$\varrho \equiv \tilde{\varrho}, \ \ \vartheta = \tilde{\vartheta}, \ \ \mathbf{u} = \tilde{\mathbf{u}} \ \textit{in} \ [0,T].$$

Here, *smooth* means that all relevant derivatives exist in the classical sense and that the functions satisfy the corresponding boundary conditions. The result can be extended to other types of boundary conditions as well as to a larger class of spatial domains.

§5. Scale analysis

By scaling the equations, meaning by choosing appropriately the system of the reference units, the parameters determining the behavior of the system become explicit. Asymptotic analysis provides a useful tool in the situations when certain of these parameters called *characteristic numbers* vanish or become infinite. The *Navier–Stokes–Fourier system* in the standard form introduced in Section 2.5 does not reveal anything more than the balance laws of certain quantities characterizing the instantaneous state of a fluid. In order to get a somewhat deeper insight into the structure of possible solutions, we identify the *characteristic values* of relevant physical quantities: the *reference time* $T_{\rm ref}$, the *reference length* $L_{\rm ref}$, the *reference density* $\varrho_{\rm ref}$, the *reference temperature* $\vartheta_{\rm ref}$, together with the *reference velocity* $U_{\rm ref}$, and the characteristic values of other composed quantities $p_{\rm ref}$, $e_{\rm ref}$, $\mu_{\rm ref}$, $\eta_{\rm ref}$, $\kappa_{\rm ref}$, and the source term $\nabla_x F_{\rm ref}$. Introducing new independent and dependent variables $X' = X/X_{\rm ref}$ and omitting the primes in the resulting equations, we arrive at the following SCALED NAVIER–STOKES–FOURIER SYSTEM:

$$\text{(63)} \qquad {\rm Sr}\ \partial_t \varrho + {\rm div}_x(\varrho \mathbf{u}) = 0,$$

$$\text{(64)} \qquad {\rm Sr}\ \partial_t(\varrho \mathbf{u}) + {\rm div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{{\rm Ma}^2} \nabla_x p = \frac{1}{{\rm Re}} {\rm div}_x \mathbb{S} + \frac{1}{{\rm Fr}^2} \varrho \nabla_x F,$$

$$\mathrm{Sr}\ \partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \frac{1}{\mathrm{Pe}}\mathrm{div}_x\Big(\frac{\mathbf{q}}{\vartheta}\Big) = \sigma, \tag{65}$$

together with the associated total energy balance

$$\mathrm{Sr}\ \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\frac{\mathrm{Ma}^2}{2}\varrho|\mathbf{u}|^2 + \varrho e - \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\varrho F\Big)\,\mathrm{d}x = 0, \tag{66}$$

with

$$\sigma \geq \frac{1}{\vartheta}\Big(\frac{\mathrm{Ma}^2}{\mathrm{Re}}\mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}}\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big), \tag{67}$$

and the associated boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ [\mathbb{S}\mathbf{n}]\times\mathbf{n}|_{\partial\Omega} = 0,\ \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0 \tag{68}$$

(cf. Klein et al. [16]).

A sample of used dimensionless CHARACTERISTIC NUMBERS is listed below:

△ SYMBOL	△ DEFINITION	△ NAME
Sr	$L_{\mathrm{ref}}/(T_{\mathrm{ref}}U_{\mathrm{ref}})$	Strouhal number
Ma	$U_{\mathrm{ref}}/\sqrt{p_{\mathrm{ref}}/\varrho_{\mathrm{ref}}}$	Mach number
Re	$\varrho_{\mathrm{ref}}U_{\mathrm{ref}}L_{\mathrm{ref}}/\mu_{\mathrm{ref}}$	Reynolds number
Fr	$U_{\mathrm{ref}}/\sqrt{L_{\mathrm{ref}}f_{\mathrm{ref}}}$	Froude number
Pe	$p_{\mathrm{ref}}L_{\mathrm{ref}}U_{\mathrm{ref}}/(\vartheta_{\mathrm{ref}}\kappa_{\mathrm{ref}})$	Péclet number

5.1. From compressible to incompressible fluids

In many real world applications, such as atmosphere-ocean flows, fluid flows in engineering devices and astrophysics, velocities are small compared with the speed of sound proportional to $1/\sqrt{\mathrm{Ma}}$ in the *scaled Navier–Stokes–Fourier system.* This observation has a significant impact on both exact solutions to the governing equations and their numerical approximations. We consider a scaled *Navier–Stokes–Fourier* system in the form:

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0, \tag{69}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho,\vartheta) = \mathrm{div}_x\mathbb{S}(\vartheta,\nabla_x\mathbf{u}), \tag{70}$$

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}(\vartheta,\nabla_{\mathbf{x}}\vartheta)}{\vartheta}\right) = \sigma_\varepsilon, \tag{71}$$

supplemented with the total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\varepsilon}\left(\frac{\varepsilon^2}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t,\cdot)\ \mathrm{d}x = 0, \tag{72}$$

where the entropy production rate σ_ε satisfies

$$\sigma_\varepsilon \geq \frac{1}{\vartheta}\left(\varepsilon^2\mathbb{S} : \nabla_x\mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta}|\nabla_x\vartheta|^2\right) \geq 0. \tag{73}$$

The system is supplemented with conservative boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ [\mathbb{S}\mathbf{n}]\times\mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \tag{74}$$

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0. \tag{75}$$

Finally, the initial state of the fluid system is determined by the following conditions:

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon\varrho^1_{0,\varepsilon},\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon\vartheta^1_{0,\varepsilon}, \tag{76}$$

where

$$\overline{\varrho},\ \overline{\vartheta} > 0,\ \int_{\Omega_\varepsilon}\varrho^1_{0,\varepsilon}\ \mathrm{d}x = \int_{\Omega_\varepsilon}\vartheta^1_{0,\varepsilon}\ \mathrm{d}x = 0 \text{ for all } \varepsilon > 0, \tag{77}$$

and

$$\{\varrho^1_{0,\varepsilon}\}_{\varepsilon>0},\ \{\vartheta^1_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^2\cap L^\infty(\Omega_\varepsilon). \tag{78}$$

In addition, we suppose

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \tag{79}$$

where

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2\cap L^\infty(\Omega_\varepsilon;R^3). \tag{80}$$

The fluid is considered on a family of *bounded* domains Ω_ε chosen to "mimick" the behavior of the fluid in a fictitious large (unbounded) domain Ω. Pursuing the philosophy that any real physical space is always bounded but possibly "large" with respect to the speed of sound in the medium, we consider a family of *bounded* domains $\{\Omega_\varepsilon\}_{\varepsilon>0}\subset R^3$ such that $\Omega_\varepsilon \approx \Omega$ in a certain sense as $\varepsilon \to 0$. More specifically, we suppose that

(81)

$\Omega \subset R^3$ is an unbounded domain with a compact smooth boundary $\partial\Omega$,

and set

$$\Omega_\varepsilon = B_{r(\varepsilon)} \cap \Omega, \tag{82}$$

where $B_{r(\varepsilon)}$ is a ball centered at zero with a radius $r(\varepsilon)$, with $\varepsilon r(\varepsilon) \to \infty$.

Our goal will be:

(1) establish uniform bounds on the family of solution $\{\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of problem (69)–(76) independent of the parameter $\varepsilon \to 0$;
(2) show strong (pointwise a.a.) convergence

$$\left\{ \begin{array}{c} \varrho_\varepsilon \to \overline{\varrho} \\ \\ \vartheta_\varepsilon \to \overline{\vartheta} \end{array} \right\} \text{a.a. in } (0,T) \times \Omega, \tag{83}$$

and

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ a.a. in } (0,T) \times \Omega \tag{84}$$

at least for suitable subsequences.

With (83), (84) at hand, it is relatively easy to identify the limit system represented by the so-called Oberbeck–Boussinesq approximation. The details can be found in [12, Chapter 5].

5.1.1. *Stability of static equilibria in the low Mach number limit* As already observed in Section 3.2, any weak solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}$ of the Navier–Stokes–Fourier system (69)–(72) satisfies the *total dissipation balance*

$$\int_{\Omega_\varepsilon} \left(\frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\Big[H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})(\varrho_\varepsilon - \overline{\varrho}) \right.$$
$$\left. - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big] \right) (\tau, \cdot) \ dx + \frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon \Big[[0,\tau] \times \overline{\Omega}_\varepsilon \Big] =$$
$$\int_{\Omega_\varepsilon} \left(\frac{1}{2}\varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2}\Big[H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})(\varrho_{0,\varepsilon} - \overline{\varrho}) \right.$$
$$\left. - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big] \right) \ dx \tag{85}$$

for a.a. $\tau \in [0,T]$, with the ballistic free energy $H_{\overline{\vartheta}}$ introduced in (53).

Relation (85), together with the structural properties of the function $H_{\overline{\vartheta}}$ established in (54), can be used to deduce uniform bounds independent of ε. To this end, it is convenient to introduce the *essential* and *residual* parts of a function h as

$$h = [h]_{\rm ess} + [h]_{\rm res},\ [h]_{\rm ess} = \Psi(\varrho_\varepsilon, \vartheta_\varepsilon)h,\ [h]_{\rm res} = \Big(1 - \Psi(\varrho_\varepsilon, \vartheta_\varepsilon)\Big)h,$$

where

$\Psi \in C_c^\infty(0,\infty)^2$, $0 \leq \Psi \leq 1$, $\Psi \equiv 1$ in an open neighborhood of the point $[\overline{\varrho}, \overline{\vartheta}]$.

In addition, we assume that the viscosity coefficient μ obeys *Chapman's law*

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta) \tag{86}$$

under the given scaling (cf. (43)).

The total dissipation balance (85), together with the hypotheses (76)–(80) imposed on the initial data, givee rise to the following estimates:

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c,\ \operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\rm res} \right\|_{L^{5/4}(\Omega_\varepsilon)} \leq c, \tag{87}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c,\ \operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\rm ess} \right\|_{L^4(\Omega_\varepsilon)} \leq c, \tag{88}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\sqrt{\varrho}\mathbf{u}\|_{L^2(\Omega_\varepsilon;R^3)} \leq c, \text{ and } \|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega})} \leq \varepsilon^2 c \tag{89}$$

where the generic constant c is independent of ε.

In addition, as a direct consequence of (73), the previously established bounds, the structural properties of the transport coefficients, and Korn's and Poincare's inequalities, we obtain

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)}\ {\rm d}t \leq c, \text{and } \int_0^T \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)} {\rm d}t \leq c \tag{90}$$

(see [12, Chapter 5.2] for details).

These uniform bounds reflect *stability* of the static state $\overline{\varrho}, \overline{\vartheta}$ in the low Mach number regime. In particular, we deduce immediately the pointwise convergence claimed in (83). A similar result for the velocity field $\mathbf{u}_\varepsilon$ is less trivial and will be discussed in detail in the remaining part of this text.

5.2. Acoustic waves

The strong convergence of the velocity (84) is related to propagation and attenuation of acoustic waves. As a matter of fact, (84) is not expected to hold on *bounded* domains with acoustically hard boundary, where large amplitude rapidly oscillating waves are generated in the limit $\varepsilon \to 0$ (see, for instance, Lions and Masmoudi [23], or Schochet [29]). Accordingly, for (84) to hold it is necessary that the target domain Ω be *unbounded* (cf. hypotheses (81), (82)), more specifically, the two closely related properties must be satisfied:

- the point spectrum of the associated wave operator must be empty;
- the *local* acoustic energy decays in time.

We remark that problems related to propagation of acoustic waves in R^3 were studied by Desjradins and Grenier [5].

5.2.1. *Lighthill's acoustic equation* The forthcoming analysis primarily rests on the approach proposed by Lighthill [22], where the original Navier–Stokes–Fourier system is rewritten in the form of a wave equation with a source term usually called *Lighthill's tensor.*

We begin by introducing a "time lifting" Σ_ε of the measure σ_ε through formula

$$< \Sigma_\varepsilon; \varphi > = < \sigma_\varepsilon; I[\varphi] >,$$

where we have set

$$(91) \qquad < \Sigma_\varepsilon; \varphi > = < \sigma_\varepsilon; I[\varphi] >, \ I[\varphi](t,x) = \int_0^t \varphi(z,x) \ \mathrm{d}z$$
$$\text{for any } \varphi \in L^1(0,T; C(\overline{\Omega}_\varepsilon)).$$

Lighthill's idea [22] is to rewrite the Navier–Stokes–Fourier system (69)–(71) in the form:

$$(92) \qquad \varepsilon \partial_t Z_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = \varepsilon \mathrm{div}_x \mathbf{F}^1_\varepsilon,$$

$$(93) \qquad \varepsilon \partial_t \mathbf{V}_\varepsilon + \omega \nabla_x Z_\varepsilon = \varepsilon \Big(\mathrm{div}_x \mathbb{F}^2_\varepsilon + \nabla_x F^3_\varepsilon + \frac{A}{\varepsilon^2 \omega} \nabla_x \Sigma_\varepsilon \Big),$$

supplemented with the homogeneous Neumann boundary conditions

$$(94) \qquad \mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,$$

where

$$(95)\quad Z_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega}\varrho_\varepsilon\left(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\right) + \frac{A}{\varepsilon\omega}\Sigma_\varepsilon,\ \mathbf{V}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon,$$

(96)

$$\mathbf{F}^1_\varepsilon = \frac{A}{\omega}\varrho_\varepsilon\left(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\right)\mathbf{u}_\varepsilon + \frac{A}{\omega}\frac{\kappa\nabla_x\vartheta_\varepsilon}{\varepsilon\vartheta_\varepsilon},\ \mathbb{F}^2_\varepsilon = \mathbb{S}_\varepsilon - \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon,$$

and

(97)

$$F^3_\varepsilon = \omega\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^2}\right) + A\varrho_\varepsilon\left(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right).$$

Here the constants A and ω has to be chosen to eliminate the first order term in the (formal) asymptotic expansion of the forcing term (97) expressed in terms of the quantities $(\varrho_\varepsilon - \overline{\varrho})/\varepsilon$, $(\vartheta_\varepsilon - \overline{\vartheta})/\varepsilon$, more specifically,

$$(98)\qquad A\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta} = \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta},\ \omega + A\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho} = \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}.$$

Note that the *wave speed* ω is strictly positive as a direct consequence of *hypothesis of thermodynamic stability.*

5.2.2. *Regularization and finite speed of propagation* Our ultimate goal is to show the strong (pointwise a.a.) convergence of the velocities $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ claimed in (84). Since we have assumed (82), the distance to the "outer" boundary dominates the speed of sound proportional to $1/\varepsilon$, and we may therefore replace Ω_ε by Ω.

For (84) to hold it is enough to show

$$(99)\quad \left[t\mapsto\int_\Omega \mathbf{V}_\varepsilon(t,\cdot)\cdot\mathbf{w}\ \mathrm{d}x\right] \to \left[t\mapsto\int_\Omega \mathbf{V}(t,\cdot)\cdot\mathbf{w}\ \mathrm{d}x\right] \text{ in } L^1(0,T)$$

for any fixed $\mathbf{w}\in C^\infty_c(K;R^3)$, where $\mathbf{V}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon$.

Since our task has been reduced to showing (99), we may assume, with help of a simple approximation, that all quantities appearing in the acoustic equations are smooth. Thus our task may be reduced to the following:

Show that the family

$$(100)\qquad \left[t\mapsto\int_\Omega \mathbf{V}_\varepsilon(t,\cdot)\cdot\mathbf{w}\ \mathrm{d}x\right] \textit{ is precompact in } L^1(0,T)$$

for any $\mathbf{w} \in C_c^\infty(K; R^3)$, $K \subset \overline{K} \subset \Omega$ *a bounded ball, provided that*

$$\varepsilon \partial_t Z_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = \varepsilon \mathrm{div}_x \mathbf{F}_\varepsilon^1 \ in\ (0,T) \times \Omega, \tag{101}$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \omega \nabla_x Z_\varepsilon = \varepsilon \mathrm{div}_x \mathbb{F}_\varepsilon^2 \ in\ (0,T) \times \Omega, \tag{102}$$

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{103}$$

$$Z_\varepsilon(0,\cdot) = Z_{0,\varepsilon},\ \mathbf{V}_\varepsilon(0,\cdot) = \mathbf{V}_{0,\varepsilon} \ in\ \Omega, \tag{104}$$

where

$$Z_{0,\varepsilon} \in C_c^\infty(\Omega),\ \{Z_{0,\varepsilon}\}_{\varepsilon>0} \ bounded\ in\ L^2(\Omega),$$

$$\mathbf{V}_{0,\varepsilon} \in C_c^\infty(\Omega; R^3),\ \{\mathbf{V}_{0,\varepsilon}\}_{\varepsilon>0} \ bounded\ in\ L^2(\Omega; R^3),$$

and

$$\mathbf{F}_\varepsilon^1 \in C_c^\infty((0,T) \times \Omega; R^3),\ \mathbb{F}_\varepsilon^2 = \in C_c^\infty((0,T) \times \Omega; R^{3\times 3}),$$

$$\begin{array}{l} \{\mathbf{F}_\varepsilon^1\}_{\varepsilon>0} \ bounded\ in\ L^2(0,T; L^2(\Omega; R^3)), \\ \{\mathbb{F}_{\varepsilon,0}^2\}_{\varepsilon>0} \ bounded\ in\ L^2(0,T; L^2(\Omega; R^{3\times 3})). \end{array} \tag{105}$$

5.2.3. *Compactness of the solenoidal part* Consider $\psi \in W^{1,2} \cap W^{1,\infty}(\Omega; R^3)$, $\mathrm{div}_x \psi = 0$, $\psi \cdot \mathbf{n}|_{\partial\Omega} = 0$. Multiplying equation (102) on ψ and integrating by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathbf{V}_\varepsilon \cdot \psi \ \mathrm{d}x = - \int_\Omega \mathbb{F}_\varepsilon^2 : \nabla_x \psi \ \mathrm{d}x,\ \int_\Omega \mathbf{V}_\varepsilon(0,\cdot) \cdot \psi \ \mathrm{d}x = \int_\Omega \mathbf{V}_{0,\varepsilon} \cdot \psi \ \mathrm{d}x,$$

in particular the family

$$\left[t \mapsto \int_\Omega \mathbf{V}_\varepsilon \cdot \psi \ \mathrm{d}x \right] \text{ is precompact in } C[0,T]. \tag{106}$$

Relation (106) may be viewed as (weak) precompactness of the *solenoidal* component of the vector field $\mathbf{V}_\varepsilon$.

5.2.4. *Abstract variational formulation* Our aim is to rewrite system (101), (102) in terms of an abstract differential operator

$$\Delta_N,\ \Delta_N[v] = \Delta v,\ \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0,\ v(x) \to 0 \text{ as } |x| \to \infty,$$

with

$$\mathcal{D}(\Delta_N) = \{w \in L^2(\Omega) \mid w \in W^{2,2}(\Omega),\ \nabla_x w \cdot \mathbf{n}|_{\partial\Omega} = 0\}.$$

It can be shown that $-\Delta_N$ is a self-adjoint, non-negative operator in $L^2(\Omega)$, with an absolutely continuous spectrum $[0,\infty)$. Moreover, Δ_N satisfies the *limiting absorption principle*

(107)
$$\sup_{\lambda \in C, 0<\alpha \leq \mathrm{Re}[\lambda] \leq \beta < \infty,\ \mathrm{Im}[\lambda] \neq 0} \left\| \mathcal{V} \circ (-\Delta_N - \lambda)^{-1} \circ \mathcal{V} \right\|_{\mathcal{L}[L^2(\Omega);L^2(\Omega)]} \leq c_{\alpha,\beta},$$

where

$$\mathcal{V}(x) = (1 + |x|^2)^{-\frac{s}{2}},\ s > 1$$

(see Leis [18]).

Introducing the *acoustic potential*

$$\Phi_\varepsilon = \Delta_N^{-1}[\mathrm{div}_x \mathbf{V}_\varepsilon], \tag{108}$$

we can rewrite equations (101), (102) in the form

$$\varepsilon \partial_t Z_\varepsilon + \Delta_N \Phi_\varepsilon = \varepsilon \mathrm{div}_x \mathbf{F}_\varepsilon^1,\ \varepsilon \partial_t \Phi_\varepsilon + \omega Z_\varepsilon = \Delta_N^{-1} \mathrm{div}_x \mathrm{div}_x \mathbb{F}_\varepsilon^2. \tag{109}$$

Consequently, the *acoustic potential* Φ_ε may be expressed by means of the standard *Duhamel's formula*:

$$\begin{aligned} \Phi_\varepsilon(t,\cdot) \quad (110) \\ = \exp\left(\pm \mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N}\right) \Bigg[\Delta_N[h_\varepsilon^1] + \frac{1}{\sqrt{-\Delta_N}}[h_\varepsilon^2] \\ \pm \mathrm{i}\left(\Delta_N[h_\varepsilon^3] + \frac{1}{\sqrt{-\Delta_N}}[h_\varepsilon^4]\right)\Bigg] \\ + \int_0^t \exp\left(\pm \mathrm{i}\frac{t-s}{\varepsilon}\sqrt{-\Delta_N}\right) \Bigg[\Delta_N[H_\varepsilon^1] + \frac{1}{\sqrt{-\Delta_N}}[H_\varepsilon^2] \\ \pm \mathrm{i}\left(\Delta_N[H_\varepsilon^3] + \frac{1}{\sqrt{-\Delta_N}}[H_\varepsilon^4]\right)\Bigg] \, \mathrm{d}s, \end{aligned}$$

with certain functions

(111)
$$\{h_\varepsilon^i\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega),\ \{H_\varepsilon^i\}_{\varepsilon>0} \text{ is bounded in } L^2((0,T)\times\Omega),$$

5.2.5. *An abstract result of Kato* In order to show strong convergence of the gradient component of the velocity field, we invoke the space-time decay estimates for the group $\exp(\mathrm{i}t\sqrt{-\Delta_N})$ obtained by Kato [15].

Theorem 8. [Reed and Simon [28, Theorem XIII.25 and Corollary]] *Let A be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that*

$$\Gamma = \sup_{\lambda \notin R,\ v \in \mathcal{D}(A^*),\ \|v\|_X = 1} \|A \circ R_H[\lambda] \circ A^*[v]\|_X < \infty. \tag{112}$$

Then

$$\sup_{w \in X,\ \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \|A \exp(-\mathrm{i}tH)[w]\|_X^2 \ \mathrm{d}t \leq \Gamma^2.$$

The desired conclusion (100) follows by applying Theorem 8 to

$$X = L^2(\Omega),\ H = \sqrt{-\Delta_N},\ A[v] = \varphi G(-\Delta_N)[v],\ v \in X,$$

where

$$G \in C_c^\infty(0, \infty),\ \varphi \in C_c^\infty(\Omega) \text{ are given functions.}$$

5.2.6. *Convergence via RAGE theorem* Kato's result is applicable provided the limit domain and the associated Neumann Laplacean obey the limiting absorption principle (107). This assumption can be ralexed by means of the celebrated RAGE theorem, see Cycon et al. [4, Theorem 5.8]:

Theorem 9. *Let H be a Hilbert space, $A : \mathcal{D}(A) \subset H \to H$ a self-adjoint operator, $C : H \to H$ a compact operator, and P_c the orthogonal projection onto the space of continuity H_c of A, specifically,*

$$H = H_c \oplus \mathrm{cl}_H \Big\{ \mathrm{span}\{ w \in H \mid w \textit{ an eigenvector of } A \} \Big\}.$$

Then

$$\left\| \frac{1}{\tau} \int_0^\tau \exp(-\mathrm{i}tA) C P_c \exp(\mathrm{i}tA) \ \mathrm{d}t \right\|_{\mathcal{L}(H)} \to 0 \textit{ as } \tau \to \infty. \tag{113}$$

RAGE theorem represents both necessary and sufficient condition for the local pointwise converegence of the acoustic waves, namely, the absence of eigenvalues of the Neumann Laplacean in the domain Ω. More detailed discussion concerning propagation of acoustic waves and its characterization by means of spectral measures can be found in [9].

5.3. Inviscid incompressible limits

An interesting situation appears when the fluid is asymptotically incompressible, and, at the same time, the transport coefficients—the viscosity and the heat conductivity—are small. This is the situation when the Mach number is small but Reynolds and Peclet numbers are high. The associated scaled system, in the absence of external forces, reads

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{114}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho, \vartheta) = \varepsilon^a \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}), \tag{115}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \varepsilon^b \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_\mathbf{x} \vartheta)}{\vartheta} \right) = \sigma_\varepsilon, \tag{116}$$

supplemented with the total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\varepsilon} \left(\frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) (t, \cdot)\ \mathrm{d}x = 0, \tag{117}$$

where the entropy production rate σ_ε satisfies

$$\sigma_\varepsilon \geq \frac{1}{\vartheta} \left(\varepsilon^{a+2} \mathbb{S} : \nabla_x \mathbf{u} + \varepsilon^b \frac{\kappa(\vartheta)}{\vartheta} |\nabla_x \vartheta|^2 \right) \geq 0. \tag{118}$$

To avoid problems related to the boundary conditions, and, at the same time, to guarantee the dispersive estimates for the acoustic equation, we consider the problem in $\Omega = R^3$, prescribing the "far field" boundary conditions

$$\varrho \to \overline{\varrho} > 0,\ \vartheta \to \overline{\vartheta},\ \mathbf{u} \to 0 \text{ as } |x| \to \infty.$$

Furthermore, we consider the initial data in the form

$$\varrho(0, \cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \vartheta(0, \cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon}. \tag{119}$$

Under these circumstances, the limit (target) problem can be identified as the incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0, \tag{120}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \tag{121}$$

supplemented with a transport equation for the temperature deviation T,

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0. \tag{122}$$

Here, the function $\mathbf{v}$ is the limit velocity while $T \approx \frac{\vartheta - \overline{\vartheta}}{\varepsilon}$. Note that the system (120)–(122) can be obtained as a hydrodynamic limit of the Boltzmann equation, see Golse [14].

System (120)–(122) is known to possess a smooth solution at least on a short time interval that may depend on the size of the initial data. Solutions of the primitive Navier–Stokes–Fourier system are expected to converge to solutions of (120)–(122) on the interval of existence of the latter.

Let $\mathbf{H}$ denote the standard Helmholtz projection onto the space of solenoidal functions. We report the following result, [11, Theorem 3.1]:

Theorem 10. *Let the thermodynamic functions p, e, and s as well as the transport coefficients μ and κ comply with the hypotheses of Theorem 1, with $\alpha = 1$. Let*

$$b > 0, \ 0 < a < \frac{10}{3}. \tag{123}$$

Furthermore, take the initial data (119) in such a way that

$$\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0}, \ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^2 \cap L^\infty(R^3),$$

$$\varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0, \ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(R^3),$$

and

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2(R^3; R^3), \ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(R^3; R^3),$$

where

$$\varrho^{(1)}_0, \ \vartheta^{(1)}_0 \in W^{1,2} \cap W^{1,\infty}(R^3), \ \mathbf{H}[\mathbf{u}_0] = \mathbf{v}_0 \in W^{2,k}(R^3; R^3)$$

$$\text{for a certain } k > \frac{5}{2}.$$

Let $T_{\max} \in (0, \infty]$ denote the maximal life-span of the regular solution $\mathbf{v}$ to the Euler system (120), (121) satisfying $\mathbf{v}(0, \cdot) = \mathbf{v}_0$. Finally, let $\{\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}$ be a very weak solution of the Navier–Stokes–Fourier system in $(0, T) \times R^3$, $T < T_{\max}$.

Then

$$\operatorname{ess}\sup_{t\in(0,T)} \| \varrho_\varepsilon(t, \cdot) - \overline{\varrho} \|_{L^2 + L^{5/3}(R^3)} \leq \varepsilon c,$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}}\ \mathbf{v}\ \textit{in}\ L^\infty_{\rm loc}((0,T];L^2_{\rm loc}(R^3;R^3))$$
$$\textit{and weakly-}(*)\ \textit{in}\ L^\infty(0,T;L^2(R^3;R^3)),$$

and

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to T\ \textit{in}\ L^\infty_{\rm loc}((0,T];L^q_{\rm loc}(R^3;R^3))$$
$$\textit{and weakly-}(*)\ \textit{in}\ L^\infty(0,T;L^2 + L^q(R^3)),\ 1 \le q < 2,$$

where $\mathbf{v}$, T *is the unique solution of the Euler–Boussinesq system* (120)–(122), *with the initial data*

$$\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0],\ T_0 = \overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta_0^{(1)} - \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\varrho_0^{(1)}.$$

The proof of Theorem 10 is done by means of the relative entropy inequality (62) and illustrates the strength of the relative entropy method delineated in Section 4. Results of this type for a simpler compressible Navier–Stokes system (without temperature) were obtained by Masmoudi [25], [26].

The leading idea is to take

$$\mathbf{U} = \nabla_x \Phi_\varepsilon + \mathbf{v},\ r = \overline{\varrho} + \varepsilon R_\varepsilon,\ \Theta = \overline{\vartheta} + \varepsilon T_\varepsilon$$

in the relative entropy inequality (62), where where $\mathbf{v}$ is the solution to the incompressible Euler system, while R_ε, T_ε, and Φ_ε solve the *acoustic equation*:

$$\varepsilon\partial_t(\alpha R_\varepsilon + \beta T_\varepsilon) + \omega\Delta\Phi_\varepsilon = 0,$$
$$\varepsilon\partial_t\nabla_x\Phi_\varepsilon + \nabla_x(\alpha R_\varepsilon + \beta T_\varepsilon) = 0,$$
$$\alpha = \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho},\ \beta = \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta},\ \omega = \overline{\varrho}\left(\alpha + \frac{\beta^2}{\delta}\right).$$

Noting that the functions R_ε, T_ε are not uniquely determined, we introduce the *transport equation*

(124) $\partial_t(\delta T_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x(\delta T_\varepsilon - \beta R_\varepsilon) + (\delta T_\varepsilon - \beta R_\varepsilon)\mathrm{div}_x\mathbf{U}_\varepsilon = 0,$

with

$$\delta = \overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}.$$

Equation (124) is nothing other than a convenient linearization of the entropy balance. The resulting system of equations is now well-posed.

Acknowledgements. The work of E. F. was supported by Grant 201/09/0917 of GAČR as a part of the general research programme of the Academy of Sciences of the Czech Republic, RVO: 67985840.

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