Advanced Studies in Pure Mathematics 63, 2012
Galois–Teichmüller Theory and Arithmetic Geometry
pp. 601–627

Lie algebras of Galois representations on fundamental groups

Zdzisław Wojtkowiak

Abstract.

In this paper we are studying Lie algebras associated with Galois representations on the fundamental groups of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_{2^n})$ and $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_{3^n})$.

§0. Introduction

The Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_n))$ acts on the étale fundamental group $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_n); \overrightarrow{01})$. One of the most interesting problems is to describe the image of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_n))$ in the group of automorphisms of $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_n); \overrightarrow{01})$. To simplify the situation one usually considers pro-l quotient of π_1 and then an infinitesimal version of the problem (see [1] and [7]).

Let $Lie(X, Y_0, \ldots, Y_{n-1})$ be a free Lie algebra over $\mathbb{Q}_l$ on $n+1$ free generators $X, Y_0, \ldots, Y_{n-1}$. We equipped it with the Ihara bracket $\{\,,\}$ and we denote the resulting Lie algebra by $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$.

In the infinitesimal version of the problem we get a representation of the associated graded Lie algebra, denoted by $L_l(\mathbb{Z}[\mu_n][\frac{1}{n}])$, of a certain weighted Tate completion of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_n))$ into the Lie algebra $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$. In Section 2, as the first step to understand the Galois action on π_1, we are studying the Lie algebra $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$. The results of Section 2 are very elementary and very likely well known.

In Sections 1 and 3 there are collected some general facts about Galois actions on fundamental groups of the projective line minus a finite number of points taken from the previous papers of the author.

Received March 30, 2011.
Revised December 15, 2011.
The travel expenses to participate in the Kyoto conferences were supported by JSPS KAKENHI 21340009.

In Sections 4 and 5 we are studying the infinitesimal version of Galois actions on fundamental groups of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_{2^n})$ and $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_{3^n})$. In the case of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_{2^n})$ we extend slightly our previous result (see [13, Corollary 15.6.3.]). We show that the image of the Lie subalgebra of $L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])$ in $Lie(X, Y_0, \dots, Y_{2^n-1})_{\{\}}$ generated by all generators in degrees greater than one and 3/4 of generators in degree one is free. We prove also the analogous result for $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_{3^n})$.

Finally, in the last section we show that the Lie algebra representation associated with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\pi_1^{\text{ét}}(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \overrightarrow{0p})$, where p is a prime number, has a big kernel.

Acknowledgments. This research was inspired by the talk given by P. Deligne in Schloss Ringberg (see [2]). Parts of this paper were written during our visit in Max-Planck-Institut für Mathematik in Bonn. We would like to thank very much MPI for support. Section 6 was added after receiving the referee report, though we mentioned this example in our Kyoto talk.

0.1 Notation Let L be a Lie algebra. We define

$$\Gamma^2 L := [L, L] .$$

Let X and Y belong to L. We shall use the following inductively defined short hand notation:

$$[Y, X^{(m)}] := \begin{cases} Y & \text{if } m = 0 \\ [[Y, X^{(m-1)}], X] & \text{for } m > 0. \end{cases}$$

We denote by $\chi : G_K \to \mathbb{Z}_l^\times$ the l-adic cyclotomic character. We denote by $\mathbb{N}$ the set of positive integers $1, 2, 3, \dots$.

§1. Galois action on π_1

In this section we review some results and constructions from our previous papers (see [11], [12], [13], [14] and [16]).

Let K be a number field. Let $a_1, \dots, a_n \in K$. Let us set

$$V := \mathbb{P}^1_K \setminus \{a_1, \dots, a_n, \infty\}.$$

Let v and z be K-points of V or tangential K-points of V. Let l be a fixed prime. We denote by

$$\pi_1(V_{\bar{K}}; v)$$

the maximal pro-l quotient of the étale fundamental group of $V_{\bar{K}}$ based at v and by $\pi(V_{\bar{K}}; z, v)$ the $\pi_1(V_{\bar{K}}; v)$-torsor of l-adic paths from v to z. Let v_i be a tangential K-point on $V_{\bar{K}}$ at a_i for $i = 1, \dots, n$. Let $s_i \in \pi_1(V_{\bar{K}}; v_i)$ be a generator of the inertia group of a place over a_i and let $\gamma_i \in \pi(V_{\bar{K}}; v_i, v)$.

We set

$$x_i := \gamma_i^{-1} \cdot s_i \cdot \gamma_i$$

for $i = 1, \dots, n$. The elements $x_1, \dots, x_n$ are free generators of $\pi_1(V_{\bar{K}}; v)$.

Let γ be a path from v to z. For any $\sigma \in G_K$ we set

$$\mathfrak{f}_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi_1(V_{\bar{K}}; v).$$

Proposition 1.1. (see [11, Proposition 2.2.1.]) The action of G_K on $\pi_1(V_{\bar{K}}; v)$ is given by the formulas

$$\sigma(x_i) = \mathfrak{f}_{\gamma_i}(\sigma)^{-1} \cdot x_i^{\chi(\sigma)} \cdot \mathfrak{f}_{\gamma_i}(\sigma)$$

for $i = 1, \dots, n$.

Let $\mathbb{Q}_l\{\{X_1, \dots, X_n\}\}$ be a $\mathbb{Q}_l$-algebra of formal power series on non-commuting variables $X_1, \dots, X_n$ equipped with the topology of the projective limit. We denote by

$$\mathrm{Aut}(\mathbb{Q}_l\{\{X_1, \dots, X_n\}\})$$

the group of continuous automorphisms of the $\mathbb{Q}_l$-algebra $\mathbb{Q}_l\{\{X_1, \dots, X_n\}\}$ and by

$$\mathrm{GL}(\mathbb{Q}_l\{\{X_1, \dots, X_n\}\})$$

the group of continuous linear automorphisms of the $\mathbb{Q}_l$-vector space $\mathbb{Q}_l\{\{X_1, \dots, X_n\}\}$.

Let $\mathbb{Q}_l\{X_1, \dots, X_n\}$ be a $\mathbb{Q}_l$-algebra of polynomials on non-commuting variables $X_1, \dots, X_n$. We denote by

$$\mathrm{Der}(\mathbb{Q}_l\{X_1, \dots, X_n\})$$

the Lie algebra of derivations of the $\mathbb{Q}_l$-algebra $\mathbb{Q}_l\{X_1, \dots, X_n\}$ and by

$$\mathrm{End}(\mathbb{Q}_l\{X_1, \dots, X_n\})$$

the $\mathbb{Q}_\ell$-vector space of endomorphisms of the $\mathbb{Q}_l$-vector space $\mathbb{Q}_l\{X_1, \dots, X_n\}$.

Observe that $\mathrm{Der}(\mathbb{Q}_l\{X_1, \dots, X_n\})$ is the associated graded Lie algebra of the Lie algebra of $\mathrm{Aut}(\mathbb{Q}_l\{\{X_1, \dots, X_n\}\})$ and $\mathrm{End}(\mathbb{Q}_l\{X_1, \dots, X_n\})$ is the associated graded Lie algebra of the Lie algebra of $\mathrm{GL}(\mathbb{Q}_l\{\{X_1, \dots, X_n\}\})$.

We define a continuous multiplicative embedding

$$E : \pi_1(V_{\bar{K}}; v) \to \mathbb{Q}_l\{\{X_1, \dots, X_n\}\}$$

setting $E(x_i) = \exp(X_i)$ for $i = 1, \dots, n$.

Let set

$$\Lambda_\gamma(\sigma) := E(\mathfrak{f}_\gamma(\sigma)).$$

The action of G_K on $\pi_1(V_{\bar{K}}; v)$ induces an action of G_K on $\mathbb{Q}_l\{\{X_1, \dots, X_n\}\}$ by automorphisms of $\mathbb{Q}_l$-algebra. Hence we get the representation

$$\varphi_v : G_K \to \mathrm{Aut}(\mathbb{Q}_l\{\{X_1, \dots, X_n\}\}) .$$

It follows from Proposition 1.1 that

$$\varphi_v(\sigma)(X_i) = \Lambda_{\gamma_i}(\sigma)^{-1} \cdot (\chi(\sigma)X_i) \cdot \Lambda_{\gamma_i}(\sigma)$$

for $i = 1, \dots, n$.

Let $t_\gamma : \pi(V_{\bar{K}}; z, v) \to \pi_1(V_{\bar{K}}; v)$ be given by $t_\gamma(\delta) = \gamma^{-1} \cdot \delta$. Composing t_γ with the embedding E we get an embedding of $\pi(V_{\bar{K}}; z, v)$ into $\mathbb{Q}_l\{\{X_1, \dots, X_n\}\}$. The action of G_K on the torsor of paths $\pi(V_{\bar{K}}; z, v)$ induces a linear action of G_K on $\mathbb{Q}_l\{\{X_1, \dots, X_n\}\}$. Hence we get the representation

$$\psi_\gamma : G_K \to \mathrm{GL}(\mathbb{Q}_l\{\{X_1, \dots, X_n\}\}) .$$

It follows from [11, Lemma 1.0.2] that for any $w \in \mathbb{Q}_l\{\{X_1, \dots, X_n\}\}$ we have

$$\psi_\gamma(\sigma)(w) = \Lambda_\gamma(\sigma) \cdot (\varphi_v(\sigma)(w)) .$$

We shall write also $\varphi_{V,v}$ and $\psi_{V,\gamma}$ instead of φ_v and ψ_γ if we want to indicate the dependence of representations on the algebraic variety V.

We denote by

$$\{\mathfrak{l} \mid l\}_K$$

the set of finite places of K lying over the prime ideal (l) of $\mathbb{Z}$.

The representations φ_v and ψ_γ are weighted Tate representations (see [15, Proposition 1.0.3, Theorem 2.1 and Proposition 2.3]). We recall below the definition of weighted Tate representations (see [5] and [6]).

Definition 1.2. Let K be a number field and let S be a finite set of finite places of K. Let W be a finite dimensional vector space over $\mathbb{Q}_l$ equipped with an increasing filtration

$$\{ W_i \}_{n \le i \le m}$$

such that $W_n = 0$, $W_m = W$ and $W_i = W_{i-1}$ for i even. A continuous representation

$$\phi : G_K \to \mathrm{GL}(W)$$

is called a weighted Tate representation unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$ if

i) $\phi(W_i) \subset W_i$ for all i;
ii) ϕ acts on W_{2i}/W_{2i-2} by the $(-i)$th power of the l-adic cyclotomic character;
iii) ϕ is unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$.

Projective limits of weighted Tate representations unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$ we shall call also weighted Tate representations.

Let us assume that the pair (V, v) and the triple (V, z, v) have good reduction outside a finite set S of finite places of K. Then the representations φ_v and ψ_γ are unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$, hence they factor through the weighted Tate $\mathbb{Q}_l$-completion $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ of $\pi_1(\mathrm{Spec}\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; \mathrm{Spec}\bar{K})$ (see [5] and [6]).

Let $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ be the associated graded Lie algebra with respect to the weight filtration of the affine prounipotent proalgebraic group

$$\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) := \mathrm{Ker}\big(\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathbb{G}_m\big).$$

The representations φ_v and ψ_γ induce morphisms of graded Lie algebras

$$\Phi_v : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathrm{Der}(\mathbb{Q}_l\{X_1, \dots, X_n\})$$

and

$$\Psi_{z,v} : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathrm{End}(\mathbb{Q}_l\{X_1, \dots, X_n\})$$

respectively.

In degree one the Lie algebra $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ has more generators than the corresponding Lie algebra of the motivic fundamental group of mixed Tate motives over $\mathrm{Spec}\mathcal{O}_{K,S}$. These additional generators come from divisors of (l) which are not in S. Hain and Matsumoto showed in [6], using the notion of crystalline representations, how to construct weighted Tate completion, when S does not contain $\{\mathfrak{l} \mid l\}_K$.

In [16] we gave a different, elementary construction of the corresponding graded Lie algebra when S does not contain $\{\mathfrak{l} \mid l\}_K$. We recall briefly our construction of this Lie algebra (see [16, Section 1]).

Let $u \in \mathcal{O}^\times_{K,S\cup\{\mathfrak{l}|l\}_K}$. The Kummer character of u induces

$$\kappa(u) : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_1 \to \mathbb{Q}_l\,,$$

where $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_1$ is the subspace of elements of degree one in $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$. Let us set

$$(\mathfrak{l}\mid l)_{K,S} := \bigcap_{u\in\mathcal{O}_{K,S}^\times} \big(\mathrm{Ker}\big(\kappa(u) : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_1 \to \mathbb{Q}_l\big)\big).$$

We define

$$\langle\mathfrak{l}\mid l\rangle_{K,S}$$

to be the Lie ideal of $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ generated by $(\mathfrak{l}\mid l)_{K,S}$. We define

$$L_l(\mathcal{O}_{K,S}) := L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)/\langle\mathfrak{l}\mid l\rangle_{K,S}.$$

The Lie algebra $L_l(\mathcal{O}_{K,S})$ is graded. We have

$$L_l(\mathcal{O}_{K,S}) = \bigoplus_{i=1}^{\infty} L_l(\mathcal{O}_{K,S})_i\,,$$

where $L_l(\mathcal{O}_{K,S})_i$ is the vector subspace of elements of degree i.

The graded Lie algebra $L_l(\mathcal{O}_{K,S})$ has the same number of generators as the Lie algebra of the motivic fundamental group of mixed Tate motives over $\mathrm{Spec}\mathcal{O}_{K,S}$ (see [16, Proposition 1.3]). The construction of the graded Lie algebra $L_l(\mathcal{O}_{K,S})$ is functorial (see [16, Proposition 2.3]).

It follows from [16, Theorem 3.1] that the representations Φ_v and $\Psi_{z,v}$ factor through the graded Lie algebra $L_l(\mathcal{O}_{K,S})$. The induced representations we shall also denote by

$$\Phi_v : L_l(\mathcal{O}_{K,S}) \to \mathrm{Der}(\mathbb{Q}_l\{X_1,\dots,X_n\})$$
$$\text{and } \Psi_{z,v} : L_l(\mathcal{O}_{K,S}) \to \mathrm{End}(\mathbb{Q}_l\{X_1,\dots,X_n\})$$

or by

$$\Phi_{V,v} \text{ and } \Psi_{V,z,v}$$

if we want to indicate the dependence on the algebraic variety V.

Finally we recall the definition of l-adic polylogarithms from [12]. Let γ be a path on $\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,\infty\}$ from $\overrightarrow{01}$ to a K-point of $\mathbb{P}^1_K\setminus\{0,1,\infty\}$ or to a tangential point defined over K. After the standard embedding of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,\infty\};\overrightarrow{01})$ into $\mathbb{Q}_l\{\{X,Y\}\}$ we get a formal power series

$$\Lambda_\gamma(\sigma) \in \mathbb{Q}_l\{\{X,Y\}\}.$$

Let I_2 be an ideal of $\mathbb{Q}_l\{\{X,Y\}\}$ generated by monomials with two or more Y's. The l-adic polylogarithms $l_n(z)_\gamma$ and the l-adic logarithm

$l(z)_\gamma$ are functions from G_K to $\mathbb{Q}_l$. They are coefficients of the formal power series $\log \Lambda_\gamma(\sigma)$. They are defined by the following congruence

$$\log \Lambda_\gamma(\sigma) \equiv l(z)_\gamma(\sigma)X + \sum_{n=1}^{\infty} l_n(z)_\gamma(\sigma)[Y, X^{(n-1)}] \mod I_2.$$

The l-adic logarithm $l(z)_\gamma$ is the Kummer character associated to z and $l_1(z)_\gamma$ is the Kummer character associated to $1-z$. The l-adic polylogarithms satisfy the same functional equations as the classical complex polylogarithms. In [12] we showed that the inversion relation and the distribution relations are satisfied by l-adic polylogarithms. We shall use them in the last two sections.

§2. Lie algebras

Let K be a field. The set of Lie polynomials in the K-algebra $K\{\{X, Y_0, \dots, Y_{n-1}\}\}$ of formal power series on non-commuting variables $X, Y_0, \dots, Y_{n-1}$ we denote by

$$Lie(X, Y_0, \dots, Y_{n-1}).$$

It is a free Lie algebra over a field K on $n+1$ free generators $X, Y_0, \dots, Y_{n-1}$. The Lie bracket we denote by $[\,,]$.

Let $A = A(X, Y_0, \dots, Y_{n-1}) \in Lie(X, Y_0, \dots, Y_{n-1})$. We define a derivation

$$D_A : Lie(X, Y_0, \dots, Y_{n-1}) \to Lie(X, Y_0, \dots, Y_{n-1})$$

by the formulas

$$D_A(X) = 0 \text{ and } D_A(Y_i) = [Y_i, A(X, Y_i, Y_{i+1}, \dots, Y_{i+n-1})]$$

for $i = 0, 1, \dots, n-1$. The sum $a+b$ is calculated modulo n. Observe that

$$D_A \circ D_B - D_B \circ D_A = D_{[A,B]+D_A(B)-D_B(A)}. \tag{2.1}$$

We denote by $\mathrm{Der}(Lie(X, Y_0, \dots, Y_{n-1}))$ the Lie algebra of all derivations of $Lie(X, Y_0, \dots, Y_{n-1})$. It follows from (2.1) that

$$\mathrm{Der}^*_{\mathbb{Z}/n}(Lie(X, Y_0, \dots, Y_{n-1})) :=$$

$$\{D_A \in \mathrm{Der}(Lie(X, Y_0, \dots, Y_{n-1})) \mid A \in Lie(X, Y_0, \dots, Y_{n-1})\}$$

is a Lie subalgebra of $\mathrm{Der}(Lie(X, Y_0, \dots, Y_{n-1}))$.

Let $\langle a\rangle$ be a one-dimensional vector subspace of $Lie(X, Y_0, \dots, Y_{n-1})$ generated by a. The map

$$Lie(X, Y_0, \dots, Y_{n-1})/\langle Y_0\rangle \to \mathrm{Der}^*_{\mathbb{Z}/n}(Lie(X, Y_0, \dots, Y_{n-1})), \quad A \mapsto D_A$$

is an isomorphism of vector spaces.

We define a new bracket $\{\,,\}$, called the Ihara bracket (see [8]), on the vector space $Lie(X, Y_0, \dots, Y_{n-1})/\langle Y_0\rangle$ by the formula

$$\{A, B\} := [A, B] + D_A(B) - D_B(A). \tag{2.2}$$

It follows from (2.1) that the bracket $\{\,,\}$ satisfies the Jacobi identity. Hence the vector space $Lie(X, Y_0, \dots, Y_{n-1})/\langle Y_0\rangle$ equipped with $\{\,,\}$ is a Lie algebra, which we shall denote by

$$Lie(X, Y_0, \dots, Y_{n-1})_{\{\,\}}.$$

Observe that the one-dimensional vector subspace $\langle X\rangle$ is a Lie ideal of the Lie algebra $Lie(X, Y_0, \dots, Y_{n-1})_{\{\,\}}$. Hence

$$Lie(X, Y_0, \dots, Y_{n-1})_{\{\,\}} = \langle X\rangle \oplus (Lie(X, Y_0, \dots, Y_{n-1})_{\{\,\}})/\langle X\rangle$$

as Lie algebras.

If n is a prime number greater than 3 then the Lie algebra $(Lie(X, Y_0, \dots, Y_{n-1})_{\{\,\}})/\langle X\rangle$ is not free (see [16, Proposition 8.1] and also [14, Proposition 20.5] for $n = 5$ and [4, Theorem 4.1] for $n = 7$).

The main method to show that a family of elements of the Lie algebra $Lie(X, Y_0, \dots, Y_{n-1})_{\{\,\}}$ generates a free Lie subalgebra of $Lie(X, Y_0, \dots, Y_{n-1})_{\{\,\}}$ is to show that the Lie bracket $\{\,,\}$ on these elements modulo some vector subspace reduces to the standard Lie bracket $[\,,]$ of the Lie algebra $Lie(X, Y_0, \dots, Y_{n-1})$ (see [2] and [3]). Hence first we shall study some useful Lie ideals and Lie subalgebras of the Lie algebras $Lie(X, Y_0, \dots, Y_{n-1})$ and $Lie(X, Y_0, \dots, Y_{n-1})_{\{\,\}}$.

We denote by

$$I_r$$

the Lie ideal of $Lie(X, Y_0, \dots, Y_{n-1})$ generated by Lie brackets in generators $X, Y_0, \dots, Y_{n-1}$, which contain at least r elements (possible with repetitions) of the set $\{Y_0, \dots, Y_{n-1}\}$.

It is clear that $I_{r+1} \subset I_r$. The filtration $\{I_r\}_{r\in\mathbb{N}}$ of $Lie(X, Y_0, \dots, Y_{n-1})$ is called the depth filtration. Observe that

$$Lie(X, Y_0, \dots, Y_{n-1})/I_2 \approx \langle X\rangle \oplus \bigoplus_{i=0}^{n-1}\bigoplus_{k=0}^{\infty}\langle [Y_i, X^{(k)}]\rangle \tag{2.3}$$

as vector spaces. Observe that in the quotient Lie algebra $Lie(X, Y_0, \ldots, Y_{n-1})/I_2$ the classes of $[Y_i, X^{(k)}]$ and $[Y_j, X^{(l)}]$ commute and the Lie bracket of the class of $[Y_i, X^{(k)}]$ and of the class of X is the class of $[Y_i, X^{(k+1)}]$, i.e. $[[Y_i, X^{(k)}], X] = [Y_i, X^{(k+1)}]$.

It is clear that I_r is also a Lie ideal of $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\ \}}$. The quotient Lie algebra

$$Lie(X, Y_0, \ldots, Y_{n-1})_{\{\ \}}/I_2 \approx \langle X \rangle \oplus \Big(\bigoplus_{i=0}^{n-1} \bigoplus_{k=0}^{\infty} \langle [Y_i, X^{(k)}] \rangle\Big)/\langle Y_0 \rangle$$

is commutative. We do not know if the natural surjection

$$\begin{aligned}&Lie(X, Y_0, \ldots, Y_{n-1})_{\{\ \}}/\Gamma^2\big(Lie(X, Y_0, \ldots, Y_{n-1})_{\{\ \}}\big) \\ &\longrightarrow Lie(X, Y_0, \ldots, Y_{n-1})_{\{\ \}}/I_2\end{aligned}$$

is an isomorphism.

Let S be a subset of $\{0, 1, \ldots, n-1\}$. Let $r \geq 2$. We denote by

$$I_r(S)$$

the Lie ideal of $Lie(X, Y_0, \ldots, Y_{n-1})$ generated by Lie brackets in generators $X, Y_0, \ldots, Y_{n-1}$, which contain at least r elements (possible with repetitions) of the set $\{Y_0, Y_1, \ldots, Y_{n-1}\}$ and at least one of these elements is Y_s with $s \in S$ and at least one of these elements is Y_t with $t \notin S$.

It is clear that $I_{r+1}(S) \subset I_r(S)$ for any $r \geq 2$. Hence we have a filtration $\{I_r(S)\}_{r\geq 2}$ of $Lie(X, Y_0, \ldots, Y_{n-1})$ by Lie ideals. Notice that $I_r(S)$ is not a Lie ideal of the Lie algebra $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\ \}}$.

Let $A(S)$ be a Lie subalgebra of $Lie(X, Y_0, \ldots, Y_{n-1})$ generated by elements X and Y_s with $s \in S$. We set

$$I_r(A(S)) := I_r \cap A(S).$$

We identify the set $\{0, 1, \ldots, n-1\}$ with $\mathbb{Z}/n$. If $S \subset \{0, 1, \ldots, n-1\}$ then the subsets $S + S := \{a + b \in \mathbb{Z}/n \mid a, b \in S\}$ and $-S := \{-a \in \mathbb{Z}/n \mid a \in S\}$ are well defined.

Lemma 2.4. Let S be a subset of $\{0, 1, \ldots, n-1\}$ such that $(S+S)\cap S = \emptyset$. Let $r, r_1 \geq 2$ and $p, p_1 \geq 1$. Let $w \in I_r(S)$, $w_1 \in I_{r_1}(S)$ and $a \in I_p(A(S))$, $a_1 \in I_{p_1}(A(S))$. Then

$$\begin{aligned}&[w, w_1] \in I_{r+r_1}(S),\ \ D_w(w_1) \in I_{r+r_1}(S),\ \ [a, w] \in I_{r+p}(S), \\ &D_a(w) \in I_{r+p}(S), D_w(a) \in I_{r+p}(S),\ \ D_a(a_1) \in I_{p+p_1}(S).\end{aligned}$$

Proof. It is clear that $[w, w_1]$ and $D_w(w_1)$ belong to $I_{r+r_1}(S)$ as well as that $[a, w]$ and $D_a(w)$ belong to $I_{r+p}(S)$.

Let $s, s_1 \in S$. Observe that $D_{Y_{s_1}}(Y_s) = [Y_s, Y_{s+s_1}]$. The element $s \in S$ and $s + s_1 \notin S$ by the assumption that $(S + S) \cap S = \emptyset$. Hence it follows that $D_w(a) \in I_{r+p}(S)$ and $D_a(a_1) \in I_{p+p_1}(S)$. Q.E.D.

Corollary 2.5. The assumptions are the same as in Lemma 2.4. Then

$$\{a, w\} = [a, w] + D_a(w) - D_w(a) \in I_{r+p}(S),$$
$$\{w, w_1\} = [w, w_1] + D_w(w_1) - D_{w_1}(w) \in I_{r+r_1}(S)$$

and

$$\{a, a_1\} \equiv [a, a_1] \bmod I_{p+p_1}(S).$$

Let $V(S)$ be a vector subspace of $Lie(X, Y_0, \ldots, Y_{n-1})$ generated by Lie brackets in $X, Y_0, \ldots, Y_{n-1}$ which contain at least one Y_s with $s \in S$. $V(S)$ is clearly a Lie ideal of $Lie(X, Y_0, \ldots, Y_{n-1})$.

Proposition 2.6. Let S be a subset of $\{0, 1, \ldots, n-1\}$ such that $(S + S) \cap S = \emptyset$. Then

i) $V(S)$ is a Lie subalgebra of the Lie algebra $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\ \}}$.
ii) For any $r \geq 2$, $I_r(S) \subset V(S)$.
iii) $I_r(S)$ is a Lie ideal of $V(S)$ considered as a Lie subalgebra of the Lie algebra $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\ \}}$ and $\{I_r(S), V(S)\} \subset I_{r+1}(S)$.

Proof. Let $a = a(Y_s \ldots)$ and $b = b(Y_{s_1} \ldots)$ belong to $V(S)$. Then $\{a, b\} = [a, b] + D_a(b) - D_b(a) \in V(S)$, hence $V(S)$ is a Lie subalgebra of $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\ \}}$. It is clear that $I_r(S) \subset V(S)$. If $a \in V(S)$ and $w \in I_r(S)$ then $[a, w] \in I_{r+1}(S)$, $D_a(w) \in I_{r+1}(S)$. The assumption $(S+S) \cap S = \emptyset$ implies that $D_w(a) \in I_{r+1}(S)$. Hence $\{a, w\} \in I_{r+1}(S) \subset I_r(S)$. Q.E.D.

Let $\mathcal{L}$ be a Lie algebra over a field K. Let $\mathcal{Z} = \{z_i \in \mathcal{L} \mid i \in \mathbb{N}\}$ be a linearly independent subset of $\mathcal{L}$. We denote by

$$\mathcal{L}(\mathcal{Z})$$

a Lie subalgebra of $\mathcal{L}$ generated by the subset $\mathcal{Z}$.

Let $Lie(Z_i \mid i \in \mathbb{N})$ be a free Lie algebra over K on symbols Z_i, $i \in \mathbb{N}$. We denote by

$$Hallbasis(Z_i \mid i \in \mathbb{N})$$

the set of basic Lie elements of $Lie(Z_i \mid i \in \mathbb{N})$ formed from the sequence $Z_1, Z_2, \ldots, Z_n, \ldots$ following the rules described in [9] on pages 322–327. The set $Hallbasis(Z_i \mid i \in \mathbb{N})$ is a basis of the vector space $Lie(Z_i \mid i \in \mathbb{N})$ (see [9, Theorem 5.8.]).

Let

$$\mathcal{P} : Lie(Z_i \mid i \in \mathbb{N}) \to \mathcal{L}$$

be a morphism of Lie algebras given by $\mathcal{P}(Z_i) = z_i$ for $i \in \mathbb{N}$. Observe that the image of $\mathcal{P}$ is the Lie algebra $\mathcal{L}(\mathcal{Z})$. We denote by

$$\mathcal{HB}(\mathcal{Z})_{\mathcal{L}}$$

the image of the set $Hallbasis(Z_i \mid i \in \mathbb{N})$ by the morphism $\mathcal{P}$.

Lemma 2.7. The Lie subalgebra $\mathcal{L}(\mathcal{Z})$ of $\mathcal{L}$ is free, freely generated by the subset $\mathcal{Z}$ of $\mathcal{L}$ if and only if the set $\mathcal{HB}(\mathcal{Z})_{\mathcal{L}}$ is linearly independent over K.

Proof. If the Lie algebra $\mathcal{L}(\mathcal{Z})$ is free, freely generated by the subset $\mathcal{Z}$ of $\mathcal{L}$ then it follows from [9, Theorem 5.8.] that the subset $\mathcal{HB}(\mathcal{Z})_{\mathcal{L}}$ of $\mathcal{L}$ is linearly independent.

If the subset $\mathcal{HB}(\mathcal{Z})_{\mathcal{L}}$ of $\mathcal{L}$ is linearly independent then the morphism of Lie algebras $\mathcal{P} : Lie(Z_i \mid i \in \mathbb{N}) \to \mathcal{L}(\mathcal{Z})$ is an isomorphism. Hence the Lie algebra $\mathcal{L}(\mathcal{Z})$ is free, freely generated by $\mathcal{Z}$. Q.E.D.

In the next proposition we indicate how to construct various free Lie subalgebras of the Lie algebra $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$.

Proposition 2.8. Let $S \subset \{0, 1, \ldots, n-1\}$ be such that $(S+S) \cap S = \emptyset$. Let $z_s^k \in Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$ for $s \in S$ and $k \in \mathbb{N}$ be such that

$$z_s^k \equiv [Y_s, X^{(k-1)}] \mod I_2. \tag{2.8.1}$$

Then the elements z_s^k for $s \in S$ and $k \in \mathbb{N}$ generate freely a free Lie subalgebra of the Lie algebra $Lie(X, Y_0, \ldots, Y_{n-1})_{\{\,\}}$.

Proof. Let us set $y_s^k := [Y_s, X^{(k-1)}]$. Let $\mathcal{Z} := \{z_s^k \mid s \in S,\, k \in \mathbb{N}\}$ and $\mathcal{Y} := \{y_s^k \mid s \in S,\, k \in \mathbb{N}\}$. Observe that the subset $\mathcal{Y}$ of $Lie(X, Y_0, \ldots, Y_{n-1})$ is linearly independent. It follows from the congruences (2.8.1) that the subset $\mathcal{Z}$ is also linearly independent.

The Lie subalgebra of $Lie(X, Y_0, \ldots, Y_{n-1})$ generated by $\mathcal{Y}$ is free, freely generated by $\mathcal{Y}$ by the Shirshov–Witt theorem (see [9, page 331]). Hence Lemma 2.7 implies that the subset $\mathcal{HB}(\mathcal{Y})_{Lie(X,Y_0,\ldots,Y_{n-1})}$ of $Lie(X, Y_0, \ldots, Y_{n-1})$ is linearly independent.

It follows from the congruences (2.8.1) that for any arrangements of brackets of length m

$$(2.8.2) \qquad \{..\{z_{s_1}^{k_1}, z_{s_2}^{k_2}\}.., z_{s_m}^{k_m}\} \equiv \{..\{y_{s_1}^{k_1}, y_{s_2}^{k_2}\}.., y_{s_m}^{k_m}\} \mod I_{m+1}.$$

Observe that

$$\{y_{s_1}^{k_1}, y_{s_2}^{k_2}\} \equiv [y_{s_1}^{k_1}, y_{s_2}^{k_2}] \mod I_2(S).$$

It follows by induction from Corollary 2.5 that for any arrangements of brackets of length m

$$(2.8.3) \qquad \{..\{y_{s_1}^{k_1}, y_{s_2}^{k_2}\}.., y_{s_m}^{k_m}\} \equiv [..[y_{s_1}^{k_1}, y_{s_2}^{k_2}].., y_{s_m}^{k_m}] \mod I_m(S).$$

The set $\mathcal{HB}(\mathcal{Y})_{Lie(X,Y_0,\dots,Y_{n-1})}$ is linearly independent. Hence it follows from the congruences (2.8.3) that the set $\mathcal{HB}(\mathcal{Y})_{Lie(X,Y_0,\dots,Y_{n-1})_{\{\ \}}}$ is also linearly independent. Therefore it follows from the congruences (2.8.2) that the set $\mathcal{HB}(\mathcal{Z})_{Lie(X,Y_0,\dots,Y_{n-1})_{\{\ \}}}$ is linearly independent.

It follows from Lemma 2.7 that the elements of $\mathcal{Z}$ generate freely a free Lie subalgebra of $Lie(X, Y_0, \dots, Y_{n-1})_{\{\ \}}$. Q.E.D.

We finish this section with a definition which will be very useful in the last two sections.

Definition 2.9. We define

$$Pol(X, Y_0, \dots, Y_{n-1})$$

to be a K-vector subspace of $Lie(X, Y_0, \dots, Y_{n-1})$ generated by X and by Lie brackets $[Y_i, X^{(k-1)}]$ for $0 \le i < n$ and $k = 1, 2, \dots$. We define

$$\mathcal{P}r : Lie(X, Y_0, \dots, Y_{n-1}) \to Pol(X, Y_0, \dots, Y_{n-1})$$

to be a projection such that $\operatorname{Ker} \mathcal{P}r = I_2$.

§3. Galois action on the fundamental group of the projective line minus 0, ∞ and the nth roots of 1

From now on let

$$V := \mathbb{P}^1_{\mathbb{Q}(\mu_n)} \setminus (\{0, \infty\} \cup \mu_n).$$

The generators

$$x,\ y_0, \dots,\ y_{n-1}$$

of $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$ we choose as in [13]. Let p be the standard path from $\overrightarrow{01}$ to $\overrightarrow{10}$ on $V_{\bar{\mathbb{Q}}}$. The Galois group $G_{\mathbb{Q}(\mu_n)}$ acts on $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$. The action is described in the next proposition.

Proposition 3.1. (see [13, Proposition 15.1.7]). Let $\sigma \in G_{\mathbb{Q}(\mu_n)}$. We have

$$\sigma(x) = x^{\chi(\sigma)},$$

$$\sigma(y_0) = \big(\mathfrak{f}_p(\sigma)(x, y_0, \dots, y_{n-1})\big)^{-1} \cdot y_0^{\chi(\sigma)} \cdot \mathfrak{f}_p(\sigma)(x, y_0, \dots, y_{n-1})$$

and

$$\sigma(y_k) = x^{\frac{-(\chi(\sigma)-1)k}{n}} \cdot \big(\mathfrak{f}_p(\sigma)(x, y_k, \dots, y_{n-1}, x^{-1}y_0x, \dots, x^{-1}y_{k-1}x)\big)^{-1}$$
$$\cdot\, y_k^{\chi(\sigma)} \cdot \mathfrak{f}_p(\sigma)(x, y_k, \dots, y_{n-1}, x^{-1}y_0x, \dots, x^{-1}y_{k-1}x) \cdot x^{\frac{(\chi(\sigma)-1)k}{n}}$$

for $1 < k < n$.

As usual we embed $\pi_1(V_{\bar{\mathbb{Q}}}; \vec{01})$ into the $\mathbb{Q}_l$-algebra $\mathbb{Q}_l\{\{X, Y_0, \dots, Y_{n-1}\}\}$ by the continuous multiplicative map

$$E : \pi_1(V_{\bar{\mathbb{Q}}}; \vec{01}) \to \mathbb{Q}_l\{\{X, Y_0, \dots, Y_{n-1}\}\}$$

such that $E(x) = \exp X$ and $E(y_i) = \exp Y_i$ for $i = 0, 1, \dots, n-1$.

Let S_n be the finite set of finite places of $\mathbb{Q}(\mu_n)$ lying over prime divisors of n. Then the ring $\mathcal{O}_{\mathbb{Q}(\mu_n), S_n}$ is equal

$$\mathbb{Z}[\mu_n][\frac{1}{n}].$$

The pair $(V, \vec{01})$ has good reduction outside the prime divisors of n. Hence it follows from [16, Theorem 3.1] that the action of $G_{\mathbb{Q}(\mu_n)}$ on $\pi_1(V_{\bar{\mathbb{Q}}}; \vec{01})$ induces a morphism of graded Lie algebras

$$\Phi_{V,\vec{01}} : L_l(\mathbb{Z}[\mu_n][\frac{1}{n}]) \to Lie(X, Y_0, \dots, Y_{n-1})_{\{\,\}}.$$

Let us set $\xi_n := e^{\frac{2\pi\sqrt{-1}}{n}}$. The next result follows immediately from [13, Lemma 15.3.1].

Proposition 3.2.

i) Let $k > 1$ and let $\sigma \in L_l(\mathbb{Z}[\mu_n][\frac{1}{n}])_k$. Then

$$\Phi_{V,\vec{01}}(\sigma) \equiv \sum_{i=0}^{n-1} l_k(\xi_n^{-i})(\sigma)[Y_i, X^{(k-1)}] \mod I_2.$$

ii) Let $\sigma \in L_l(\mathbb{Z}[\mu_n][\frac{1}{n}])_1$. Then we have

$$\Phi_{V,\vec{01}}(\sigma) = \sum_{i=1}^{n-1} l(1 - \xi_n^{-i})(\sigma)Y_i.$$

§4. Projective line minus 0, ∞ and the 2^nth roots of 1

Let

$$V := \mathbb{P}^1_{\mathbb{Q}(\mu_{2^n})} \setminus (\{0,\infty\} \cup \mu_{2^n}).$$

The pair $(V, \overrightarrow{01})$ has good reduction everywhere outside (2). Notice that the ring $\mathcal{O}_{\mathbb{Q}(\mu_{2^n}),(2)} = \mathbb{Z}[\mu_{2^n}][\frac{1}{2}]$. The action of $G_{\mathbb{Q}(\mu_{2^n})}$ on $\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})$ induces the morphism of graded Lie algebras

$$\Phi_{V,\overrightarrow{01}} : L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}]) \to Lie(X, Y_0, \ldots, Y_{2^n-1})_{\{\}}.$$

The number of complex places of the field $\mathbb{Q}(\mu_{2^n})$ is 2^{n-2}. Hence the Lie algebra $L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])$ has 2^{n-2} free generators in each degree greater than one. In degree one, the Lie algebra $L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])$ has $\dim_{\mathbb{Q}}(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}]^\times \otimes \mathbb{Q}) = 2^{n-2}$ generators. Generators in degree one are constructed in the following way. The 2-units $(1-\xi^i_{2^n})$ for $0 < i < 2^{n-1}$ and $(i,2) = 1$ generate freely $(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])^\times \otimes \mathbb{Q}$. Therefore the Kummer characters

$$l(1-\xi^i_{2^n})$$

for $0 < i < 2^{n-1}$ and i odd form a base of the $\mathbb{Q}_l$-vector space

$$\mathrm{Hom}_{\mathbb{Q}_l}\left(L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])_1, \mathbb{Q}_l\right).$$

Let

$$\sigma^{(1)}_i$$

for $0 < i < 2^{n-1}$ and i odd be the dual base of $L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])_1$, i.e.

$$l(1-\xi^i_{2^n})(\sigma^{(1)}_j) = \delta^j_i .$$

Observe that we have the following equalities between 2-units:

$$(1-\xi^i_{2^n}) = -\xi^i_{2^n} \cdot (1-\xi^{-i}_{2^n});$$

$$(1-\xi^1_4) \cdot (1-\xi^3_4) = 2$$

and

$$(1-\xi^i_{2^k}) \cdot (1-\xi^{i+2^{k-1}}_{2^k}) = (1-\xi^i_{2^{k-1}})$$

for $0 < i < 2^{n-1}$, i odd and $k = n, n-1, \ldots, 3$. Hence we get the following relations between the Kummer characters $l(1-\xi^i_{2^n})$ on $L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])_1$:

$$(4.1) \qquad l(1-\xi^i_{2^n}) = l(1-\xi^{-i}_{2^n});$$

$$l(1-\xi_4^1)+l(1-\xi_4^3)=l(2) \tag{4.2}$$

and

$$l(1-\xi_{2^k}^i)+l(1-\xi_{2^k}^{i+2^{k-1}})=l(1-\xi_{2^{k-1}}^i) \tag{4.3}$$

for $0<i<2^{n-1}$, i odd and $k=n,n-1,\dots,3$.

Lemma 4.4. We have

$$\Phi_{V,\vec{01}}(\sigma_i^{(1)})=Y_i+Y_{2^n-i}+\sum_{a=1}^{n-2}(Y_{2^a i}+Y_{2^n-2^a i})+2Y_{2^{n-1}}.$$

Proof. The lemma follows from Proposition 3.2, the equalities (4.1), (4.2) and (4.3) and the very definition of the elements $\sigma_i^{(1)}$. Q.E.D.

Generators of the Lie algebra $L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])$ in degree k greater than one we denote by

$$\sigma_i^{(k)}$$

for $0<i<2^{n-1}$ and i odd and we choose them to be dual to l-adic polylogarithms

$$l_k(\xi_{2^n}^j)$$

for $0<j<2^{n-1}$ and j odd in the sense that

$$l_k(\xi_{2^n}^j)(\sigma_i^{(k)})=\delta_i^j.$$

We recall that l-adic polylogarithms satisfy the inversion relations (see [12, Corollary 11.2.6])

$$l_k(\xi_{2^n}^j)+(-1)^k l_k(\xi_{2^n}^{-j})=0 \tag{4.5}$$

and the distribution relations (see [12, Corollary 11.2.3])

$$(2^n)^{k-1}\cdot\sum_{i=0}^{2^n-1} l_k(\xi_{2^n}^i)=l_k(1) \tag{4.6}$$

and

$$2^{k-1}\big(l_k(\xi_{2^j}^i)+l_k(\xi_{2^j}^{i+2^{j-1}})\big)=l_k(\xi_{2^{j-1}}^i) \tag{4.7}$$

for $j=n,n-1,\dots,2$ and i odd.

We recall from Section 2, Definition 2.9 that

$$\mathcal{P}r: Lie(X,Y_0,\dots,Y_{2^n-1})\to Pol(X,Y_0,\dots,Y_{2^n-1})$$

is the natural projection.

Lemma 4.8. Let k be greater than 1. Then we have

$$\Phi_{V,\vec{01}}(\sigma_i^{(k)}) \equiv \mathcal{P}r(\Phi_{V,\vec{01}}(\sigma_i^{(k)})) \mod I_2$$

and

$$\mathcal{P}r(\Phi_{V,\vec{01}}(\sigma_i^{(k)})) \equiv [Y_i, X^{(k-1)}] + [Y_{2^n-i}, X^{(k-1)}] \mod 2\,.$$

Proof. The first congruence is clear from the definition of the projection $\mathcal{P}r$. Proposition 3.2 implies that

$$\Phi_{V,\vec{01}}(\sigma) \equiv \sum_{i=0}^{2^n-1} l_k(\xi_{2^n}^{-i})(\sigma)[Y_i, X^{(k-1)}] \mod I_2\,.$$

Observe that

$$l_k(\xi_{2^n}^{-j})(\sigma_j^{(k)}) = (-1)^{k-1},\ \ l_k(\xi_{2^n}^{j})(\sigma_j^{(k)}) = 1 \ \text{ and } \ l_k(\xi_{2^n}^{i})(\sigma_j^{(k)}) = 0$$

for i odd and $i \notin \{j, -j\}$. It follows from the relations (4.5), (4.6) and (4.7) that $l_k(\xi_{2^n}^{-j})$ for j even is equal 2 times a linear combination with $\mathbb{Z}_{(2)}$-coefficients of $l_k(\xi_{2^n}^{i})$ for $0 < i < 2^{n-1}$ and i odd. Hence $l_k(\xi_{2^n}^{i})(\sigma_j^{(k)})$ for i even is a multiple of 2 in $\mathbb{Z}_{(2)}$. Therefore

$$\mathcal{P}r(\Phi_{V,\vec{01}}(\sigma_j^{(k)})) = \sum_{i=0}^{2^n-1} l_k(\xi_{2^n}^{-i})(\sigma_j^{(k)})[Y_i, X^{(k-1)}]$$

and

$$\sum_{i=0}^{2^n-1} l_k(\xi_{2^n}^{-i})(\sigma_j^{(k)})[Y_i, X^{(k-1)}] \equiv [Y_j, X^{(k-1)}] + [Y_{2^n-j}, X^{(k-1)}] \mod 2\,.$$

Q.E.D.

We define elements $\sigma_i^{(1)}$ for any i odd setting

$$\sigma_i^{(1)} := \sigma_{2^n-i}^{(1)} \ \text{ for } \ 2^{n-1} < i < 2^n$$
$$\text{and } \ \sigma_i^{(1)} := \sigma_{i\pm 2^n}^{(1)} \ \text{ for } \ i < 0 \ \text{ or } \ i > 2^n\,.$$

Definition 4.9. Let

$$\mathcal{L}_{2^n}$$

be a Lie subalgebra of $L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}])$ generated by differences

$$\sigma_i^{(1)} - \sigma_{-i+2^{n-1}}^{(1)},\ \sigma_{i+2^{n-2}}^{(1)} - \sigma_{-i+2^{n-1}}^{(1)},\ \sigma_{-i+2^{n-2}}^{(1)} - \sigma_{-i+2^{n-1}}^{(1)}$$

for $0 < i < 2^{n-3}$ and i odd and by the elements

$$\sigma_i^{(k)}$$

for $k > 1$ and $0 < i < 2^{n-1}$ and i odd.
(The four elements $\sigma_i^{(1)}$, $\sigma_{-i+2^{n-2}}^{(1)}$, $\sigma_{i+2^{n-2}}^{(1)}$ and $\sigma_{-i+2^{n-1}}^{(1)}$ are such that in their images by $\Phi_{V,\vec{01}}$ there appear exactly the same Y_t's with $t \equiv 0$ modulo 4.)

Theorem 4.10. The morphism of graded Lie algebras

$$\Phi_{V,\vec{01}} : L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}]) \to Lie(X, Y_0, \ldots, Y_{2^n-1})_{\{\}}.$$

restricted to the Lie subalgebra $\mathcal{L}_{2^n}$ is injective.

Proof. We define elements of the Lie algebra $Lie(X, Y_0, \ldots, Y_{2^n-1})_{\{\}}$ by

$$\begin{aligned} z_i^{(1)} &:= Y_i + Y_{2^n-i} + Y_{2i} + Y_{2^n-2i} \\ &\quad - (Y_{2^{n-1}-i} + Y_{2^n-2^{n-1}+i} + Y_{2^n-2i} + Y_{2i}), \\ z_{-i+2^{n-2}}^{(1)} &:= Y_{2^{n-2}-i} + Y_{2^n-2^{n-2}+i} + Y_{2^{n-1}-2i} + Y_{2^n-2^{n-1}+2i} \\ &\quad - (Y_{2^{n-1}-i} + Y_{2^n-2^{n-1}+i} + Y_{2^n-2i} + Y_{2i}), \\ z_{i+2^{n-2}}^{(1)} &:= Y_{i+2^{n-2}} + Y_{2^n-2^{n-2}-i} + Y_{2i+2^{n-1}} + Y_{2^n-2^{n-1}-2i} \\ &\quad - (Y_{2^{n-1}-i} + Y_{2^n-2^{n-1}+i} + Y_{2^n-2i} + Y_{2i}), \end{aligned}$$

for $0 < i < 2^{n-3}$ and i odd and

$$z_i^{(k)} := [Y_i, X^{(k-1)}] + [Y_{2^n-i}, X^{(k-1)}]$$

for $k > 1$ and $0 < i < 2^{n-1}$ and i odd. Let us denote by

$$\mathcal{Z}$$

the set of all these elements.

Observe that if i is odd then one of the two numbers i and $2^n - i$ is congruent to 1 modulo 4 and the other is congruent to 3 modulo 4.

Let us write

$$z_j^{(k)} = y_j^{(k)} + d_j^{(k)}, \tag{4.10.1}$$

where in $y_j^{(k)}$ there appear only Y_α's with $\alpha \equiv 1$ modulo 4 and in $d_j^{(1)}$ there appear only Y_β's with $\beta \equiv 3$ modulo 4 and Y_β's with $\beta \equiv 2$ modulo 4. Let

$$\mathcal{Y}$$

be the set of all elements $y_j^{(k)}$. Let us denote by

$$I(0,2,3)$$

a Lie ideal of $Lie(X, Y_0, \ldots, Y_{2^n-1})$ generated by Y_t's with $t \equiv \varepsilon$ modulo 4, where $\varepsilon \in \{0,2,3\}$.

We shall show by induction the following two statements:
A(r): for any Lie bracket of length r and any r elements of $\mathcal{Z}$, $z_1 = y_1 + d_1, \ldots, z_r = y_r + d_r$ we have

$$\{..\{z_1, z_2\}.., z_r\} \equiv [..[y_1, y_2].., y_r] \mod I(0,2,3), \tag{4.10.2}$$

B(r): a Lie bracket only with Y_α's such that $\alpha \equiv 0$ modulo 4 does not appear in the decomposition of $\{..\{z_1, z_2\}.., z_r\}$.

The statements **A(1)** and **B(1)** are true. For any two elements $z = y + d$ and $z_1 = y_1 + d_1$ of the set $\mathcal{Z}$ one checks that

$$\{z, z_1\} \equiv [y, y_1] \mod I(0,2,3)\,.$$

Observe also that a Lie bracket only with Y_α's such that $\alpha \equiv 0$ mod 4 does not appear in the decomposition of $\{z, z_1\}$ in a Hall base of $Lie(X, Y_0, \ldots, Y_{2^n-1})$. Hence the statements **A(2)** and **B(2)** are also true.

Let us assume that the statements **A(k)** and **B(k)** are proved for all $k \le m$. Let $1 \le r \le m$ and $1 \le s \le m$. Let $z_i = y_i + d_i$, $i = 1, \ldots, r$ and $z'_j = y'_j + d'_j$, $j = 1, \ldots, s$ be elements of $\mathcal{Z}$ decomposed accordingly to (4.10.1). Let

$$Z = \{..\{z_1, z_2\}.., z_r\}\,, \quad Y = [..[y_1, y_2].., y_r]\,,$$

$$Z' = \{..\{z'_1, z'_2\}.., z'_s\}\,, \quad Y' = [..[y'_1, y'_2].., y'_s]\,.$$

By the inductive hypothesis we have

$$Z = Y + d \text{ and } Z' = Y' + d'\,,$$

where $d, d' \in I(0,2,3)$. Calculating the Lie bracket $\{Z, Z'\}$ we get

$$\begin{gathered}\{Z, Z'\} = [Y, Y'] + [Y, d'] + [d, Y'] + [d, d'] + \\ D_Y(Y') + D_Y(d') + D_d(Y') + D_d(d') - D_{Y'+d'}(Y + d)\,.\end{gathered} \tag{4.10.3}$$

The assumption $\mathbf{B(r)}$ (resp. $\mathbf{B(s)}$) implies that $D_d(Y') \in I(0,2,3)$ (resp. $D_{d'}(Y) \in I(0,2,3)$). It is clear that all other terms on the left hand side of the equality (4.10.3) except $[Y,Y']$ belong to $I(0,2,3)$. Hence it follows that

$$\{Z,Z'\} \equiv [Y,Y'] \mod I(0,2,3)\,.$$

Therefore we have proved the statement $\mathbf{A(m+1)}$. From the form of all terms on the right hand side of (4.10.3) it is clear that $\mathbf{B(m+1)}$ is also true.

Let K be a field. At this stage of the proof we shall consider Lie algebras $Lie(X,Y_0,\dots,Y_{2^n-1})$, $Lie(X,Y_0,\dots,Y_{2^n-1})_{\{\,\}}$ to be Lie algebras over K. We fix a Hall base of the Lie algebra $Lie(X,Y_0,\dots,Y_{2^n-1})$. The coefficients of elements of $\mathcal{Z}$ and $\mathcal{Y}$ are integers with respect to the Hall base. Observe that the subset $\mathcal{Y}$ of $Lie(X,Y_0,\dots,Y_{2^n-1})$ is linearly independent over any field K including the field $\mathbb{F}_2 = \mathbb{Z}/2$. Therefore the Lie subalgebra $Lie(X,Y_0,\dots,Y_{2^n-1})(\mathcal{Y})$ of $Lie(X,Y_0,\dots,Y_{2^n-1})$ is free freely generated by the set $\mathcal{Y}$. It follows from Lemma 2.7 that the set $\mathcal{HB}(\mathcal{Y})_{Lie(X,Y_0,\dots,Y_{2^n-1})}$ is linearly independent over K. Hence it follows from the congruence (4.10.2) that the set $\mathcal{HB}(\mathcal{Z})_{Lie(X,Y_0,\dots,Y_{2^n-1})_{\{\,\}}}$ is also linearly independent over any field K including the field $\mathbb{F}_2$.

Let us set

$$\zeta_i^{(k)} := \Phi_{V,\vec{01}}(\sigma_i^{(k)}) \text{ and } w_i^{(k)} := \mathcal{P}r(\zeta_i^{(k)})$$

for $k>1$ and $0<i<2^{n-1}$ and i odd. For $k=1$ and $0<i<2^{n-3}$ and i odd we set

$$\zeta_i^{(1)} := \Phi_{V,\vec{01}}(\sigma_i^{(1)} - \sigma_{-i+2^{n-1}}^{(1)})\,,\ \ w_i^{(1)} := \zeta_i^{(1)}\,,$$

$$\zeta_{i+2^{n-2}}^{(1)} := \Phi_{V,\vec{01}}(\sigma_{i+2^{n-2}}^{(1)} - \sigma_{-i+2^{n-1}}^{(1)})\,,\ \ w_{i+2^{n-2}}^{(1)} := \zeta_{i+2^{n-2}}^{(1)}\,,$$

$$\zeta_{-i+2^{n-2}}^{(1)} := \Phi_{V,\vec{01}}(\sigma_{-i+2^{n-2}}^{(1)} - \sigma_{-i+2^{n-1}}^{(1)})\,,\ \ w_{-i+2^{n-2}}^{(1)} := \zeta_{-i+2^{n-2}}^{(1)}\,.$$

Observe that

$$\zeta_i^{(1)} = w_i^{(1)} = z_i^{(1)} \tag{4.10.4}$$

for i odd and $0<i<2^{n-2}+2^{n-3}$. It follows from Lemma 4.8 that for $k>1$ we have

$$\zeta_i^{(k)} \equiv w_i^{(k)} \mod I_2 \tag{4.10.5}$$

and

$$w_i^{(k)} \equiv z_i^{(k)} \mod 2\,. \tag{4.10.6}$$

Let

$$\Sigma$$

be the set of all elements $\zeta_i^{(k)}$ and let

$$\mathcal{W}$$

be the set of all elements $w_i^{(k)}$.

Let $\zeta_1, \dots, \zeta_r \in \Sigma$, $w_1, \dots, w_r \in \mathcal{W}$ and $z_1, \dots, z_r \in \mathcal{Z}$ be such

$$\zeta_i \equiv w_i \mod I_2 \text{ and } w_i \equiv z_i \mod 2\,.$$

Then it follows from (4.10.4) and (4.10.5) that for any Lie bracket of length r in the Lie algebra $Lie(X, Y_0, \dots, Y_{2^n-1})_{\{\,\}}$ over $\mathbb{Q}_l$ we have

$$\{..\{\zeta_1, \zeta_2\}.., \zeta_r\} \equiv \{..\{w_1, w_2\}.., w_r\} \mod I_{r+1}\,. \tag{4.10.7}$$

The elements of $\mathcal{W}$ and $\mathcal{Z}$ have coefficients in $\mathbb{Z}_{(2)}$ in a Hall base. It follows from (4.10.4) and (4.10.6) that

$$\{..\{w_1, w_2\}.., w_r\} \equiv \{..\{z_1, z_2\}.., z_r\} \mod 2\,.$$

We have seen already that the set $\mathcal{HB}(\mathcal{Z})_{Lie(X,Y_0,\dots,Y_{2^n-1})_{\{\,\}}}$ is linearly independent in the Lie algebra $Lie(X, Y_0, \dots, Y_{2^n-1})_{\{\,\}}$ over $\mathbb{F}_2$. Therefore the set $\mathcal{HB}(\mathcal{W})_{Lie(X,Y_0,\dots,Y_{2^n-1})_{\{\,\}}}$ is linearly independent in the Lie algebra $Lie(X, Y_0, \dots, Y_{2^n-1})_{\{\,\}}$ over $\mathbb{F}_2$, hence also over $\mathbb{Q}$ and $\mathbb{Q}_l$. Hence it follows from the congruence (4.10.7) that the set

$$\mathcal{HB}(\Sigma)_{Lie(X,Y_0,\dots,Y_{2^n-1})_{\{\,\}}}$$

is linearly independent over $\mathbb{Q}_l$. Therefore the Lie subalgebra of $Lie(X, Y_0, \dots, Y_{2^n-1})_{\{\,\}}$ generated by the set Σ is free, freely generated by the set Σ. Hence it follows that the morphism

$$\Phi_{V,\vec{01}} : L_l(\mathbb{Z}[\mu_{2^n}][\frac{1}{2}]) \to Lie(X, Y_0, \dots, Y_{2^n-1})_{\{\,\}}$$

restricted to the Lie subalgebra $\mathcal{L}_{2^n}$ is injective. Q.E.D.

§5. Projective line minus 0, ∞ and the 3^nth roots of 1

Let

$$V := \mathbb{P}^1_{\mathbb{Q}(\mu_{3^n})} \setminus (\{0, \infty\} \cup \mu_{3^n}).$$

The action of $G_{\mathbb{Q}(\mu_{3^n})}$ on $\pi_1(V_{\bar{\mathbb{Q}}};\overrightarrow{01})$ induces the morphism of graded Lie algebras

$$\Phi_{V,\overrightarrow{01}} : L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}]) \to Lie(X, Y_0, \dots, Y_{3^n-1})_{\{\}}.$$

The 3-units $(1-\xi_{3^n}^i)$ for $0 < i < \frac{3^n}{2}$ and $(i,3) = 1$ generate freely $(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])^{\times} \otimes \mathbb{Q}$. Therefore the Kummer characters $l(1-\xi_{3^n}^i)$ for $0 < i < 3^{n-1}$ and $(i,3) = 1$ form a base of $\mathrm{Hom}_{\mathbb{Q}_l}\big(L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])_1; \mathbb{Q}_l\big)$. Let

$$\sigma_i^{(1)}$$

for $0 < i < \frac{3^n}{2}$ and $(i,3) = 1$ be the dual base of $L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])_1$.

For farther applications we introduce the following convention

$$\sigma_{3^n-i}^{(1)} := \sigma_i^{(1)} \text{ and } \sigma_{i\pm 3^n}^{(1)} := \sigma_i^{(1)}.$$

There are the following relations between 3-units. For any $0 < i < \frac{3^n}{2}$, $(i,3) = 1$ and $k = n, n-1, \dots, 2$ we have

$$(1-\xi_{3^k}^i)\cdot(1-\xi_{3^k}^{i+3^{k-1}})\cdot(1-\xi_{3^k}^{i+2\cdot 3^{k-1}}) = (1-\xi_{3^{k-1}}^i)$$

and

$$(1-\xi_{3^k}^i) = -\xi_{3^k}^i(1-\xi_{3^k}^{-i}).$$

Hence we get the following relations between the Kummer characters $l(1-\xi_{3^k}^i)$ on $L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])_1$

$$l(1-\xi_{3^k}^i) + l(1-\xi_{3^k}^{i+3^{k-1}}) + l(1-\xi_{3^k}^{i+2\cdot 3^{k-1}}) = l(1-\xi_{3^{k-1}}^i) \tag{5.1}$$

and

$$l(1-\xi_{3^k}^i) = l(1-\xi_{3^k}^{-i}). \tag{5.2}$$

Lemma 5.3. We have

$$\Phi_{V,\overrightarrow{01}}(\sigma_i^{(1)}) = Y_i + Y_{3^n-i} + \sum_{k=1}^{n-1}(Y_{3^k i} + Y_{3^n-3^k i})$$

for $0 < i < \frac{3^n}{2}$ and $(i,3) = 1$.

Proof. The lemma follows from Proposition 3.2, the relations (5.1) and (5.2) and the very definition of the elements $\sigma_i^{(1)}$. Q.E.D.

The field $\mathbb{Q}(\mu_{3^n})$ has 3^{n-1} complex places. Hence in degree k greater than one, the graded Lie algebra $L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])$ has 3^{n-1} free generators. We denote by

$$\sigma_i^{(k)}$$

for $0 < i < \frac{3^n}{2}$ and $(i,3) = 1$, generators in degree $k > 1$. We chose them to be dual to l-adic polylogarithms $l_k(\xi_{3^n}^j)$ for $0 < j < \frac{3^n}{2}$ and $(j,3) = 1$, in the sense that

$$l_k(\xi_{3^n}^j)(\sigma_i^{(k)}) = \delta_i^j \, .$$

The l-adic polylogarithms $l_k(\xi_{3^n}^j)$ satisfy the inversion relation

$$l_k(\xi_{3^n}^i) + (-1)^k l_k(\xi_{3^n}^{-i}) = 0 \, , \tag{5.4}$$

and the distribution relations

$$3^{n(k-1)}(\sum_{i=0}^{3^n-1} l_k(\xi_{3^n}^i)) = l_k(1) \tag{5.5}$$

and

$$3^{k-1} \cdot (l_k(\xi_{3^j}^i) + l_k(\xi_{3^j}^{i+3^{j-1}}) + l_k(\xi_{3^j}^{i+2\cdot 3^{j-1}})) = l_k(\xi_{3^{j-1}}^i) \tag{5.6}$$

for $0 < i < \frac{3^n}{2}$, $(i,3) = 1$ and $j = n, n-1, \dots, 2$. It follows from the relations (5.4), (5,5) and (5.6) that

$$3^{n(k-1)}(\sum_{i=1\ (3,i)=1}^{3^n-1} d_i l_k(\xi_{3^n}^i)) = (1 - 3^{n(k-1)}) l_k(1) \tag{5.7}$$

for certain $d_i \in \mathbb{Z}$.

Lemma 5.8. We have

$$\Phi_{V,\vec{01}}(\sigma_i^{(k)}) \equiv \mathcal{P}r(\Phi_{V,\vec{01}}(\sigma_i^{(k)})) \mod I_2$$

for $k > 1$,

$$\mathcal{P}r(\Phi_{V,\vec{01}}(\sigma_i^{(k)})) \equiv [Y_i, X^{(k-1)}] + [Y_{3^n-i}, X^{(k-1)}] \mod 3$$

for $k > 1$ and odd,

$$\mathcal{P}r(\Phi_{V,\vec{01}}(\sigma_i^{(k)})) \equiv -[Y_i, X^{(k-1)}] + [Y_{3^n-i}, X^{(k-1)}] \mod 3$$

for $k > 1$ and even.

Proof. Lemma follows from Proposition 3.2, the inversion relation (5.4), the distribution relations (5.5), (5.6), (5.7) and the definition of the generators $\sigma_i^{(k)}$ as duals to l-adic polylogarithms. Q.E.D.

Unfortunately once more we are not able to show that the morphism

$$\Phi_{V,\vec{01}} : L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}]) \to Lie(X, Y_0, \dots, Y_{3^n-1})_{\{\,\}}$$

is injective. We shall define a certain Lie subalgebra of $L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])$ and we shall show that the map $\Phi_{V,\vec{01}}$ restricted to this subalgebra is injective.

Definition 5.9. Let

$$\mathcal{L}_{3^n}$$

be a Lie subalgebra of $L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}])$ generated by differences

$$\sigma_i^{(1)} - \sigma_{i+2\cdot 3^{n-1}}^{(1)},\ \sigma_{i+3^{n-1}}^{(1)} - \sigma_{i+2\cdot 3^{n-1}}^{(1)}$$

for $0 < i < \frac{3^n}{2\cdot 3}$ and $(i,3) = 1$ and by elements $\sigma_j^{(k)}$ for $k > 1$, $0 < j < \frac{3^n}{2}$ and $(j,3) = 1$.

Notice that $\mathcal{L}_{3^n}$ is a free Lie algebra, freely generated by the elements indicated in the definition.

Theorem 5.10. The morphism of graded Lie algebras

$$\Phi_{V,\vec{01}} : L_l(\mathbb{Z}[\mu_{3^n}][\frac{1}{3}]) \to Lie(X, Y_0, \dots, Y_{3^n-1})_{\{\,\}}$$

restricted to the Lie subalgebra $\mathcal{L}_{3^n}$ is injective.

Proof. Let us set

$$z_i^{(1)} := Y_i + Y_{3^n-i} - Y_{i+2\cdot 3^{n-1}} - Y_{3^n-(i+2\cdot 3^{n-1})}\,,$$

$$z_{i+3^{n-1}}^{(1)} := Y_{i+3^{n-1}} + Y_{3^n-(i+3^{(n-1)})} - Y_{i+2\cdot 3^{n-1}} - Y_{3^n-(i+2\cdot 3^{n-1})}$$

for $0 < i < \frac{3^n}{2\cdot 3}$, $(i,3) = 1$ and

$$z_i^{(k)} := (-1)^{k-1}[Y_i, X^{(k-1)}] + [Y_{3^n-i}, X^{(k-1)}]$$

for $k > 1$ and $0 < i < \frac{3^n}{2}$ and $(i,3) = 1$. Let us denote by

$$\mathcal{Z}$$

the set of all these elements.

Let $k > 1$. We shall write

$$z_i^{(k)} = y_i^{(k)} + d_i^{(k)},$$

where $y_i^{(k)} = (-1)^{k-1}[Y_i, X^{(k-1)}]$ and $d_i^{(k)} = [Y_{3^n-i}, X^{(k-1)}]$ if $i \equiv 1$ (3) and vice versa if $i \equiv 2$ (3). Similarly we decompose

$$z_i^{(1)} = y_i^{(1)} + d_i^{(1)},$$

where in $y_i^{(1)}$ there appear only Y_α's with $\alpha \equiv 1$ (3) and in $d_i^{(1)}$ there appear only Y_β's with $\beta \equiv 2$ (3). Let

$$\mathcal{Y}$$

be the set of all elements $y_i^{(k)}$.

We denote by

$$I(0,2)$$

a Lie ideal of $Lie(X, Y_0, \ldots, Y_{3^n-1})$ generated by Y_t's with $t \equiv 2$ (3) and by Y_t's with $t \equiv 0$ (3).

The rest of the proof is same as in the proof of Theorem 4.10 in Section 4. Q.E.D.

§6. An example when the Lie algebra representation associated with Galois action has a big kernel

Let p be a prime number. The pair $(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}, \overrightarrow{0p})$ has good reduction everywhere outside p. Hence the action of $G_{\bar{\mathbb{Q}}}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \overrightarrow{0p})$ induces

$$\Phi_{\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}, \overrightarrow{0p}} : L_l(\mathbb{Z}[\frac{1}{p}]) \to Lie(X,Y)_{\{\ \}}\,.$$

We shall show that the Lie algebra representation $\Phi_{\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}, \overrightarrow{0p}}$ has a very big kernel. First we describe the action of $G_{\mathbb{Q}}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \overrightarrow{0p})$. Let γ be the standard path from $\overrightarrow{01}$ to $\overrightarrow{10}$ and let α be the standard path from $\overrightarrow{0p}$ to $\overrightarrow{01}$. Then $\delta = \gamma \cdot \alpha$ is a path from $\overrightarrow{0p}$ to $\overrightarrow{10}$. Let x and y be the standard generators of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \overrightarrow{01})$. Then

$$x_1 := \alpha^{-1} \cdot x \cdot \alpha \quad \text{and} \quad y_1 := \alpha^{-1} \cdot y \cdot \alpha$$

are generators of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \overrightarrow{0p})$.

Lemma 6.1. Let $\sigma \in G_{\mathbb{Q}}$. Then we have

$$\sigma(x_1) = x_1^{\chi(\sigma)}\,,$$

$$\sigma(y_1) = x_1^{-l(1/p)(\sigma)} \cdot \alpha^{-1} \cdot \mathfrak{f}_\gamma(\sigma)^{-1} \cdot \alpha \cdot y_1^{\chi(\sigma)} \cdot \alpha^{-1} \cdot \mathfrak{f}_\gamma(\sigma) \cdot \alpha \cdot x_1^{l(1/p)(\sigma)}\,.$$

Proof. Observe that $\sigma(y_1) = \mathfrak{f}_\delta(\sigma)^{-1} \cdot y_1^{\chi(\sigma)} \cdot \mathfrak{f}_\delta(\sigma)$ by [11, Proposition 2.1.]. It follows from [11, Lemma 1.0.6.] that $\mathfrak{f}_\delta(\sigma) = \alpha^{-1} \cdot \mathfrak{f}_\gamma(\sigma) \cdot \alpha \cdot \mathfrak{f}_\alpha(\sigma)$. Hence it rests to calculate $\mathfrak{f}_\alpha(\sigma)$.

Let t (resp. z) be a local parameter at 0 corresponding to the tangential point $\overrightarrow{0p}$ (resp. $\overrightarrow{01}$). Then $\mathfrak{f}_\alpha(\sigma) = \alpha^{-1} \cdot \sigma \cdot \alpha \cdot \sigma^{-1}$ acts on t^{1/l^n} as follows

$$t^{1/l^n} \xrightarrow{\sigma^{-1}} t^{1/l^n} \xrightarrow{\alpha} (1/p^{1/l^n})z^{1/l^n} \xrightarrow{\sigma} \xi_{l^n}^{l(1/p)(\sigma)}(1/p^{1/l^n})z^{1/l^n}$$
$$\xrightarrow{\alpha^{-1}} \xi_{l^n}^{l(1/p)(\sigma)} t^{1/l^n}\,.$$

Therefore we get that $\mathfrak{f}_\alpha(\sigma) = x_1^{l(1/p)(\sigma)}$. Q.E.D.

Observe that $\dim_{\mathbb{Q}}(\mathbb{Z}[\frac{1}{p}]^\times \otimes \mathbb{Q}) = 1$. It follows from [10, Theorem 1] that $H^1(G_{\mathbb{Q}}; \mathbb{Q}_l(k)) = 0$ for $k > 0$ and even and $H^1(G_{\mathbb{Q}}; \mathbb{Q}_l(k)) = \mathbb{Q}_l$ for $k > 1$ and odd. Hence the Lie algebra $L_l(\mathbb{Z}[\frac{1}{p}])$ has one generator in each odd degree. We denote them by

$$\sigma^k$$

for $k = 1, 3, 5, \ldots$.

The Kummer character associated to p induces an isomorphism

$$l(p) : L_l(\mathbb{Z}[\frac{1}{p}])_1 \to \mathbb{Q}_l\,.$$

The generator σ^1 of $L_l(\mathbb{Z}[\frac{1}{p}])$ in degree one we choose such that $l(1/p)(\sigma^1) = 1$. The generator σ^k for $k > 1$ we choose to be dual to the l-adic polylogarithm $l_k(\overrightarrow{10})$ in the sense that $l_k(\overrightarrow{10})(\sigma^k) = 1$.

Definition 6.2. We denote by

$$I(\sigma^1)$$

the Lie ideal of $L_l(\mathbb{Z}[\frac{1}{p}])$ generated by all brackets

$$[..[\sigma^{i_1}, \sigma^{i_2}].., \sigma^{i_r}]$$

in generators σ^1, σ^3, $\sigma^5, \ldots$ of length r, for all r greater than one and such that at least one of the elements σ^{i_1}, $\sigma^{i_2} \ldots$, σ^{i_r} is σ^1.

Proposition 6.3. Let

$$\Phi_{\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,\infty\},\vec{0p}} : L_l(\mathbb{Z}[\frac{1}{p}]) \to Lie(X,Y)_{\{\ \}}.$$

be the morphism of Lie algebras induced by the action of $G_{\mathbb{Q}}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,\infty\};\vec{0p})$. Then we have

$$\Phi_{\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,\infty\},\vec{0p}}(\sigma^1) = X$$

and

$$\Phi_{\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,\infty\},\vec{0p}}(\sigma^k) \equiv [Y, X^{(k-1)}] \mod I_2$$

for $k > 1$ and odd.

Proof. It follows immediately from Lemma 6.1 that $\Phi_{\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,\infty\};\vec{0p}}(\sigma^1) = X$. The element $\mathfrak{f}_\gamma(\sigma)$ was studied in [1] and in [7]. Hence it follows the description of $\Phi_{\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,\infty\};\vec{0p}}(\sigma^k)$ for $k > 1$. Q.E.D.

Corollary 6.4. We have

$$I(\sigma^1) \subset \operatorname{Ker} \Phi_{\mathbb{P}^1_{\bar{\mathbb{Q}}}\setminus\{0,1,\infty\},\vec{0p}}.$$

Proof. We have already observed in Section 2 that in the Lie algebra $Lie(X,Y)_{\{\ \}}$ the element X commutes with any other element. Q.E.D.

References

[1] P. Deligne, Le groupe fondamental de la droite projective moins trois points, In: Galois Groups over Q, (eds. Y.Ihara, K.Ribet and J.-P. Serre), Math. Sci. Res. Inst. Publ., **16**, Cambridge Univ. Press, 1989, pp. 79–297.

[2] P. Deligne, Lecture on the Conference in Schloss Ringberg, 1998.

[3] P. Deligne, Le Groupe fondamental de $\mathbb{G}_m \setminus \mu_N$, pour $N = 2, 3, 4, 6$ ou 8, http://www.math.ias.edu/people/faculty/deligne/preprints.

[4] J.-C. Douai and Z. Wojtkowiak, On the Galois Actions on the Fundamental Group of $\mathbb{P}^1_{\mathbb{Q}(\mu_n)} \setminus \{0, \mu_n, \infty\}$, Tokyo J. Math., **27** (2004), 21–34.

[5] R. Hain and M. Matsumoto, Weighted completion of Galois groups and Galois actions on the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, Compositio Math., **139** (2003), 119–167.

[6] R. Hain and M. Matsumoto, Tannakian fundamental groups associated to Galois groups, In: Galois Groups and Fundamental Groups, (ed. L. Schneps), Math. Sci. Res. Inst. Publ., **41**, Cambridge Univ. Press, 2003, pp. 183–216.

[7] Y. Ihara, Profinite braid groups, Galois representations and complex multiplications, Ann. of Math. (2), **123** (1986), 43–106.

[8] Y. Ihara, The Galois representations arising from $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and Tate twists of even degree, In: Galois Groups over Q, (eds. Y. Ihara, K. Ribet and J.-P. Serre), Math. Sci. Res. Inst. Publ., **16**, Cambridge Univ. Press, 1989, pp. 299–313.

[9] W. Magnus, A. Karras and D. Solitar, Combinatorial Group Theory, Interscience Publishers, 1966.

[10] Ch. Soulé, On higher p-adic regulators, In: Algebraic K-Theory, Lecture Notes in Math., **854**, Springer-Verlag, 1981, pp. 372–401.

[11] Z. Wojtkowiak, On l-adic iterated integrals. I. Analog of Zagier conjecture, Nagoya Math. J., **176** (2004), 113–158.

[12] Z. Wojtkowiak, On l-adic iterated integrals. II. Functional equations and ℓ-adic polylogarithms, Nagoya Math. J., **177** (2005), 117–153.

[13] Z. Wojtkowiak, On l-adic iterated integrals. III. Galois actions on fundamental groups, Nagoya Math. J., **178** (2005), 1–36.

[14] Z. Wojtkowiak, On l-adic iterated integrals. IV. Ramifications and generators of Galois actions on fundamental groups and on torsors of paths, Math. J. Okayama Univ., **51** (2009), 47–69.

[15] Z. Wojtkowiak, On the Galois actions on torsors of paths. I. Descent of Galois representations, J. Math. Sci. Univ. Tokyo, **14** (2007), 177–259.

[16] Z. Wojtkowiak, Periods of mixed Tate motives, examples, l-adic side, Max-Planck-Institut für Math., preprint series, 2010(80), to appear in Arithmetic and Geometry around Galois Theory, Lecture Notes of GTEM/TÜBİTAK Summer Schools 2008 and 2009, Galatasaray Univ., Istanbul, (eds. P. Debes, M. Emsalem, M. Romagny and A. M. Uludag).

Université de Nice-Sophia Antipolis
Département de Mathématiques
Laboratoire Jean Alexandre Dieudonné
U.R.A. au C.N.R.S., N° 168
Parc Valrose – B.P. N° 71
06108 Nice Cedex 2
France
E-mail address: wojtkow@math.unice.fr