Advanced Studies in Pure Mathematics 63, 2012
Galois–Teichmüller Theory and Arithmetic Geometry
pp. 189–220

Noether's problem for transitive permutation groups of degree 6

Kiichiro Hashimoto and Hiroshi Tsunogai

Abstract.

Suppose that a finite group G is realized in the Cremona group $\mathrm{Cr}_m(k)$, the group of k-automorphisms of the rational function field K of m variables over a constant field k. The most general version of Noether's problem is then to ask, whether the subfield K^G consisting of G-invariant elements is again rational or not. This paper treats Noether's problem for various subgroups G of $\mathfrak{S}_6$, the symmetric group of degree 6, acting on the function field $\boldsymbol{Q}(s,t,z)$ over $k = \boldsymbol{Q}$ of the moduli space $\mathcal{M}_{0,6}$ of $\mathbb{P}^1$ with ordered six marked points. We shall show that this version of Noether's problem has an affirmative answer for all but two conjugacy classes of transitive subgroups G of $\mathfrak{S}_6$, by exhibiting explicitly a system of generators of the fixed field $\boldsymbol{Q}(s,t,z)^G$. In the exceptional cases $G \cong \mathfrak{A}_6, \mathfrak{A}_5$, the problem remains open.

§0. Introduction

0.1. General version of Noether's problem (GNP)

Let k be a field and $x_1, \dots, x_n$ be independent variables. The group of k-automorphisms of the rational function field $K = k(x_1, \dots, x_n)$ is called the *Cremona group* of dimension n over k, and denoted as $\mathrm{Cr}_n(k)$ in case we need not specify the variables. Suppose now a finite subgroup G of $\mathrm{Cr}_n(k)$ is given. Then one can ask whether the subfield K^G consisting of G-invariant elements of K is again *rational*, i.e., purely transcendental, over k or not. This question will be referred to as the

Received May 23, 2011.
Revised April 4, 2012.
2010 *Mathematics Subject Classification.* Primary 12F12; Secondary 12F10, 14H05, 14H10, 20B25, 20B35.
Key words and phrases. Galois theory, invariant theory, permutation groups, Cremona group, cross ratios, hyperelliptic curves.
This work was partially supported by JSPS KAKENHI 18540048, 22540032.

most general version of *Noether's problem*, GNP for short, while its original version (Perm-NP, for short) is the case where the action of G on K is induced from the permutation of $x_1, \ldots, x_n$ (cf. [Noe], [Sw1]). If GNP has an affirmative answer, one can find n algebraically independent elements $u_1, \ldots, u_n \in K$ which satisfy $K^G = k(u_1, \ldots, u_n)$ and a polynomial $f(X) \in k(u_1, \ldots, u_n)[X]$ whose splitting field is K. Thus $f(X)$ is a G-polynomial with n free parameters. In some cases including that of Perm-NP, it turns out that $f(X)$ is a *generic* G-polynomial over k in the sense of [De],[KM]. This is the most significant motivation to study Noether's problem.

Unfortunately, however, there are many finite groups for which the answer to Noether's problem is either unknown yet or negative, even for Perm-NP. Cyclic groups of order $8m$ ($m \in \boldsymbol{N}$) are among those groups for which the answer to Perm-NP over $\boldsymbol{Q}$ is negative. Here it is noteworthy that the answer to Noether's problem depends on the base field k and the way by which G acts on K. As an illustration, we note that the original version of Noether's problem for finite abelian groups always has an affirmative answer over an algebraically closed field of characteristic 0 (cf. [Fi]). Let α be a $\boldsymbol{Q}$-automorphism of $\boldsymbol{Q}(x, y)$ defined by

$$\alpha : (x, y) \mapsto \left(y, \frac{1+x}{1-x} \right).$$

Then $\langle \alpha \rangle$ is a cyclic subgroup of $\mathrm{Cr}_2(\boldsymbol{Q})$ of order 8. In contrast to the above mentioned fact, however, one can show that $\boldsymbol{Q}(x, y)^{\langle \alpha \rangle} = \boldsymbol{Q}(u, v)$ for suitable $u, v \in \boldsymbol{Q}(x, y)$ (see [HHR]).

Thus an important question arises: in how many ways a given finite group G can be realized as a subgroup of the Cremona group $\mathrm{Cr}_n(k)$, up to conjugation, and how the answer to GNP depends on its realization?

Obviously, among all possible realizations of G in $\mathrm{Cr}_n(k)$, the case of smaller dimension n should be more interesting, since n is the number of parameters of a G-polynomial $f(X)$ obtained from the extension K/K^G as explained above. Even more interesting are the cases in which $f(X)$ is a generic G-polynomial. This is in fact the case we treat in this paper for the realization of the symmetric group $\mathfrak{S}_m$ of degree m in $\mathrm{Cr}_{m-3}(k)$ (cf.[HT]).

0.2. Cross-ratio Noether problem (CR-NP)

Here we assume that k is an arbitrary field of characteristic 0, and $n \geq 4$. Recall first that $\mathrm{Cr}_1(k) \cong \mathrm{PGL}(2, k)$, i.e., any k-automorphism of $k(x)$ is expressed as a linear fractional transformation

$$A : x \mapsto A(x) := \frac{ax+b}{cx+d}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,k).$$

One can extend this action diagonally to $L_n = k(x_1, \dots, x_n)$, by putting $x_i \mapsto A(x_i)$ $(1 \le i \le n)$, and denote by $K_n := L_n{}^{\mathrm{PGL}(2,k)}$ the subfield of L_n consisting of the invariant elements under this action. One can describe the field K_n in a geometric way as follows. Let $\mathcal{M}_{0,n}$ be the moduli space of $\mathbb{P}^1$ over k with ordered n-marked points:

$$\begin{aligned}\mathcal{M}_{0,n} &: \text{moduli space of } \mathbb{P}^1 \text{ with ordered } n\text{-marked points} \\ &= \left((\mathbb{P}^1)^n \setminus (\text{weak diagonal})\right)/\mathrm{PGL}(2) \\ &= \{(x_1, \dots, x_n) | x_i \in \mathbb{P}^1, x_i \ne x_j (i \ne j)\}/\mathrm{PGL}(2).\end{aligned}$$

Then K_n is the function field of $\mathcal{M}_{0,n}$ over k. Since $\mathcal{M}_{0,n}$ is of dimension $(n-3)$, K_n/k is an extension of transcendental degree $(n-3)$. In fact, K_n/k is a purely transcendental extension, and is generated by the $(n-3)$ cross-ratios

$$y_i := \frac{x_i - x_1}{x_i - x_2} \Big/ \frac{x_3 - x_1}{x_3 - x_2} \qquad (4 \le i \le n).$$

To see this, we take a normalized representative for $P = [x_1, \dots, x_n] := (x_1, \dots, x_n) \bmod \mathrm{PGL}(2) \in \mathcal{M}_{0,n}$. Since the action of PGL(2) on $\mathbb{P}^1$ is sharply 3-transitive, there exists a unique $A \in \mathrm{PGL}(2)$ such that

$$A(x_1) = 0, \quad A(x_2) = \infty, \quad A(x_3) = 1.$$

A simple computation shows that $A(x_i) = y_i$ is a cross ratio of x_1, x_2, x_3, x_i as given above, for $4 \le i \le n$. Thus each point P can be uniquely represented in a normalized form $[0, \infty, 1, y_4, \dots, y_n]$, where y_i $(4 \le i \le n)$ are regarded as functions on $\mathcal{M}_{0,n}$. These y_i's give an isomorphism $\mathcal{M}_{0,n} \simeq (\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus$ (weak diagonal). It follows that we have $K_n = k(y_4, \dots, y_n)$.

Now a crucial point is that the permutations of $x_1, \dots, x_n$ commute with the diagonal action of PGL(2) on L_n, hence they induce the action on K_n as k-automorphisms. Thus we have a realization of the symmetric group $\mathfrak{S}_n$ of degree n in $\mathrm{Cr}_{n-3}(k)$. We shall study Noether's problem in this context, calling it CR-NP for short, in the case $n = 6$ and G is a transitive permutation group of degree 6.

We remark that the same problem for the case $n = 5$ has been already settled by present authors [HT], in which there are five transitive permutation groups up to isomorphism.

We also remark that the original Noether problem is related to CR-NP as follows (cf. [Tsu]). Let B (resp. U) be the Borel (resp. unipotent) subgroup of $\mathrm{PGL}(2,k)$:

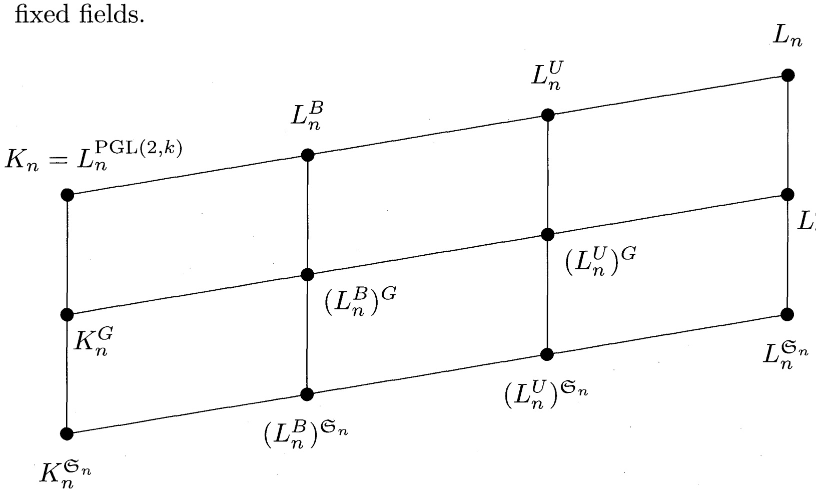
$$\mathrm{PGL}(2,k) \supset B := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \supset U := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \supset \{1\},$$

and denote its fixed field by L_n^B (resp. L_n^U). We easily find that

$$L_n^U = k\,(x_i - x_1 \,|\, i = 2, \ldots, n) \quad (\text{“the field of differences”}),$$

$$L_n^B = k\left(\frac{x_i - x_1}{x_i - x_2} \,\middle|\, i = 3, \ldots, n\right) \quad (\text{“the field of ratios of differences”}).$$

Let G be a subgroup of $\mathfrak{S}_n$. Then we have the following picture of the fixed fields.



Thus, if K_n^G is rational (CR-NP holds) and if all steps of the successive extensions $L_n^G \supset (L_n^U)^G \supset (L_n^B)^G \supset K_n^G$ of relative dimension one are relatively rational, we can conclude that L_n^G is rational (the original NP holds).

Transitive permutation groups of degree n have been classified in [B-MK] for $n \leq 11$. There are 16 classes in the case $n = 6$, which are labelled by $6T1, \ldots, 6T16$ respectively (see Table 1, and Fig. 1 of §3). We shall show, for each of these subgroups G except for $6T15$ and $6T12$, that the answer to Noether's problem is affirmative by exhibiting a system of generators of K_n^G. For the whole group $\mathfrak{S}_6 = 6T16$, we shall work with the geometry of abelian surfaces with the level 2 structure, while for small subgroups we shall employ a more direct computational method. For simplicity the labels $6Tn$ will denote here specified subgroups although they usually really denote conjugacy classes of subgroups.

The answer to CR-NP for $6T1$, $6T2$, $6T3$, $6T9$, $6T13$ (resp. $6T4$, $6T6$, $6T7$, $6T8$, $6T11$) were obtained by K. Kusumori [Kus] (resp. Y. Ohta [Oh]) in his master's thesis in Sophia University under the suggestion of the second author. Although the arguments as well as the description of the fixed fields presented here have been simplified, the basic ideas are due to them.

§1. Geometric background

We describe here an interpretation of the action of $\mathfrak{S}_n$ on $K_n := k(\mathcal{M}_{0,n})$ in the setting of arithmetic geometry.

Let $f(x) \in k[X]$ be a monic separable polynomial of degree $n \geq 5$, and let W_f be the set of its roots in $\overline{k}$, the algebraic closure of k. To $f(x)$ one can associate the hyperelliptic curve X_f defined by the equation

$$X_f : \quad y^2 = f(x). \tag{1.1}$$

Then X_f is viewed as a double cover of $\mathbb{P}^1$ by the map

$$\pi : X_f \longrightarrow \mathbb{P}^1, \quad (x, y) \longmapsto x$$

which ramifies exactly at the following points

$$\begin{cases} \{(x,0) \mid x \in W_f\} & (n : \text{even}) \\ \{(x,0) \mid x \in W_f\} \cup \{(\infty,\infty)\} & (n : \text{odd}). \end{cases}$$

It follows that the genus g of X_f is determined by

$$n = 2g + 1 \quad (n : \text{odd}), \qquad n = 2g + 2 \quad (n : \text{even}).$$

Now we recall the well-known fact that if $g(x) \in k[X]$ is also a monic separable polynomial of degree n having W_g as the set of its roots, then the following three conditions are equivalent:

- $X_f \cong X_g$,
- $g(x) = (cx + d)^n f(A(x))$ for some $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \overline{k})$,
- $W_f \cong W_g \bmod \mathrm{PGL}(2, \overline{k})$.

To connect the above argument with $K_n := k(\mathcal{M}_{0,n})$, we need to introduce an order, or a labelling, $x_1, .., x_n$ into W_f. This is given by the *level* 2 *structure* on the jacobian variety $J_f := \mathrm{Jac}(X_f)$ of the curve X_f with the Galois action. To describe it in a concrete way, we shall identify J_f with $\mathrm{Pic}^0(X_f)$:

$$J_f \cong \mathrm{Pic}^0(X_f) := \mathrm{Div}^0(X_f)/\sim \text{ (linear equivalence)}.$$

Our first observation is the following description of the 2-torsion subgroup $\mathrm{Pic}^0(X_f)[2]$ of $\mathrm{Pic}^0(X_f)$ (see [Dol2]). We put $P_i = (x_i, 0)$ and $P_\infty = (\infty, \infty)$.

• If $n = 2g+1$ is odd, the principal divisors of the functions $(x - a_i)$, y are given by

$$(1.2)\qquad \begin{cases} \mathrm{div}(x - a_i) = 2(P_i - P_\infty) & (1 \le i \le 2g+1), \\ \mathrm{div}(y) = (P_1 + \cdots + P_{2g+1}) - (2g+1)P_\infty. \end{cases}$$

It follows that

$$(1.3)\qquad e_i := [P_i - P_\infty] \quad \in \mathrm{Pic}^0(X_f)[2] \qquad (1 \le i \le 2g+1).$$

For $S \subseteq \{1, 2, \ldots, n = 2g+1\}$ we put

$$e_S := \sum_{i \in S} e_i.$$

From the second equation of (1.2) we have

$$e_1 + \cdots + e_{2g+1} = 0, \qquad e_S = e_{\overline{S}}.$$

• If $n = 2g+2$ is even, we need a slight modification. We see from

$$\mathrm{div}\left(\frac{x - a_i}{x - a_j}\right) = 2(P_i - P_j) \qquad (1 \le i, j \le n),$$

that the class $e_{\{i,j\}} \in \mathrm{Pic}^0(X_f)$ defined by

$$(1.4)\qquad e_{\{i,j\}} := [P_i - P_j] \quad \in \mathrm{Pic}^0(X_f) \qquad (1 \le i, j \le 2g+2)$$

is a 2-torsion element. For each subset $S \subseteq \{1, 2, \ldots, n = 2g+2\}$ satisfying $|S| = 2m \equiv 0 \mod 2$, let $S = S_1 \cup \cdots \cup S_m$ be a disjoint decomposition with $|S_i| = 2$ $(1 \le i \le m)$, and put

$$e_S := \sum_{k=1}^{m} e_{S_i}.$$

One can show that this definition does not depend on the decomposition of S, and $e_S = e_{\overline{S}}$. Thus we have the following concrete description of the 2-torsion subgroup of $J_f[2] \cong \mathrm{Pic}^0(X_f)$ (cf. [Mum]):

Proposition 1.1. *The 2-torsion subgroup* $\mathrm{Pic}^0(X_f)[2]$ *of* $\mathrm{Pic}^0(X_f)$ *is given by*

$$\mathrm{Pic}^0(X_f)[2] = \{e_S \mid |S| \equiv 0 \bmod 2\} \cong \boldsymbol{F}_2{}^{\oplus 2g}, \tag{1.5}$$

$$\textit{with} \quad e_S + e_T = e_{S\triangle T} \qquad (S\triangle T := S \cup T - S \cap T).$$

Proposition 1.2. *The field generated by 2-torsion points* $J_f[2]$ *of* $\mathrm{Jac}(X_f)$ *is the splitting field of* $f(x)$ *over* k*:*

$$k(J_f[2]) = k(x_1, .., x_n). \tag{1.6}$$

Suppose now that $f(x) \in k[x]$ is a polynomial of degree $n = 2g + 2$ and regard its Galois group $\mathrm{Gal}(f, k)$ over k as a subgroup of $\mathfrak{S}_{2g+2}$, then the Galois representation

$$\overline{\rho}_{f,2} : \mathrm{Gal}(\overline{k}/k) \longrightarrow \mathrm{GSp}(2g, \boldsymbol{F}_2) \;=\; \mathrm{Sp}(2g, \boldsymbol{F}_2) \tag{1.7}$$

attached to $J_f[2]$ factors through the inclusion

$$\mathrm{Gal}(k(J_f[2])/k) \cong \mathrm{Gal}(f, k) \quad \rightarrowtail \quad \mathfrak{S}_{2g+2}, \tag{1.8}$$

where $\mathrm{GSp}(2g, \boldsymbol{F}_2)$ denotes the symplectic similitude group of degree $2g$ over $\boldsymbol{F}_2$, which is nothing but $\mathrm{Sp}(2g, \boldsymbol{F}_2)$, the symplectic group of degree $2g$ over $\boldsymbol{F}_2$. Since $\mathrm{Gal}(f, k)$ is isomorphic to $\mathfrak{S}_{2g+2}$ for general f, this implies that there is an embedding of the groups for $g \geq 2$

$$\mathfrak{S}_{2g+2} \rightarrowtail \mathrm{Sp}(2g, \boldsymbol{F}_2) \tag{1.9}$$

which is an isomorphism for $g = 2$:

$$\mathfrak{S}_6 \cong \mathrm{Sp}(4, \boldsymbol{F}_2). \tag{1.10}$$

Remark 1.3. For $g = 1$ $(n = 4)$, since the cross ratios of x_1, x_2, x_3, x_4 are invariant under the Klein four group $V_4 := \{id., (12)(34), (13)(24), (14)(23)\}$, the action of $\mathfrak{S}_4$ on $K_4 = \boldsymbol{Q}(s)$ factors through its quotient $\mathfrak{S}_4/V_4 \cong \mathfrak{S}_3$ so that we have an injection

$$\mathfrak{S}_3 \rightarrowtail \mathrm{Sp}(2, \boldsymbol{F}_2) \;=\; \mathrm{SL}(2, \boldsymbol{F}_2), \tag{1.11}$$

which is, in fact, an isomorphism.

In our study of CR-NP it is important to have a concrete description of the above embedding (1.9) in terms of the geometry over $\boldsymbol{F}_2$. Asyzygetic systems give an answer to this question.

§2. Realization of $\mathfrak{S}_{2g+2}$ in $\mathrm{Sp}(2g, \boldsymbol{F}_2)$ by Asyzygetic systems

Let $V := \boldsymbol{F}_2^{\oplus 2g}$ be the vector space of dimension $2g$ over $\boldsymbol{F}_2$ with the standard skew symmetric form

$$F(\vec{x}, \vec{y}) = \sum_{i=1}^{g} x_i y_{i+g} - y_i x_{i+g}. \tag{2.1}$$

Thus we have

$$\mathrm{Aut}(V, F) = \mathrm{GSp}(2g, \boldsymbol{F}_2) = \mathrm{Sp}(2g, \boldsymbol{F}_2).$$

We introduce a notion of Asyzygetic systems (e.g. [Me]).

Definition 2.1. A matrix $\mathcal{X} := (\vec{x}_{i,j})_{1 \le i,j \le 2g+2}$ of degree $(2g+2)$ with entries in V is called an Asyzygetic system, A-system in short, if it satisfies the following conditions

$$\begin{cases} \text{(i)} \quad \vec{x}_{i,j} + \vec{x}_{j,k} + \vec{x}_{k,i} = \vec{0} & \text{(for any } i, j, k), \\ \text{(ii)} \quad F(\vec{x}_{i,j}, \vec{x}_{i,k}) = 1 & \text{(for any distinct } i, j, k). \end{cases}$$

Some immediate consequences of (i), (ii) are $\vec{x}_{i,i} = \vec{0}$, $\vec{x}_{i,j} = \vec{x}_{j,i}$ and

$$\vec{x}_{i,j} = \vec{x}_{i,2g+2} + \vec{x}_{j,2g+2} \ \text{(for all } 1 \le i, j \le 2g+2).$$

Putting $\vec{x}_i := \vec{x}_{i,2g+2}$ $(1 \le i \le 2g+1)$, we see that the Gram matrix of $\{\vec{x}_1, \ldots, \vec{x}_{2g}\}$ is expressed as

$$G := \Big(F(\vec{x}_i, \vec{x}_j) \Big)_{1 \le i,j \le 2g} = \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{pmatrix}$$

and it satisfies $G^2 = I_{2g}$. It follows that $\{\vec{x}_1, \ldots, \vec{x}_{2g}\}$ form a basis of V. We also have

$$F(\vec{x}_1 + \ldots + \vec{x}_{2g+1}, \vec{x}_i) = 2g = 0 \qquad (1 \le i \le 2g).$$

Since F is nondegenerate it follows that

$$\vec{x}_1 + \ldots + \vec{x}_{2g+1} = \vec{0}, \quad \text{hence} \quad \vec{x}_{2g+1} = \vec{x}_1 + \ldots + \vec{x}_{2g}.$$

The above arguments can be reversed to recover an A-system from a given basis $\{\vec{x}_1, \ldots, \vec{x}_{2g}\}$ of V.

Now for each $\sigma \in \mathfrak{S}_{2g+2}$ we put

$$\sigma(\mathcal{X}) := (\vec{x'}_{i,j}), \quad \vec{x'}_{i,j} = \vec{x}_{\sigma(i),\sigma(j)}.$$

Then any linear relation between some of the $\vec{x}_{i,j}$ is satisfied by the $\vec{x'}_{i,j}$. In particular, it follows that $\sigma(\mathcal{X})$ is again an A-system, namely the system $(\vec{x'}_{i,j})$ satisfies the relation (i) (ii) of Definition 2.1. Since the entries of $\mathcal{X}$ contains a basis $\{\vec{x}_1, \ldots, \vec{x}_{2g}\}$ of V, the map σ is extended uniquely to a linear automorphism M_σ of V which preserves the skew symmetric form F. Thus we obtain an embedding:

$$\text{(2.2)} \qquad \overline{\rho_{\mathcal{X}}} : \mathfrak{S}_{2g+2} \rightarrowtail \mathrm{Aut}(V, F) \cong \mathrm{Sp}(2g, \boldsymbol{F}_2), \quad \sigma \longmapsto \phi_\sigma.$$

One can apply the above argument to the vector space $\mathrm{Pic}^0(X_f)[2] \cong J_f[2]$ over $\boldsymbol{F}_2$ equipped with the Weil pairing, and obtain the following

Theorem 2.2. *For a separable polynomial $f(x)$ of degree $(2g+2)$, let X_f be the hyperelliptic curve defined by (1.1), and put*

$$\text{(2.3)} \quad \vec{x}_{i,j} := [P_i - P_j] \in \mathrm{Pic}^0(X_f)[2] \cong J_f[2] \qquad (1 \le i, j \le 2g+2).$$

Then the matrix $\mathcal{X} := (\vec{x}_{i,j})_{1 \le i,j \le 2g+2}$ forms an A-system in $J_f[2]$ with respect to the Weil pairing. Furthermore, the action of the Galois group $\mathrm{Gal}(f, k)$ on $J_f[2]$ is realized via the embedding (1.8), (2.2).

Now we assume that $g = 2$. Following the above argument we can give an explicit from of the isomorphism (1.10). For this purpose we first note the following fact which can be proved by a direct computation.

Lemma 2.3. *Suppose that $\mathcal{X} := (\vec{x}_{i,j})_{1 \le i,j \le 6}$ is an A-system. Then the vectors*

$$\text{(2.4)} \qquad \begin{cases} \vec{a}_1 = \vec{x}_{1,6}, & \vec{a}_2 = \vec{x}_{1,6} + \vec{x}_{2,6} + \vec{x}_{3,6}, \\ \vec{a}_3 = \vec{x}_{2,6}, & \vec{a}_4 = \vec{x}_{1,6} + \vec{x}_{2,6} + \vec{x}_{4,6} \end{cases}$$

form a symplectic basis with respect to F i.e., we have

$$\Big(F(\vec{a}_i, \vec{a}_j) \Big)_{1 \le i,j \le 4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It follows that, if we express the linear automorphism $\phi_\sigma \in \mathrm{Aut}(V, F)$ by a matrix M_σ using the basis $\{\vec{a}_1, \ldots, \vec{a}_4\}$, for each $\sigma \in \mathfrak{S}_6$, then the map $\sigma \mapsto M_\sigma$ gives the desired isomorphism. We tabulate below some of $\sigma \in \mathfrak{S}_6$ and corresponding $M_\sigma \in \mathrm{Sp}(4, \boldsymbol{F}_2)$.

$\sigma \in \mathfrak{S}_6$	M_σ	$\sigma \in \mathfrak{S}_6$	M_σ
(12)	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	(13)	$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$
(14)	$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	(15)	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$
(16)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$		
$\alpha = (123456)$	$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$	$\beta = (14)(23)(56)$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
$\theta = (14)(25)(36)$	$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\gamma_1 = (135)$	$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$
$\gamma_2 = (246)$	$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\delta = (14)(2563)$	$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
$\tau_1 = (14)(25)$	$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$	$\tau_2 = (14)(36)$	$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$
$\varphi = (15243)$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$		

§3. Transitive permutation groups of degree 6

Transitive subgroups of $\mathfrak{S}_6$ have been classified long time ago. Up to conjugacy there are 16 such groups, as tabulated below. The symbol $+$ indicates that it is contained in $\mathfrak{A}_6$.

[B-MK]	order	sign	structure	generators	(CR-NP)
$6T1$	6		C_6	α	○
$6T2$	6		$\mathfrak{S}_3(6)$	α^2, β	○
$6T3$	12		D_6	α, β	○
$6T4$	12	$+$	$\mathfrak{A}_4$	α^2, τ_1, τ_2	○
$6T5$	18		$\mathfrak{S}_3 \times C_3$	$6T2, \gamma_1$	○
$6T6$	24		$\mathfrak{A}_4 \times C_2$	$6T4, \theta$	○
$6T7$	24	$+$	${\mathfrak{S}_4}^{(+)}$	$6T4, \beta\theta$	○
$6T8$	24		${\mathfrak{S}_4}^{(-)}$	$6T4, \beta$	○
$6T9$	36		$V_4 \ltimes (C_3 \times C_3)$	$6T3, \gamma_1$	○
$6T10$	36	$+$	$C_4 \ltimes (C_3 \times C_3)$	$\alpha^2, \alpha\beta, \gamma_1, \delta$	○
$6T11$	48		$\mathfrak{S}_4 \times C_2$	$6T4, \beta, \theta$	○
$6T12$	60	$+$	$\mathfrak{A}_5(6)$	$6T4, \varphi$	?
$6T13$	72		$D_4 \ltimes (C_3 \times C_3)$	$6T9, \delta$	○
$6T14$	120		$\mathfrak{S}_5(6)$	$6T8, \varphi$	○
$6T15$	360	$+$	$\mathfrak{A}_6$	$6T7, \varphi$	?
$6T16$	720		$\mathfrak{S}_6$	$6T15, \beta$	○

$$\alpha = (123456), \beta = (14)(23)(56), \theta = \alpha^3 = (14)(25)(36),$$
$$\gamma_1 = (135), \gamma_2 = (246), \delta = (14)(2563),$$
$$\tau_1 = (14)(25), \tau_2 = (14)(36), \varphi = (15243).$$

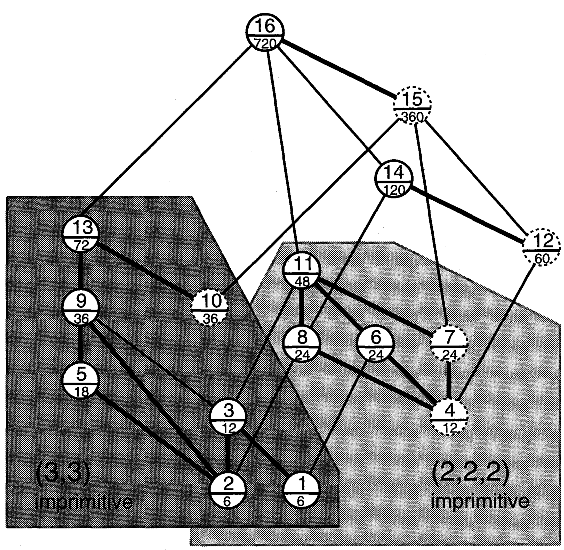


Fig. 1. The transitive groups of degree 6

In Fig. 1, the upper labels n stand for $6Tn$, and the lower labels stand for the corresponding orders.

§4. CR-NP for $6T16$, $6T15$

Our problem CR-NP for $G \subseteq \mathfrak{S}_6$ is stated in a concrete form as follows. The group $\mathfrak{S}_6$ is realized as the group of automorphisms of $\boldsymbol{Q}(\mathcal{M}_{0,6}) = \boldsymbol{Q}(s,t,z)$ which are induced from the permutations of $x_1, \ldots, x_6$, where s, t, z are cross-ratios

$$(4.1) \qquad \begin{cases} s := \dfrac{x_3 - x_1}{x_3 - x_2} \Big/ \dfrac{x_1 - x_4}{x_2 - x_4}, & t := \dfrac{x_5 - x_1}{x_5 - x_2} \Big/ \dfrac{x_3 - x_1}{x_3 - x_2}, \\ z := \dfrac{x_6 - x_1}{x_6 - x_2} \Big/ \dfrac{x_3 - x_1}{x_3 - x_2} \end{cases}$$

determined by the normalization $[0, \infty, s, 1, st, sz] = [0, \infty, 1, s^{-1}, t, z]$.

Here we make the following interpretation of CR-NP for $G \subseteq \mathfrak{S}_6$ in terms of the ring of Siegel modular forms. Let $A(\Gamma(2)) = \bigoplus_{k=0}^{\infty} M_k(\Gamma(2))$ be the graded ring of Siegel modular forms of genus two with respect to the principal congruence subgroup $\Gamma(2)$ of $\mathrm{Sp}(4, \boldsymbol{Z})$ of level 2, and denote by $F_0(A(\Gamma(2)))$ the field consisting of the quotients F_1/F_2 ($F_1, F_2 \in M_k(\Gamma(2))$) of forms of the same weight which have rational Fourier coefficients. Then the natural action of $\mathrm{Sp}(4, \boldsymbol{Z})$ on $M_k(\Gamma(2))$ induces the action of $\mathrm{Sp}(4, \boldsymbol{Z})/\Gamma(2) \cong \mathfrak{S}_6$ on $F_0(A(\Gamma(2)))$ (cf. [Ig1]).

On the other hand, by Torelli's theorem one can identify the moduli space $\mathcal{A}_{2,2}$ of principally polarized abelian surfaces with level 2 structure and the moduli space $\mathcal{M}_{0,6}$ of $\mathbb{P}^1$ with ordered six marked points. Namely to each point $[x_1, .., x_6] \in \mathcal{M}_{0,6}$ one can associate the class of abelian surface $J_f \cong \mathrm{Pic}^0(X_f)$ with level 2-structure as in Proposition 1.1, which induces the following commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_{0,6} & \longrightarrow & \mathfrak{H}_2/\Gamma(2) & \xrightarrow{\simeq} & \mathcal{A}_{2,2} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_2 & \longrightarrow & \mathfrak{H}_2/\mathrm{Sp}(4, \boldsymbol{Z}) & \xrightarrow{\simeq} & \mathcal{A}_2 \end{array}$$

where $\mathfrak{H}_2$ is the Siegel upper half plane of degree 2 and $\mathcal{A}_2$ (resp. $\mathcal{M}_2$) denotes the moduli space of principally polarized abelian surfaces (curves of genus 2). From the arguments of §1 and §2, we see that this identification preserves the action of $\mathfrak{S}_6$. In other words, we have the following

Proposition 4.1. *There is an isomorphism $\phi : F_0(A(\Gamma(2))) \to K = \boldsymbol{Q}(\mathcal{M}_{0,6})$ such that the following diagram commutes for any $\sigma \in \mathfrak{S}_6$:*

$$\begin{array}{ccc} F_0(A(\Gamma(2))) & \xrightarrow{\phi} & \boldsymbol{Q}(\mathcal{M}_{0,6}) \\ \sigma\downarrow & & \downarrow\sigma \\ F_0(A(\Gamma(2))) & \xrightarrow{\phi} & \boldsymbol{Q}(\mathcal{M}_{0,6}). \end{array}$$

We refer to [Ru] for the details. By this identification our problem CR-NP for $G \subseteq \mathfrak{S}_6$ is reduced to the rationality of the fixed field $F_0(A(\Gamma(2)))^G$. In particular we have for $G = \mathfrak{S}_6$

$$F_0(A(\Gamma(2)))^{\mathfrak{S}_6} = F_0(A(\mathrm{Sp}(4, \boldsymbol{Z}))) \tag{4.2}$$

and it is well-known that the field $F_0(A(\mathrm{Sp}(4, \boldsymbol{Z})))$ is rational (see [Ig1], and also [Dol1]). Thus we have

Theorem 4.2. *The answer to CR-NP for $G = \mathfrak{S}_6$ over $\boldsymbol{Q}$ is affirmative. Namely the field $\boldsymbol{Q}(\mathcal{M}_{0,6})^{\mathfrak{S}_6} = \boldsymbol{Q}(s,t,z)^{\mathfrak{S}_6}$ is again a rational function field $\boldsymbol{Q}(a,b,c)$ for suitable $a,b,c \in \boldsymbol{Q}(s,t,z)$.*

We shall briefly sketch how we find a,b,c in the above theorem explicitly. Let $S(2,6)$ be the ring of invariants for the binary sextics. Then there is an injective ring homomorphism $\rho : A(\mathrm{Sp}(4,\boldsymbol{Z})) \to S(2,6)$ which preserves the weight and degree ratio, such that $F_0(\mathrm{Im}(\rho)) = F_0(S(2,6))$. Recall now from the classical invariant theory that the ring $S(2,6)$ is generated by five homogeneous elements A, B, C, D, E which are expressed by the roots $x_1, \dots, x_6$ of generic sextic as

(4.3)

$$\begin{cases} A := \sum (i_1i_2)^2(i_3i_4)^2(i_5i_6)^2 \qquad \text{(15terms)}, \\ B := \sum (i_1i_2)^2(i_2i_3)^2(i_3i_1)^2(i_4i_5)^2(i_5i_6)^2(i_6i_4)^2 \qquad \text{(10terms)}, \\ C := \sum (i_1i_2)^2(i_2i_3)^2(i_3i_1)^2(i_4i_5)^2(i_5i_6)^2(i_6i_4)^2(i_1i_4)^2(i_2i_5)^2(i_3i_6)^2 \\ \qquad \text{(60terms)}, \\ D := \prod_{1\le i<j\le 6} (ij)^2, \\ E := \prod \det \begin{pmatrix} 1 & x_{i_1}+x_{i_2} & x_{i_1}x_{i_2} \\ 1 & x_{i_3}+x_{i_4} & x_{i_3}x_{i_4} \\ 1 & x_{i_5}+x_{i_6} & x_{i_5}x_{i_6} \end{pmatrix}. \end{cases}$$

where (ij) denotes $(x_i - x_j)$, and the sum (resp. product) for A (resp. E) is taken over the 15 decompositions $\{i_1, i_2\} \cup \{i_3, i_4\} \cup \{i_5, i_6\} = \{1, \ldots, 6\}$. Furthermore, it is known that A, B, C, D are algebraically independent, and E^2 can be expressed by a polynomial in A, B, C, D. Now put

$$a := B/A^2, \quad b := C/A^2, \quad c := D/A^5, \quad d := E^2/D^3, \quad e := E/(BD).$$

Then we see that a, b, c, d are homogeneous of degree 0, while e is not. In fact a, b, c, d can be expressed as rational functions in three cross-ratios s, t, z defined above, so that $F_0(S(2,6)) = \boldsymbol{Q}(a, b, c, d)$. Moreover, from results of [Ig1] we can deduce that they satisfy the following relation

(4.4)

$$\begin{aligned} &40310784(d-3125)c^3 - 8748(27+2200a-58000a^2-12000b+240000ab)c^2 \\ &+324\big(3a^2+239a^3-1828a^4+128a^5-18ab-2688a^2b+14640a^3b+9540ab^2 \\ &+27b^2-28800a^2b^2-10800b^3\big)c-(1+80a-384b)\big(a^3-(a-3b)^2\big)^2 = 0. \end{aligned}$$

Since this equation is linear in d, it follows that $\boldsymbol{Q}(\mathcal{M}_{0,6})^{\mathfrak{S}_6} = F_0(S(2,6)) = \boldsymbol{Q}(a, b, c)$. This proves the rationality of the field $\boldsymbol{Q}(s, t, z)^{\mathfrak{S}_6}$.

Since D is the discriminant of the generic binary sextic whose roots are $x_1, \ldots, x_6$, the CR-NP for the alternating group $\mathfrak{A}_6$ is reduced to the rationality of a hypersurface in $\boldsymbol{A}^4$:

Corollary 4.3. *$\boldsymbol{Q}(\mathcal{M}_{0,6})^{\mathfrak{A}_6}$ is isomorphic to the function field of the hypersurface in $\boldsymbol{A}^4$ defined over $\boldsymbol{Q}$ by the equation obtained from (4.4), replacing a, b, c, d by x, y, u, w^2 respectively.*

§5. CR-NP for smaller subgroups (preparation)

In the following sections, we shall give an affirmative answer of CR-NP for smaller transitive subgroups by a more computational method. Here in this section, we choose a system of generators (cross-ratios) of $\boldsymbol{Q}(\mathcal{M}_{0,6})$ used in our actual computation, and explain how to compute the action of $\mathfrak{S}_6$ on these generators.

We take a system of generators a, b, c of $K := \boldsymbol{Q}(\mathcal{M}_{0,6})$ by the following symmetric uniformization:

(5.1)
$$\begin{aligned} [x_1, x_2, x_3, x_4, x_5, x_6] &= \left[0, a, 1, \frac{1}{1-b}, \infty, \frac{c-1}{c}\right] \\ &= \left[\infty, \frac{a-1}{a}, 0, b, 1, \frac{1}{1-c}\right] = \left[1, \frac{1}{1-a}, \infty, \frac{b-1}{b}, 0, c\right]. \end{aligned}$$

In terms of $x_1, \ldots, x_6$, a, b, c are expressed as

$$a = \frac{x_2 - x_1}{x_2 - x_5} \Big/ \frac{x_3 - x_1}{x_3 - x_5}, \quad b = \frac{x_4 - x_3}{x_4 - x_1} \Big/ \frac{x_5 - x_3}{x_5 - x_1},$$
$$c = \frac{x_6 - x_5}{x_6 - x_3} \Big/ \frac{x_1 - x_5}{x_1 - x_3}.$$

We have $K = \boldsymbol{Q}\left(a, \frac{1}{1-b}, \frac{c-1}{c}\right) = \boldsymbol{Q}(a, b, c)$. The action of $\mathfrak{S}_6$ on $\boldsymbol{Q}(a, b, c)$ can be calculated by the following renormalization technique. For example, consider the action of the element $\alpha = (1\ 2\ 3\ 4\ 5\ 6)$. Put

$$P = [x_1, \ldots, x_6] = \left[0, a, 1, \frac{1}{1-b}, \infty, \frac{c-1}{c}\right].$$

Recall that the action of $\mathfrak{S}_6$ on K is defined as the pull-back of the action on $\mathcal{M}_{0,6}$, i.e. for $\sigma \in \mathfrak{S}_6$ and $f \in K = \boldsymbol{Q}(\mathcal{M}_{0,6})$, $\sigma(f)$ is defined to make the following diagram compatible:

$$\begin{array}{ccc} \mathcal{M}_{0,6} & \xrightarrow{f} & \mathbb{P}^1 \\ {\scriptstyle\sigma}\downarrow & & \| \\ \mathcal{M}_{0,6} & \xrightarrow{\sigma(f)} & \mathbb{P}^1, \end{array}$$

so that $\sigma(f)(P) = f(\sigma^{-1}(P))$ for $P \in \mathcal{M}_{0,6}$. We can calculate $\alpha^{-1}(P)$ as

$$\alpha^{-1}(P) = [x_2, x_3, x_4, x_5, x_6, x_1] = \left[a, 1, \frac{1}{1-b}, \infty, \frac{c-1}{c}, 0\right]$$
$$= \left[0, \frac{(1-a)(1-b+bc)}{1-a+ab}, 1, \frac{1-b+bc}{c(1-a+ab)}, \infty, \frac{-a(1-b+bc)}{(1-c)(1-a+ab)}\right],$$

where the renormalization is given by

$$\xi \mapsto \frac{\xi - a}{\xi - \frac{c-1}{c}} \Big/ \frac{\frac{1}{1-b} - a}{\frac{1}{1-b} - \frac{c-1}{c}} = \frac{(1-b+bc)(\xi - a)}{(1-a+ab)(1-c+c\xi)}.$$

Thus we obtain

$$\alpha(a) = \frac{(1-a)(1-b+bc)}{1-a+ab},$$

and also

$$\alpha\left(\frac{1}{1-b}\right) = \frac{1-b+bc}{c(1-a+ab)}, \quad \alpha\left(\frac{c-1}{c}\right) = \frac{-a(1-b+bc)}{(1-c)(1-a+ab)},$$

from which we can easily derive the expressions for $\alpha(b), \alpha(c)$ (see below).

§6. CR-NP for $6T3$ and its subgroups

The subgroup named $6T3$ in [B-MK] is the stabilizer of the necklace permutation $(1,2,3,4,5,6)$, which is generated by $\alpha := (1\ 2\ 3\ 4\ 5\ 6), \beta := (1\ 4)(2\ 3)(5\ 6)$, and is isomorphic to the dihedral group D_6 of degree 6.

The action of $6T3$ on $\boldsymbol{Q}(\mathcal{M}_{0,6})$ is described as follows:

$$\alpha : \begin{cases} a \mapsto \dfrac{(1-a)(1-b+bc)}{1-a+ab} \\ b \mapsto \dfrac{(1-b)(1-c+ca)}{1-b+bc} \\ c \mapsto \dfrac{(1-c)(1-a+ab)}{1-c+ca}, \end{cases} \qquad \beta : \begin{cases} a \mapsto \dfrac{b(1-c+ca)}{1-a+ab} \\ b \mapsto \dfrac{a(1-b+bc)}{1-c+ca} \\ c \mapsto \dfrac{c(1-a+ab)}{1-b+bc}. \end{cases}$$

The action of α^2 and $\alpha\beta$ is quite simple:

$$\alpha^2 : \begin{cases} a \mapsto b \\ b \mapsto c \\ c \mapsto a, \end{cases} \qquad \alpha\beta : \begin{cases} a \mapsto 1-b \\ b \mapsto 1-a \\ c \mapsto 1-c. \end{cases}$$

Put

$$x := a(1-b), \quad y := b(1-c), \quad z := c(1-a), \quad \text{and} \quad p := abc.$$

Note that x, y, z are determined respectively by the following normalizations of the generic point $P = [x_1, \dots, x_6] \in \mathcal{M}_{0,6}$

$$[0, x, *, 1, \infty, *], \qquad [\infty, *, 0, y, *, 1], \qquad [*, 1, \infty, *, 0, z],$$

and that $\boldsymbol{Q}(x,y,z)$ is a subfield of $K = \boldsymbol{Q}(a,b,c)$ with $[K : \boldsymbol{Q}(x,y,z)] = 2$, over which K is generated by p. To show this, we observe first the following identities

$$(1-y)a = x + p, \quad (1-z)b = y + p, \quad (1-x)c = z + p,$$

so that $K = \boldsymbol{Q}(x,y,z,p)$. By eliminating a, b, c we obtain the quadratic equation of p over $\boldsymbol{Q}(x,y,z)$:

$$p^2 + (x+y+z-1)p + xyz = 0. \tag{6.1}$$

Next one can express the action of α, β on these elements as follows:

$$\beta : \begin{cases} x \mapsto y \\ y \mapsto x \\ z \mapsto z \\ p \mapsto p \end{cases}, \quad \alpha : \begin{cases} x \mapsto z \\ y \mapsto x \\ z \mapsto y \\ p \mapsto (1-a)(1-b)(1-c) = 1 - (x+y+z) - p. \end{cases}$$

In this section, we deal with the following subgroups of $6T3$:

$$Z(6T3) = \langle\theta\rangle \simeq C_2, \tag{6.2}$$

$$6T1 := \langle\alpha\rangle \simeq C_6, \tag{6.3}$$

$$6T2 := \langle\alpha^2, \beta\rangle = \mathfrak{S}_3, \tag{6.4}$$

$$6T3 = \langle\alpha, \beta\rangle = D_6, \tag{6.5}$$

$$6T3 \cap \mathfrak{A}_6 = \langle\alpha^2, \alpha\beta\rangle = \mathfrak{S}_3, \tag{6.6}$$

where we put $\theta := \alpha^3 = (1\ 4)(2\ 5)(3\ 6)$ and denote the center of $6T3$ by $Z(6T3)$. Though the groups $Z(6T3)$ and $6T3 \cap \mathfrak{A}_6$ are not transitive, they will play some important roles in our computation.

6.1. $Z(6T3)$

The action of θ on the elements above is as follows:

$$\theta : \begin{cases} a \mapsto \dfrac{(1-b)(1-c+ca)}{1-b+bc} \\ b \mapsto \dfrac{(1-c)(1-a+ab)}{1-c+ca} \\ c \mapsto \dfrac{(1-a)(1-b+bc)}{1-a+ab}, \end{cases} \qquad \begin{cases} x \mapsto x \\ y \mapsto y \\ z \mapsto z \\ p \mapsto 1-(x+y+z)-p. \end{cases}$$

It follows that θ fixes x, y and z, so $\boldsymbol{Q}(x, y, z) \subset K^{\langle\theta\rangle}$. On the other hand, we have $[K : \boldsymbol{Q}(x, y, z)] = 2$ as is observed above. We have thus shown that $K^{\langle\theta\rangle} = \boldsymbol{Q}(x, y, z)$ and, in particular, $K^{\langle\theta\rangle}$ is rational over $\boldsymbol{Q}$.

6.2. $6T3$

We note that

$$6T3/Z(6T3) = \langle\overline{\alpha}, \overline{\beta}\rangle \simeq \mathfrak{S}_3$$

and that the action of this group on $K^{Z(6T3)} = \boldsymbol{Q}(x, y, z)$ is nothing but the standard permutation. Hence we have

$$K^{6T3} = (K^{Z(6T3)})^{6T3/Z(6T3)} = \boldsymbol{Q}(s, t, u),$$

where $s := x + y + z, t := xy + yz + zx, u := xyz$ are the fundamental symmetric polynomials in x, y, z.

6.3. $6T1$

By what we have proved in the previous subsection, the problem for $6T1$ is reduced to the rationality of

$$K^{6T1/Z(6T3)} = \boldsymbol{Q}(x, y, z)^{\langle\overline{\alpha}\rangle}$$

where $\langle\overline{\alpha}\rangle$ is isomorphic to $\mathfrak{A}_3$ as a subgroup of $6T3/Z(6T3) \cong \mathfrak{S}_3$. So the fixed field is given as

$$K^{6T1} = \boldsymbol{Q}(x,y,z)^{\langle\overline{\alpha}\rangle} = K^{6T3}(d),$$
$$d := (x-y)(y-z)(z-x),$$

and this field is known to be rational, since the original Noether's problem (Perm-NP) for $\mathfrak{A}_3$ is settled affirmatively. Here we review this classical result, because it is referred to again and again in this article. Let $D(s,t,u)$ be the discriminant form of $T^3 - sT^2 + tT - u$:

$$D(s,t,u) = -27u^2 + 18stu + s^2t^2 - 4s^3u - 4t^3.$$

Our basic ingredient to Perm-NP for $\mathfrak{A}_3$ is the following

Lemma 6.4. *By a birational change of variables*

$$\begin{cases} s = \dfrac{3s_1}{t_1} \\ t = \dfrac{3s_1^2 - u_1}{t_1^2} \\ u = \dfrac{s_1^3 - u_1}{t_1^3} \\ d = \dfrac{d_1 u_1}{t_1^3} \end{cases} \quad \textit{and} \quad \begin{cases} s_1 = \dfrac{3s(s^2-3t)}{s^3-27u} \\ t_1 = \dfrac{9(s^2-3t)}{s^3-27u} \\ u_1 = \dfrac{27(s^2-3t)^3}{(s^3-27u)^2} \\ d_1 = \dfrac{27d}{s^3-27u}, \end{cases}$$

$D(s,t,u)$ is expressed as

$$D(s,t,u) = \frac{u_1^2(4u_1 - 27(s_1-1)^2)}{t_1^6}.$$

Then we have $K^{6T3} = \boldsymbol{Q}(s_1,t_1,u_1)$ and $K^{6T1} = K^{6T3}(d_1)$ where d_1 satisfies a relation $d_1^2 = 4u_1 - 27(s_1-1)^2$. Hence we obtain $K^{6T1} = \boldsymbol{Q}(s_1,t_1,d_1)$, which is rational over $\boldsymbol{Q}$.

6.5. $6T2$

We can show that $K^{6T2} = K^{6T3}(p)$ by observing that $6T2$ fixes p and that $\alpha(p) \neq p$. Since we also have $p^2 + (s-1)p + u = 0$, $K^{6T2} = \boldsymbol{Q}(s,t,p) = \boldsymbol{Q}(t,u,p)$.

6.6. $6T3 \cap \mathfrak{A}_6$

Put

$$a' := 2a - 1, \qquad b' := 2b - 1, \quad \text{and} \quad c' := 2c - 1.$$

Then the action of α^2 and $\alpha\beta$ on them is

$$\alpha^2 : \begin{cases} a' \mapsto b' \\ b' \mapsto c' \\ c' \mapsto a', \end{cases} \qquad \alpha\beta : \begin{cases} a' \mapsto -b' \\ b' \mapsto -a' \\ c' \mapsto -c'. \end{cases}$$

This is the permutation action by $\mathfrak{S}_3$ twisted with the signature. The difference product $d' := (a' - b')(b' - c')(c' - a') = 8(a-b)(b-c)(c-a)$ is fixed by this action. We can easily show that

$$\boldsymbol{Q}(a, b, c)^{\langle \alpha^2, \alpha\beta \rangle} = \boldsymbol{Q}\left(\frac{t'}{s'^2}, \frac{u'}{s'^3}, \frac{d'}{s'^2}\right),$$

where $s' := a' + b' + c', t' := a'b' + b'c' + c'a', u' := a'b'c'$ are the fundamental symmetric polynomials in a', b' and c'.

§7. $6T13$ and its subgroups—$(3,3)$-imprimitive subgroups

The subgroup named $6T13$ in [B-MK] is the stabilizer of the partition $\{\{1,3,5\},\{2,4,6\}\}$. Hence $6T13$ is a maximal $(3,3)$-imprimitive subgroup of $\mathfrak{S}_6$, which is isomorphic to $C_2 \wr \mathfrak{S}_3 = C_2 \ltimes (\mathfrak{S}_3)^2$.

Adding to the notation in the previous section, we put $\gamma_1 := (1\ 3\ 5)$, $\gamma_2 := (2\ 4\ 6)$ and $N := \langle \gamma_1, \gamma_2 \rangle \simeq C_3 \times C_3$. Then we see that $\alpha^2 = \gamma_1\gamma_2 \in N$, and that N is the unique (normal) 3-Sylow subgroup of $6T13$. We also put

$$\delta := (1\ 4)(2\ 5\ 6\ 3).$$

Then it satisfies the relations

$$\delta^4 = \theta^2 = 1, \quad \theta\delta = \delta^{-1}\theta$$

where $\theta = \alpha^3 = (1\ 4)(2\ 5)(3\ 6)$. It follows that δ and θ generate a subgroup isomorphic to D_4, which is in $6T13$ a complementary group to N:

$$6T13 = \langle \theta, \delta \rangle \ltimes N, \tag{7.1}$$

$$6T13/N = \langle \overline{\theta}, \overline{\delta} \rangle \simeq D_4. \tag{7.2}$$

In this section, we deal with four transitive subgroups of $\mathfrak{S}_6$.

$$6T5 := \langle \theta\delta^2 \rangle \ltimes N = \langle \alpha^2, \beta, \gamma_1 \rangle, \tag{7.3}$$
$$(\delta(6T5)\delta^{-1} = \langle \theta \rangle \ltimes N = \langle \alpha, \gamma_1 \rangle,)$$
$$6T9 := \langle \theta, \delta^2 \rangle \ltimes N = \langle \alpha, \beta, \gamma_1 \rangle, \tag{7.4}$$
$$6T10 := \langle \delta \rangle \ltimes N = \langle \alpha^2, \alpha\beta, \gamma_1, \delta \rangle = 6T13 \cap \mathfrak{A}_6, \tag{7.5}$$
$$6T13 = \langle \theta, \delta \rangle \ltimes N = \langle \alpha, \beta, \gamma_1, \delta \rangle. \tag{7.6}$$

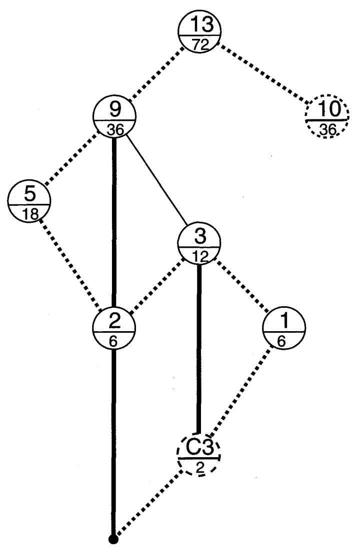


Fig. 2. $6T13$ and its subgroups

7.1. $6T9$

The group $6T9$ has $6T2 = \langle \alpha^2, \beta \rangle = \langle \beta \rangle \ltimes \langle \gamma_1\gamma_2 \rangle$ as its normal subgroup. The quotient group $6T9/6T2$ is isomorphic to $\mathfrak{S}_3$:

$$6T9/6T2 = \langle \overline{\gamma_2}, \overline{\theta} \rangle \simeq \mathfrak{S}_3.$$

Recall that $K^{6T2} = \boldsymbol{Q}(s,t,u,p)$ is a quadratic extension of $\boldsymbol{Q}(s,t,u)$ generated by p which satisfies $p^2 + (s-1)p + u = 0$ by (6.1). The action of $\overline{\gamma_2}, \overline{\theta}$ on K^{6T2} is as follows:

$$\overline{\gamma_2} : \begin{cases} s \mapsto -\dfrac{s}{p} \\ t \mapsto \dfrac{t}{p^2} \\ u \mapsto -\dfrac{u}{p^3} \\ p \mapsto -\dfrac{u}{p^2}, \end{cases} \qquad \overline{\theta} : \begin{cases} s \mapsto s \\ t \mapsto t \\ u \mapsto u \\ p \mapsto 1 - s - p = \dfrac{u}{p}. \end{cases}$$

We can also observe that

$$\overline{\gamma_2}(1 - s + t - u) = \frac{1 - s + t - u}{p^2}.$$

Put

$$r_1 := \frac{p}{1 - s + t - u}, \qquad r_2 := -\frac{u}{1 - s + t - u}, \qquad r_3 := \frac{1 - s - p}{1 - s + t - u}.$$

Then we have $K^{6T2} = \boldsymbol{Q}(r_1, r_2, r_3)$ and

$$\overline{\gamma_2} : \begin{cases} r_1 \mapsto r_2 \\ r_2 \mapsto r_3 \\ r_3 \mapsto r_1, \end{cases} \qquad \overline{\theta} : \begin{cases} r_1 \mapsto r_3 \\ r_2 \mapsto r_2 \\ r_3 \mapsto r_1. \end{cases}$$

Hence the action of $6T9/6T2 \simeq \mathfrak{S}_3$ on $K^{6T2} = \boldsymbol{Q}(r_1, r_2, r_3)$ is determined by the natural permutation of its generators r_1, r_2 and r_3. Putting

$$s_9 := r_1 + r_2 + r_3, \quad t_9 := r_1 r_2 + r_3 r_1 + r_2 r_3, \quad u_9 := r_1 r_2 r_3,$$

we conclude that $K^{6T9} = \boldsymbol{Q}(s_9, t_9, u_9)$, which is rational over $\boldsymbol{Q}$.

7.2. $6T5$

The group $6T5$ is an intermediate group of $6T9/6T2$, which corresponds with $\mathfrak{A}_3$ via $6T9/6T2 \simeq \mathfrak{S}_3$. The rationality of K^{6T5} follows from this. In fact, if we put $d_5 := (r_1 - r_2)(r_2 - r_3)(r_3 - r_1)$, $K^{6T5} = K^{6T9}(d_5)$ with $d_5^2 = D(s_9, t_9, u_9)$, which is rational over $\boldsymbol{Q}$ by Lemma 6.4.

7.3. $6T13$

The group $6T13$ is generated by $6T9$ together with $\delta = (1\ 4)(2\ 5\ 6\ 3)$. The action of δ on K^{6T9} is as follows:

$$\delta : \begin{cases} s_9 \mapsto 1 - s_9 \\ t_9 \mapsto t_9 \\ u_9 \mapsto -u_9. \end{cases}$$

Hence we have $K^{6T13} = \boldsymbol{Q}(s_{13}, t_{13}, u_{13})$, where we put $s_{13} := \dfrac{2s_9 - 1}{u_9}$, $t_{13} := t_9$, $u_{13} := u_9^2$. In particular, K^{6T13} is rational over $\boldsymbol{Q}$.

7.4. $6T10$

Consider the fixed fields of the subgroups of

$$6T13/N = \langle \overline{\theta}, \overline{\delta} \rangle \simeq D_4.$$

First look at $6T9/N = \langle \overline{\theta}, \overline{\delta}^2 \rangle \simeq C_2 \times C_2$. Three intermediate fields between K^{6T9} and K^N are $K^{6T5} = K^{6T9}(d_5)$, $K^{\delta(6T5)\delta^{-1}} = \delta(K^{6T5}) = K^{6T9}(\delta(d_5))$ and K^{N_2}, where $N_2 := \langle \delta^2 \rangle \ltimes N = 6T9 \cap \mathfrak{A}_6$. Hence we have $K^{N_2} = K^{6T9}(d_{10})$, where $d_{10} := d_5\delta(d_5)$. Next look at $6T13/N_2 = \langle \overline{\theta}, \overline{\delta} \rangle / \langle \overline{\delta}^2 \rangle \simeq C_2 \times C_2$. Since d_{10} is δ-invariant and

$$d_{10}^2 = D(s_9, t_9, u_9) D(1 - s_9, t_9, -u_9) \in (K^{6T9})^{\langle \overline{\delta} \rangle} = K^{6T13},$$

we have $K^{6T10} = K^{6T13}(d_{10}) = \boldsymbol{Q}(s_{13}, t_{13}, u_{13}, d_{10})$. The relation among these generators is

$$\begin{aligned} d^2 &= 1296su^2 - 1152st^4u - 4s^5t^2u^3 + 136s^3t^3u^2 - 4u - 144s^4tu^3 \\ &\quad + 11664u^2 - 32t^5 + 120st^3u + 12s^2u^2 - 4st^2u + 4s^6u^4 - 12s^4u^3 \\ &\quad - 2s^2t^4u - 32s^2t^5u + 1080s^2t^2u^2 + 256t^6 + 8s^3t^2u^2 + s^4t^4u^2 \\ &\quad + 144tu - 7776stu^2 - 1512t^2u + 3456t^3u + 432s^3u^3 + t^4, \end{aligned}$$

where in the equation above we omit the subscripts for simplicity. Unfortunately this equation is so complicated that we can not determine whether K^{6T10} is rational or not, at least, at the present time via this equation. Hence we must take a different way for $6T10$, though the way above looks natural.

Put $f_6 := \dfrac{b - a}{1 - a + ab}$. The $6T10$-orbit of f_6 is $\{f_1, \ldots, f_6\}$, where

$$\begin{aligned} f_1 &:= \frac{1}{1 - c + ca} - \frac{b}{1 - a + ab} + \frac{bc}{1 - b + bc}, & f_2 &:= \frac{c - b}{1 - b + bc}, \\ f_3 &:= \frac{1}{1 - a + ab} - \frac{c}{1 - b + bc} + \frac{ca}{1 - c + ca}, & f_4 &:= \frac{a - c}{1 - c + ca}, \\ f_5 &:= \frac{1}{1 - b + bc} - \frac{a}{1 - c + ca} + \frac{ab}{1 - a + ab}, & f_6 &= \frac{b - a}{1 - a + ab}, \end{aligned}$$

and we can see that $6T10$ acts on this set as permutations of indices. Since

$$a = -\frac{(1-f_4)(1-f_5)+f_5f_6}{(1-f_1)(1-f_6)+f_1f_4},$$
$$b = -\frac{(1-f_6)(1-f_1)+f_1f_2}{(1-f_3)(1-f_2)+f_3f_6},$$
$$c = -\frac{(1-f_2)(1-f_3)+f_3f_4}{(1-f_5)(1-f_4)+f_5f_2},$$

we have $\boldsymbol{Q}(f_1,\dots,f_6)=\boldsymbol{Q}(a,b,c)=K$. Put

$$F_1(T)=T^3-h_{11}T^2+h_{12}T-h_{13}:=(T-f_1)(T-f_3)(T-f_5),$$
$$F_2(T)=T^3-h_{21}T^2+h_{22}T-h_{23}:=(T-f_2)(T-f_4)(T-f_6).$$

Then a direct calculation shows that

$$h_{11}=h_{12},\qquad h_{21}=h_{22}\qquad\text{and}\qquad h_{11}+h_{21}=3.$$

Denote the difference product of their roots by $d_1:=(f_1-f_3)(f_3-f_5)(f_5-f_1), d_2:=(f_2-f_4)(f_4-f_6)(f_6-f_2)$ and put $d_0:=d_1/d_2$. We consider the fixed fields of N and N_2. First we claim that

Claim 7.4.1. *$K^N=\boldsymbol{Q}(h_{11},h_{13},h_{23},d_1,d_2)$ and $K^{N_2}=\boldsymbol{Q}(h_{11},h_{13},h_{23},d_0)$.*

Proof. Since γ_1 (resp. γ_2) permutes the roots of F_1 (resp. F_2) cyclically and fixes the roots of F_2 (resp. F_1), $\boldsymbol{Q}(h_{11},h_{13},h_{23},d_1,d_2)$ is included in the fixed field K^N of $N=\langle\gamma_1,\gamma_2\rangle$. On the other hand, since the Galois groups of both F_1 and F_2 over $\boldsymbol{Q}(h_{11},h_{13},h_{23},d_1,d_2)$ are cyclic and K is generated by their roots $f_1,\dots,f_6$, we have $[K:\boldsymbol{Q}(h_{11},h_{13},h_{23},d_1,d_2)]\le 9$. Hence we obtain $K^N=\boldsymbol{Q}(h_{11},h_{13},h_{23},d_1,d_2)$.

Since $\delta^2=(2\ 6)(3\ 5)$ fixes h_{11},h_{13},h_{23} and maps d_1 (resp. d_2) to $-d_1$ (resp. $-d_2$), $\boldsymbol{Q}(h_{11},h_{13},h_{23},d_0)$ is included in the fixed field K^{N_2}. On the other hand, since $d_1^2=D(h_{11},h_{11},h_{13})\in\boldsymbol{Q}(h_{11},h_{13},h_{23})$ and $d_2=d_1/d_0$, we have $[K^N:\boldsymbol{Q}(h_{11},h_{13},h_{23},d_0)]\le 2$. Since $[K^N:K^{N_2}]=2$, we obtain $K^{N_2}=\boldsymbol{Q}(h_{11},h_{13},h_{23},d_0)$. Q.E.D.

Next we shall show the rationality of $K^{N_2}=\boldsymbol{Q}(h_{11},h_{13},h_{23},d_0)$. We have a relation

$$\begin{aligned}d_0^2&=\frac{D(h_{11},h_{11},h_{13})}{D(3-h_{11},3-h_{11},h_{23})}\\&=\frac{-27h_{13}^2+18h_{13}h_{11}^2+h_{11}^4-4h_{11}^3h_{13}-4h_{11}^3}{-27h_{23}^2+54h_{23}-18h_{23}h_{11}^2-27+18h_{11}^2-8h_{11}^3+h_{11}^4+4h_{11}^3h_{23}}\end{aligned}$$

among h_{11}, h_{13}, h_{23} and d_0. By a birational change of variables

$$\begin{cases} h_{11} = 3g_1 \\ h_{13} = g_1(g_1g_3 - g_3 + 1) \\ h_{23} = (g_1 - 1)(g_1g_2 - 1), \end{cases} \quad \text{and} \quad \begin{cases} g_1 = \dfrac{h_{11}}{3} \\ g_2 = \dfrac{3(3 - h_{11} - 3h_{23})}{h_{11}(3 - h_{11})} \\ g_3 = \dfrac{3(h_{11} - 3h_{13})}{h_{11}(3 - h_{11})}, \end{cases}$$

we have $K^{N_2} = \boldsymbol{Q}(h_{11}, h_{13}, h_{23}, d_0) = \boldsymbol{Q}(g_1, g_2, g_3, d_0)$ with

$$g_1 = \frac{d_0^2(g_2 + 1)^2 - (g_3 - 1)^2}{4(d_0^2 g_2 + g_3)}.$$

Hence we obtain $K^{N_2} = \boldsymbol{Q}(g_2, g_3, d_0)$, which is rational over $\boldsymbol{Q}$.

Now we determine $K^{6T10} = (K^N)^{\langle\delta\rangle}$ and show its rationality. The action of δ on K^N is

$$\begin{cases} h_{11} \mapsto 3 - h_{11} \\ h_{13} \mapsto h_{23} \\ h_{23} \mapsto h_{13} \\ d_1 \mapsto -d_2 \\ d_2 \mapsto d_1, \end{cases} \quad \text{hence} \quad \begin{cases} g_2 \mapsto g_3 \\ g_3 \mapsto g_2 \\ d_0 \mapsto -\dfrac{1}{d_0}. \end{cases}$$

Hence we finally obtain

$$K^{6T10} = \boldsymbol{Q}\left(g_2 + g_3, (g_2 - g_3)(d_0 + \frac{1}{d_0}), d_0 - \frac{1}{d_0}\right),$$

which is rational over $\boldsymbol{Q}$.

§8. $6T11$ and its subgroups—$(2,2,2)$-imprimitive subgroups

The subgroup named $6T11$ in [B-MK] is the stabilizer of the partition $\{\{1,4\},\{2,5\},\{3,6\}\}$. Hence $6T11$ is a maximal $(2,2,2)$-imprimitive subgroup of $\mathfrak{S}_6$, which is isomorphic to $\mathfrak{S}_3 \wr C_2 = \mathfrak{S}_3 \ltimes (C_2)^3 \simeq \mathfrak{S}_4 \times C_2$. The center $Z(6T11)$ of $6T11$ is $\langle\theta\rangle \simeq C_2$, which coincides with the center $Z(6T3)$ of $6T3$ discussed in 6.1, whose fixed field $K^{Z(6T11)} = \boldsymbol{Q}(x, y, z)$.

Adding to the notation in the previous section, we put $\tau_1 := (1\ 4)(2\ 5)$, $\tau_2 := (1\ 4)(3\ 6)$ and $M := \langle\tau_1, \tau_2, \theta\rangle \simeq (C_2)^3$. Then we have $6T11 = 6T2 \ltimes M$, where $6T2 = \langle\alpha^2, \beta\rangle \simeq \mathfrak{S}_3$. We also put $M^+ := M \cap \mathfrak{A}_6 = \langle\tau_1, \tau_2\rangle$.

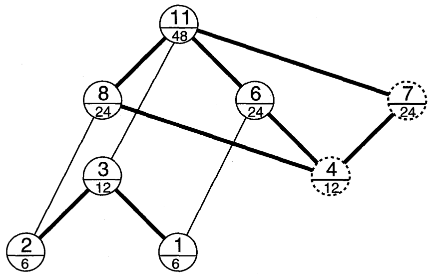


Fig. 3. $6T11$ and its subgroups

In this section, we deal with five transitive subgroups of $\mathfrak{S}_6$.

$$6T4 := \langle \alpha^2 \rangle \ltimes M^+ = \langle \alpha^2, \tau_1, \tau_2 \rangle, \tag{8.1}$$

$$6T6 := \langle \alpha^2 \rangle \ltimes M = \langle \alpha, \tau_1, \tau_2 \rangle = \langle 6T4, \theta \rangle, \tag{8.2}$$

$$\begin{aligned} 6T7 &:= \langle \alpha^2, \alpha\beta \rangle \ltimes M^+ = \langle \alpha^2, \alpha\beta, \tau_1, \tau_2 \rangle = \langle 6T4, \beta\theta \rangle \\ &= 6T11 \cap \mathfrak{A}_6, \end{aligned} \tag{8.3}$$

$$6T8 := 6T2 \ltimes M^+ = \langle \alpha^2, \beta, \tau_1, \tau_2 \rangle = \langle 6T4, \beta \rangle, \tag{8.4}$$

$$6T11 = 6T2 \ltimes M = \langle \alpha, \beta, \tau_1, \tau_2 \rangle = \langle 6T4, \beta, \theta \rangle. \tag{8.5}$$

Three subgroups $6T6, 6T7, 6T8$ correspond to three proper subgroups of $6T11/6T4 = \langle \overline{\beta}, \overline{\theta} \rangle \simeq C_2 \times C_2$. Both $6T7$ and $6T8$ are isomorphic to $\mathfrak{S}_4$, but they are not conjugate in $\mathfrak{S}_6$. We also have $6T11 = 6T7 \times Z(6T11)$.

8.1. $6T11$

Put $x' := 2x-1, y' := 2y-1, z' := 2z-1$: $K^{Z(6T11)} = \boldsymbol{Q}(x,y,z) = \boldsymbol{Q}(x',y',z')$. Note that we can consider these elements through the following normalization:

$$[0, x, *, 1, \infty, *] = [-1, x', *, 1, \infty, *], \tag{8.6}$$

$$[\infty, *, 0, y, *, 1] = [\infty, *, -1, y', *, 1], \tag{8.7}$$

$$[*, 1, \infty, *, 0, z] = [*, 1, \infty, *, -1, z']. \tag{8.8}$$

The induced action of $6T11/Z(6T11) \simeq 6T7 = \mathfrak{S}_4$ on $K^{Z(6T11)} = \boldsymbol{Q}(x',y',z')$ is canonically identified with the action of $6T7$ restricted on it. Then we have the following remarkable proposition:

Proposition 8.2. *The action of $6T11/Z(6T11) \simeq \mathfrak{S}_4$ on $\boldsymbol{Q}(x', y', z')$ can be identified with the standard action of $\mathfrak{S}_4$ on the regular octahedron in the 3-dimansional Euclidean space. Hence the fixed field $\boldsymbol{Q}(x', y', z')^{6T11/Z(6T11)} = K^{6T11}$ is rational over $\boldsymbol{Q}$.*

Proof. We have

$$6T7 = \langle \alpha^2, \alpha\beta \rangle \ltimes \langle \tau_1, \tau_2 \rangle \simeq \mathfrak{S}_3 \ltimes V_4 \simeq \mathfrak{S}_4,$$

where their action on x, y, z etc. are as follows:

$$\alpha^2 = (1\ 3\ 5)(2\ 4\ 6) : \begin{cases} x \mapsto y \\ y \mapsto z \\ z \mapsto x, \end{cases} \qquad \begin{cases} x' \mapsto y' \\ y' \mapsto z' \\ z' \mapsto x', \end{cases}$$

$$\alpha\beta = (1\ 5)(2\ 4) : \begin{cases} x \mapsto x \\ y \mapsto z \\ z \mapsto y, \end{cases} \qquad \begin{cases} x' \mapsto x' \\ y' \mapsto z' \\ z' \mapsto y', \end{cases}$$

$$\tau_1 = (1\ 4)(2\ 5) : \begin{cases} x \mapsto x \\ y \mapsto 1 - y \\ z \mapsto 1 - z, \end{cases} \qquad \begin{cases} x' \mapsto x' \\ y' \mapsto -y' \\ z' \mapsto -z', \end{cases}$$

$$\tau_2 = (1\ 4)(3\ 6) : \begin{cases} x \mapsto 1 - x \\ y \mapsto y \\ z \mapsto 1 - z, \end{cases} \qquad \begin{cases} x' \mapsto -x' \\ y' \mapsto y' \\ z' \mapsto -z'. \end{cases}$$

Thus we have $K^{6T11} = \boldsymbol{Q}(x', y', z')^{\mathfrak{S}_4} = \boldsymbol{Q}(s_{11}, t_{11}, u_{11})$, where

$$s_{11} = x'^2 + y'^2 + z'^2, \quad t_{11} = x'^2 y'^2 + y'^2 z'^2 + z'^2 x'^2,$$
$$u_{11} = x'y'z'.$$

Q.E.D.

We have another typical $6T11$-invariant element

$$w_{11} = xyz(1-x)(1-y)(1-z) \in K^{6T11}.$$

Since it holds that $t_{11} = 64w_{11} - 1 + s_{11} + u_{11}^2$, we have also $K^{6T11} = \boldsymbol{Q}(s_{11}, u_{11}, w_{11})$.

8.3. $6T6$

The subgroup $6T6$ corresponds with $\mathfrak{A}_4 \subset \mathfrak{S}_4$ via $6T11/Z(6T11) \simeq \mathfrak{S}_4$. Thus $K^{6T6} = (K^{Z(6T11)})^{6T6/Z(6T11)} = \boldsymbol{Q}(x', y', z')^{\mathfrak{A}_4}$ is rational by a classical result.

Presicely, it is easy to see that $K^{6T6} = K^{6T11}(d)$, where

$$d = ({x'}^2 - {y'}^2)({y'}^2 - {z'}^2)({z'}^2 - {x'}^2).$$

Since d^2 is the discriminant of $T^3 - s_{11}T^2 + t_{11}T - u_{11}^2$, there is an algebraic relation

$$d^2 = -27u_{11}^4 + 18u_{11}^2 s_{11} t_{11} + s_{11}^2 t_{11}^2 - 4s_{11}^3 u_{11}^2 - 4t_{11}^3. \tag{8.9}$$

By a birational change of variables

$$\begin{cases} s_6 = \dfrac{3s_{11}(s_{11}^2 - 3t_{11})}{s_{11}^3 - 27u_{11}^2} \\ t_6 = \dfrac{9(s_{11}^2 - 3t_{11})}{s_{11}^3 - 27u_{11}^2} \\ u_6 = -\dfrac{(s_{11}^2 - 3t_{11})u_{11}}{s_{11}t_{11} - 9u_{11}^2} \\ d_6 = \dfrac{3d}{s_{11}t_{11} - 9u_{11}^2}, \end{cases} \quad \text{and} \quad \begin{cases} s_{11} = \dfrac{3s_6}{t_6} \\ t_{11} = \dfrac{(s_6 - 3)(t_6 u_6^2 (s_6 - 3) - s_6^2)}{t_6^2} \\ u_{11} = \dfrac{u_6(s_6 - 3)}{t_6} \\ d = \dfrac{d_6(s_6 - 3)(t_6 u_6^2 (s_6 - 3)^2) - s_6^3}{t_6^3} \end{cases}$$

we have $K^{6T11} = \boldsymbol{Q}(s_{11}, t_{11}, u_{11}) = \boldsymbol{Q}(s_6, t_6, u_6)$ and $K^{6T6} = K^{6T11}(d_6)$ with $t_6 = \dfrac{4s_6 - 3 - d_6^2}{4u_6^2}$. Hence we obtain $K^{6T6} = \boldsymbol{Q}(s_6, u_6, d_6)$, so that it is rational over $\boldsymbol{Q}$.

8.4. $6T8$

Put $v_8 := 2(p - \theta(p)) = \mathrm{tr}_{\langle \tau_1, \tau_2 \rangle}(p) + 1$, where $p = abc$. The stabilizer of v_8 in $6T11$ is $6T8$. Hence $K^{6T8} = K^{6T11}(p) = \boldsymbol{Q}(s_{11}, t_{11}, u_{11}, v_8)$. Moreover we have $\theta(v_8) = -v_8$, hence $v_8^2 \in K^{6T11}$. In fact, $v_8^2 = s_{11} - 2u_{11} - 1$. Therefore we obtain that $K^{6T8} = \boldsymbol{Q}(t_{11}, u_{11}, v_8)$.

8.5. $6T4$

The quotient group $6T11/6T4$ is isomorphic to $C_2 \times C_2$, and has three proper subgroups corresponding $6T6 = \langle 6T4, \theta \rangle, 6T7 = \langle 6T4, \beta\theta \rangle$ and $6T8 = \langle 6T4, \beta \rangle$. Since $6T4 = 6T6 \cap 6T8$, we have $K^{6T4} = K^{6T6}K^{6T8} = K^{6T6}(v_8) = \boldsymbol{Q}(s_6, u_6, d_6)(v_8)$. Since

$$t_6 = \frac{4s_6 - 3 - d_6^2}{4u_6^2} = \frac{3s_6 - 2u_6(s_6 - 3)}{v_8^2 + 1},$$

it follows that

$$s_6 = \frac{(v_8^2+1)(d_6^2+3)+24u_6^3}{4(v_8^2+1-3u_6^2+2u_6^3)}.$$

Hence we have $K^{6T4} = \boldsymbol{Q}(s_6, u_6, d_6, v_8) = \boldsymbol{Q}(u_6, d_6, v_8)$.

8.6. $6T7$

We look at the extension K^{6T4}/K^{6T11} whose Galois group is $\mathrm{Gal}(K^{6T4}/K^{6T11}) = \langle\overline{\theta}, \overline{\beta}\rangle \simeq V_4$. The action of $\overline{\theta}, \overline{\beta}$ on K^{6T4} is as follows:

$$\overline{\theta} : \begin{cases} u_6 \mapsto u_6 \\ d_6 \mapsto d_6 \\ v_8 \mapsto -v_8, \end{cases} \qquad \overline{\beta} : \begin{cases} u_6 \mapsto u_6 \\ d_6 \mapsto -d_6 \\ v_8 \mapsto v_8. \end{cases}$$

In fact, we already have $\boldsymbol{Q}(u_6, d_6^2, v_8^2) = K^{6T11}, \boldsymbol{Q}(u_6, d_6, v_8^2) = K^{6T6}$ and $\boldsymbol{Q}(u_6, d_6^2, v_8) = K^{6T8}$. By the observation above, we obtain that $K^{6T7} = (K^{6T4})^{\langle\overline{\beta\theta}\rangle} = \boldsymbol{Q}(u_6, d_6^2, d_6 v_8)$ is rational over $\boldsymbol{Q}$.

Thus we have shown that CR-NP for all solvable transitive subgroups of $\mathfrak{S}_6$ are affirmative.

§9. $6T14$—$\mathfrak{S}_5$ transitively embedded in $\mathfrak{S}_6$

9.1. The exotic $\mathfrak{S}_6$-action

The subgroup named $6T14$ in [B-MK] is isomorphic to the symmetric group $\mathfrak{S}_5$ of degree 5. There are two conjugacy classes of subgroups of $\mathfrak{S}_6$ which are isomorphic to $\mathfrak{S}_5$. One consists of the stabilizers of one of the indices among $\{1, \ldots, 6\}$, and the other consists of the conjugates of $6T14$, where they are interchanged by a non-trivial outer automorphism of $\mathfrak{S}_6$. We hope to understand $6T14$ in the context of the $\mathfrak{S}_6$-action twisted by its non-trivial outer automorphism.

Let $S_6 := \{1, \ldots, 6\}$ be the set consisting of six points with the standard $\mathfrak{S}_6$-action. To describe the $\mathfrak{S}_6$-action twisted by an outer automorphism, we introduce the *exotic* $\mathfrak{S}_6$*-set*

$$X_6 := \{\iota : \mathbb{P}^1(\boldsymbol{F}_5) \to S_6 : \text{bijective}\}/\mathrm{PGL}(2, \boldsymbol{F}_5), \tag{9.1}$$

where the projective linear group $\mathrm{PGL}(2, \boldsymbol{F}_5)$ of degree 2 over $\boldsymbol{F}_5$, which is isomorphic to $\mathfrak{S}_5$, acts from the right: $g : \iota \mapsto \iota \circ g$, and $\mathfrak{S}_6$ acts on X_6 from the left: $\sigma : [\iota] \mapsto [\sigma \circ \iota]$. Then, $6T14$ (resp. $6T8$, $6T2$) is the stabilizer of one point (resp. ordered two points, ordered three points) of X_6.

To combine the calculation in previous sections, we take a point $[\iota_0] \in X_6$ where ι_0 maps $(-2, 0, +2, +1, -1, \infty)$ to $(1, 2, 3, 4, 5, 6)$ in this order, and define $6T14$ to be the stabilizer of $[\iota_0]$. Then each element $\sigma \in 6T14$ induces an element $g \in \mathrm{PGL}(2, \boldsymbol{F}_5)$ which satisfies $\sigma \circ \iota_0 = \iota_0 \circ g$. Moreover we put $\iota_1 : (+2, 0, -1, +1, -2, \infty) \mapsto (1, 2, 3, 4, 5, 6)$ and $\iota_2 : (-1, 0, -2, +1, +2, \infty) \mapsto (1, 2, 3, 4, 5, 6)$. Then $6T8$ (resp. $6T2$) in the previous sections coincides with the stabilizer of the ordered two points $([\iota_0], [\iota_1])$ (resp. the ordered three points $([\iota_0], [\iota_1], [\iota_2])$). In fact, the generators $\alpha^2 = (1\ 3\ 5)(2\ 4\ 6), \beta = (1\ 4)(2\ 3)(5\ 6), \tau_1 = (1\ 4)(2\ 5)$ and $\tau_2 = (1\ 4)(3\ 6)$ of $6T8$ induces linear fractional transformations $\xi \mapsto \dfrac{1}{1-\xi}, \dfrac{-\xi+2}{\xi+1}, -\xi-1$ and $\dfrac{2\xi}{\xi-2}$ on $\mathbb{P}^1(\boldsymbol{F}_5)$ respectively. The quotient set $6T14/6T8$ can be identified with the orbit $X_6 \smallsetminus \{[\iota_0]\}$ of $[\iota_1]$, on which $6T14 \simeq \mathfrak{S}_5$ acts by permutations. The group $\mathrm{PGL}(2, \boldsymbol{F}_5)$ has a translation $\xi \mapsto \xi + 1$ of order 5, which is not included in $6T8$. This corresponds to $\varphi := (1\ 5\ 2\ 4\ 3) \in 6T14$. The group $6T14$ is generated by $6T8$ together with φ, and $\{\varphi^i \mid i = 0, \ldots, 4\}$ forms a system of representatives of $6T14/6T8$.

9.2. $6T14$

By the consideration in the preceding subsection, we know that $6T14 \simeq \mathfrak{S}_5$ acts by permutations on the $\langle\varphi\rangle$-orbit of an element in K^{6T8} which is not fixed by φ. We shall find an element in K^{6T8} suitable for our computation. First we observe the action of φ:

$$\varphi = (1\ 5\ 2\ 4\ 3) : \begin{cases} a \mapsto -\dfrac{a(1-b)}{1-a+ab} \\ b \mapsto \dfrac{1}{a} \\ c \mapsto \dfrac{1-c+ca}{1-c}. \end{cases}$$

Put $w_0 := \dfrac{2}{2-v_8} \in K^{6T8}$ and $w_i := \varphi^i(w_0)$ for $i = 1, \ldots, 4$. Then we have

$$\begin{aligned} w_0 &= \frac{1}{1-p+\theta(p)} = \frac{1}{2-a-b-c+ab+bc+ca-2abc}, \\ w_1 &= (1-c)(1-a+ab)w_0, \qquad w_2 = -abcw_0, \\ w_3 &= (1-b)(1-c+ca)w_0, \qquad w_4 = (1-a)(1-b+bc)w_0. \end{aligned}$$

Since

$$a = -\frac{1-w_0-w_3}{1-w_2-w_1}, \qquad b = -\frac{1-w_0-w_1}{1-w_2-w_4}, \qquad c = -\frac{1-w_0-w_4}{1-w_2-w_3},$$

we have $\boldsymbol{Q}(w_0,\dots,w_4)=\boldsymbol{Q}(a,b,c)=K$. Put

$$F(T):=\prod_{i=0}^{4}(T-w_i)=:T^5+\sum_{i=0}^{4}C_iT^i.$$

Then direct calculation shows that $C_4=-2$ and $C_2+C_3=1$. Hence, by putting $s_{14}:=C_2, t_{14}:=C_1$ and $u_{14}:=C_0$, we have $\boldsymbol{Q}(s_{14},t_{14},u_{14})\subset K^{6T14}$ and
(9.2)
$$F(T)=X^5-2X^4+(1-s_{14})X^3+s_{14}X^2+t_{14}X+u_{14}\in\boldsymbol{Q}(s_{14},t_{14},u_{14})[T].$$

Moreover since the roots $w_0,\dots,w_4$ generate $K=\boldsymbol{Q}(a,b,c)$, we have $K^{6T14}=\boldsymbol{Q}(s_{14},t_{14},u_{14})$, which is rational over $\boldsymbol{Q}$.

We give a remark on the fixed field of $6T12\simeq\mathfrak{A}_5$. Since $6T14\simeq\mathfrak{S}_5$ acts on the roots $w_0,\dots,w_4$ of $F(T)$ by permutations, K^{6T12} is generated by the difference product

$$d_{14}:=\prod_{0\le i<j\le 4}(w_i-w_j)$$

of the roots of $F(T)$ over K^{6T14}. Hence the CR-NP for $6T12$ is reduced to the rationality of a hypersurface in $\boldsymbol{A}^4$:

Corollary 9.3. *$\boldsymbol{Q}(\mathcal{M}_{0,6})^{6T12}$ is isomorphic to the function field of the hypersurface in $\boldsymbol{A}^4$ defined over $\boldsymbol{Q}$ by the following equation:*

$$\begin{aligned} d^2 \;=\;& 3125u^4-(750s^2-450s-108-5000t)u^3\\ &-(108s^5+3s^4+110s^3-900s^3t+27s^2+470s^2t-432st\\ &+2000st^2-108t-1800t^2)u^2-2(s^2-4t)(8s^4+36s^3t+2s^3\\ &-35s^2t-9st-160st^2-4t^2)u+t^2(s^2-4t)^2(4s+16t+1).\end{aligned}$$

References

[B-MK] G. Butler and J. McKay, The symmetric groups of degree up to eleven, Comm. Algebra, **11** (1983), 893–911.
[De] F. DeMeyer, Generic polynomials, J. Algebra, **84** (1982), 441–448.
[Dol1] I. Dolgachev, Lectures on Invariant Theory, London Math. Soc. Lecture Note Ser., **296**, Cambridge Univ. Press, 2003.
[Dol2] I. Dolgachev, Topics in Classical Algebraic Geometry, Part I, available at http://www.math.lsa.umich.edu/˜idolga/lecturenotes.html.
[EM] S. Endô and T. Miyata, Invariants of finite abelian groups, J. Math. Soc. Japan, **25** (1973), 7–26.

[Fi] E. Fischer, Die Isomorphie der Invariantenkörper der endlichen Abelschen Gruppen lineare Transformationen, Nachr. Königl. Ges. Wiss. Göttingen, **1915** (1915), 77–80.

[GMS] S. Garibaldi, A. Merkurjev and J. P. Serre, Cohomological Invariants in Galois Cohomology, Univ. Lecture Ser., **28**, Amer. Math. Soc., Providence, RI, 2003.

[Ha] M. Hall, Jr., The Theory of Groups, The Macmillan Co., New York, NY, 1959.

[Has] K. Hashimoto, On Brumer's family of RM-curves of genus two, Tohoku Math. J. (2), **52** (2000), 475–488.

[HHR] K. Hashimoto, A. Hoshi and Y. Rikuna, Noether's problem and $\boldsymbol{Q}$-generic polynomials for the normalizer of the 8-cycle in S_8 and its subgroups, Math. Comp., **77** (2008), 1153–1183.

[HT] K. Hashimoto and H. Tsunogai, Generic polynomials over $\boldsymbol{Q}$ with two parameters for the transitive groups of degree five, Proc. Japan Acad. Ser. A Math. Sci., **79** (2003), 142–145.

[Ib] T. Ibukiyama, On symplectic Euler factors of genus two, J. Fac. Sci. Univ. Tokyo Sect IA Math. **30** (1984), 587–614.

[Ig1] J. Igusa, Modular forms and projective invariants, Amer. J. Math., **89** (1967), 817–855.

[Ig2] J. Igusa, Arithmetic variety of moduli for genus two, Ann. of Math. (2), **72** (1960), 612–649.

[JLY] C. Jensen, A. Ledet and N. Yui, Generic Polynomials, Constructive Aspects of the Inverse Galois Problem, MSRI Publications, Cambridge, 2002.

[Kem] G. Kemper, A constructive approach to Noether's problem, Manuscripta Math., **90** (1996), 343–363.

[KM] G. Kemper and E. Mattig, Generic polynomials with few parameters, J. Symbolic Comput., **30** (2000), 843–857.

[Kus] K. Kusumori, On cross-ratio Noether problem of degree 6 (Japanese), master's thesis, Sophia Univ., 2005.

[Kuy] W. Kuyk, Over het omkeerprobleem van de Galoistheorie (Dutch), Vrije Univ. te Amsterdam, Amsterdam 1960.

[Le] H. W. Lenstra, Rational functions invariant under a finite abelian group, Invent. Math., **25** (1974), 299–325.

[Mae] T. Maeda, Noether's problem for A_5, J. Algebra, **125** (1989), 418–430.

[Mas] K. Masuda, Application of the theory of the group of classes of projective modules to the existance problem of independent parameters of invariant, J. Math. Soc. Japan, **20** (1968), 223–232.

[Matt] A. Mattuck, The field of multisymmetric functions, Proc. Amer. Math. Soc., **19** (1968), 764–765.

[MM] G. Malle and B. H. Matzat, Inverse Galois Theory, Springer-Verlag, 1999.

[Matz] B. H. Matzat, Konstruktive Galoistheorie, Lect. Notes in Math., **1284**, Springer-Verlag, 1986.

[Me] W. Meyer, Signaturdefekte, Teichmüllergruppen und Hyperelliptische Faserungen, Hablitationsschrift, Univ. Bonn, 1979.

[Mum] D. Mumford, Abelian Varieties, Tata Inst. for Fund. Research, Corrected Reprint (2008).

[Noe] E. Noether, Gleihungen mit vorgeschriebener Gruppe, Math. Ann., **78** (1916), 221–229.

[Oh] Y. Ohta, Cross-ratio Noether problem for $(2, 2, 2)$-imprimitive subgroups of degree 6 (Japanese), master's thesis, Sophia Univ., 2008.

[Pl] B. Plans, On Noether's problem for central extensions of symmetric and alternating groups, J. Algebra, **321** (2009), 3704–3713.

[Ru] B. Runge, Level-two-structures and hyperelliptic curves, Osaka J. Math., **34** (1997), 21–51.

[Sa] D. J. Saltman, Generic Galois extensions and problems in field theory, Adv. in Math., **43** (1982), 250–283.

[Sw1] R. G. Swan, Noether's problem in Galois theory, In: Emmy Noether in Bryn Mawr, (eds. J. D. Sally and B. Srinivasan), Springer-Verlag, 1983, pp. 21–40.

[Sw2] R. G. Swan, Invariant rational functions and a problem of Steenrod, Invent. Math., **7** (1969), 148–158.

[Tsu] H. Tsunogai, On a relative rationality problem related to Noether's problem, in preparation.

Kiichiro Hashimoto
Department of Pure and Applied Mathematics
Graduate School of Fundamental Science and Engineering
Waseda University
Japan

Hiroshi Tsunogai
Department of Mathematics
Faculty of Science and Technology
Sophia University
Japan

E-mail address: `khasimot@waseda.jp`,
`tsuno-h@cc.sophia.ac.jp`