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Remarks on the Milnor conjecture over schemes

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Abstract.

The Milnor conjecture has been a driving force in the theory of quadratic forms over fields, guiding the development of the theory of cohomological invariants, ushering in the theory of motivic cohomology, and touching on questions ranging from sums of squares to the structure of absolute Galois groups. Here, we survey some recent work on generalizations of the Milnor conjecture to the context of schemes (mostly smooth varieties over fields of characteristic $\neq 2$). Surprisingly, a version of the Milnor conjecture fails to hold for certain smooth complete p-adic curves with no rational theta characteristic (this is the work of Parimala, Scharlau, and Sridharan). We explain how these examples fit into the larger context of the unramified Milnor question, offer a new approach to the question, and discuss new results in the case of curves over local fields and surfaces over finite fields.

The first cases of the (as of yet unnamed) Milnor conjecture were studied in Pfister's Habilitationsschrift [89] in the mid 1960s. As Pfister [90, p. 3] himself points out, "[the Milnor conjecture] stimulated research for quite some time." Indeed, it can be seen as one of the driving forces in the theory of quadratic forms since Milnor's original formulation [71] in the early 1970s.

The classical cohomological invariants of quadratic forms (rank, discriminant, and Clifford–Hasse–Witt invariant) have a deep connection with the history and development of the subject. In particular, they are used in the classification (Hasse–Minkowski local-global theorem) of quadratic forms over local and global fields. The first "higher invariant" was described in Arason's thesis [1], [3]. The celebrated results

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of Merkurjev [66] and Merkurjev–Suslin [69] settled special cases of the Milnor conjecture in the early 1980s, and served as a starting point for Voevodsky's development of the theory of motivic cohomology. Other special cases were settled by Arason–Elman–Jacob [5] and Jacob–Rost [50]. Voevodsky's motivic cohomology techniques [106] ultimately led to a complete solution of the Milnor conjecture, for which he was awarded the 2002 Fields Medal.

The consideration of quadratic and symmetric bilinear forms over rings (more general than fields) has its roots in the number theoretic study of lattices (i.e. symmetric bilinear forms over $\mathbb{Z}$) by Gauss as well as the algebraic study of division algebras and hermitian forms (i.e. symmetric bilinear forms over algebras with involution) by Albert. A general framework for the study of quadratic and symmetric bilinear forms over rings was established by Bass [19], with the case of (semi)local rings treated in depth by Baeza [11]. Bilinear forms over Dedekind domains (i.e. unimodular lattices) were studied in a number theoretic context by Fröhlich [38], while the consideration of quadratic forms over algebraic curves (and their function fields) was initiated by Geyer, Harder, Knebusch, Scharlau [48], [42], [59], [57]. The theory of quadratic and symmetric bilinear forms over schemes was developed by Knebusch [56], [58], and utilized by Arason [2], Dietel [30], Parimala [81], [82], Fernández-Carmena [35], Sujatha [100], [88], Arason–Elman–Jacob [6], [7], and others. A theory of symmetric bilinear forms in additive and abelian categories was developed by Quebbemann–Scharlau–Schulte [92], [93]. Further enrichment came eventually from the triangulated category techniques of Balmer [12], [13], [14], and Walter [109]. This article will focus on progress in generalizing the Milnor conjecture to these contexts.

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Conventions. A graded abelian group or ring $\bigoplus_{n\geq 0} M^n$ will be denoted by $M^\bullet$. If $0 \subset \cdots \subset N^2 \subset N^1 \subset N^0 = M$ is a decreasing filtration of a ring M by ideals, denote by $N^\bullet/N^{\bullet+1} = \bigoplus_{n\geq 0} N^n/N^{n+1}$ the associated graded ring. Denote by ${}_2M$ the elements of order 2 in an abelian group M. All abelian groups will be written additively.

§1. The Milnor conjecture over fields

Let F be a field of characteristic $\neq 2$. The total Milnor K-ring $K_{\mathrm{M}}^\bullet(F) = T^\bullet(F^\times)/\langle a \otimes (1-a) : a \in F^\times\rangle$ was introduced in [71]. The total Galois cohomology ring $H^\bullet(F, \boldsymbol{\mu}_2^{\otimes\bullet}) = \bigoplus_{n\geq 0} H^n(F, \boldsymbol{\mu}_2^{\otimes n})$ is canonically isomorphic, under our hypothesis on the characteristic of F, to the total Galois cohomology ring $H^\bullet(F, \mathbb{Z}/2\mathbb{Z})$ with coefficients in the trivial Galois module $\mathbb{Z}/2\mathbb{Z}$. The Witt ring $W(F)$ of nondegenerate quadratic forms modulo hyperbolic forms has a decreasing filtration $0 \subset \cdots \subset I^1(F) \subset I^0(F) = W(F)$ generated by powers of the *fundamental ideal* $I(F)$ of even rank forms. The Milnor conjecture relates these three objects: Milnor K-theory, Galois cohomology, and quadratic forms.

The quotient map $K_{\mathrm{M}}^1(F) = F^\times \to F^\times/F^{\times 2} \cong H^1(F, \boldsymbol{\mu}_2)$ induces a graded ring homomorphism $h^\bullet : K_{\mathrm{M}}^\bullet(F)/2 \to H^\bullet(F, \boldsymbol{\mu}_2^{\otimes\bullet})$ called the *norm residue symbol* by Bass–Tate [20]. The *Pfister form* map $K_{\mathrm{M}}^1(F) = F^\times \to I(F)$ given by $a \mapsto \ll a \gg = < 1, -a >$ induces a group homomorphism $K_{\mathrm{M}}^1(F)/2 \to I^1(F)/I^2(F)$ (see Scharlau [97, 2 Lemma 12.10]), which extends to a surjective graded ring homomorphism $s^\bullet : K_{\mathrm{M}}^\bullet(F)/2 \to I^\bullet(F)/I^{\bullet+1}(F)$, see Milnor [71, Thm. 4.1].

Theorem 1 (Milnor conjecture). *Let F be a field of characteristic $\neq 2$. There exists a graded ring homomorphism $e^\bullet : I^\bullet(F)/I^{\bullet+1}(F) \to H^\bullet(F, \boldsymbol{\mu}_2^{\otimes\bullet})$ called the* higher invariants *of quadratic forms, which fits into the following diagram*

$$\begin{tikzcd} K_M^\bullet(F)/2 \arrow[r, "h^\bullet"] \arrow[d, "s^\bullet"'] & H^\bullet(F, \boldsymbol{\mu}_2^{\otimes\bullet}) \\ I^\bullet(F)/I^{\bullet+1}(F) \arrow[ur, "e^\bullet"'] & \end{tikzcd}$$

of isomorphisms of graded rings.

Many excellent introductions to the Milnor conjecture and its proof exist in the literature. For example, see the surveys of Friedlander [36], Friedlander–Rapoport–Suslin [37], Kahn [51], Merkurjev [68], Morel [73], and Pfister [91].

The conjecture breaks up naturally into three parts: the conjecture for the norm residue symbol $h^\bullet$, the conjecture for the Pfister form map $s^\bullet$, and the conjecture for the higher invariants $e^\bullet$. Milnor [71, Question

4.3, §6] originally made the conjecture for $h^\bullet$ and $s^\bullet$, which was already known for finite, local, global, and real closed fields, see [71, Lemma 6.2]. For general fields, the conjecture for h^1 follows from Hilbert's theorem 90, and for s^1 and e^1 by elementary arguments. The conjecture for s^2 is easy, see Pfister [89]. Merkurjev [66] proved the conjecture for h^2 (hence for e^2 as well), with alternate proofs given by Arason [4], Merkurjev [67], and Wadsworth [108]. The conjecture for h^3 was settled by Merkurjev–Suslin [69] (and independently by Rost [94]). The conjecture for $e^\bullet$ can be divided into two parts: to show the existence of maps $e^n : I^n(F) \to H^n(F, \boldsymbol{\mu}_2^{\otimes n})$ (which are *a priori* only defined on generators, the Pfister forms), and then to show they are surjective. The existence of e^3 was proved by Arason [1], [3]. The existence of e^4 was proved by Jacob–Rost [50] and independently Szyjewski [99]. Voevodsky [106] proved the conjecture for $h^\bullet$. Orlov–Vishik–Voevodsky [78] proved the conjecture for $s^\bullet$, with different proofs given by Morel [74] and Kahn–Sujatha [52].

1.1. Classical invariants of quadratic forms

The theory of quadratic forms over a general field has its genesis in Witt's famous paper [112]. Because of the assumption of characteristic $\neq 2$, we do not distinguish between quadratic and symmetric bilinear forms. The *orthogonal sum* $(V, b) \perp (V', b') = (V \oplus V', b + b')$ and *tensor product* $(V, b) \otimes (V', b') = (V \otimes V', b \otimes b')$ give a semiring structure on the set of isometry classes of symmetric bilinear forms over F. The *hyperbolic plane* is the symmetric bilinear form (H, h), where $H = F^2$ and $h((x, y), (x', y')) = xy' + x'y$. The *Witt ring* of symmetric bilinear forms is the quotient of the Grothendieck ring of nondegenerate symmetric bilinear forms over F with respect to $\perp$ and $\otimes$, modulo the ideal generated by the hyperbolic plane, see Scharlau [97, Ch. 2].

The *rank* of a bilinear form (V, b) is the F-vector space dimension of V. Since the hyperbolic plane has rank 2, the rank modulo 2 is a well defined invariant of an element of the Witt ring, and gives rise to a surjective ring homomorphism

$$e^0 : W(F) = I^0(F) \to \mathbb{Z}/2\mathbb{Z} = H^0(F, \mathbb{Z}/2\mathbb{Z})$$

whose kernel is the *fundamental ideal* $I(F)$.

The *signed discriminant* of a non-degenerate bilinear form (V, b) is defined as follows. Choosing an F-vector space basis $v_1, \dots, v_r$ of V, we consider the Gram matrix M_b of b, i.e. the matrix given by $M_b = (b(v_i, v_j))$. Then b is given by the formula $b(v, w) = v^t M_b w$, where $v, w \in F^r \cong V$. The Gram matrix of b, with respect to a different basis for V with change of basis matrix T, is $T^t M_b T$. Thus $\det M_b \in F^\times$, which depends on the choice of basis, is only well-defined up to squares. For $a \in F^\times$, denote by (a) its class in the abelian group $F^\times/F^{\times 2}$. The *signed discriminant* of (V, b) is defined as $d_\pm(V, q) = (-1)^{r(r-1)/2} \det M_b \in F^\times/F^{\times 2}$. Introducing the sign into the signed discriminant ensures its vanishing on the ideal of hyperbolic forms, hence it descents to the Witt group. While the signed discriminant is not additive on $W(F)$, its restriction to $I(F)$ gives rise to a group homomorphism

$$e^1 : I(F) \to F^\times/F^{\times 2} \cong H^1(F, \boldsymbol{\mu}_2)$$

which is easily seen to be surjective. It's then not difficult to check that its kernel coincides with the square $I^2(F)$ of the fundamental ideal. See Scharlau [97, §2.2] for more details.

The *Clifford invariant* of a non-degenerate symmetric bilinear form (V, b) is defined in terms of its Clifford algebra. The *Clifford algebra* $C(V, b)$ of (V, b) is the quotient of the tensor algebra $T(V) = \bigoplus_{r \geq 0} V^{\otimes r}$ by the two-sided ideal generated by $\{v \otimes v - b(v, v) : v \in V\}$. Since the relations are between elements of degree 2 and 0 (in the tensor algebra), the parity of an element is well-defined, and $C(V, b) = C_0(V, b) \oplus C_1(V, b)$ inherits the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded F-algebra, where $C_0(V, b)$ is the subalgebra of elements of even degree. By the structure theory of the Clifford algebra, if (V, b) has rank r, then $C(V, b)$ is a semisimple F-algebra of F-dimension 2^r, and $C(V, b)$ or $C_0(V, b)$ is a central simple F-algebra depending on whether r is even or odd, respectively, see Scharlau [97, §9.2]. The *Clifford invariant* $c(V, b) \in \mathrm{Br}(F)$ is defined as the class of $C(V, b)$ or $C_0(V, b)$, respectively, in the Brauer group of finite-dimensional central simple F-algebras. Since the Clifford algebra and its even subalgebra carry canonical involutions of the first kind, their respective classes in the Brauer group are of order 2, see Knus [60, §IV.7.8]. While the Clifford invariant is not additive on $W(F)$, its restriction to $I^2(F)$ gives rise to a group homomorphism

$$e^2 : I^2(F) \to {}_2\mathrm{Br}(F) \cong H^2(F, \boldsymbol{\mu}_2) \cong H^2(F, \boldsymbol{\mu}_2^{\otimes 2}),$$

see Knus [60, IV Prop. 8.1.1].

Any symmetric bilinear form (V, b) over a field of characteristic $\neq 2$ can be diagonalized, i.e. a basis can be chosen for V so that the Gram matrix M_b is diagonal. For $a_1, \ldots, a_r \in F^\times$, we write $<a_1, \ldots, a_r>$ for the standard symmetric bilinear form with associated diagonal Gram matrix. For $a, b \in F^\times$, denote by $(a, b)_F$ the (quaternion) F-algebra generated by symbols x and y subject to the relations $x^2 = a$, $y^2 = b$, and $xy = -yx$. For example, $(-1, -1)_\mathbb{R}$ is Hamilton's ring of quaternions. Then the discriminant and Clifford invariant can be conveniently calculated in terms of a diagonalization. For $(V,b) \cong < a_1, \dots, a_r >$, we have

$$d_\pm(V,b) = ((-1)^{r(r-1)/2} a_1 \cdots a_r) \in F^\times / F^{\times 2}$$

and

(1)

$$c(V,b) = \alpha(r)(-1, a_1 \cdots a_r)_F + \beta(r)(-1,-1)_F + \sum_{i<j} (a_j, a_j)_F \in {}_2\mathrm{Br}(F)$$

where

$$\alpha(r) = \frac{(r-1)(r-2)}{2}, \qquad \beta(r) = \frac{(r+1)r(r-1)(r-2)}{24},$$

see Lam [63], Scharlau [97, II.12.7], or Esnault–Kahn–Levine–Viehweg [33, §1].

§2. Globalization of cohomology theories

Generalizations (what we will call *globalizations*) of the Milnor conjecture to the context of rings and schemes have emerged from many sources, see Parimala [80], Colliot-Thélène–Parimala [25], Parimala–Sridharan [84], Monnier [72], Pardon [79], Elbaz-Vincent–Müller-Stach [31], Gille [44], and Kerz [53]. To begin with, one must ask for appropriate globalizations of the objects in the conjecture: Milnor K-theory, Galois cohomology theory, and the Witt group with its fundamental filtration. While there are many possible choices of such globalizations, we will focus on two types: global and unramified.

2.1. Global globalization

Let F be a field of characteristic $\neq 2$. Let Field_F (resp. Ring_F) be the category of fields (resp. commutative unital rings) with an F-algebra structure together with F-algebra homomorphisms. Let Sch_F be the category of separated F-schemes, and Sm_F the category of smooth F-schemes. We will denote, by the same names, the associated (large) Zariski sites. Let Ab (resp. $\mathsf{Ab}^\bullet$) be the category of abelian groups (resp. graded abelian groups), we will always consider Ab as embedded in $\mathsf{Ab}^\bullet$ in degree 0.

Let $M^\bullet : \mathsf{Field}_F \to \mathsf{Ab}^\bullet$ be a functor. A *globalization* of $M^\bullet$ to rings (resp. schemes) is a functor $\tilde{M}^\bullet : \mathsf{Ring}_F \to \mathsf{Ab}^\bullet$ (resp. contravariant functor $\mathcal{M}^\bullet : \mathsf{Sch}_F \to \mathsf{Ab}^\bullet$) extending $M^\bullet$. If $\tilde{M}^\bullet$ is a globalization of $M^\bullet$ to rings, then we can define a globalization to schemes by taking the sheaf $\mathcal{M}^\bullet$ associated to the presheaf $U \mapsto \tilde{M}^\bullet(\Gamma(U, \mathscr{O}_U))$ on Sch_F (always considered with the Zariski topology).

"Naïve" Milnor K-theory. For a commutative unital ring R, mimicking Milnor's tensorial construction (with the additional relation that $a \otimes (-a) = 0$, which is automatic for fields) yields a graded ring $K_{\mathrm{M}}^{\bullet}(R)$, which should be referred to as "naïve" Milnor K-theory. This already appears in Guin [46, §3] and later studied by Elbaz-Vincent–Müller-Stach [31]. Naïve Milnor K-theory has some bad properties when R has small finite residue fields, see Kerz [54] who also provides a improved Milnor K-theory repairing these defects. Thomason [103] has shown that there exists no globalization of Milnor K-theory to smooth schemes that satisfies $\mathbb{A}^1$-homotopy invariance and has a functorial homomorphism to algebraic K-theory.

Étale cohomology. Étale cohomology provides a natural globalization of Galois cohomology to schemes. We will thus consider the functor $X \mapsto H^{\bullet}_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes \bullet})$ on Sch_F.

Global Witt group. For a scheme X with 2 invertible, the *global* Witt group $W(X)$ of regular symmetric bilinear forms (modulo metabolic forms) introduced by Knebusch [58] provides a natural globalization of the Witt group functor to schemes. A *regular symmetric bilinear form* $(\mathscr{V}, b)$ on X consists of a locally free $\mathscr{O}_X$-module $\mathscr{V}$ of finite constant rank together with an $\mathscr{O}_X$-module homomorphism $b : S^2\mathscr{V} \to \mathscr{O}_X$ such that the induced map $\psi_b : \mathscr{V} \to \mathscr{V}^{\vee}$ is an $\mathscr{O}_X$-module isomorphism. Metabolic is the correct globalization to schemes of the notion of hyperbolic, see Knebusch [58, I.3].

Other possible globalizations are obtained from the Witt groups of triangulated category with duality introduced by Balmer [12], [13], [14], [15]. These include: the *derived* Witt group of the bounded derived category of coherent locally free $\mathscr{O}_X$-modules; the *coherent* Witt group of the bounded derived category of quasicoherent $\mathscr{O}_X$-modules with coherent cohomology (assuming X has a dualizing complex, see Gille [43, §2.5], [44, §2]); the *perfect* Witt group of the derived category of perfect complexes of $\mathscr{O}_X$-modules. The global and derived Witt groups are canonically isomorphic by Balmer [14, Thm. 4.7]. All of the above Witt groups are isomorphic (though not necessarily canonically) if X is assumed to be regular.

Fundamental filtration and the classical invariants. Globalizations of the classical invariants of symmetric bilinear forms (which are briefly reviewed in the context of global Witt groups in Balmer [15, §1.3]) are defined as follows. Let $(\mathscr{V}, b)$ be a regular symmetric bilinear form of rank n on X.

Taking the rank (modulo 2) of $\mathscr{V}$ gives rise to a functorial homomorphism

$$e^0 : W(X) \to \mathrm{Hom}_{\mathrm{cont}}(X, \mathbb{Z}/2\mathbb{Z}) = H^0_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}),$$

whose kernel $I^1(X)$ is called the *fundamental ideal* of $W(X)$.

Taking the *signed discriminant* form $(\det \mathscr{V}, (-1)^{n(n-1)/2} \det \psi_b)$ gives rise to a functorial homomorphism

$$e^1 : I^1(X) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$$

see Knus [60, III §4.2]. Here, we identify $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$ with the group (under tensor product) of regular symmetric bilinear forms of rank 1, as in Milne [70, III §4]. Alternatively, let $\mathscr{C}(\mathscr{V}, b) = \mathscr{C}_0(\mathscr{V}, b) \oplus \mathscr{C}_1(\mathscr{V}, b)$ be the Clifford algebra of $(\mathscr{V}, b)$, which is a locally free $\mathbb{Z}/2\mathbb{Z}$-graded $\mathscr{O}_X$-algebra whose definition is analogous to the classical one, see for example [32, §1.9]. Then the center of $\mathscr{C}(\mathscr{V}, b)$ or $\mathscr{C}_0(\mathscr{V}, b)$ is an étale quadratic $\mathscr{O}_X$-algebra, depending on the whether $\mathscr{V}$ has odd or even rank, respectively. This center defines a class in $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ called the *Arf* or *discriminant* invariant, which coincides with the signed discriminant under the canonical morphism $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$, see Knus [60, IV Prop. 4.6.3] or Parimala–Srinivas [87, §2.2]. Denote the kernel of e^1 by $I^2(X)$, which is an ideal of $W(X)$. Note that $I^2(X)$ may not be the square of the ideal $I^1(X)$.

Taking the Clifford $\mathscr{O}_X$-algebra $\mathscr{C}(\mathscr{V}, b)$, which is an Azumaya algebra if $(\mathscr{V}, n)$ has even rank as in §1.1, gives rise to a functorial homomorphism

$$e^2 : I^2(X) \to {}_2\mathrm{Br}(X)$$

called the *Clifford invariant*, see Knus–Ojanguren [61] and Parimala–Srinivas [87, §2]. Here, $\mathrm{Br}(X)$ is the group of Brauer equivalence classes of Azumaya $\mathscr{O}_X$-algebras ($\mathscr{O}_X$-algebras which are étale locally isomorphic to endomorphism algebras of locally free $\mathscr{O}_X$-modules) and there is a canonical injective homomorphism $\mathrm{Br}(X) \to H^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$, see Milne [70, IV §2]. Denote the kernel of e^2 by $I^3(X)$, which is an ideal of $W(X)$.

Classical invariants for Grothendieck–Witt groups. As Parimala–Srinivas [87, p. 223] point out, there is no functorial map $I^2(X) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ lifting the Clifford invariant. Instead, we can work with the Grothendieck–Witt group $GW(X)$ of regular symmetric bilinear forms modulo the equivalence relation splitting all metabolic forms, see Knebusch [58, I.4] or Walter [109] for precise definitions. This group sits in an exact sequence

$$K_0(X) \xrightarrow{H} GW(X) \to W(X) \to 0$$

where $K_0(X)$ is the Grothendieck group of locally free $\mathscr{O}_X$-modules of finite rank and H is the hyperbolic form functor $\mathscr{V} \mapsto H(\mathscr{V})$. The *hyperbolic form* $H(\mathscr{V})$ is defined as $\big(\mathscr{V} \oplus \mathscr{V}^\vee, ((v,f),(w,g)) \mapsto f(w) + g(x)\big)$.

Taking the rank (modulo 2) gives rise to a functorial homomorphism

$$ge^0 : GW(X) \to H^0_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$$

whose kernel is denoted by $GI^1(X)$. We write $GI^0(X) = GW(X)$.

Taking the signed discriminant gives rise to a functorial homomorphism

$$ge^1 : GI^1(X) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$$

whose kernel is denoted by $GI^2(X)$.

Taking the class of the Clifford $\mathscr{O}_X$-algebra, together with it's canonical involution (via the "involutive" Brauer group construction of Parimala–Srinivas [87, §2]), gives rise to a functorial homomorphism

$$ge^2 : GI^2(X) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$$

also see Knus–Parimala–Sridharan [62]. Denote the kernel of ge^2 by $GI^3(X)$, which is an ideal of $GW(X)$.

Lemma 2.1. *Let X be a scheme with 2 invertible. Under the quotient map $GW(X) \to W(X)$, the image of the ideal $GI^n(X)$ is precisely the ideal $I^n(X)$ for $n \leq 3$.*

Proof. For $n = 1, 2$ this is a consequence of the following diagram

$$\begin{tikzcd}
 & K_0(X) \arrow[r, equal] \arrow[d, "H"] & K_0(X) \arrow[d, "H"] & & \\
0 \arrow[r] & GI^n(X) \arrow[r] \arrow[d, dotted] & GI^{n-1}(X) \arrow[r, "ge^{n-1}"] \arrow[d] & H^{n-1}_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I^n(X) \arrow[r] \arrow[d, dotted] & I^{n-1}(X) \arrow[r, "e^{n-1}"] \arrow[d] & H^{n-1}_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

whose rows are exact by definition, whose upper left square (hence lower right square) is commutative since hyperbolic spaces have even rank and trivial signed discriminant, and whose lower left vertical arrows exist by a diagram chase (first considering $n = 1$ then $n = 2$).

For $n = 3$, we have the formula $ge^2(H(\mathscr{V})) = c_1(\mathscr{V}, \boldsymbol{\mu}_2)$, which follows from Esnault–Kahn–Viehweg [32, Prop. 5.5] combined with (1).

Here $c_1(-,\boldsymbol{\mu}_2)$ is the 1st Chern class modulo 2, defined as the first coboundary map in the long-exact sequence in étale cohomology

$$\begin{aligned}\cdots \to \mathrm{Pic}(X) \xrightarrow{2} \mathrm{Pic}(X) \xrightarrow{c_1} H^2_{\text{ét}}(X,\boldsymbol{\mu}_2)\\ \to H^2_{\text{ét}}(X,\mathbb{G}_{\mathrm{m}}) \xrightarrow{2} H^2_{\text{ét}}(X,\mathbb{G}_{\mathrm{m}}) \to \cdots\end{aligned}$$

arising from the étale Kummer exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbb{G}_{\mathrm{m}} \xrightarrow{2} \mathbb{G}_{\mathrm{m}} \to 1,$$

see Grothendieck [45]. The claim then follows by considering the following diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & K_0'(X) \arrow[r] \arrow[d, "H"] & K_0(X) \arrow[r, "\det"] \arrow[d, "H"] & \mathrm{Pic}(X)/2 \arrow[r] \arrow[d, "c_1"] & 0 \\
0 \arrow[r] & GI^3(X) \arrow[r] \arrow[d, dotted] & GI^2(X) \arrow[r, "ge^2"] \arrow[d] & H^2_{\text{ét}}(X,\boldsymbol{\mu}_2) \arrow[d] & \\
0 \arrow[r] & I^3(X) \arrow[r] \arrow[d, dotted] & I^2(X) \arrow[r, "e^2"] \arrow[d] & {}_2H^2_{\text{ét}}(X,\mathbb{G}_{\mathrm{m}}) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

whose right vertical column arises from (and is exact by) the Kummer sequence (here e^2 is considered as a map $I^2(X) \to {}_2H^2_{\text{ét}}(X,\mathbb{G}_{\mathrm{m}})$ via the canonical injection ${}_2\mathrm{Br}(X) \to {}_2H^2_{\text{ét}}(X,\mathbb{G}_{\mathrm{m}})$, see for example Milne [70, IV Thm. 2.5]), whose central column is exact by the diagram in the case $n = 2$, whose top horizontal rows are exact by definition (here $K_0'(X)$ is the subgroup of $K_0(X)$ generated by locally free $\mathscr{O}_X$-modules whose determinant is a square), and whose lower left vertical arrows exists by a diagram chase. Q.E.D.

Remark 2.1. In fact ${}_2\mathrm{Br}(X) = {}_2H^2_{\text{ét}}(X,\mathbb{G}_{\mathrm{m}})$ is satisfied if X is a quasi-compact quasi-separated scheme admitting an ample invertible sheaf by de Jong's extension [28] (see also Lieblich [65, Th. 2.2.2.1]) of a result of Gabber [39].

The existence of global globalizations of the higher invariants (e.g. a globalization of the Arason invariant) remains a mystery. Esnault–Kahn–Levine–Viehweg [33] have shown that for a regular symmetric bilinear form $(\mathscr{V}, b)$ that represents a class in $GI^3(X)$, the obstruction to having an Arason invariant in $H^3_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ is precisely the 2nd Chern class $c_2(\mathscr{V}) \in CH^2(X)/2$ in the Chow group modulo 2 (note that the invariant $c(\mathscr{V}) \in \mathrm{Pic}(X)/2$ of [33] is trivial if $(\mathscr{V}, b)$ represents a class in $GI^3(X)$). They also provide examples where this obstruction does not vanish. On the other hand, higher cohomological invariants always exist in unramified cohomology.

2.2. Unramified globalization

A functorial framework for the notion of "unramified element" is established in Colliot-Thélène [23, §2]. See also the survey by Zainoulline [113, §3]. Rost [95, Rem. 5.2] gives a different perspective in terms of cycle modules, also see Morel [74, §2]. Assume that X has finite Krull dimension and is equidimensional over a field F. For simplicity of exposition, assume that X is integral. Denote by $X^{(i)}$ its set of codimension i points.

Denote by Local_F the category of local F-algebras together with local F-algebra morphisms. Given a functor $M^\bullet : \mathsf{Local}_F \to \mathsf{Ab}^\bullet$, call

$$M^\bullet_{\mathrm{ur}}(X) = \bigcap_{x \in X^{(1)}} \mathrm{im}\big(M^\bullet(\mathscr{O}_{X,x}) \to M^\bullet(F(X))\big)$$

the group of *unramified elements* of $M^\bullet$ over X. Then $X \mapsto M^\bullet_{\mathrm{ur}}(X)$ is a globalization of $M^\bullet$ to schemes.

Given a functor $M^\bullet : \mathsf{Sch}_F \to \mathsf{Ab}^\bullet$, there is a natural map $M^\bullet(X) \to M^\bullet_{\mathrm{ur}}(X)$. If this map is injective, surjective, or bijective we say that the *injectivity*, *weak purity*, or *purity* property hold, respectively. Whether these properties hold for various functors $M^\bullet$ and schemes X is the subject of many conjectures and open problems, see Colliot-Thélène [23, §2.2] for examples.

Unramified Milnor K-theory. Define the *unramified Milnor K-theory (resp. modulo 2)* of X to be the graded ring of unramified elements $K^\bullet_{\mathrm{M,ur}}(X)$ (resp. $K^\bullet_{\mathrm{M,ur}}/2(X)$) of the "naïve" Milnor K-theory (resp. modulo 2) functor $K^\bullet_{\mathrm{M}}$ (resp. $K^\bullet_{\mathrm{M}}/2$) restricted to Local_F, see §2.1. Let $\mathcal{K}^\bullet_{\mathrm{M}}$ be the Zariski sheaf on Sch_F associated to "naïve" Milnor K-theory and $\mathcal{K}^\bullet_{\mathrm{M}}/2$ the associate sheaf quotient, which is also the Zariski sheaf associated to the presheaf $U \mapsto K^\bullet_{\mathrm{M}}(\Gamma(U, \mathscr{O}_U))/2$, see Morel [74, Lemma 2.7]. Then $K^\bullet_{\mathrm{M,ur}}(X) = \Gamma(X, \mathcal{K}^\bullet_{\mathrm{M}})$ and $K^\bullet_{\mathrm{M,ur}}/2(X) = \Gamma(X, \mathcal{K}^\bullet_{\mathrm{M}}/2)$ when X is smooth over an infinite field (compare with the remark in §2.1) by the Bloch–Ogus–Gabber theorem for Milnor K-theory, see Colliot-Thélène–Hoobler–Kahn [24, Cor. 5.1.11, §7.3(5)]. Also, see Kerz [53]. Note that the long exact sequence in Zariski cohomology associated to the short exact sequence

$$0 \to \mathcal{K}_{\mathrm{M}}^{\bullet} \xrightarrow{2} \mathcal{K}_{\mathrm{M}}^{\bullet} \to \mathcal{K}_{\mathrm{M}}^{\bullet}/2 \to 0$$

of sheaves on Sch_F yields a short exact sequence

$$0 \to K_{\mathrm{M,ur}}^{\bullet}(X)/2 \to K_{\mathrm{M,ur}}^{\bullet}/2(X) \to {}_2H^1(X, \mathcal{K}_{\mathrm{M}}^{\bullet}) \to 0$$

assuming X is smooth over an infinite field.

Unramified cohomology. Define the *unramified étale cohomology (modulo 2)* of X to be the graded ring of unramified elements $H_{\mathrm{ur}}^{\bullet}(X, \boldsymbol{\mu}_2^{\otimes \bullet})$ of the functor $H_{\mathrm{\acute{e}t}}^{\bullet}(-, \boldsymbol{\mu}_2^{\otimes \bullet})$. Letting $\mathcal{H}_{\mathrm{\acute{e}t}}^{\bullet}$ be the Zariski sheaf on Sch_F associated to the functor $H_{\mathrm{\acute{e}t}}^{\bullet}(-, \boldsymbol{\mu}_2^{\otimes \bullet})$, then $\Gamma(X, \mathcal{H}_{\mathrm{\acute{e}t}}^{\bullet}) = H_{\mathrm{ur}}^{\bullet}(X, \mathbb{Z}/2\mathbb{Z})$ when X is smooth over a field of characteristic $\neq 2$ by the exactness of the Gersten complex (also known as the "arithmetic resolution") for étale cohomology, see Bloch–Ogus [22, Thm. 4.2, Ex. 2.1, Rem. 4.7].

Unramified fundamental filtration of the Witt group. Assume that F has characteristic $\neq 2$. Define the *unramified Witt group* of X to be the abelian group of unramified elements $W_{\mathrm{ur}}(X)$ of the global Witt group functor W. Letting $\mathcal{W}$ be the Zariski sheaf associated to the global Witt group functor, then $W_{\mathrm{ur}}(X) = \Gamma(X, \mathcal{W})$ when X is regular over a field of characteristic $\neq 2$ by Ojanguren–Panin [76] (also see Morel [74, Thm. 2.2]). Writing $I_{\mathrm{ur}}^n(X) = I^n(F(X)) \cap W_{\mathrm{ur}}(X)$, then the functors $I_{\mathrm{ur}}^n(-)$ are Zariski sheaves on Sch_F, denoted by $\mathcal{I}^n$, which form a filtration of $\mathcal{W}$, see Morel [74, Thm. 2.3].

Note that the long exact sequence in Zariski cohomology associated to the short exact sequence

$$0 \to \mathcal{I}^{n+1} \to \mathcal{I}^n \to \mathcal{I}^n/\mathcal{I}^{n+1} \to 0$$

of sheaves on Sch_F yields a short exact sequence

$$0 \to I_{\mathrm{ur}}^n(X)/I_{\mathrm{ur}}^{n+1}(X) \to \mathcal{I}^n/\mathcal{I}^{n+1}(X) \to H^1(X, \mathcal{I}^{n+1})' \to 0$$

where $H^1(X, \mathcal{I}^{n+1})' = \ker\big(H^1(X, \mathcal{I}^n) \to H^1(X, \mathcal{I}^{n+1})\big)$ assuming X is regular (over a field of characteristic $\neq 2$). If the obstruction group $H^1(X, \mathcal{I}^{n+1})'$ is nontrivial, then not every element of $\mathcal{I}^n/\mathcal{I}^{n+1}(X)$ is represented by a quadratic form on X. If X is the spectrum of a regular local ring, then $I_{\mathrm{ur}}^n(X)/I_{\mathrm{ur}}^{n+1}(X) = \mathcal{I}^n/\mathcal{I}^{n+1}(X)$, see Morel [74, Thm. 2.12].

Remark 2.2. As before, the notation $I_{\mathrm{ur}}^n(X)$ does not necessarily mean the nth power of $I_{\mathrm{ur}}(X)$. This is true, however, when X is the spectrum of a regular local ring containing an infinite field of characteristic $\neq 2$, see Kerz–Müller-Stach [55, Cor. 0.5].

2.3. Gersten complexes

Gersten complexes (Cousin complexes) exists in a very general framework. For the purposes of defining unramified globalizations of the norm residue symbol, Pfister form map, and higher cohomological invariants, we will only need Gersten complexes for Milnor K-theory, étale cohomology, and (the fundamental filtration of) the Witt group.

Gersten complex for Milnor K-theory. Let X be a regular excellent integral F-scheme. Let $C(X, K_{\mathrm{M}}^n)$ denote the Gersten complex for Milnor K-theory

$$\begin{aligned} 0 \quad \longrightarrow \quad & K_{\mathrm{M}}^n(F(X)) \xrightarrow{\partial^{K_{\mathrm{M}}^n}} \bigoplus_{x \in X^{(1)}} K_{\mathrm{M}}^{n-1}(F(x)) \\ \xrightarrow{\partial^{K_{\mathrm{M}}^{n-1}}} \quad & \bigoplus_{y \in X^{(2)}} K_{\mathrm{M}}^{n-2}(F(y)) \longrightarrow \cdots \end{aligned}$$

where $\partial^{K_{\mathrm{M}}}$ is the "tame symbol" homomorphism defined in Milnor [71, Lemma 2.1]. We have that $H^0(C(X, K_{\mathrm{M}}^n)) = K_{\mathrm{M,ur}}^n(X)$. See Rost [95, §1] or Fasel [34, Ch. 2] for more details. We will also consider the Gersten complex $C(X, K_{\mathrm{M}}^n/2)$ for Milnor K-theory modulo 2, for which we have that $H^0(C(X, K_{\mathrm{M}}^n/2)) = K_{\mathrm{M,ur}}^n/2(X)$.

Gersten complex for étale cohomology. Let X be a smooth integral F-scheme, with F of characteristic $\neq 2$. Let $C(X, H^n)$ denote the Gersten complex for étale cohomology

$$\begin{aligned} 0 \quad \longrightarrow \quad & H^n(F(X)) \xrightarrow{\partial^{H^n}} \bigoplus_{x \in X^{(1)}} H^{n-1}(F(x)) \\ \xrightarrow{\partial^{H^{n-1}}} \quad & \bigoplus_{y \in X^{(2)}} H^{n-2}(F(y)) \longrightarrow \cdots \end{aligned}$$

where $H^n(-) = H^n(-, \boldsymbol{\mu}_2^{\otimes n})$ and ∂^H is the homomorphism induced from the spectral sequence associated to the coniveau filtration, see Bloch–Ogus [22]. Then we have that $C(X, H^n)$ is a resolution of $H_{\mathrm{ur}}^n(X, \boldsymbol{\mu}_2^{\otimes n})$.

Gersten complex for Witt groups. Let X be a regular integral scheme (of finite Krull dimension) over a field F of characteristic $\neq 2$. Let $C(X, W)$ denote the Gersten–Witt complex

$$\begin{aligned} 0 \quad \longrightarrow \quad & W(F(X)) \xrightarrow{\partial^{W}} \bigoplus_{x \in X^{(1)}} W(F(x)) \\ \xrightarrow{\partial^{W}} \quad & \bigoplus_{y \in X^{(2)}} W(F(y)) \longrightarrow \cdots \end{aligned}$$

where ∂^W is the homomorphism induced from the second residue map for a set of choices of local parameters, see Balmer–Walter [18]. Because of these choices, $C(X,W)$ is only defined up to isomorphism, though there is a canonical complex defined in terms of Witt groups of finite length modules over the local rings of points. Over the local ring of a regular point, the sequence is exact (after the zeroth term) by Balmer–Gille–Panin–Walter [17]. We have that $H^0(C(X,W)) = W_{\mathrm{ur}}(X)$.

Fundamental filtration. The filtration of the Gersten complex for Witt groups induced by the fundamental filtration was first studied methodically by Arason–Elman–Jacob [5], see also Parimala–Sridharan [84], Gille [44], and Fasel [34, §9].

The differentials of the Gersten complex for Witt groups respect the fundamental filtration as follows:

$$\partial^{I^n}\Big(\bigoplus_{x\in X^{(p)}} I^n(F(x))\Big) \subset \bigoplus_{y\in X^{(p+1)}} I^{n-1}(F(y)),$$

see Fasel [34, Thm. 9.2.4] and Gille [44]. Thus for all $n \geq 0$ we have complexes $C(X,I^n)$

$$\begin{aligned} 0 \longrightarrow\ & I^n(F(X)) \xrightarrow{\partial^{I^n}} \bigoplus_{x\in X^{(1)}} I^{n-1}(F(x)) \\ \xrightarrow{\partial^{I^{n-1}}}\ & \bigoplus_{y\in X^{(2)}} I^{n-2}(F(y)) \longrightarrow \cdots \end{aligned}$$

which provide a filtration of $C(X,W)$ in the category of complexes of abelian groups. Here we write $I^n(-) = W(-)$ for $n \leq 0$. We have that $H^0(C(X,I^n)) = I^n_{\mathrm{ur}}(X)$.

The canonical inclusion $C(X,I^{n+1}) \to C(X,I^n)$ respects the differentials, and so defines a cokernel complex $C(X,I^n/I^{n+1})$

$$\begin{aligned} 0 \longrightarrow\ & I^n/I^{n+1}(F(X)) \longrightarrow \bigoplus_{x\in X^{(1)}} I^{n-1}/I^n(F(x)) \\ \longrightarrow\ & \bigoplus_{y\in X^{(2)}} I^{n-2}/I^{n-1}(F(y)) \longrightarrow \cdots \end{aligned}$$

see Fasel [34, Déf. 9.2.10], where $I^n/I^{n+1}(L) = I^n(L)/I^{n+1}(L)$ for a field L. We have that $H^0(C(X,I^n/I^{n+1})) = \mathcal{I}^n/\mathcal{I}^{n+1}(X)$

For the rest of this section, we assume X is a smooth integral scheme (of finite Krull dimension) over a field F of characteristic $\neq 2$.

Unramified norm residue symbol. The norm residue symbol for fields provides a morphism of complexes $h^n : C(X, K^n_{\mathrm{M}}/2) \to C(X, H^n)$, where the map on terms of degree j is h^{n-j}. By the Milnor conjecture for fields, this is an isomorphism of complexes. Upon restriction, we have an isomorphism

$$h^n_{\mathrm{ur}} : K^n_{\mathrm{M,ur}}/2(X) \to H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n}), \tag{2}$$

further restricting to an injection $h^n_{\mathrm{ur}} : K^n_{\mathrm{M,ur}}(X)/2 \to H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n})$.

Unramified Pfister form map. The Pfister form map for fields provides a morphism of complexes $s^n : C(X, K^n_{\mathrm{M}}/2) \to C(X, I^n/I^{n+1})$, where the map on terms of degree j is s^{n-j}. By the Milnor conjecture for fields, this is an isomorphism of complexes. Upon restriction, we have an isomorphism

$$s^n_{\mathrm{ur}} : K^n_{\mathrm{M,ur}}/2(X) \to \mathcal{I}^n/\mathcal{I}^{n+1}(X). \tag{3}$$

See Fasel [34, Thm. 10.2.6].

Unramified higher cohomological invariants. By the Milnor conjecture for fields, there exists a higher cohomological invariant morphism of complexes $e^n : C(X, I^n) \to C(X, H^n)$, where the map on terms of degree j is e^{n-j}. Upon restriction, we have homomorphisms $e^n_{\mathrm{ur}} : I^n_{\mathrm{ur}}(X) \to H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n})$ factoring through to $e^n_{\mathrm{ur}} : I^n_{\mathrm{ur}}(X)/I^{n+1}_{\mathrm{ur}}(X) \to H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n})$.

Furthermore, on the level of complexes, the higher cohomological invariant morphism factors through to a morphism of complexes $e^n : C(X, I^n/I^{n+1}) \to C(X, H^n)$, which by the Milnor conjecture over fields, is an isomorphism. Upon restriction, we have isomorphisms

$$e^n_{\mathrm{ur}} : \mathcal{I}^n/\mathcal{I}^{n+1}(X) \to H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n}). \tag{4}$$

Also see Morel [74, §2.3].

2.4. Motivic globalization

There is another important globalization of Milnor K-theory and Galois cohomology, but we only briefly mention it here. Conjectured to exist by Beĭlinson [21] and Lichtenbaum [64], and then constructed by Voevodsky [106], motivic complexes modulo 2 give rise to Zariski and étale motivic cohomology groups $H^n_{\mathrm{Zar}}(X, \mathbb{Z}/2\mathbb{Z}(m))$ and $H^n_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}(m))$ modulo 2.

For a field F, Nesterenko–Suslin [75] and Totaro [104] establish a canonical isomorphism $H^n_{\mathrm{Zar}}(\operatorname{Spec} F, \mathbb{Z}/2\mathbb{Z}(n)) \cong K^n_{\mathrm{M}}(F)/2$. The work of Bloch, Gabber, and Suslin (see the survey by Geisser [40, §1.3.1]) establishes an isomorphism $H^n_{\text{ét}}(\operatorname{Spec} F, \mathbb{Z}/2\mathbb{Z}(n)) \cong H^n(F, \boldsymbol{\mu}_2^{\otimes n})$ for F of characteristic $\neq 2$. The natural pullback map

$$\varepsilon^* : H^n_{\text{Zar}}(\operatorname{Spec} F, \mathbb{Z}/2\mathbb{Z}(n)) \to H^n_{\text{ét}}(\operatorname{Spec} F, \mathbb{Z}/2\mathbb{Z}(n))$$

induced from the change of site $\varepsilon : X_{\text{ét}} \to X_{\text{Zar}}$ is then identified with the norm residue homomorphism. Thus $H^n_{\text{Zar}}(-, \mathbb{Z}/2\mathbb{Z}(n))$ and $H^n_{\text{ét}}(-, \mathbb{Z}/2\mathbb{Z}(n))$ provide *motivic* globalizations of the mod 2 Milnor K-theory and Galois cohomology functors, respectively. On the other hand, it is not clear if there exists a direct motivic globalization of the Witt group or its fundamental filtration.

§3. Globalization of the Milnor conjecture

Unramified. Let F be an infinite field of characteristic $\neq 2$. Combining the results of §2.2–2.3, we have that for any smooth integral F-scheme X, the maps (2), (3), and (4) yield a commuting triangle

$$\begin{tikzcd}
\Gamma(X, \mathcal{K}^\bullet_{\mathrm{M}}/2) = K^\bullet_{\mathrm{M,ur}}/2(X) \arrow[r, "h^\bullet_{\mathrm{ur}}"] \arrow[d, "s^\bullet_{\mathrm{ur}}"'] & H^\bullet_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^\bullet) = \Gamma(X, \mathcal{H}^\bullet_{\text{ét}}) \\
\Gamma(X, \mathcal{I}^\bullet/\mathcal{I}^{\bullet+1}) = \mathcal{I}^\bullet/\mathcal{I}^{\bullet+1}(X) \arrow[ur, "e^\bullet_{\mathrm{ur}}"'] &
\end{tikzcd}$$

of functorial isomorphisms of graded abelian groups. Thus there's a commutative triangle of isomorphisms

$$\begin{tikzcd}
\mathcal{K}^\bullet_{\mathrm{M}}/2 \arrow[r, "h^\bullet"] \arrow[d, "s^\bullet"'] & \mathcal{H}^\bullet_{\text{ét}} \\
\mathcal{I}^\bullet/\mathcal{I}^{\bullet+1} \arrow[ur, "e^\bullet"'] &
\end{tikzcd}$$

of sheaves of graded abelian groups on Sm_F. What we will consider as a globalization of the Milnor conjecture—the *unramified Milnor question*—is a refinement of the above triangle of global sections.

Question 3.1 (Unramified Milnor question). *Let X be a smooth integral scheme over an infinite field of characteristic $\neq 2$. Consider the following diagram:*

$$\begin{tikzcd}
K^\bullet_{M,\mathrm{ur}}(X)/2 \arrow[r, hook, "i^\bullet_K"] \arrow[d, dashed, "?"'] & \mathcal{K}^\bullet_M/2(X) \arrow[r, "h^\bullet_{\mathrm{ur}}"] \arrow[d, "s^\bullet_{\mathrm{ur}}"'] & H^\bullet_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes\bullet}) \\
I^\bullet_{\mathrm{ur}}(X)/I^{\bullet+1}_{\mathrm{ur}}(X) \arrow[r, hook, "i^\bullet_I"] & \mathcal{I}^\bullet/\mathcal{I}^{\bullet+1}(X) \arrow[ur, "e^\bullet_{\mathrm{ur}}"'] &
\end{tikzcd}$$

(1) *Is the inclusion $i_I^\bullet$ surjective?*
(2) *Is the inclusion $i_K^\bullet$ surjective?*
(3) *Does the restriction of $s_{\mathrm{ur}}^\bullet$ to $K_{M,\mathrm{ur}}^\bullet(X)/2$ have image contained in $I_{\mathrm{ur}}^\bullet(X)/I_{\mathrm{ur}}^{\bullet+1}(X)$? If so, is it an isomorphism?*

Note that in degree n, Questions 3.1 (1), (2), and (3) can be rephrased in terms of the obstruction groups, respectively: does $H^1(X, \mathcal{I}^{n+1})'$ vanish; does ${}_2H^1(X, \mathcal{K}_{\mathrm{M}}^n)$ vanish; and does the restriction of s_{ur}^n yield a map ${}_2H^1(X, \mathcal{K}_{\mathrm{M}}^n) \to H^1(X, \mathcal{I}^{n+1})'$ and is it an isomorphism?

From now on we shall focus mainly on the unramified Milnor question for quadratic forms (i.e. Question 3.1(1)), which was already explicitly asked by Parimala–Sridharan [84, Question Q]. Note that for this question, since we avoid Milnor K-theory, we not require the hypothesis that F be infinite.

Global Grothendieck–Witt. We mention a global globalization of the Milnor conjecture for quadratic forms. Because of the conditional definition of the global cohomological invariants, we restrict ourselves to the classical invariants on Grothendieck–Witt groups defined in §2.1.

Question 3.2 (Global Merkurjev question). *Let X be a regular scheme with 2 invertible. For $n \leq 2$, consider the homomorphisms*

$$ge^n : GI^n(X)/GI^{n+1}(X) \to H_{\text{ét}}^n(X, \mathbb{Z}/2\mathbb{Z})$$

induced from the (classical) cohomological invariants on Grothendieck–Witt groups. Are they isomorphisms?

For $n = 0$ and $n = 1$, this question has a positive answer. For $n = 2$, this question can be viewed as a natural globalization of Merkurjev's theorem (though to the author's knowledge, Merkurjev never posed the question): is every 2-torsion Brauer class represented by the Clifford algebra of a regular quadratic form over X? Indeed, a consequence of Lemma 2.1 is that $ge^2 : GI^2(X) \to H_{\text{ét}}^2(X, \mathbb{Z}/2\mathbb{Z})$ is surjective if and only $e^2 : I^2(X) \to {}_2\mathrm{Br}(X)$ is surjective. Question 3.2 is thus a consequence of a positive answer to Question 3.1(1) for a smooth scheme X over a field of characteristic $\neq 2$ that satisfies weak purity for the Witt group, see §3.1 for details.

Motivic. Finally, we mention a globalization of the Milnor conjecture for the norm residue symbol using Zariski and étale motivic cohomology modulo 2 (see §2.4). This is the (n, n) modulo 2 case of the Beĭlinson–Lichtenbaum conjecture: for a smooth variety X over a field, the canonical map $H_{\mathrm{Zar}}^n(X, \mathbb{Z}/2\mathbb{Z}(m)) \to H_{\text{ét}}^n(X, \mathbb{Z}/2\mathbb{Z}(m))$ is an isomorphism for $n \leq m$. The combined work of Suslin–Voevodsky [102] and

Geisser–Levine [41] show the Beĭlinson–Lichtenbaum conjecture to be a consequence of the Bloch–Kato conjecture, which is now proved using ideas of Voevodsky [107] and Rost [96] with various details being filled in by Haesemeyer–Weibel [47], Suslin–Joukhovitski [101], and Weibel [110], [111].

3.1. Some purity results

In this section we review some of the purity results (see §2.2) relating the global and unramified Witt groups and cohomology.

Purity for Witt groups. For a survey on purity results for Witt groups, see Zainoulline [113]. Purity for the global Witt group means that the natural map $W(X) \to W_{\mathrm{ur}}(X)$ is an isomorphism.

Theorem 2. *Let X be a regular integral noetherian scheme with 2 invertible. Then purity holds for the global Witt group functor under the following hypotheses:*

(1) *X has dimension ≤ 3,*
(2) *X is the spectrum of a regular local ring of dimension ≤ 4,*
(3) *X is the spectrum of a regular local ring containing a field.*

Weak purity for the global Witt group functor holds for X of dimension ≤ 4.

For part (1), the case of dimension ≤ 2 is due to Colliot-Thélène–Sansuc [27, Cor. 2.5], the case of dimension 3 and X affine is due to Ojanguren–Parimala–Sridharan–Suresh [77], and for the general case (as well as (2) and the final assertion) see Balmer–Walter [18]. For (3), see Ojanguren–Panin [76].

Purity for étale cohomology. For X geometrically locally factorial and integral, purity holds for étale cohomology in degree ≤ 1, i.e.

$$H^0_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) = H^0_{\mathrm{ur}}(X, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}, \text{ and } H^1_{\text{ét}}(X, \boldsymbol{\mu}_2) = H^1_{\mathrm{ur}}(X, \boldsymbol{\mu}_2)$$

see Colliot-Thélène–Sansuc [26, Cor. 3.2, Prop. 4.1].

For X smooth over a field of characteristic $\neq 2$, weak purity holds for étale cohomology in degree 2. Moreover, there's a canonical identification ${}_2\mathrm{Br}(X) = H^2_{\mathrm{ur}}(X, \boldsymbol{\mu}_2)$ by Bloch–Ogus [22] such that the canonical map $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \to H^2_{\mathrm{ur}}(X, \boldsymbol{\mu}_2) = {}_2\mathrm{Br}(X)$ arises from the Kummer exact sequence already considered in the proof of Lemma 2.1.

Purity for the classical invariants. Considering the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I^3(X) \arrow[r] \arrow[d] & I^2(X) \arrow[r, "e^2"] \arrow[d] & {}_2\mathrm{Br}(X) \arrow[d] \\ 0 \arrow[r] & I^3_{\mathrm{ur}}(X) \arrow[r] & I^2_{\mathrm{ur}}(X) \arrow[r, "e^2_{\mathrm{ur}}"] & H^2_{\mathrm{ur}}(X, \boldsymbol{\mu}_2) \end{tikzcd}$$

(and similar diagrams for $n = 0$ and 1) note that weak purity for the Witt group (resp. purity for the Witt group and injectivity for étale cohomology) implies weak purity (resp. purity) for the global fundamental filtration $I^n(X) \subset W(X)$ for $n \leq 3$.

3.2. Positive results

We now survey some of the known positive cases of the unramified Milnor question in the literature.

Theorem 3 (Kerz–Müller-Stach [55, Cor. 0.8], Kerz [53, Thm. 1.2])**.** *Let R be a local ring with infinite residue field of characteristic $\neq 2$. Then the unramified Milnor question (all parts of Question 3.1) has a positive answer over* $\operatorname{Spec} R$.

Hoobler [49] had already proved this in degree 2.

The following result was communicated to us by Stefan Gille (who was inspired by Totaro [105]).

Theorem 4. *Let X be a proper smooth integral variety over an infinite field F of characteristic $\neq 2$. If X is F-rational then the unramified Milnor question (all parts of Question 3.1) has a positive answer over X.*

Proof. The groups $K^n_{\mathrm{M,ur}}(X)$, $H^n_{\mathrm{ur}}(X, \boldsymbol{\mu}_2^{\otimes n})$, and $I^n_{\mathrm{ur}}(X)$ are birational invariants of smooth proper F-varieties. To see this, one can use Colliot-Thélène [23, Prop. 2.1.8e] and the fact that the these functors satisfy weak purity for regular local rings (see Theorem 2). Another proof uses the fact that the complexes $C(X, K^n_{\mathrm{M}})$, $C(X, H^n)$, and $C(X, I^n)$ are cycle modules in the sense of Rost, see [95, Cor. 12.10]. In any case, by Colliot-Thélène [23, Prop. 2.1.9] the pullback induces isomorphisms $K^n_{\mathrm{M}}(F) \cong K^n_{\mathrm{M,ur}}(\mathbb{P}^m)$ (first proved by Milnor [71, Thm. 2.3] for $\mathbb{P}^1$), $H^n(F, \boldsymbol{\mu}_2^{\otimes n}) \cong H^n_{\mathrm{ur}}(\mathbb{P}^m, \boldsymbol{\mu}_2^{\otimes n})$, and $I^n_{\mathrm{ur}}(F) \cong I^n_{\mathrm{ur}}(\mathbb{P}^m)$ for all $n \geq 0$ and $m \geq 1$. In particular, $K^n_{\mathrm{M}}(F)/2 \cong K^n_{\mathrm{M,ur}}(X)/2$ and $I^n_{\mathrm{ur}}(F)/I^{n+1}_{\mathrm{ur}}(F) \cong I^n_{\mathrm{ur}}(X)/I^{n+1}_{\mathrm{ur}}(X)$, and the theorem follows from the Milnor conjecture over fields. The hypothesis that F is infinite may be dropped when only considering Question 3.1(1). Q.E.D.

The following positive results are known for low dimensional schemes. Recall the notion of *cohomological dimension* $cd(F)$ of a field (see Serre [98, I §3.1]), *virtual cohomological dimension* $vcd(F) = cd(F(\sqrt{-1}))$ and their 2-primary versions. Denoting by $d(F)$ any of these notions of dimension, note that if $d(F) \leq k$ and $\dim X \leq l$ then $d(F(X)) \leq k + l$.

Theorem 5 (Parimala–Sridharan [84], Monnier [72]). *Let X be a smooth integral curve over a field F of characteristic $\neq 2$. Then the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer over X in the following cases:*

(1) $cd_2(F) \leq 1$,
(2) $vcd(F) \leq 1$,
(3) $cd_2(F) = 2$ *and X is affine,*
(4) $vcd(F) = 2$ *and X is affine.*

Proof. For (1), this follows from Parimala–Sridharan [84, Lemma 4.1] and the fact that e^1 is always surjective. For (2), the case $vcd(F) = 0$ (i.e. F is real closed) is contained in Monnier [72, Cor. 3.2] and the case $vcd(F) = 1$ follows from a straightforward generalization to real closed fields of the results in [84, §5] for the real numbers. For (3), see [84, Lemma 4.2]. For (4), the statement follows from a generalization of [84, Thm. 6.1]. Q.E.D.

We wonder whether vcd can be replaced by vcd_2 in Theorem 5. Parimala–Sridharan [84, Rem. 4] ask whether there exist affine curves (over a well-chosen field) over which the unramified Milnor question has a negative answer.

For surfaces, there are positive results in the case of $vcd(F) = 0$, i.e. F is real closed. If F is algebraically closed, then the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer by a direct computation, see Fernández-Carmena [35]. If F is real closed, one has the following result.

Theorem 6 (Monnier [72, Thm. 4.5]). *Let X be smooth integral surface over a real closed field F. If the number of connected components of $X(F)$ is ≤ 1 (i.e. in particular if $X(F) = \varnothing$), then the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer over X.*

Examples of surfaces with many connected components over a real closed field, and over which the unramified Milnor question still has a positive answer, are also given in Monnier [72].

Finally, by [10] the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer over any scheme X such that

${}_2\mathrm{Br}(X)$ is generated by quaternion Azumaya algebras. In particular, this recovers the known cases of curves over finite fields (using class field theory) and surfaces over algebraically closed fields (using Artin [8] or de Jong [29]).

§4. Negative results

Alex Hahn asked if there exists a ring R over which the global Merkurjev question (Question 3.2) has a negative answer, i.e. $e^2 : I^2(R) \to {}_2\mathrm{Br}(R)$ is not surjective. The results of Parimala, Scharlau, and Sridharan [83], [84], [85], show that there exist smooth complete curves X (over p-adic fields F) over which the unramified Milnor question (Question 3.1(1)) in degree 2 (and hence, by purity, the global Merkurjev question) has a negative answer.

Remark 4.1. The assertion (in Gille [44, §10.7] and Pardon [79, §5]) that the unramified Milnor question (Question 3.1(1)) has a positive answer over any smooth scheme (over a field of characteristic $\neq 2$) is incorrect. In these texts, the distinction between the groups $I^n_{\mathrm{ur}}(X)/I^{n+1}_{\mathrm{ur}}(X)$ and $\mathcal{I}^n/\mathcal{I}^{n+1}(X)$ is not made clear.

Definition 4.1 (Parimala–Sridharan [84]). A scheme X over a field F has the *extension property* for symmetric bilinear forms if there exists $x_0 \in X(F)$ such that every regular symmetric bilinear form on $X \smallsetminus \{x_0\}$ extends to a regular symmetric bilinear form on X.

Proposition 4.1 (Parimala–Sridharan [84, Lemma 4.3]). *Let F be a field of characteristic $\neq 2$ and with $cd_2F \leq 2$ and X a smooth integral F-curve. Then the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer for X if and only if X has the extension property.*

The extension property holds for any smooth curve X having an F-rational point and whose Witt group has a *residue theorem* (or *reciprocity law*): an F-vector space T together with a nontrivial F-linear map $t_x : W(F(x)) \to T$ for each $x \in X^{(1)}$ and satisfying $\sum_{x \in X^{(1)}} t_x \circ \partial_x = 0$. Indeed, if $x_0 \in X(F)$ and b is a regular symmetric bilinear form on $U = X \smallsetminus \{x_0\}$, then considering $b \in W(F(X))$ we have $\partial_x(b) = 0$ for all $x \in U$, hence $t_{x_0}(\partial_{x_0}(b)) = 0$. The reciprocity property implies that $\partial_{x_0}(b) = 0$ since $t_{x_0} : F(x_0) \cong F \to T$ is injective. Thus $b \in W_{\mathrm{ur}}(X)$ extends to X by purity.

There is a standard residue theorem for $X = \mathbb{P}^1$ due to Milnor [71, §5] (independently proven by Harder [48, Satz 3.5] and Knebusch [56, §13]). For nonrational curves, the choice of local parameters inherent in defining the residue maps is eliminated by considering quadratic forms with values in the canonical bundle $\omega_{X/F}$.

Definition 4.2. Let X be a scheme and $\mathscr{L}$ an invertible $\mathscr{O}_X$-module. A *regular ($\mathscr{L}$-valued) symmetric bilinear form* on X is a triple $(\mathscr{V}, b, \mathscr{L})$, where $\mathscr{V}$ is a locally free $\mathscr{O}_X$-module of finite rank and $b : S^2\mathscr{V} \to \mathscr{L}$ is an $\mathscr{O}_X$-module morphism such that the canonical map $\psi_b : \mathscr{V} \to \mathscr{H}om(\mathscr{V}, \mathscr{L})$ is an $\mathscr{O}_X$-module isomorphism.

The Witt group $W(X, \mathscr{L})$ of regular $\mathscr{L}$-valued symmetric bilinear forms is the quotient of the Grothendieck group of such forms under orthogonal sum by the subgroup generated by $\mathscr{L}$-valued metabolic forms (having a half-dimensional isotropic $\mathscr{O}_X$-submodule), see [9, §1.7]. See Balmer [12], [13], [14], [15] for generalizations. We remark that any choice of isomorphism $\varphi : \mathscr{L}' \otimes \mathscr{N}^{\otimes 2} \cong \mathscr{L}$ induces a group isomorphism $W(X, \mathscr{L}') \to W(X, \mathscr{L})$ via $(\mathscr{V}, b, \mathscr{L}') \mapsto (\mathscr{V} \otimes \mathscr{N}, \varphi \circ (b \otimes \mathrm{id}_{\mathscr{N}}), \mathscr{L})$, see also Balmer–Calmès [16].

Theorem 7 (Geyer–Harder–Knebusch–Scharlau [42]). *Let X be a smooth proper integral curve over a perfect field F of characteristic $\neq 2$. Then there is a canonical complex (which is exact at the first two terms)*

$$\begin{aligned} 0 \quad &\longrightarrow \quad W(X, \omega_{X/F}) \longrightarrow W(F(X), \omega_{F(X)/F}) \\ &\xrightarrow{\partial^{\omega_X}} \quad \bigoplus_{x \in X^{(1)}} W(F(x), \omega_{F(x)/F}) \xrightarrow{\mathrm{Tr}_{X/F}} W(F) \end{aligned}$$

and thus in particular $W(X, \omega_{X/F})$ has a residue theorem.

Now any choice of *theta-characteristic* (i.e. isomorphism $\mathscr{N}^{\otimes 2} \cong \omega_{X/F}$) induces a group isomorphism $W(X) \to W(X, \omega_{X/F})$. Thus in particular, if $\omega_{X/F}$ is a square in $\mathrm{Pic}(X)$, then X has a residue theorem (thus the extension property), hence Question 3.1(1) has a positive answer for X. Conversely we have the following.

Theorem 8 (Parimala–Sridharan [85, Thm. 3]). *Let F be a local field of characteristic $\neq 2$ and X a smooth integral hyperelliptic F-curve of genus ≥ 2 with $X(F) \neq \varnothing$. Then the unramified Milnor question for quadratic forms (Question 3.1(1)) holds over X if and only if $\omega_{X/F}$ is a square.*

Example 4.1. Let X be the smooth proper hyperelliptic curve over $\mathbb{Q}_3$ with affine model $y^2 = (x^2 + 3)(x^4 + x^3 + x^2 + x + 1)$. One can show using [83, Thm. 2.4] that $\omega_{X/\mathbb{Q}_3}$ is not a square. The point $(x, y) = (-1, 2)$ defines a $\mathbb{Q}_3$-rational point of X. Hence by Theorem 8, the unramified Milnor question has a negative answer over X. Compare with Parimala–Sridharan [85, Rem. 3].

Note that possible counter examples that are surfaces could be extracted from the following result.

Theorem 9 (Monnier [72, Thm. 4.5]). *Let X be a smooth integral surface over a real closed field F. Then the unramified Milnor question for quadratic forms (Question 3.1(1)) has a positive answer over X if and only if the cokernel of the mod 2 signature homomorphism is 4-torsion.*

§5. Total invariants of line bundle-valued quadratic forms

Let X be a smooth scheme over a field F of characteristic $\neq 2$. Let $W_{\mathrm{tot}}(X) = \bigoplus_{\mathscr{L}} W(X, \mathscr{L})$ be the *total* Witt group of regular line bundle-valued symmetric bilinear forms, where the sum is taken over a set of representative invertible $\mathscr{O}_X$-modules $\mathscr{L}$ of $\mathrm{Pic}(X)/2$. While this group is only defined up to non-canonical isomorphism depending on our choice of representatives, none of our cohomological invariants depend on such isomorphisms, see [10] for details. Furthermore, we will not consider any ring structure on this group. Thus we will not need to descend into the important considerations of Balmer–Calmès [16].

Taking the rank modulo 2 gives rise to a functorial homomorphism

$$e^0_{\mathrm{tot}} : W_{\mathrm{tot}}(X) \to \mathbb{Z}/2\mathbb{Z} = H^0_{\mathrm{ur}}(X, \mathbb{Z}/2\mathbb{Z})$$

whose kernel is denoted by $I^1_{\mathrm{tot}}(X) = \oplus_{\mathscr{L}} I^1(X, \mathscr{L})$.

Taking the generalization to regular line bundle-valued forms of the signed discriminant (see Parimala–Sridharan [86] or [9, Def. 1.11]) gives rise to a functorial homomorphism

$$e^1_{\mathrm{tot}} : I^1_{\mathrm{tot}}(X) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2) = H^1_{\mathrm{ur}}(X, \boldsymbol{\mu}_2).$$

Denote by $I^2(X, \mathscr{L}) \subset I^1(X, \mathscr{L})$ the subgroup generated by forms of trivial signed discriminant and $I^2_{\mathrm{tot}}(X) = \oplus_{\mathscr{L}} I^2(X, \mathscr{L})$.

As defined in [10], there exists a *total Clifford invariant* for regular line bundle-valued quadratic forms, which gives rise to a functorial homomorphism

$$e^2_{\mathrm{tot}} : I^2_{\mathrm{tot}}(X) \to {}_2\mathrm{Br}(X) = H^2_{\mathrm{ur}}(X, \boldsymbol{\mu}_2)$$

that coincides with the Clifford invariant (see §2.1) of ($\mathscr{O}_X$-valued) quadratic forms when restricted to $I^2(X) = I^2(X, \mathscr{O}_X) \subset I^2_{\mathrm{tot}}(X)$. The surjectivity of the total Clifford invariant can be viewed as a version of the global Merkurjev question (Question 3.2) for line bundle-valued quadratic forms.

Theorem 10 ([10]). *Let X be a smooth integral curve over a local field of characteristic $\neq 2$ or a smooth integral surface over a finite field of characteristic $\neq 2$. Then the total Clifford invariant*

$$e^2_{\mathrm{tot}} : I^2_{\mathrm{tot}}(X) \to {}_2\mathrm{Br}(X)$$

is surjective.

The surjectivity of the total Clifford invariant can also be reinterpreted as the statement that while not every class in $\mathcal{I}^2/\mathcal{I}^3(X) \cong H^2_{\mathrm{ur}}(X)$ is represented by a quadratic form on X, every class is represented by a line bundle-valued quadratic form on X.

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