Advanced Studies in Pure Mathematics 62, 2012
Arrangements of Hyperplanes—Sapporo 2009
pp. 261–291

Resonance webs of hyperplane arrangements

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Abstract.

Each irreducible component of the first resonance variety of a hyperplane arrangement naturally determines a codimension one foliation on the ambient space. The superposition of these foliations define what we call the resonance web of the arrangement. In this paper we initiate the study of these objects with emphasis on their spaces of abelian relations.

§1. Introduction

Let $\mathsf{A} = \{H_1, \ldots, H_r\}$ be an arrangement of $r \geq 1$ hyperplanes in $\mathbb{P}^n$. The complement of A is an affine variety that will be denoted by $M = M(\mathsf{A})$. It is a result of Arnold [2] (for the braid arrangement) and Brieskorn [5] (for and arbitrary hyperplane arrangement) that the cohomology ring of M, $H^\bullet(M, \mathbb{Z})$, is torsion free and generated, as a graded algebra, by the degree one elements determined by the classes of the logarithmic differential forms

$$(2\pi i)^{-1}\left(d\log\frac{h_i}{h_r}\right) \qquad \text{for } i \in \{1, \ldots, r-1\}$$

where $h_1, \ldots, h_r$ are linear polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ defining the hyperplanes in A.

Given $a \in H^1(M) = H^1(M, \mathbb{C})$, consider the complex $(H^\bullet(M), a)$ with arrows given by multiplication by a:

$$0 \longrightarrow H^0(M) \longrightarrow H^1(M) \longrightarrow H^2(M) \longrightarrow \cdots \longrightarrow H^n(M) \longrightarrow 0\,.$$

Received March 31, 2010.
Revised December 24, 2010.
2010 *Mathematics Subject Classification.* 52C35, 53A60.
Key words and phrases. Web geometry, abelian relations, hyperplane arrangements, resonance varieties.

The resonance varieties of M, or A, are defined as

$$\mathsf{R}^i(M) = \mathsf{R}^i(\mathsf{A}) = \{a \in H^1(M),\ | h^i(H^\bullet(M), a) \neq 0\}\,.$$

The paper [14] provides the following nice description of the first resonance variety $\mathsf{R}^1(M)$. The irreducible components of $\mathsf{R}^1(M)$ are precisely the maximal isotropic subspaces of $H^1(M)$ for the quadratic form

$$\wedge : H^1(M, \mathbb{C}) \otimes H^1(M, \mathbb{C}) \longrightarrow H^2(M, \mathbb{C})$$

having dimension at least two. Moreover, the irreducible components of $\mathsf{R}^1(M)$ of dimension k are in correspondence with pencils of hypersurfaces on $\mathbb{P}^n$ having exactly $k+1$ elements with support contained in the arrangement. In particular, for each irreducible component Σ of the resonance variety there is a unique (singular holomorphic) foliation $\mathcal{F}_\Sigma$ on $\mathbb{P}^n$ defined by the corresponding pencil of hypersurfaces.

The interest of the study of the resonance varieties of a hyperplane arrangement is amplified by its relation with the cohomology jumping loci of rank one local systems on M. The characteristic variety $\mathrm{Char}^i(M)$ of M is the subvariety of $\mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ defined as

$$\mathrm{Char}^i(M) = \{\rho \in \mathrm{Hom}(\pi_1(M), \mathbb{C}^*) \,|\, h^i(M, \mathbb{C}_\rho) \neq 0\}\,,$$

where $\mathbb{C}_\rho$ is the rank one local system determined by ρ. The above mentioned relation is given by the following Theorem [1, 11]: the exponential map

$$\begin{aligned} \exp : H^1(M) &\longrightarrow \mathrm{Hom}(\pi_1(M), \mathbb{C}^*) \\ a &\longmapsto \left(\gamma \mapsto \exp\left(2\pi i \int_\gamma a\right)\right) \end{aligned}$$

defines isomorphisms between the germs $(\mathsf{R}^i(M), 0)$ and $(\mathrm{Char}^i(M), \mathbf{1})$, where $\mathbf{1} \in \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ is the trivial representation.

The study of the foliation $\mathcal{F}_\Sigma$ for irreducible components $\Sigma \subset \mathsf{R}^1(M)$ led the author and S. Yuzvinsky (see [23]) to bounds for the dimension of Σ. Although there is now (especially after [29]) a reasonably clear picture about each of the irreducible components of $\mathsf{R}^1(M)$, it is not very clear how the totality of them sit inside $H^1(M, \mathbb{C})$. In this paper we propose an approach to produce invariants for arrangements that may turn out to be useful to the study of this question. The underlying idea is fairly simple: instead of looking at the foliations associated to the irreducible components of $\mathsf{R}^1(M)$ one at a time, we should look at all of them at the same time. More precisely, we will associate to an

arrangement A what we call its resonance web $\mathcal{W}(\mathsf{A})$—the superposition of all the foliations $\mathcal{F}_\Sigma$ associate to irreducible components of $\mathsf{R}^1(\mathsf{A})$—and will study its space of abelian relations.

Conversely, many relevant examples for web geometry, especially in what concerns the dimension of the space of abelian relations, appear as resonance webs of certain arrangements. The list starts with the Bol exceptional 5-web [4], contains the Spence–Kummer exceptional 9-web [27, 25], and ends with some other exceptional webs presented in [27, 25]. This provides further motivation to pursue the study of resonance webs.

Our main result is Theorem 4.1 which determines the rank of the resonance webs for the braid arrangements which, as we will see in Section 4, correspond to the $\binom{n+3}{4}$-web on $\mathfrak{M}_{0,n+3}$ (the moduli space of $n+3$ distinct ordered points on $\mathbb{P}^1$) defined by the $\binom{n+3}{4}$ natural maps

$$\mathfrak{M}_{0,n+3} \longrightarrow \mathfrak{M}_{0,4} \simeq \mathbb{C} \setminus \{0,1\}\,.$$

We will also draw some general considerations about the abelian relations of resonance webs and use them to study some of the examples of exceptional webs mentioned above. Although we have no major results on the structure of the space of abelian relations of resonance webs for arbitrary arrangements, the blurry picture delineated by these examples is considerably intricate and, we believe, invites further investigation.

Plan of the paper. In Section 2 we define webs, their spaces of abelian relations, and show how to bound the rank of arbitrary codimension one webs. We also review the algebraization results for webs of maximal rank, define exceptional webs, and present Bol's example of exceptional planar 5-web. In Section 3 we define resonance webs and initiate the study of their spaces of abelian relations, more specifically the subspace of polylogarithmic abelian relations generated by collections of iterated integrals of logarithmic 1-forms with poles on the arrangement. The reader will also find in Section 3 a brief presentation of a couple of basic results from Chen's theory of iterated integrals relevant to our study. Section 4 is devoted to the statement and proof of our main result: the determination of the rank of the resonance webs of the braid arrangements. Section 5 studies some of the exceptional planar webs found by Pirio and Robert as resonance webs of line arrangements in $\mathbb{P}^2$.

§2. Web geometry

For us a **germ of codimension one k-web** $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ on $(\mathbb{C}^n, 0)$ is a collection of k germs of smooth codimension one holomorphic foliations subjected to the condition that any two distinct foliations $\mathcal{F}_i, \mathcal{F}_j$ have distinct tangent spaces at zero.

Usually in the literature a stronger condition is imposed on the tangent spaces at zero. In the terminology of [7] the tangent spaces are usually assumed to be in **strong general position**, meaning that for any $1 \leq m \leq n$ the intersection of tangent spaces at zero of m distinct foliations $\mathcal{F}_i$ have codimension m.

Perhaps the most studied invariant of a germ of codimension one web $\mathcal{W}$ is its **space of abelian relations** $\mathcal{A}(\mathcal{W})$. If we choose integrable 1-forms ω_i inducing the foliations $\mathcal{F}_i$ then $\mathcal{A}(\mathcal{W})$ is equal to

$$\left\{ (\eta_i)_{i=1}^k \in (\Omega^1(\mathbb{C}^n, 0))^k \;\middle|\; \forall i \; d\eta_i = 0\,,\; \eta_i \wedge \omega_i = 0 \;\text{ and }\; \sum_{i=1}^k \eta_i = 0 \right\}.$$

If $u_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are local submersions defining the foliations $\mathcal{F}_i$ then, after integration, the abelian relations can be read as functional equations of the form $\sum_{i=1}^k g_i(u_i) = 0$ for some germs of holomorphic functions $g_i : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. Thus we can interpret the abelian relations of $\mathcal{W}$ as functional equations (of a rather special kind) among the first integrals of the foliations defining it.

2.1. Rank of webs

Clearly $\mathcal{A}(\mathcal{W})$ is a vector space and its dimension is commonly called the **rank** of $\mathcal{W}$ and is denoted by $\operatorname{rank}(\mathcal{W})$. We will now explain how one can bound the rank of arbitrary codimension one webs. This is a classical subject in web geometry and has been treated by Bol ($n = 2$) and Chern ($n \geq 3$ for webs in strong general position) in the decade of 1930, and more recently by Cavalier–Lehmann for ordinary webs, see the definition below. Here we will deal with arbitrary codimension one webs. This section is a summary of [21, Section 2.2].

For every $i \in \{1, \ldots, k\}$ let ω_i be a germ of 1-form defining $\mathcal{F}_i$ and satisfying $\omega_i(0) \neq 0$. For any positive integer j define $\mathcal{L}^j(\mathcal{W})$ as the subspace of the $\mathbb{C}$-vector space $\operatorname{Sym}^j(\Omega^1(\mathbb{C}^n, 0))$ generated by the j-th symmetric powers of the exterior forms $\omega_i(0)$ with $i \in \{1, \ldots, k\}$. Set $\ell^j(\mathcal{W}) = \dim \mathcal{L}^j(\mathcal{W})$.

Alternatively, if $u_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are local submersions defining the foliations $\mathcal{F}_i$, and h_i are their linear terms then

$$\ell^j(\mathcal{W}) = \dim \left(\mathbb{C}h_1^j + \cdots + \mathbb{C}h_k^j \right) . \tag{1}$$

Notice that the integer $\ell^j(\mathcal{W})$ is bounded by k and by the dimension of the vector space of homogeneous polynomials of degree j in n variables, i.e.

$$\ell^j(\mathcal{W}) \leq \max \left\{ k, \binom{n+j-1}{n-1} \right\} . \tag{2}$$

In the terminology of [7], a germ of k-web $\mathcal{W}$ on $(\mathbb{C}^n, 0)$ is **ordinary** if and only if $\ell^j(\mathcal{W}) = \max \left\{ k, \binom{n+j-1}{n-1} \right\}$ for every positive integer j.

A good lower bound is harder to obtain. For webs in strong general position there is a lemma by Castelnuovo [21, Proposition 2.2.2] which says that

$$\ell^j(\mathcal{W}) \geq \min(k, j(n-1)+1). \tag{3}$$

For arbitrary webs, no longer in strong general position, the best possible lower bound is $\ell^j(\mathcal{W}) \geq \min(k, j+1)$.

The argument used to prove the proposition below is borrowed from Trépreau's proof [28] of Chern's bound for the rank of webs in strong general position.

Proposition 2.1. *If $\mathcal{W}$ is an arbitrary k-web on $(\mathbb{C}^n, 0)$ then*

$$\operatorname{rk}(\mathcal{W}) \leq \sum_{j=1}^{\infty} \max(0, k - \ell^j(\mathcal{W}))$$

and the sum involves only finitely many non-zero terms.

Proof. The space of abelian relations of $\mathcal{W}$ admits a natural filtration $\mathcal{A}(\mathcal{W}) = \mathcal{A}^0(\mathcal{W}) \supseteq \mathcal{A}^1(\mathcal{W}) \supseteq \ldots \supseteq \mathcal{A}^j(\mathcal{W}) \supseteq \ldots$, where

$$\mathcal{A}^j(\mathcal{W}) = \ker \left\{ \mathcal{A}(\mathcal{W}) \longrightarrow \left(\frac{\Omega^1(\mathbb{C}^n, 0)}{\mathfrak{m}^j \cdot \Omega^1(\mathbb{C}^n, 0)} \right)^k \right\},$$

and $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}\{x_1, \ldots, x_n\}$.

One can easily verify [21, Lemma 2.2.6] that

$$\dim \frac{\mathcal{A}^j(\mathcal{W})}{\mathcal{A}^{j+1}(\mathcal{W})} \leq k - \dim \left(\mathbb{C} \cdot h_1^{j+1} + \cdots + \mathbb{C} \cdot h_k^{j+1} \right) \tag{4}$$

where h_i is as in (1). The bound follows. Moreover, as $\ell^j(\mathcal{W}) \geq \min(k, j+1)$ there are only finitely many non-zero terms at the summation above. Q.E.D.

The proposition above combined with the lower bounds previously discussed allows us to recover the bounds for the rank of germs of codimension one webs available elsewhere.

Corollary 2.1. Let $\mathcal{W}$ be a germ of k-web on $(\mathbb{C}^n, 0)$. The assertions below hold true.

(1) (**Bol's bound**) If $n = 2$ then

$$\operatorname{rank}(\mathcal{W}) \leq \frac{(k-1)(k-2)}{2}.$$

(2) (**Chern's bound**) If $n \geq 3$ and $\mathcal{W}$ is in strong general position then

$$\operatorname{rank}(\mathcal{W}) \leq \sum_{j=1}^{\infty} \max(0, k - j(n-1) - 1).$$

(3) (**Cavalier-Lehmann's bound**) If $n \geq 3$ and $\mathcal{W}$ is an ordinary web then

$$\operatorname{rank}(\mathcal{W}) \leq \sum_{j=1}^{\infty} \max\left(0, k - \binom{n+j-1}{n-1}\right).$$

The number at right-hand side of Chern's bound is Castelnuovo's bound $\pi(n, k)$ for the arithmetic genus of irreducible, non-degenerated curves of degree k on $\mathbb{P}^n$.

2.2. Algebraic and algebraizable webs

An important class of examples of webs is the class of **algebraic webs** which are webs dual to projective curves. If C is a reduced degree k projective curve on $\mathbb{P}^n$ then for every general hyperplane H_0 a germ of codimension one k-web $\mathcal{W}_C$ is naturally defined on $(\check{\mathbb{P}}^n, H_0)$ through projective duality. More precisely $\mathcal{W}_C$ is defined by the germs of submersions $p_i : (\check{\mathbb{P}}^n, H_0) \to C$ characterized by

$$H \cdot C = p_1(H) + p_2(H) + \cdots + p_k(H)$$

for every H sufficiently close to H_0.

If ω_C denotes the sheaf of abelian differentials on C, see for instance [21, Chapter 3, Section 2.2] for the definition, then Abel's addition Theorem says that for every $p_0 \in C$ and every regular 1-form $\omega \in H^0(C, \omega_C)$ the sum

$$\int_{p_0}^{p_1(H)} \omega + \int_{p_0}^{p_2(H)} \omega + \cdots + \int_{p_0}^{p_k(H)} \omega$$

does not depend on H. One can reformulate this statement as

$$\sum_{i=1}^{k} p_i^* \omega = 0\,.$$

It follows that $(p_1^*\omega, \ldots, p_k^*\omega)$ can be interpreted as an abelian relation of the algebraic web $\mathcal{W}_C$. Consequently there is an injection of $H^0(C, \omega_C)$ into $\mathcal{A}(\mathcal{W}_C)$. There is a converse to Abel's addition Theorem (due to Lie, Poincaré, Darboux, Wirtinger see [21, Chapter 4]) which implies that this injection is indeed an isomorphism.

Since $h^0(C, \omega_C) = (k-1)(k-2)/2$ for any reduced plane curve of degree k, it follows that every algebraic planar web has maximal rank. The same is no longer true in higher dimensions as the (arithmetic) genus of curves is not determined by theirs degrees. The curves giving rise to maximal rank webs are exactly those which have maximal genus in a given degree, the so called Castelnuovo curves.

2.3. Exceptional webs

For k-webs in strong general position on $(\mathbb{C}^n, 0)$, $n \geq 3$, the maximality of rank implies that the web is **algebraizable** (biholomorphic to a web obtained from a projective curve through duality as explained above) when $k \leq n+1$ or $k \geq 2n$. This was proved by Bol for $n = 3$, and for $n > 3$ is a recent result of Trépreau, see [28]. The planar case ($n = 2$) is rather special in what concerns the classification of webs of maximal rank. For $k \leq 4$ it is well-known that planar webs of maximal rank are algebraizable, the proof for $k = 4$ can be traced back to Lie's work on double translation surfaces, and for $k = 3$ is due to Blaschke–Dubourdieu. In sharp contrast, [19] exhibits examples of non-algebraizable planar k-webs of maximal rank for every $k \geq 5$. Further infinite families of examples appear in [20]. Despite recent advances, see for instance [15, 16, 19, 20], the classification of **exceptional** (non-algebraizable and of maximal rank) planar k-webs is wide open. For a short review of these results see [22]. A more leisurely account can be found in [21].

So far the focus was on germs of webs, but we can consider webs globally defined on a complex variety X. For our pourposes it will

be sufficient to consider completely decomposable webs, that is webs $\mathcal{W}$ which can be globally presented as the superposition of k pairwise distinct global foliations $\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$. For the general definition of global webs see [21, Chapter 1].

Given a global web $\mathcal{W}$, there exists a subvariety $\Lambda = \Lambda(\mathcal{W}) \subset X$ such that for every $x \in X \setminus \Lambda$ the germ of web $\mathcal{W}_x$ obtained by localizing $\mathcal{W}$ at x is a germ of codimension one web and the rank of $\mathcal{W}_x$ is independent of $x \in X \setminus \Lambda$, see [26, Theorem 1.2.2]. More precisely, over $X \setminus \Lambda$ the space of abelian relations of $\mathcal{W}$ is a local system of k-uples of germs of closed 1-forms. Therefore it still makes sense to talk about the rank for global webs.

The first example of exceptional web dates back to the 1936 and was found by Bol, see [4]. It is the global 5-web $\mathcal{B}_5$ on $\mathbb{P}^2$ formed by the superposition of 4 pencils of lines with base points in general position and one pencil of conics through these four base points. We will explain below that this web is naturally associated to an arrangement of lines on $\mathbb{P}^2$.

§3. Resonance webs

Let $\mathsf{A} = \{H_1, \ldots, H_r\}$ be an arrangement of $r \geq 1$ hyperplanes in $\mathbb{P}^n$. Recall from the introduction that $\mathsf{R}^1(\mathsf{A})$, the first resonance variety of A, is the union of the maximal isotropic subspaces of $(H^1(M), \wedge)$ of dimension at least two.

For $i \in \{1, \ldots, r\}$, let $h_i \in \mathbb{C}[x_0, \ldots, x_n]$ be a linear polynomial defining H_i. From now on we will identify $H^\bullet(M)$ with the algebra generated by the logarithmic 1-forms $(d\log h_i - d\log h_r)$ with $i \in \{1, \ldots, r-1\}$.

Before defining our main object of study, let us give a brief idea on how one can associate a pencil of hypersurfaces with $d+1$ completely decomposable fibers to an irreducible component Σ of $\mathsf{R}^1(\mathsf{A})$ of dimension d. Let $\omega_1, \ldots, \omega_d$ be a basis of Σ. As Σ is isotropic there exists non-constant rational functions $h_{ij} \in \mathbb{C}(x_0, \ldots, x_n)$ $(i \neq j)$ such that $\omega_i = h_{ij}\omega_j$. Differentiating this last expression one obtains $0 = dh_{ij} \wedge \omega_j$. Thus the level sets of the functions h_{ij} are tangent to the distribution determined by ω_i for every $i \in \{1, \ldots, d\}$. Stein factorization theorem ensures the existence of a rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$ such that $df \wedge dh_{ij} = df \wedge \omega_i = 0$ for every $i, j \in \{1, \ldots, d\}$. Moreover, there exists d linearly independent logarithmic 1-forms on $\mathbb{P}^1$, say $\eta_1, \ldots, \eta_d$, such that $f^*(\eta_i)_\infty \subset \mathsf{A}$, and $f^*(\eta_i) = \omega_i$ for every $i \in \{1, \ldots, d\}$. The

maximality of Σ implies that the cardinality of

$$\mathsf{P}_\Sigma = \bigcup_{i \in \underline{d}} (\eta_i)_\infty$$

is equal to $d+1$. Thus the rational map $f = f_\Sigma$ determines a pencil of hypersurfaces with $d+1$ fibers contained in the support of the arrangement. Moreover, the restriction of f to $M = \mathbb{P}^n \setminus \mathsf{A}$ is a regular morphism

$$f_{|M} : M \to C_\Sigma \,,$$

where $C_\Sigma = \mathbb{P}^1 \setminus \mathsf{P}_\Sigma$.

Let $\mathcal{F}_\Sigma$ be the foliation on M (or on $\mathbb{P}^n$) determined by the level sets of f_Σ. We define $\mathcal{W}(\mathsf{A})$, the **resonance web** of A as the global web on M (or on $\mathbb{P}^n$) obtained by the superposition of the foliations $\mathcal{F}_\Sigma$ with Σ ranging over the irreducible components of $\mathsf{R}^1(\mathsf{A})$.

Example 3.1. Consider the arrangement on $\mathbb{P}^2$ defined by the polynomial $\{xyz(x-z)(y-z)(x-y) = 0\}$. The points of its complement can be interpreted as isomorphism classes of 5 ordered points on $\mathbb{P}^1$. To wit, the point $(x:y:1) \in \mathbb{P}^2$ satisfying $x - y \neq 0, x \neq 0, 1$, and $y \neq 0, 1$ naturally correspond to the 5-tuple $(0, 1, \infty, x, y) \in (\mathbb{P}^1)^5$. We will denote this arrangement by $\mathsf{A}_{0,5}$ and its complement in $\mathbb{P}^2$ by $\mathfrak{M}_{0,5}$. The resonance variety of $\mathfrak{M}_{0,5}$ has five irreducible components and the associated morphisms are the five forgetting maps $\mathfrak{M}_{0,5} \longrightarrow \mathfrak{M}_{0,4}$, sending isomorphism classes of five ordered points on $\mathbb{P}^1$ to isomorphism classes of four ordered points on $\mathbb{P}^1$. It is a simple exercise to verify that fibers of four of these maps form pencils of lines with base points in general position, and that the fibers of one of them is a pencil of conics through these base points. Thus Bol's exceptional 5-web $\mathcal{B}_5$ is nothing more than $\mathcal{W}(\mathsf{A}_{0,5})$ the resonance web of $\mathsf{A}_{0,5}$.

Due to the important role of Bol's 5-web in the development of web geometry it is natural to enquire about the rank of the resonance webs for arbitrary arrangements. To determine the rank of an arbitrary web is a daunting task. The only general method dates back to Abel (see [26, 25] for a modern account) and involves lengthy algebraic manipulations which lead to linear differential equations of high order that have to be solved. Although implementable, in practice such method is not computationally efficient and cannot deal with k-webs when k is large, say $k > 10$.

An alternative approach to compute the rank of certain webs has been devised by Gilles Robert. Loosely speaking it restricts the search

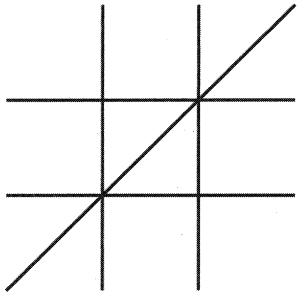

Fig. 1. Affine trace of the arrangement $\mathsf{A}_{0,5}$

of abelian relations to a certain class of differential forms defined from iterated integrals of logarithmic differentials. Here we will explore this approach, adding some topology/combinatorics of arrangements to the picture.

3.1. Logarithmic abelian relations

Consider the morphism

$$\begin{aligned} \Psi_1 : \bigoplus_I H^1(C_\Sigma) &\longrightarrow H^1(M) \\ (\eta_\Sigma) &\mapsto \sum f_\Sigma^* \eta_\Sigma \,, \end{aligned}$$

where Σ ranges over all the irreducible components of $\mathsf{R}^1(\mathsf{A})$. Let $\mathrm{Log}^1 \mathcal{W}(\mathsf{A})$ be its kernel, i.e., $\mathrm{Log}^1 \mathcal{W}(\mathsf{A}) = \ker \Psi_1$.

Proposition 3.1. *The vector space* $\mathrm{Log}^1 \mathcal{W}(A)$ *embeds into the space of abelian relations of* $\mathcal{W}(A)$.

Proof. Let $(\eta_\Sigma) \in \ker \Psi_1$ be a non-zero element. Each η_Σ corresponds to a logarithmic 1-form on $\mathbb{P}^1$. The pull-backs under the corresponding rational map $f_\Sigma^* \eta_\Sigma$ are closed logarithmic 1-forms on $\mathbb{P}^n$, and $f_\Sigma^* \eta_\Sigma$ defines a distribution tangent to the foliation $\mathcal{F}_\Sigma$. Thus $(f_\Sigma^* \eta_\Sigma) \in \mathsf{A}(\mathcal{W}(\mathsf{A}))$ as wanted. Q.E.D.

A natural place to look for further abelian relations is to consider differential forms with logarithmic coefficients, that is, differential forms like

$$\log(z) \frac{dz}{z-1}.$$

A convenient formalism to deal with such objects is Chen's theory of iterated integrals.

3.2. Chen's theory of iterated integrals

In this paragraph M will be an arbitrary connected complex manifold. Given a path $\gamma : [0,1] \to M$ and a collections of 1-forms $\omega_1, \ldots, \omega_k$ the iterated integral of $\omega_1 \otimes \cdots \otimes \omega_k$ along γ is defined as

$$\int_\gamma \omega_1 \otimes \cdots \otimes \omega_k = \int_{\Delta_k} p_1^* \omega_1 \wedge \cdots \wedge p_k^* \omega_k$$

where $p_i : M^k \to M$ is the projection on the i-th factor and Δ_k is the image on M^k of the k-simplex in $\mathbb{R}^k$ given by $0 \leq x_1 \leq \cdots \leq x_k \leq 1$ under the map $\underbrace{\gamma \times \cdots \times \gamma}_{k \text{ times}}$.

Even if the 1-forms ω_i are closed this integral does depend on the path and not just on its homotopy class. It is a result of Chen [8, Theorem 4.1.1] that the elements of $\Omega^1(M)^{\otimes k}$ for which the corresponding iterated integral does not depend on the representative in a given homotopy class are in the intersection of the kernels of the linear maps, $i \in \{1, \ldots, k-1\}$,

$$\begin{array}{rcl} \Omega^1(M)^{\otimes k} & \longrightarrow & \Omega^1(M)^{\otimes i-1} \otimes \Omega^2(M) \otimes \Omega^1(M)^{\otimes (k-i-1)} \\ \omega_1 \otimes \cdots \otimes \omega_k & \mapsto & \omega_1 \otimes \cdots \otimes \omega_{i+1} \wedge \omega_{i+2} \otimes \omega_{i+3} \otimes \cdots \otimes \omega_k \,, \end{array}$$

with the kernels of the linear maps, $i \in \{1, \ldots, k\}$,

$$\begin{array}{rcl} \Omega^1(M)^{\otimes k} & \longrightarrow & \Omega^1(M)^{\otimes i-1} \otimes \Omega^2(M) \otimes \Omega^1(M)^{\otimes (k-i)} \\ \omega_1 \otimes \cdots \otimes \omega_k & \mapsto & \omega_1 \otimes \cdots \otimes d\omega_{i+1} \otimes \omega_{i+2} \otimes \cdots \otimes \omega_k \,. \end{array}$$

If $B^k(M)$ denotes this intersection then every element of $B^k(M)$ gives rise to a function on the universal covering of M through (iterated) integration. Thus we can interpret the elements of $B^k(M)$ as closed 1-forms on the universal covering of M by considering the differential of this function.

Moreover, Chen also proved that if we consider a vector subspace V of $\Omega^1(M)$ formed by closed 1-forms with no non-zero exact forms then the iterated integrals define an injection of $\oplus_{k \geq 1} V^{\otimes k} \cap B^k(M)$ into the space of holomorphic functions on the universal covering of M. In particular, when M is the complement of a hyperplane arrangement, this is the case for $H^1(M)$ seen as a vector subspace of $\Omega^1(M)$. For

more about iterated integrals of logarithmic 1-forms on the complement of arrangements see [18].

It is also interesting to observe that $B(M) = \oplus_{k=1}^{\infty} B^k(M)$ admits a natural structure of $\pi_1(M)$-module, with action defined by analytic continuation of the iterated integrals. Notice that the summands $B^k(M)$ are not $\pi_1(M)$-invariant when $k \geq 2$, but the terms of the filtration $F^\bullet : F^k(M) = \bigoplus_{i=1}^{k} B^i(M)$ are.

3.3. Polylogarithmic abelian relations

It is natural to extend the construction of Section 3.1 to arbitrary iterated integrals of logarithmic 1-forms. For each $i \geq 1$, consider the morphism

$$\begin{aligned} \Psi_i : \bigoplus_I H^1(C_\Sigma)^{\otimes i} &\longrightarrow H^1(M)^{\otimes i} \\ (\eta_\Sigma) &\mapsto \sum f_\Sigma^* \eta_\Sigma \,, \end{aligned}$$

where, as before, Σ ranges over all the irreducible components of $\mathsf{R}^1(\mathsf{A})$. Define $\mathrm{Log}^i \mathcal{W}(\mathsf{A})$ as its kernel, i.e., $\mathrm{Log}^i \mathcal{W}(\mathsf{A}) = \ker \Psi_i$. Define also $\mathrm{Log}^\infty \mathcal{W}(\mathsf{A})$ as the direct sum

$$\mathrm{Log}^\infty \mathcal{W}(\mathsf{A}) = \bigoplus_{i=1}^{\infty} \mathrm{Log}^i \mathcal{W}(\mathsf{A}) \,.$$

Proposition 3.2. *If $\mathcal{W}$ is the localization of the web $\mathcal{W}(A)$ at a generic point of M then the vector space $\mathrm{Log}^\infty \mathcal{W}(A)$ embeds into the space of abelian relations of $\mathcal{W}$. Moreover, the analytic continuation of this embedding gives rise to a local system of abelian relations globally defined on M.*

Proof. Fix $i \in \mathbb{N}$, and let $(\eta_\Sigma) \in \ker \Psi_i$ be a non-zero element. Each η_Σ corresponds to a iterated integral in C_Σ. The pull-backs $f_\Sigma^* \eta_\Sigma$ under the corresponding rational maps are iterated integrals on M, and as such can be interpreted as functions on the universal covering of M. Moreover the (multi-valued) functions on M determined by $f_\Sigma^* \eta_\Sigma$ are (locally) constant along the leaves of the foliation $\mathcal{F}_\Sigma$. As we are considering iterated integrals of logarithmic 1-forms, the properties of iterated integrals recalled on Section 3.2 imply the functions on the universal covering of M coming from the components of $(\eta_\Sigma) \in \ker \Psi_i$ will sum up to zero. Thus the differentials of these functions will define an abelian relation for $\mathcal{W}(\mathsf{A})$. To check injectivity one has just to notice that two different elements in $\oplus_i H^1(C_\Sigma)^{\otimes i}$ will define different functions

on the universal covering of C_Σ according to Chen's Theory, see Section 3.2. Q.E.D.

Let $k_d(\mathsf{A})$ be the number of irreducible components of $\mathsf{R}^1(\mathsf{A})$ of dimension d and $k(\mathsf{A})$ be the total number of irreducible components of $\mathsf{R}^1(\mathsf{A})$. Notice that the resonance web of A is a $k(\mathsf{A})$-web.

Corollary 3.1. The following inequalities hold true:

$$\begin{aligned}
\operatorname{rank}(\mathcal{W}(\mathsf{A})) &\geq \dim \operatorname{Log}^\infty(\mathsf{A})\,,\\
\dim \operatorname{Log}^\infty(\mathsf{A}) &\leq \frac{(k(\mathsf{A})-1)(k(\mathsf{A})-2)}{2}\,,\\
\dim \operatorname{Log}^i(\mathsf{A}) &\geq \sum_d d^i k_d(\mathsf{A}) - \dim B^i(M) \cap H^1(M)^{\otimes i}\,.
\end{aligned}$$

In particular $\operatorname{Log}^\infty(\mathsf{A})$ is a finite dimensional vector space.

Proof. The first inequality follows from Proposition 3.2. The second follows from the first combined with Bol's bound (Corollary 2.1) for the rank of planar webs. To prove the third inequality it suffices to notice that Chen's *integrability conditions* are trivially satisfied by collections of 1-forms on a curve. Thus the morphism Ψ_k factors as in the diagram below

$$\begin{array}{ccc}
 & & N^k(M) \\
 & \nearrow & \downarrow \\
\bigoplus_\Sigma H^1(C_\Sigma)^{\otimes k} & \xrightarrow{\Psi_k} & H^1(M)^{\otimes k} \\
 & & \downarrow \\
 & & H^2 \otimes H^{1\otimes k-2} \oplus \cdots \oplus H^{1\otimes k-2} \otimes H^2
\end{array}$$

where $H^i = H^i(M)$ and $N^k(M) = B^k(M) \cap H^1(M)^{\otimes k}$. The corollary follows. Q.E.D.

Since M is the complement of a hyperplane arrangement, $H^\bullet(M)$ is generated in degree one. Consequently

$$\dim N^2(M) = h^1(M)^2 - h^2(M)\,.$$

In general we do not know how to control the dimensions of the vector spaces $N^i(M)$ when $i \geq 3$. Nevertheless for fiber type arrangements there is the following Künneth type formula which is a corollary of [6,

Theorem 3.38]: If $\mathcal{A}$ is a fiber type arrangement on $\mathbb{P}^n$ with exponents $\{e_1, \ldots, e_n\}$ then

$$\dim N^i(M) = \sum e_1^{j_1} \cdots e_n^{j_n}$$

where the sum is over all ordered n-uples $0 \leq j_1 \leq \cdots \leq j_n$ with $j_1 + \cdots + j_n = i$.

As will be made clear by the examples in Section 5 the bound for the rank given by Corollary 3.1 is rather crude and does not capture many otherwise easily predicable abelian relations. Nevertheless, we will need not more than these crude bounds to determine the rank of the resonance webs of the braid arrangements.

§4. Resonance webs of the braid arrangements

For $n \geq 2$, let $\mathsf{A}_{0,n+3}$ be the arrangement of hyperplanes on $\mathbb{P}^n$ defined by the vanishing of the polynomial

$$\left(\prod_{i=0}^{n} x_i\right)\left(\prod_{i=1}^{n}(x_i - x_0)\right)\left(\prod_{1\leq i<j\leq n}(x_i - x_j)\right).$$

It is the quotient of the braid arrangement B_{n+2} on $\mathbb{P}^{n+2}$

$$\prod_{0\leq i\leq j\leq n+1}(y_i - y_j)$$

by its center $\{y_0 = y_1 = \ldots = y_{n+1}\}$. The resonance variety of $\mathsf{A}_{0,n+3}$ is isomorphic to the resonance variety of B_{n+2}, and the resonance web of B_{n+2} is a linear pull-back of the resonance web of $\mathsf{A}_{0,n+3}$. Consequently, both webs have isomorphic space (local system) of abelian relations.

The complement of $\mathsf{A}_{0,n+3}$ will be denoted by $\mathfrak{M}_{0,n+3}$ and can be identified with the moduli space of $(n+3)$-uples of pairwise distinct ordered points of $\mathbb{P}^1$. According to [11], the resonance variety of $\mathfrak{M}_{0,n+3}$ has $\binom{n+3}{4}$ irreducible components which are in correspondence with the forgetful maps

$$\mathfrak{M}_{0,n+3} \longrightarrow \mathfrak{M}_{0,4}.$$

Thus the resonance web of $\mathcal{W}(\mathsf{A}_{0,n+3})$ is a $\binom{n+3}{4}$-web on $\mathfrak{M}_{0,n+3} \subset \mathbb{P}^n$. The main result of this paper is the determination of the rank of $\mathcal{W}(\mathsf{A}_{0,n+3})$ given below.

Theorem 4.1. *For every $n \geq 2$ the equality*

$$\operatorname{rank}(\mathcal{W}(\mathsf{A}_{0,n+3})) = 3\binom{n+3}{4} - \binom{n+2}{3} - \binom{n+1}{2} - n$$

holds true.

The remaining of this Section is devoted to the proof of this Theorem. It will be convenient to work in the affine chart $x_0 = 1$.

4.1. Upper bound for the rank

For each ordered 4-uple of ordered integers $1 \leq \alpha < \beta < \gamma < \delta \leq n+3$ consider the map

$$\begin{aligned} \rho_{\alpha\beta\gamma\delta} : \mathfrak{M}_{0,n+3} &\longrightarrow \mathfrak{M}_{0,4} \\ (x_1, \ldots, x_{n+3}) &\longmapsto (x_\alpha, x_\beta, x_\gamma, x_\delta) \end{aligned}$$

where the points in the source and the target represent isomorphism classes.

Since $\mathfrak{M}_{0,4} = \mathbb{C} - \{0,1\}$ each of these $\binom{n+3}{4}$ maps define isotropic subspaces of $H^1(\mathfrak{M}_{0,n+3})$, namely $\rho^*_{\alpha\beta\gamma\delta} H^1(\mathfrak{M}_{0,4}) \subset H^1(\mathfrak{M}_{0,n+3})$. It can be verified that these isotropic subspaces are maximal, and that there are no other resonance varieties for $\mathsf{A}_{0,n+3}$, see [11].

Consider now $\mathcal{W} = \mathcal{W}(\mathsf{A}_{0,n+3})$, the germification of the resonance web of $\mathsf{A}_{0,n+3}$ at a generic point of $\mathfrak{M}_{0,n+3} \subset \mathbb{C}^n$. It will be useful to consider the following subwebs:

(1) For each $\alpha \in \underline{n+3}$ let $\mathcal{W}_\alpha$ be the $\binom{n+2}{3}$-subweb defined by the maps ρ_I where I ranges over all the ordered 4-uples containing α. Similarly for distinct α, β and α, β, γ in the above range let $\mathcal{W}_{\alpha,\beta}$ be the $\binom{n+1}{2}$-subweb and $\mathcal{W}_{\alpha,\beta,\gamma}$ be the n-subweb where I ranges over the ordered 4-uples that contains α, β and α, β, γ respectively.

(2) For each $\alpha \in \underline{n+3}$ let $\mathcal{W}^\alpha$ be the pull back of $\mathcal{W}(\mathsf{A}_{0,n+2})$ under the morphism

$$\rho^\alpha : \mathfrak{M}_{0,n+3} \longrightarrow \mathfrak{M}_{0,n+2}$$

that forgets the α-th point.

We combine the two constructions above and define

$$\mathcal{W}^\alpha_{\beta,\gamma} = \mathcal{W}^\alpha \cap \mathcal{W}_{\beta,\gamma}$$

where the intersection of the webs is the web formed by the common foliations of both webs.

Proposition 4.1. *For each $a \in \{1,2,3,4\}$ and every ordered subset I of $\underline{n+3}$ of cardinality $4-a$, $\mathcal{L}^a(\mathcal{W})$ is isomorphic to $\mathcal{L}^a(\mathcal{W}_I)$. Moreover, the dimension of $\mathcal{L}^a(\mathcal{W})$ is given by the formula*

$$\ell^a(\mathcal{W}) = \ell^a(\mathcal{W}_I) = \binom{n+a-1}{n-1}.$$

Proof. The proof will be by a double induction on n and a.

When $n = 2$, $\mathcal{W}_I = \mathcal{W}(\mathsf{A}_{0,5})_I$ is an $(a+1)$-web on $\mathbb{C}^2$. Therefore $\mathcal{L}^a(\mathcal{W})$ contains $\mathcal{L}^a(\mathcal{W}_I)$, and equation (2) implies

$$a + 1 \geq \ell^a(\mathcal{W}) \geq \ell^a(\mathcal{W}_I) = a + 1.$$

The result follows in this case.

When $a = 1$, we can assume that $I = (n+1, n+2, n+3)$ and therefore by normalizing the points of $\mathfrak{M}_{0,n+3}$ in such way that the last three are $0, 1, \infty$ we have that the n foliations defining $\mathcal{W}_I$ are defined by the morphisms $(x_1, \ldots, x_n, 0, 1, \infty) \mapsto (x_i, 0, 1, \infty)$. Clearly $\mathcal{L}^1(\mathcal{W}) = \mathcal{L}^1(\mathcal{W}_I) = \oplus_{i=1}^n \mathbb{C}x_i$.

Suppose now that $a \geq 2$ and $n \geq 3$. Assume $I \subset \{n+1, n+2, n+3\}$, $j \in \{1, \ldots, n\}$, and $(x_{n+1}, x_{n+2}, x_{n+3}) = (0, 1, \infty)$. Consider the linear map defined by the derivation $\frac{\partial}{\partial x_j}$ from $\mathbb{C}_a[x_1, \ldots, x_n] \simeq Sym^a\Omega^1(\mathbb{C}^n, 0)$ to $\mathbb{C}_{a-1}[x_1, \ldots, x_n] \simeq Sym^{a-1}\Omega^1(\mathbb{C}^n, 0)$. It induces the following diagram with exact rows where the unlabeled arrows are the natural inclusions.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}^a(\mathcal{W}_I^j) & \longrightarrow & \mathcal{L}^a(\mathcal{W}_I) & \xrightarrow{\frac{\partial}{\partial x_j}} & \mathcal{L}^{a-1}(\mathcal{W}_{I\cup\{j\}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L}^a(\mathcal{W}^j) & \longrightarrow & \mathcal{L}^a(\mathcal{W}) & \xrightarrow{\frac{\partial}{\partial x_j}} & \mathcal{L}^{a-1}(\mathcal{W}_j) & \longrightarrow & 0.
\end{array}$$

Notice that $\mathcal{W}^j$ is isomorphic to $\mathcal{W}(\mathsf{A}_{0,n+2})$ and therefore, by induction hypothesis, the leftmost vertical arrow is an isomorphism. The induction hypothesis also implies that the vector spaces $\mathcal{L}^{a-1}(\mathcal{W}_j)$ and $\mathcal{L}^{a-1}(\mathcal{W}_{I\cup\{j\}})$ are both isomorphic to $\mathcal{L}^{a-1}(\mathcal{W})$. Thus the rightmost arrow is also an isomorphism. It follows that the middle vertical arrow is also an isomorphism and has dimension given by

$$\ell^a(\mathcal{W}) = \binom{(n-1)+a-1}{(n-1)-1} + \binom{n+(a-1)-1}{n-1} = \binom{n+a-1}{n-1}.$$

The proposition follows. Q.E.D.

As a consequence we obtain that for every $n \geq 2$ the webs $\mathcal{W}(\mathsf{A}_{0,n+3})$ are ordinary webs.

Corollary 4.1. For every $n \geq 2$ the inequality

$$\operatorname{rank}(\mathcal{W}(\mathsf{A}_{0,n+3})) \leq 3\binom{n+3}{4} - \binom{n+2}{3} - \binom{n+1}{2} - n$$

holds true.

Proof. It suffices to combine Propositions 2.1 and 4.1. Q.E.D.

4.2. Lower bound for the rank

As there are exactly $\binom{n}{2} + 2n + 1$ hyperplanes in the (projective) arrangement $\mathsf{A}_{0,n+3}$, $h^1(\mathfrak{M}_{0,n+3}) = \binom{n}{2} + 2n$. Notice also that $h^1(\mathfrak{M}_{0,4}) = 2$. Since $k(\mathsf{A}_{0,n+3}) = k_2(\mathsf{A}_{0,n+3})$, Corollary 3.1 implies

$$\begin{aligned}(5)\qquad \dim \mathrm{Log}^1\, \mathcal{W}(\mathsf{A}_{0,n+3}) &\geq 2\binom{n+3}{4} - h^1(\mathfrak{M}_{0,n+3}) \\ &= 2\binom{n+3}{4} - \binom{n}{2} + 2n.\end{aligned}$$

Similarly,

$$\dim \mathrm{Log}^2\, \mathcal{W}(\mathcal{A}_{0,n+3}) \geq \binom{n+3}{4} \cdot h^1(\mathfrak{M}_{0,4})^2 - \dim N^2(\mathfrak{M}_{0,n+3})\,.$$

It is a result of Arnold [2] that the Poincaré polynomial of $\mathfrak{M}_{0,n+3}$ is $P(t) = (1+2t)(1+3t)\cdots(1+(n+1)t)$. Therefore the dimension of $N^2(\mathfrak{M}_{0,n+3})$ is equal to

$$h^1(\mathfrak{M}_{0,n+3})^2 - h^2(\mathfrak{M}_{0,n+3}) = \left(\binom{n}{2} + 2n\right)^2 - P''(0)/2\,.$$

Consequently

$$(6)\qquad \dim AR^2_{log}(\mathcal{A}_{0,n+3}) \geq 4\binom{n+3}{4} - h^1(M_{0,n+3})^2 + h^2(M_{0,n+3})\,.$$

4.3. Proof of Theorem 4.1

It is not hard to prove by induction that summing the right-hand side of the inequalities (5) and (6) one obtains $3\binom{n+3}{4} - \binom{n+2}{3} - \binom{n+1}{2} - n$. Therefore, Proposition 3.2 implies that

$$\begin{aligned}\mathrm{rank}(\mathcal{W}(\mathcal{A}_{0,n+3})) &\geq \dim AR^1_{log}(\mathcal{A}_{0,n+3}) + \dim AR^2_{log}(\mathcal{A}_{0,n+3}) \\ &\geq 3\binom{n+3}{4} - \binom{n+2}{3} - \binom{n+1}{2} - n\,.\end{aligned}$$

Combining this lower bound with the upper bound given in Corollary 4.1 concludes the proof of Theorem 4.1. Q.E.D.

As the webs $\mathcal{W}(\mathsf{A}_{0,n+3})$ attain Cavalier–Lehmann's bound, it is natural to ask if they are algebraizable or, more generally, linearizable. As the leaves of algebraic webs are contained in hypersurfaces, every algebraizable web is linearizable but the converse is not always true. In

[24] it is proved that the webs $\mathcal{W}(\mathsf{A}_{0,n+3})$ are not linearizable. We refer to this work and references therein for more about the linearization of webs.

It is interesting to compare our Theorem 4.1 with Damiano's determination of the rank of the (dimension one) web given by the $(n+3)$ maps [12]

$$\mathfrak{M}_{0,n+3} \longrightarrow \mathfrak{M}_{0,n+2}\,.$$

These webs turn out to attain the corresponding bound for the rank of one-dimensional webs and are also non-linearizable.

§5. Examples

This section is devoted to the study of some exceptional planar webs—first found by Pirio and Robert [25, 27]—which are resonance webs of suitable line arrangements in $\mathbb{P}^2$. We use them to recognize other sources of abelian relations besides iterated integrals with logarithmic forms with poles on the arrangement.

5.1. More polylogarithmic abelian relations

We start with a simple example. Consider A_0 as the arrangement of 8 lines on $\mathbb{P}^2$ with affine trace presented in Figure 2. If we suppose that the triple point is at the origin of $\mathbb{C}^2$ then pencil of lines through it corresponds to an irreducible component of the resonance variety of dimension two. There are other two triple points at the line at infinity and they also correspond to irreducible components of the resonance variety of dimension two. There are no other irreducible components of $\mathsf{R}^1(\mathsf{A}_0)$. The resonance web $\mathcal{W}(\mathsf{A}_0)$ is the 3-web determined by the superposition of the foliations given by the level sets of the functions $x, y, x/y$. It clearly has rank one as

$$d\log x - d\log y - d\log\frac{x}{y} = 0\,,$$

but $\dim \mathrm{Log}^{\infty}\,\mathcal{W}(\mathsf{A}_0) = 0$ as one can promptly verify.

Given an arrangement of hyperplanes A on $\mathbb{P}^n$ we define the **resonance closure** of A, denoted by $\overline{\mathsf{A}}$, as the arrangement of hypersurfaces on $\mathbb{P}^n$ characterized by the following property: $H \in \overline{\mathsf{A}}$ if and only $H \in \mathsf{A}$, or there exists two distinct irreducible components Σ_1, Σ_2 of $\mathsf{R}^1(\mathsf{A})$ such that $\dim f_{\Sigma_1}(H) = \dim f_{\Sigma_2}(H) = 0$. In other words either H belongs to the original arrangement A or it is invariant by two distinct foliations of the web $\mathcal{W}(\mathsf{A})$. The complement of $\overline{\mathsf{A}}$ will be denoted by $\underline{M}$.

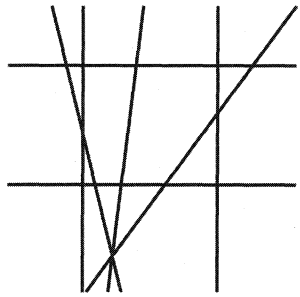

Fig. 2. $1 = \operatorname{rank}(\mathcal{W}(\mathsf{A})) > \dim \operatorname{Log}^{\infty} \mathcal{W}(\mathsf{A}) = 0$

Example 5.1. The resonance closure of the example A_0 is obtained by adding the lines $\{x = 0\}$ and $\{y = 0\}$. A more interesting (family of) examples is obtained by considering the (family of) arrangement(s) K_5 determined by the 10 lines joining 5 points in $\mathbb{P}^2$ in general position. It can be verified that the resonance variety of K_5 has 10 irreducible components: 5 of dimension 3 corresponding to pencils the lines through the points, and 5 of dimension 2 corresponding to the pencils of conics through 4 of the 5 points. If C denotes the conic through the 5 points then $\overline{\mathsf{K}_5} = \mathsf{K}_5 \cup \{C\}$.

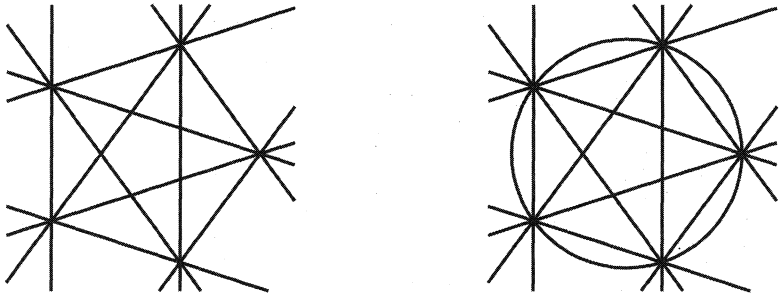

Fig. 3. On the left K_5, on the right $\overline{\mathsf{K}_5}$. The line at infinity is not included in the arrangements above

The relevance of the definition of the closure of an arrangement is put in evidence by the following specialization of [26, Theorem 1.2.2].

Proposition 5.1. *If $\mathcal{W}(\mathcal{A})$ is the resonance web of an arrangement $\mathcal{A}$ then every germ of abelian relation of $\mathcal{W}(\mathcal{A})$ extends to the universal covering of $\underline{M}$.*

If Σ is an irreducible component of $\mathsf{R}^1(\mathsf{A})$ we define its closure $\overline{\Sigma}$ as the maximal isotropic subspace of $H^1(\underline{M})$ containing the image of Σ under the natural inclusion $H^1(M) \to H^1(\underline{M})$. The **closure of the resonance variety** of A is the subvariety of $H^1(\underline{M})$ defined as

$$\overline{\mathsf{R}^1(\mathsf{A})} = \bigcup_{\Sigma \subset \mathsf{R}^1(\mathsf{A})} \overline{\Sigma}.$$

Notice that $\overline{\mathsf{R}^1(\mathsf{A})} \subset \mathsf{R}^1(\overline{\mathsf{A}})$ but the equality does not hold in general.

To each irreducible component $\overline{\Sigma} \subset \overline{\mathsf{R}^1(\mathsf{A})}$ there is a set of points of $\mathbb{P}^1$, $\mathsf{P}_{\overline{\Sigma}} \supset \mathsf{P}_\Sigma$ with complement $\underline{C_\Sigma}$ satisfying $(f_\Sigma)^* H^1(\underline{C_\Sigma}) = \overline{\Sigma}$. By analogy with what has been done in Sections 3.1 and 3.3, we can consider

$$\begin{aligned} \overline{\Psi}_i : \bigoplus_{\overline{\Sigma}} H^1(\underline{C_\Sigma})^{\otimes i} &\longrightarrow H^1(\underline{M})^{\otimes i} \\ (\eta_\Sigma) &\mapsto \sum f_\Sigma^* \eta_\Sigma \end{aligned}$$

and define, for every $i \in \mathbb{N}$, $\overline{\mathrm{Log}^i \mathcal{W}(\mathsf{A})} = \ker \overline{\Psi}_i$. We also define $\overline{\mathrm{Log}^\infty \mathcal{W}(\mathsf{A})}$ as the direct sum $\oplus_i \overline{\mathrm{Log}^i \mathcal{W}(\mathsf{A})}$.

Of course, we also have a Proposition 3.2:

Proposition 5.2. *If $\mathcal{W}$ is the localization of the web $\mathcal{W}(\mathcal{A})$ at a generic point of M then the vector space $\overline{\mathrm{Log}^\infty \mathcal{W}(\mathcal{A})}$ embeds into the space of abelian relations of $\mathcal{W}$. Moreover, the analytic continuation of this embedding gives rise to a local system of abelian relations globally defined on $\underline{M}$.*

Of course, we can write lower bounds for $\overline{\mathrm{Log}^i \mathcal{W}(\mathsf{A})}$ analogous to the ones given by Corollary 3.1.

If $\mathsf{A} = \{C_1, \ldots, C_r\}$ is an arrangement of curves on $\mathbb{P}^2$ it is still a simple matter to determine the dimension of the second cohomology group of the complement. If $t_i(\mathsf{A})$ denotes the number of singular points in the support of A through which there exactly i branches, and $\chi(\widetilde{C_i})$ ($i = 1, \ldots, r$) are the Euler characteristic of the normalizations of the curves in the arrangement then [10, Proposition 2.4],

$$(7)\quad h^1(M) = r - 1 \quad \text{and} \quad h^2(M) = 1 + \sum_{i=2}^{\infty} (i-1) t_i(\mathsf{A}) - \sum_{i=1}^{r} (\chi(\widetilde{C_i}) - 1).$$

We can use these observations (together with some computer algebra) to recover the following (unpublished) joint result of Pirio and Robert.

Theorem 5.1. *The resonance web of K_5 is exceptional.*

Proof. Notice that the closures of the irreducible components of $\mathsf{R}^1(\mathsf{K}_5)$ all have dimension 3. As $h^1(\underline{M}) = 10$, we obtain the lower bound

$$\overline{\mathrm{Log}^1\mathcal{W}}(\mathsf{K}_5) \geq 10 \cdot 3 - h^1(\overline{M}) = 30 - 10 = 20\,.$$

To control $\overline{\mathrm{Log}^2\mathcal{W}}(\mathsf{K}_5)$, we need to know the dimensions of $N^2(\underline{M}) = \ker\{H^1(\underline{M})^{\otimes 2} \to H^2(\underline{M})\}$. For an arbitrary arrangement of curves the cohomology algebra of the complement has no reason to be generated in degree 1, but for $\underline{M} = \mathbb{P}^2 \setminus \overline{\mathsf{K}_5}$, [10, Theorem 2.46] implies it is the case. Thus

$$\overline{\mathrm{Log}^2\mathcal{W}}(\mathsf{K}_5) \geq 10 \cdot 9 - h^1(\overline{M})^2 + h^2(\overline{M}) = 90 - 100 + 25 = 15\,.$$

The inequalities above turn out to be equalities. To determine the dimension of $\overline{\mathrm{Log}^3\mathcal{W}}(\mathsf{K}_5)$ one has *just* to compute the dimension of the kernel of a 3375×243 matrix. A brute-force calculation[1] shows that

$$\dim \overline{\mathrm{Log}^3\mathcal{W}}(\mathsf{K}_5) = 1.$$

Thus $\mathcal{W}(\mathsf{K}_5)$ attains Bol's bound $9 \cdot 8/2 = 36$. To prove it is non-algebraizable it suffices to apply [20, Proposition 2.1]. Q.E.D.

Notice that K_5 is indeed a 2-parameter family of 10-webs as the moduli space of isomorphisms classes of 5 point on $\mathbb{P}^2$ has dimension 2.

5.2. Rational abelian relations

The inclusion of $\overline{\mathrm{Log}^\infty\mathcal{W}}(\mathsf{A})$ into $\mathcal{A}(\mathcal{W}(\mathsf{A}))$ does not exhaust the space of abelian relations of $\mathcal{W}(\mathsf{A})$ in general, unlike when $\mathsf{A} = \mathsf{A}_{0,5}$ or $\mathsf{A} = \mathsf{K}_5$. The simplest example is when A is an arrangement of 9 lines with 3 aligned threefold intersection points and all the other intersections are ordinary. In this case the resonance variety has only three local components, each of them having dimension two. Supposing that these three points are $(0:1:0), (1:0:0)$, and $(1:1:0)$ then the corresponding foliations are defined by the 1-forms dx, dy, and $dx + dy$. As they satisfy $(dx) + (dy) - (dx + dy) = 0$, it is clear that $\mathcal{W}(\mathsf{A})$ has rank one but $\dim \overline{\mathrm{Log}^\infty\mathcal{W}}(\mathsf{A}) = 0$.

[1] Maple script available at `www.impa.br/~jvp/artigos.html`.

Define $\mathrm{Rat}\mathcal{W}(\mathsf{A})$ as the kernel of the linear map

$$\begin{aligned}\overline{\Upsilon} : \bigoplus_{\Sigma} \frac{\mathbb{C}(\mathbb{P}^1)}{\mathbb{C}} &\longrightarrow \frac{\mathbb{C}(\mathbb{P}^n)}{\mathbb{C}} \\ (g_\Sigma) &\mapsto \sum f_\Sigma^*(g_\Sigma)\,,\end{aligned}$$

where $\mathbb{C}(\mathbb{P}^n)$ stands for the field of rational functions on $\mathbb{P}^n$, and the summation Σ run over all the irreducible components of $\mathsf{R}^1(\mathsf{A})$. Coordinate-wise differentiation injects $\ker\Upsilon$ into $\mathcal{A}(\mathcal{W}(\mathsf{A}))$. Notice that its image intersects $\overline{\mathrm{Log}^\infty}\mathcal{W}(\mathsf{A})$ only at zero. Therefore, we have the following lower bound for the rank of $\mathcal{W}(\mathsf{A})$

$$\mathrm{rank}(\mathcal{W}(\mathsf{A})) \geq \dim \overline{\mathrm{Log}^\infty}\mathcal{W}(\mathsf{A}) + \dim \mathrm{Rat}\mathcal{W}(\mathsf{A})\,. \tag{8}$$

We do not know how to give general lower bounds for $\dim \mathrm{Rat}\mathcal{W}(\mathsf{A})$. We have only the following simple result.

Lemma 5.1. *Let A be a planar arrangement, and let $\{\ell_1, \ldots, \ell_m\}$ be the lines in its closure. If n_i is the number of local irreducible components $\overline{\Sigma} \subset \overline{R^1}(A)$ containing ℓ_i in its support then*

$$\dim \mathrm{Rat}\mathcal{W}(A) \geq \sum_{i=1}^{n} \frac{(n_i - 1)(n_i - 2)}{2}\,.$$

Proof. Suppose that ℓ_1 is the line at infinity. The foliations associated to the n_1 components of $\overline{\mathsf{R}^1}(\mathsf{A})$ containing ℓ_1 are defined by $dh_1, \ldots, dh_{n_1}$ where h_i is a linear form. To prove the lemma it suffices to observe that for $p \geq 1$, the kernel of the maps

$$(a_1, \ldots, a_{n_1}) \mapsto \sum_{i=0}^{n_1} a_i (h_i)^p$$

will correspond to rational abelian relations with polar set contained in ℓ_1. Q.E.D.

Of course one can do better by considering rational functions with poles on fibers of f_Σ for non-local components of $\mathsf{R}^1(\mathsf{A})$.

Example 5.2. Let B_{5+n}, $n = 1, 2, 3$, be the arrangements obtained from $\mathsf{A}_{0,5}$ by adding n generic lines, each through a distinct double point, see Figure 4. The resonance webs are the webs $\mathcal{B}_6$, $\mathcal{B}_7$ and $\mathcal{B}_8$ considered by Robert and Pirio, and proved to be exceptional by them.

The rank of $\mathcal{B}_6$ can be computed easily as follows. As it contains $\mathsf{A}_{0,5}$ it has rank at least 6. Adding the pencil of lines through a double

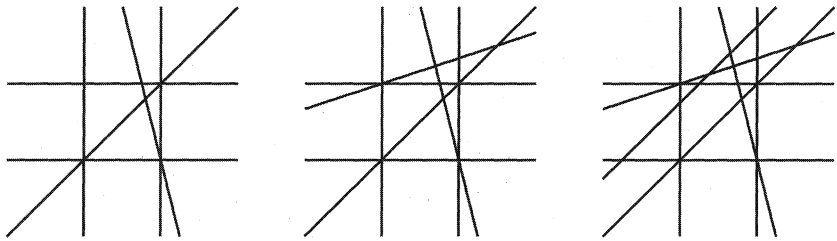

Fig. 4. Arrangements $\mathsf{B}_6, \mathsf{B}_7$, and B_8

point of the support of $\mathsf{A}_{0,5}$ we have that $\overline{\mathsf{B}_6} = \mathsf{B}_6$ contain two lines with three triple points, thus Lemma 5.1 implies $\dim \operatorname{Rat} \mathcal{W}(\mathsf{B}_6) \geq 2$. The proof of Theorem 4.1 tell us that $\Psi_1 : \oplus H^1(C_\Sigma) \to H^1(M)$ and $\Psi_2 : \oplus H^1(C_\Sigma)^{\otimes 2} \to H^1(M)^{\otimes 2}$, with $\Sigma \subset \mathsf{R}^1(A_{0,5})$, are both surjective. If h_1, h_2 are the equations of the lines intersecting at the double point under consideration then $d\log h_1 - d\log h_2$ and $(d\log h_1 - d\log h_2)^{\otimes 2}$ belong (respectively) to the image of Ψ_1 and Ψ_2. Thus

$$\dim \operatorname{Log}^\infty \mathcal{W}(\mathsf{B}_6) \geq \dim \operatorname{Log}^\infty \mathcal{W}(\mathsf{A}_{0,5}) + 2\,.$$

Putting all together we deduce that $\operatorname{rank} \mathcal{W}(\mathsf{B}_6) \geq 6+2+2 = 10$. Thus $\mathcal{B}_6 = \mathcal{W}(\mathsf{B}_6)$ is of maximal rank as it attains Bol's bound.

One can deal similarly with B_7 and B_8 but Lemma 5.1 does not suffice. One has to consider also rational first integrals for the foliation associated to the non-local component of $\mathsf{R}^1(\mathsf{B}_6)$ with poles on one (for B_7) or three (for B_8) fibers.

For general arrangements of lines, the inclusion

$$\overline{\operatorname{Log}^\infty} \mathcal{W}(\mathsf{A}) \oplus \operatorname{Rat}^\infty \mathcal{W}(\mathsf{A}) \subset \mathcal{A}(\mathcal{W}(\mathsf{A}))$$

is strict. In the next two sections we will consider two examples of line arrangements on $\mathbb{P}^2$ with resonance webs having abelian relations which support this claim.

5.3. Mixed abelian relations

In the same way that we looked for abelian relations among collections of iterated integrals of logarithmic 1-forms we can also look for them in collections of iterated integrals of arbitrary rational 1-forms. In this section we will consider a one-parameter family of arrangements with resonance webs having abelian relations of this form.

Example 5.3. For $\lambda \in \mathbb{C} \setminus \{0,1\}$, let $\mathsf{P} = \mathbb{P}_\lambda$ be the arrangement of 8 lines obtained form $\mathsf{A}_{0,5} = \{xyz(x-y)(x-z)(y-z) = 0\}$ by adding

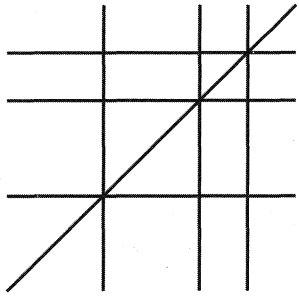

Fig. 5. The arrangement P

the two extra lines $\{(x-\lambda z)(y-\lambda z)=0\}$, see Figure 5. Its resonance variety has 8 irreducible components: two local of dimension three; three local of dimension two; and three non-local of dimension two determined by pencils of conics. It is closed arrangement in the sense that $\mathsf{P}=\overline{\mathsf{P}}$.

The family of webs $\mathcal{W}(\mathsf{P}_\lambda)$ have been studied in [25, 26]. There explicit generators for the corresponding spaces of abelian relations are presented.

From Corollary 3.1 we see that

$$\dim \mathrm{Log}^1 \mathcal{W}(\mathsf{P}) \geq 11 \text{ and } \dim \mathrm{Log}^2 \mathcal{W}(\mathsf{P}) \geq 5\,.$$

These inequalities are indeed equalities. Moreover, one can verify that $\dim \mathrm{Rat}\,\mathcal{W}(\mathsf{P})=4$. Thus we have at least a 20-dimensional subspace of the space of abelian relations.

There is still one extra abelian relation, relation G^a_{21} in Section 3.3 of [25], involving the pull-backs under f_Σ of

$$d\left(\frac{\log(x)}{1-x}\right) = -d\log(x-1)+d\log(x)+\frac{\log(x)}{(1-x)^2}dx\,.$$

The last summand is neither the differential of a rational function, nor an iterated integral of logarithmic 1-forms. Indeed it can be written as $\frac{dx}{x}\otimes\frac{dx}{(1-x)^2}$, and therefore is an iterated integral of rational 1-forms.

5.4. Twisted logarithmic abelian relations

Of course the class of abelian relations with components being iterated integrals of rational 1-forms encompass all previous classes of abelian relations considered. Note that if we consider all the abelian relations in this class we obtain an unipotent local system over the complement of the closure of the arrangement. We believe that the maximal

unipotent local system in $\mathcal{A}(\mathcal{W}(\mathsf{A}))$ is exactly the one generated by the abelian relations given by iterated integrals of rational 1-forms.

Our last example shows that the local system $\mathcal{A}(\mathcal{W}(\mathsf{A}))$ is not in general unipotent.

Example 5.4. Let F be the non-Fano arrangement presented in Figure 6. It is a closed arrangement ($\mathsf{F} = \overline{F}$) and its resonance variety has 9 irreducible components: six of them are local of dimension two, and three of them are determined pencil of conics and also have dimension two. The resonance web $\mathcal{W}(\mathsf{F})$ is the so called Spence–Kummer exceptional 9-web and was studied independently by Pirio [25] and Robert [27]. They proved that $\mathcal{W}(\mathsf{F})$ is an exceptional 9-web. The reference to Spence–Kummer comes from the fact that the foliations of the web are defined, up to a change coordinates, by the rational functions appearing in Spence–Kummer functional equation for the trilogarithm

$$\mathrm{Li}_3(z) = \int \frac{dz}{z} \otimes \frac{dz}{z} \otimes \frac{dz}{1-z}\,.$$

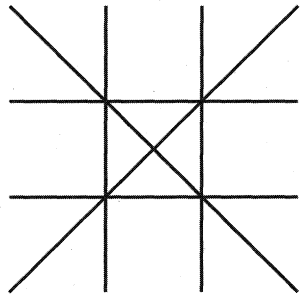

Fig. 6. The non-Fano arrangement

Corollary 3.1 implies

$$\dim \mathrm{Log}^1 \mathcal{W}(\mathsf{F}) \geq 12 \text{ and } \dim \mathrm{Log}^2 \mathcal{W}(\mathsf{F}) \geq 9\,,$$

and Lemma 5.1 implies $\dim \mathrm{Rat}\, \mathcal{W}(\mathsf{F}) \geq 4$. These bounds are indeed equalities. Moreover, brute-force computation yields $\dim \mathrm{Log}^3 \mathcal{W}(\mathsf{F}) = 2$. Thus we have a 27-dimensional subspace of the space of abelian relations.

The missing abelian relation comes from the intersection of the irreducible components of $\mathrm{Char}^1(M)$ determined by the three non-local

components. In order to explain this abelian relation we will digress a little.

§

Let A be an arrangement of hypersurfaces on $\mathbb{P}^n$ and M be its complemente. Recall from the introduction the morphism

$$\begin{aligned} \exp : H^1(M) &\longrightarrow \mathrm{Hom}(\pi_1(M), \mathbb{C}^*) \\ a &\longmapsto \left(\gamma \mapsto \exp\left(2\pi i \int_\gamma a\right)\right). \end{aligned}$$

For a 1-form $\omega \in H^1(M)$, let $\varrho_\omega : \pi_1(M) \to \mathbb{C}^*$ be the representation $\exp(\omega)$ and $\mathbb{C}_\omega$ the corresponding rank one local system.

The $\mathbb{C}$-sheaf $\mathbb{C}_\omega$ admits the following resolution

$$0 \longrightarrow \mathbb{C}_\omega \longrightarrow \mathcal{O}_M \xrightarrow{\nabla_\omega} \Omega^1(M) \xrightarrow{\nabla_\omega} \Omega^2(M) \xrightarrow{\nabla_\omega} \cdots \longrightarrow 0$$

where $\Omega^\bullet(M)$ are the sheaves of holomorphic differentials on M and $\nabla_\omega : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ is given by the formula $\nabla_\omega(\alpha) = d\alpha - \omega \wedge \alpha$. Since M is Stein, the sheaves $\Omega^\bullet(M)$ are acyclic and consequently

$$H^i(M, \mathbb{C}_\omega) = \frac{\ker \nabla_\omega : H^0(M, \Omega^i(M)) \to H^0(M, \Omega^{i+1}(M))}{\nabla_\omega(H^0(M, \Omega^{i-1}))}.$$

If $\omega_1 - \omega_2 = d\log F$ for some $F \in H^0(M, \mathcal{O}_M)$ then $\mathbb{C}_{\omega_1} \simeq \mathbb{C}_{\omega_2}$ and the corresponding resolutions relate through the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{C}_{\omega_1} & \longrightarrow & \mathcal{O}_M & \xrightarrow{\nabla_{\omega_1}} & \Omega^1(M) & \xrightarrow{\nabla_{\omega_1}} & \Omega^2(M) & \xrightarrow{\nabla_{\omega_1}} & \cdots \longrightarrow 0 \\ & & & & \downarrow{\scriptstyle F^{-1}} & & \downarrow{\scriptstyle F^{-1}} & & \downarrow{\scriptstyle F^{-1}} & & \\ 0 & \longrightarrow & \mathbb{C}_{\omega_1} & \longrightarrow & \mathcal{O}_M & \xrightarrow{\nabla_{\omega_2}} & \Omega^1(M) & \xrightarrow{\nabla_{\omega_2}} & \Omega^2(M) & \xrightarrow{\nabla_{\omega_2}} & \cdots \longrightarrow 0 \end{array} \tag{9}$$

where the vertical arrows are multiplication by F^{-1}.

For $\alpha \in \Omega^1(M)$ we have $\nabla_\omega(\alpha) = 0$ if and only if the (multi-valued) 1-form $\exp\left(\int \omega\right)\alpha$ is closed. Moreover, if ω belongs to some irreducible component Σ of $\mathsf{R}^1(\mathsf{A})$ then for every $\alpha \in \Sigma$ we have that the 1-form $\exp\left(\int \omega\right)\alpha$ is closed.

Let Σ be an irreducible component of $\mathsf{R}^1(\mathsf{A})$ and let $\alpha \in \Sigma$ be a logarithmic 1-form with residues in $\mathbb{Q} \setminus \mathbb{Z}_{>0}$. Let $P : N \to \mathbb{P}^n$ be the finite abelian covering determined by $\mathbb{C}_\alpha$. Since the monodromy is finite (residues $\in \mathbb{Q}$), N is a quasi-projective variety.

Lemma 5.2. *If $\beta \in \Sigma$ is not a complex multiple of α then $\exp\left(-\int P^*\alpha\right) P^*\beta$ is a closed rational 1-form on N which is not exact.*

Proof. Let $f = f_\Sigma : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$ be the associate rational map. Recall that for $\alpha, \beta \in \Sigma$ there exist logarithmic 1-forms $\widetilde{\alpha}$, $\widetilde{\beta}$ on $\mathbb{P}^1$ with poles of P_Σ such that $\beta = f_i^*\widetilde{\beta}$.

If $p : C \to \mathbb{P}^1 \setminus \mathsf{P}_\Sigma$ is the finite ramified covering determined by $\exp\left(-\int \widetilde{\alpha}\right)$ then it fits in the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{P} & M \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{p} & \mathbb{P}^1 \setminus \mathsf{P}_\Sigma. \end{array}$$

Clearly $\exp\left(-\int P^*\alpha\right) P^*\beta$ is equal to $\widetilde{f}^* \exp\left(-\int p^*\widetilde{\alpha}\right) \widetilde{\beta}$. Thus if the former is exact, the same holds for the latter.

Suppose $\exp\left(-\int p^*\widetilde{\alpha}\right) \widetilde{\beta} = dg$ for some rational function on C. The rational function $\tilde{h} = \exp\left(\int p^*\widetilde{\alpha}\right) g$ is invariant under the covering transformation, thus is equal to p^*h for some rational function on $\mathbb{P}^1$. It is a simple matter to verify that $\nabla_\alpha(h) = \beta$, and therefore β represents the zero class in $H^1(\mathbb{P}^1 \setminus \mathsf{P}_\Sigma, \mathbb{C}_\alpha)$. But the main result of [13] implies that the complex $(\Omega^\bullet(\mathbb{P}^1 \setminus \mathsf{P}_\Sigma), \nabla_\alpha)$ is quasi-isomorphhic to the complex $(H^\bullet(\mathbb{P}^1 \setminus \mathsf{P}_\sigma, \mathbb{C}), \wedge\alpha)$. Hence the class of β is not zero. Q.E.D.

Proposition 5.3. *Let A be an arrangement of hyperplanes on $\mathbb{P}^n$ and let $\Sigma_1, \dots, \Sigma_r$ be irreducible components of the resonance variety $R^1(A)$. Suppose $\exp(\Sigma_1), \dots, \exp(\Sigma_r)$ intersect at some $\rho \in \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ distinct from the trivial representation. If $\overline{N}$, the projective closure of the finite covering of M determined by ρ, is simply-connected then*

$$\dim \frac{\mathcal{A}(\mathcal{W}(A))}{\mathrm{Log}^\infty \mathcal{W}(A) \oplus \mathrm{Rat}\mathcal{W}(A)} \geq -h^1(M, \mathbb{C}_\omega) + \sum_{i=1}^{r} (\dim \Sigma_i - 1)\,.$$

Proof. Since ω is in the intersection of $\exp(\Sigma_1), \dots, \exp(\Sigma_r)$ there exist non-zero logarithmic 1-forms $\alpha_i \in \Sigma_i$ $(i = 1, \dots, r)$ and rational functions $f_{ij} \in \mathcal{O}(M)$ $(i, j = 1, \dots, r)$ satisfying

$$\alpha_i - \alpha_j = d \log f_{ij}\,.$$

In particular, $\mathbb{C}_{\alpha_i}$ is isomorphic to $\mathbb{C}_{\alpha_j}$ for every i, j. It is harmless to assume that all the 1-forms α_i have non-integer residues and $\omega = \alpha_1$.

Since $\Sigma_i \subset \ker \nabla_{\alpha_i}$ it follows from (9) that $(f_{ij})^{-1} \cdot \Sigma_j \in \ker \nabla_{\alpha_i}$ for every i, j. Therefore we can define the map

$$\Lambda : \bigoplus_{i=1}^{r} \frac{\Sigma_i}{\mathbb{C} \cdot \alpha_i} \longrightarrow H^1(M, \mathbb{C}_{\alpha_1})$$

$$(\beta_i) \mapsto \beta_1 + \sum_{i=2}^{r} (f_{1i})^{-1} \beta_i \, .$$

Notice that as we explained before the (multi-valued) 1-forms

$$\exp\left(-\int \alpha_1\right) (f_{1i})^{-1} \beta_i = \exp\left(-\int \alpha_i\right) \beta_i$$

are closed.

If $(\beta_1, \ldots, \beta_r)$ belongs to $\ker \Lambda$ then there exists $g \in H^0(M, \mathcal{O}_M)$ for which

$$\beta_1 + \sum_{i=2}^{r} (f_{1i})^{-1} \beta_i = dg - g\alpha_1$$

or, equivalently, for suitable choices of branches of $\exp\left(\int -\alpha_i\right)$ we have that

$$\sum_{i=1}^{r} \exp\left(\int -\alpha_1\right) (f_{1i})^{-1} \beta_i = d\left(\exp\left(\int \alpha_1\right) g\right)$$

If we pull-back this equation to N using the finite covering $P : N \to M$ then all the 1-forms involved are legitime (univalued) closed rational 1-forms. Since $\overline{N}$ is simply-connected, the pull-backs $\widetilde{f_\Sigma}$ of the maps f_Σ have as target rational curves. Thus the 1-forms $\exp\left(\int -\alpha_1\right) (f_{1i})^{-1} \beta_i$ can be uniquely written as the pull-back by $\widetilde{f_\Sigma}$ of the sum of an exact rational differential with an logarithmic 1-form. Discarding the rational component, one obtains an identity as above but with zero right-hand side. Clearly it is an abelian relation. The linear independence of the corresponding abelian relations for $(\beta_1, \ldots, \beta_r)$ varying on a basis of $\ker \Lambda$ follows from Lemma 5.2. Q.E.D.

We do not know if the hypothesis made on the topology of $\overline{N}$ is necessary to prove the proposition above.

§

Back to the analysis of Example 5.4, we have that the exponential of the three non-local components intersect at a representation ρ for which $h^1(M, \mathbb{C}_\rho) = 2$. Moreover, ρ satisfies the hypothesis of Proposition 5.3 as it is non-trivial only along two fibers of the corresponding rational maps

f_Σ, see [11, Example 10.5]. Thus Proposition 5.3 ensures the existence of the *missing* abelian relation of $\mathcal{W}(\mathcal{F})$.

5.5. Final remarks

In the table below we present the dimensions of the subspaces of the space of abelian relations of resonance webs of line arrangements studied in this paper.

	k	Log^1	$\overline{\mathrm{Log}^1}$	Log^2	$\overline{\mathrm{Log}^2}$	Log^3	$\overline{\mathrm{Log}^3}$	Rat	Mixed	Twisted
$\mathsf{A}_{0,5}$	5	5	5	1	1	0	0	0	0	0
B_6	6	6	6	2	2	0	0	2	0	0
B_7	7	7	7	3	3	0	0	5	0	0
B_8	8	8	8	4	4	0	0	9	0	0
P	8	11	11	5	5	0	0	4	1	0
F	9	12	12	9	9	2	2	4	0	1
K_5	10	16	20	5	15	0	1	0	0	0

Although we have studied resonance webs for hyperplane arrangements one can study resonance webs for arbitrary hypersurfaces arrangements on $\mathbb{P}^n$. Even more generally, if the cohomology algebra $H^\bullet(M)$ is replaced by a finite dimensional algebra of differential forms on a quasi-projective variety then one can still talk about resonance varieties. Its irreducible components are still in correspondence with rational maps to projective curves [3], and consequently one can still define the resonance webs. We are not aware of any exceptional web arising this way that are not pull-backs by rational maps of resonance webs of hyperplane arrangements.

Acknowledgements. This paper would not come to light without the warm reception given to my talk at the 2nd MSJ-SI Arrangements of Hyperplanes. I would like to thank the organizers for the opportunity to participate in this meeting. I am also indebt to Luc Pirio who, through numerous discussions, contributed a lot to this paper. Among other things, he suggested that the resonance webs of $\mathsf{A}_{0,n+3}$ should be looked at, gave a computer-assisted proof of Proposition 4.1 in the case $n = 3$, and brought reference [6] to my knowledge.

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