Perverse coherent sheaves on blow-up. I.
A quiver description

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§ Introduction

This is the first of three papers studying moduli spaces of a certain class of coherent sheaves, which we call *stable perverse coherent sheaves*, on the blowup of a projective surface. They are used to relate usual moduli spaces of stable sheaves on a surface and its blowup.

Let us give the definition for general case though we will consider only framed sheaves on the blowup $\widehat{\mathbb{P}}^2$ of the projective plane $\mathbb{P}^2$ in this paper. Let $p\colon \widehat{X} \to X$ be the blowup of a smooth projective surface X at a point. Let C be the exceptional divisor. A *stable perverse coherent sheaf* E on $\widehat{X}$, with respect to an ample line bundle H on X, is

   (1) E is a coherent sheaf on $\widehat{X}$,
   (2) $\mathrm{Hom}(E(-C), \mathcal{O}_C) = 0$,
   (3) $\mathrm{Hom}(\mathcal{O}_C, E) = 0$,
   (4) $p_* E$ is slope stable with respect to H.

We will consider the moduli spaces of coherent sheaves E on $\widehat{X}$ such that $E(-mC)$ is stable perverse coherent for $m \geq 0$. This depends on m.

Let us first explain how we find the definition of stable perverse coherent sheaves. Our definition comes from two sources. The first one is Bridgeland's paper [1], from which we take the name "perverse coherent" sheaves. He introduced a new t-structure on the derived category $\mathrm{D}^b(\mathrm{Coh}\, Y)$ of coherent sheaves on Y for a birational morphism $f\colon Y \to X$ such that (1) $\mathbf{R}f_*(\mathcal{O}_Y) = \mathcal{O}_X$ and (2) $\dim f^{-1}(x) \leq 1$ for any $x \in X$. An object $E \in \mathrm{D}^b(\mathrm{Coh}\, Y)$ is called *perverse coherent* if it is in the heart $\mathrm{Per}(Y/X)$ of the t-structure. Then he considered the moduli spaces of perverse ideal sheaves which are subobjects of the structure

sheaf $\mathcal{O}_Y$ in the heart. His interest was in the case when $\dim Y = 3$ (and to describe the flop of $f \colon Y \to X$ as moduli space of perverse ideal sheaves). But his definition of the t-structure makes sense for the blowup $p \colon \widehat{X} \to X$ of a smooth quasi-projective surface X at a point. Then we get conditions similar as above, but we further impose the stability condition (4), then it implies that E is a coherent sheaf, not an object in the derived category. (This will be explained in the subsequent paper.)

The second source, which is explained in detail in this paper, is the quiver description of the framed moduli spaces of locally free sheaves on the blowup $\widehat{\mathbb{P}}^2$ of the projective plane due to King [6], which is based on an earlier work by Buchdahl [3]. Here the framing is a trivialization of the sheaf at the line at infinity. He described the relevant moduli spaces as GIT (geometric invariant theory) quotients of representations of a certain quiver with a relation. (See §1 for a precise statement.) The groups taking quotients are the products of two copies of general linear groups of possibly different sizes. A particular linearization was used to define the GIT quotients in the original paper, but we can consider more general linearization. Then we will show that more general GIT quotients parametrize framed perverse coherent sheaves (after twisting by $\mathcal{O}(-mC)$), where we impose conditions (1)~(3) above, but we replace (4) by

(4)' E is torsion free outside C.

This is because the stability is implicit in the existence of the framing. Note also that (3) is equivalent to the torsion freeness of p_*E at the point which we blowup (see the proof of Lemma 6.1). Therefore (3) and (4)' are combined to the condition that p_*E is torsion free. But we keep them separated, as (3) is be altered by a twist by $\mathcal{O}(C)$ while (4)' is not.

From either view point, we have a chamber structure on the stability parameter. In the first view point, it is the choice of m. In the second, we have a parameter $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2$. We are interested in the domain $\zeta_0 + \zeta_1 < 0$, $\zeta_0 < 0$ in this paper. This domain is separated by walls defined by $m\zeta_0 + (m+1)\zeta_1 = 0$ for $0 \le m \le N$ for some N depending on Chern classes of sheaves. (See Figure 1.)

The wall $m\zeta_0 + (m+1)\zeta_1 = 0$ corresponds to the sheaf $\mathcal{O}_C(-m-1)$. When we cross the wall, sheaves given by extensions $0 \to E' \to E^+ \to \mathcal{O}_C(-m-1)^{\oplus p} \to 0$ are replaced by extensions in the opposite way $0 \to \mathcal{O}_C(-m-1)^{\oplus p} \to E^- \to E' \to 0$. This is very similar to the wall-crossing phenomenon appearing when we change polarization in the moduli spaces of sheaves. We study this in detail in the subsequent paper.
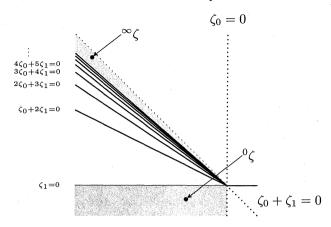
Fig. 1. Chamber structure

There are two distinguished chambers in our domain, the gray regions in the figure. The chamber adjacent to the plane $\zeta_0 + \zeta_1 = 0$ (containing the parameter $^\infty\zeta$ in the figure) gives moduli spaces of usual framed sheaves (and $(p^*H - \varepsilon C)$-stable sheaves for an ample line bundle H on X and sufficiently small ε for the projective case). King's stability condition, corresponding to framed *locally free* sheaves, lies on the plane $\zeta_0 + \zeta_1 = 0$. On the other hand, the chamber $\zeta_0 < 0$, $\zeta_1 < 0$ (containing the parameter $^0\zeta$ in the figure) gives moduli spaces of framed perverse coherent sheaves. We will show that these moduli spaces are isomorphic to those of framed sheaves on $\mathbb{P}^2$, instead of $\widehat{\mathbb{P}}^2$ (and H-stable sheaves on X for the projective case) if the first Chern class vanishes in §7.2. (We have a similar statement for general first Chern classes.) The simplest (nontrivial) case of this statement says that the framed moduli space of perverse ideal sheaves on $\widehat{\mathbb{P}}^2$ of colength 1 is isomorphic to $\mathbb{C}^2$, which is the framed moduli space of ideal sheaves on $\mathbb{P}^2$. This statement is natural in view of Bridgeland's result [1]. In the 3-dimensional case $f: Y \to X$, the moduli space of perverse ideal sheaves of colength 1 is the flop of Y. However the derived categories $\mathrm{D}^b(\mathrm{Coh}\, X)$ and $\mathrm{D}^b(\mathrm{Coh}\, \widehat{X})$, are *not* equivalent like the flop case, so our analogy breaks at this point.

Our result should have generalization to various other situations. For example, we can consider a similar problem for the minimal resolution $p: Y \to X$ where X has a quotient singularity of the form $\mathbb{C}^2/\Gamma$ for a finite subgroup Γ of $\mathrm{GL}_2(\mathbb{C})$. In particular, when Γ is a subgroup of $\mathrm{SL}_2(\mathbb{C})$, there is a quiver description of framed sheaves on the minimal resolution of $\mathbb{C}^2/\Gamma$ ([8, 14]). We have a chamber structure on the stability condition, where the space of parameters is identified with the Cartan

subalgebra of the affine Lie algebra corresponding to Γ, and walls are root hyperplanes. Moduli spaces parametrize perverse coherent sheaves twisted by line bundles, as in this paper. We again have two distinguished chambers. As in this paper, $^{\infty}\zeta$ gives the moduli of framed torsion free sheaves, while $^{0}\zeta$ gives the moduli of framed Γ-equivariant torsion free sheaves on $\mathbb{C}^2$. These moduli spaces are studied, in the name of *quiver varieties*, by the first named author [10, 12]. When sheaves have rank 1, moduli spaces of perverse coherent sheaves were constructed and studied by Toda [20]. However there is a *crucial difference* in the wall-crossing. In this case, all moduli spaces are diffeomorphic to each other, as proved in [10]. This is not true in our case, as framed moduli spaces of sheaves on $\widehat{\mathbb{P}}^2$ and $\mathbb{P}^2$ have different Betti numbers. (See [16].) If we see Figure 1, it looks like the Cartan subalgebra for the affine $\mathfrak{sl}_2$ with root hyperplanes. However if we draw all walls without imposing the condition $\zeta_0 + \zeta_1 < 0$, $\zeta_0 < 0$, then it becomes different. (See Remark 2.17.) It should be also noticed that $\mathcal{O}_C(-m-1)$ is an exceptional object (i.e., has no higher self-extensions) while the corresponding objects in the $\mathrm{SL}_2(\mathbb{C})$-setting are spherical objects. If Γ is not contained in $\mathrm{SL}_2(\mathbb{C})$, there should be difference from $\mathrm{SL}_2(\mathbb{C})$ cases, but we do not understand the picture so far.

It is also interesting to study the problem in Bridgeland's 3-dimensional case [1], and also in more general situation [2], when we could expect an explicit description of all *walls* as in this paper.

Our motivation to this research comes from the authors' study on the 'instanton counting'. (See [15, 16, 17] and the references therein.) An understanding of relations between moduli spaces of sheaves on X and $\widehat{X}$ was one of the most essential ingredient there. An application of our result to the instanton counting will be discussed in a separate paper.

Notations

Let $[z_0 : z_1 : z_2]$ be the homogeneous coordinates on $\mathbb{P}^2$ and $\ell_{\infty} = \{z_0 = 0\}$ the line at infinity. Let $p \colon \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at $[1 : 0 : 0]$. Then $\widehat{\mathbb{P}}^2$ is the closed subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ defined by

$$\left\{ ([z_0 : z_1 : z_2], [z : w]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid z_1 w = z_2 z \right\},$$

where the map p is the projection to the first factor. We denote $p^{-1}(\ell_\infty)$ also by ℓ_∞ for brevity. Let C denote the exceptional divisor given by $z_1 = z_2 = 0$. Let $\mathcal{O}$ denote the structure sheaf of $\widehat{\mathbb{P}}^2$, $\mathcal{O}(C)$ the line bundle associated with the divisor C, and $\mathcal{O}(mC)$ its mth tensor product $\mathcal{O}(C)^{\otimes m}$ when $m > 0$, $(\mathcal{O}(C)^{\otimes -m})^\vee$ if $m < 0$, and $\mathcal{O}$ if $m = 0$. And we use the similar notion $\mathcal{O}(mC + n\ell_\infty)$ for tensor products of $\mathcal{O}(mC)$ and tensor powers of the line bundle corresponding to ℓ_∞ or its dual.

The structure sheaf of the exceptional divisor C is denoted by $\mathcal{O}_C$. If we twist it by the line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$ over $C \cong \mathbb{P}^1$, we denote the resulted sheaf by $\mathcal{O}_C(n)$. Since C has the self-intersection number -1, we have $\mathcal{O}_C \otimes \mathcal{O}(C) = \mathcal{O}_C(-1)$.

§1. ADHM description—Main result

1.1. Preliminary

Let $M(r,n)$ be the framed moduli space of torsion free sheaves E on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$ with rank $E = r > 0$, $c_2(E)[\mathbb{P}^2] = n \geq 0$, where E is assumed to be locally free along ℓ_∞ and the framing is a trivialization on the line at infinity $\Phi \colon E|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}^{\oplus r}$. The ADHM description identifies $M(r,n)$ with the space of the following data $X = (B_1, B_2, i, j)$ defined for vector spaces V, W with $\dim V = n$, $\dim W = r$ modulo the action of $\mathrm{GL}(V)$:

- $B_1, B_2 \in \mathrm{End}(V)$, $i \in \mathrm{Hom}(W, V)$, $j \in \mathrm{Hom}(V, W)$,
- $[B_1, B_2] + ij = 0$,
- (stability condition)
  a subspace $T \subset V$ with $B_\alpha(T) \subset T$ ($\alpha = 1, 2$), $\mathrm{Im}\, i \subset T$ must be $T = V$.

The quotient space of data by the action by $\mathrm{GL}(V)$ can be understand as the GIT (geometric invariant theory) quotient with respect to the trivial line bundle with the $\mathrm{GL}(V)$-action given by $\det \colon \mathrm{GL}(V) \to \mathbb{C}^*$. The above stability condition is nothing but one in GIT. If we do not impose the last condition, but take the affine algebro-geometric quotient $/\!\!/ \, \mathrm{GL}(V)$, we get the Uhlenbeck (partial) compactification

$$M_0(r,n) = \bigsqcup_{m=0}^{n} M_0^{\mathrm{reg}}(r, n-m) \times S^m \mathbb{C}^2,$$

where $M_0^{\mathrm{reg}}(r, n-m)$ is the framed moduli space of locally free sheaves, which is identified with the data $X = (B_1, B_2, i, j)$ satisfying the above conditions together with

- (co-stability condition)
  a subspace $S \subset V$ with $B_\alpha(S) \subset S$ $(\alpha = 1, 2)$, $\operatorname{Ker} j \supset S$ must be $S = 0$.
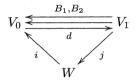
(See [13, Chap. 2] for detail.)

There is an ADHM type description of a framed moduli space $\widehat{M}_0^{\mathrm{reg}}(r, k, n)$ of locally free sheaves E on the blowup $\widehat{\mathbb{P}}^2$ for rank $E = r > 0$, $(c_1(E), [C]) = -k$, $(c_2(E) - (r-1)c_1(E)^2/(2r), [\widehat{\mathbb{P}}^2]) = n \geq 0$ due to King [6]. We take vector spaces V_0, V_1, W with

$$r = \dim W = r, \quad k = -\dim V_0 + \dim V_1, \quad n + \frac{k^2}{2r} = \frac{1}{2}(\dim V_0 + \dim V_1).$$

Theorem 1.1 ([6]). *The framed moduli space $\widehat{M}_0^{\mathrm{reg}}(r, k, n)$ is bijective to the space of the following data $X = (B_1, B_2, d, i, j)$ modulo the action of $\mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$:*

- $B_1, B_2 \in \operatorname{Hom}(V_1, V_0)$, $d \in \operatorname{Hom}(V_0, V_1)$, $i \in \operatorname{Hom}(W, V_0)$, $j \in \operatorname{Hom}(V_1, W)$,

- $\mu(B_1, B_2, d, i, j) = B_1 d B_2 - B_2 d B_1 + ij = 0$,
- *(stability condition)*
  - **(S1):** *for subspaces $S_0 \subset V_0$, $S_1 \subset V_1$ such that $B_\alpha(S_1) \subset S_0$ $(\alpha = 1, 2)$, $d(S_0) \subset S_1$, $\operatorname{Ker} j \supset S_1$, we have $\dim S_0 > \dim S_1$ or $S_0 = S_1 = 0$.*
  - **(S2):** *for subspaces $T_0 \subset V_0$, $T_1 \subset V_1$ such that $B_\alpha(T_1) \subset T_0$ $(\alpha = 1, 2)$, $d(T_0) \subset T_1$, $\operatorname{Im} i \subset T_0$, we have $\operatorname{codim} T_1 > \operatorname{codim} T_0$ or $(T_0, T_1) = (V_0, V_1)$.*

The quotient space can be again understood as the GIT quotient with respect to the trivial line bundle with the $\mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$-action given by $\det_{\mathrm{GL}(V_1)} / \det_{\mathrm{GL}(V_0)}$. See [7, (3.1)], [11, (6.2)] for the equivalence between the usual GIT-stability with respect to the line bundle and the above stability condition in the context of quiver representations.

A framed locally-free sheaf (E, Φ) is constructed from X as follows. Let us consider coordinates z_0, z_1, z_2 (resp. z, w) as sections of $\mathcal{O}(\ell_\infty)$ (resp. $\mathcal{O}(-C + \ell_\infty)$). The ratio $s := z_1/z = z_2/w$ is a section of $\mathcal{O}(C)$ which vanishes on C. Then the locally free sheaf E corresponding to

(B_1, B_2, d, i, j) is the middle cohomology of the complex
(1.2)

$$
\begin{array}{ccccc}
 & & \mathbb{C}^2 \otimes V_0 \otimes \mathcal{O} & & \\
V_0 \otimes \mathcal{O}(C - \ell_\infty) & & \oplus & & V_0 \otimes \mathcal{O}(\ell_\infty) \\
\oplus & \xrightarrow{\ \alpha\ } & \mathbb{C}^2 \otimes V_1 \otimes \mathcal{O} & \xrightarrow{\ \beta\ } & \oplus \\
V_1 \otimes \mathcal{O}(-\ell_\infty) & & \oplus & & V_1 \otimes \mathcal{O}(-C + \ell_\infty) \\
 & & W \otimes \mathcal{O} & &
\end{array},
$$

with

$$
\alpha = \begin{bmatrix} z & z_0 B_1 \\ w & z_0 B_2 \\ 0 & z_1 - z_0 dB_1 \\ 0 & z_2 - z_0 dB_2 \\ 0 & z_0 j \end{bmatrix}, \qquad
\beta = \begin{bmatrix} z_2 & -z_1 & B_2 z_0 & -B_1 z_0 & i z_0 \\ dw & -dz & w & -z & 0 \end{bmatrix}.
$$

The equation $\mu(B_1, B_2, d, i, j) = B_1 dB_2 - B_2 dB_1 + ij = 0$ is equivalent to $\beta\alpha = 0$. The stability condition (S2) is equivalent to the surjectivity of β. The stability condition (S1) is equivalent to that α is injective and the image is a subbundle. (See [6, 4.1.3] and also Lemma 5.1.) Therefore $E = \operatorname{Ker}\beta / \operatorname{Im}\alpha$ is a locally free sheaf as expected.

Let $\widehat{\mathbb{C}}^2$ be the blowup of $\mathbb{C}^2$ at the origin. The Uhlenbeck (partial) compactification $\widehat{M}_0(r, k, n)$, which is defined as a set by

$$
\widehat{M}_0(r, k, n) = \bigsqcup_{m=0}^{[n]} \widehat{M}_0^{\mathrm{reg}}(r, k, n - m) \times S^m \widehat{\mathbb{C}}^2
$$

is identified with the GIT quotient associated with the line bundle associated with the above homomorphism. Namely we replace $>$ by $\geq$ in the two inequalities in the stability condition, and then divide by the S-equivalence relation (see [7, 2.3], [11, 2.3]).

We can generalize this description to the case when $E = \operatorname{Ker}\beta / \operatorname{Im}\alpha$ is only assumed to be *torsion free*. The corresponding generalization for $\mathbb{P}^2$ was given in [13, Chap. 2], and it is given by allowing α to be injective possibly except finitely many points. This is equivalent to that α is injective as a sheaf homomorphism and $\operatorname{Ker}\beta / \operatorname{Im}\alpha$ is torsion free. The same works in the blowup case. (The proof of this statement, as well as that of Theorem 1.1, will be given in this paper in more general setting.) The corresponding equivariant line bundle for the GIT stability is given by the homomorphism $\det_{\mathrm{GL}(V_1)}^N / \det_{\mathrm{GL}(V_0)}^{N+1}$ for sufficiently large N. Then the condition (S2) is unchanged, but we replace $>$ in (S1) by $\geq$. Let us denote this condition by (S1)'.

This observation naturally leads to consider more general stability conditions. Let $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2$.

Definition 1.3. Suppose $\dim W = r \neq 0$ as above. We say $X = (B_1, B_2, d, i, j)$ is ζ-*semistable* if

(1) for subspaces $S_0 \subset V_0$, $S_1 \subset V_1$ such that $B_\alpha(S_1) \subset S_0$ ($\alpha = 1, 2$), $d(S_0) \subset S_1$, $\operatorname{Ker} j \supset S_1$, we have $\zeta_0 \dim S_0 + \zeta_1 \dim S_1 \leq 0$.

(2) for subspaces $T_0 \subset V_0$, $T_1 \subset V_1$ such that $B_\alpha(T_1) \subset T_0$ ($\alpha = 1, 2$), $d(T_0) \subset T_1$, $\operatorname{Im} i \subset T_0$, we have $\zeta_0 \operatorname{codim} T_0 + \zeta_1 \operatorname{codim} T_1 \geq 0$.

We say X is ζ-*stable* if the inequalities are strict unless $(S_0, S_1) = (0, 0)$ and $(T_0, T_1) = (V_0, V_1)$ respectively.

Thus the condition (S1),(S2) is equivalent to ζ-stability with $\zeta_0 + \zeta_1 = 0$, $\zeta_0 < 0$. In Figure 1 it is a parameter on the line $\zeta_0 + \zeta_1 = 0$ which is on the boundary on the domain $\zeta_0 + \zeta_1 < 0$, $\zeta_0 < 0$ we are interested in this paper.

This stability condition come from a $\mathbb{Q}$-line bundle with an $\operatorname{GL}(V_0) \times \operatorname{GL}(V_1)$-action if $(\zeta_0, \zeta_1) \in \mathbb{Q}^2$, i.e., the trivial line bundle with the equivariant structure given by the $\mathbb{Q}$-homomorphism $\det_{\operatorname{GL}(V_0)}^{\zeta_0} \det_{\operatorname{GL}(V_1)}^{\zeta_1}$. As we can move the parameter in a chamber (see §4.3) without changing the stability condition, the condition $(\zeta_0, \zeta_1) \in \mathbb{Q}^2$ is not essential.

We consider two types of quotient spaces:

$$\widehat{M}_\zeta^{\mathrm{s}}(r, k, n) := \left\{ (B_1, B_2, d, i, j) \in \mu^{-1}(0) \,\middle|\, \zeta\text{-stable} \right\} / \operatorname{GL}(V_0) \times \operatorname{GL}(V_1),$$

$$\widehat{M}_\zeta^{\mathrm{ss}}(r, k, n) := \left\{ (B_1, B_2, d, i, j) \in \mu^{-1}(0) \,\middle|\, \zeta\text{-semistable} \right\} / \sim,$$

where $\sim$ denotes the S-equivalence relation. (Again, see [7, 2.3], [11, 2.3] more detail.)

1.2. The statement

We fix $m \in \mathbb{Z}_{\geq 0}$ and assume further

(1.4) $\zeta_0 < 0, \qquad 0 < -(m\zeta_0 + (m+1)\zeta_1) \ll 1.$

In §4.3 we will prove that this condition specifies a chamber, where the ζ-stability and ζ-semistability is equivalent. In Figure 1, the point is in the chamber just below the line $m\zeta_0 + (m+1)\zeta_1 = 0$.

The following is the main result of this paper.

Theorem 1.5. *The space* $\widehat{M}_\zeta^{\mathrm{s}}(r, k, n) = \widehat{M}_\zeta^{\mathrm{ss}}(r, k, n)$ *is bijective to the space of isomorphism classes of framed sheaves* (E, Φ) *on* $\widehat{\mathbb{P}}^2$ *such that* $E(-mC)$ *is perverse coherent, i.e., it satisfies*

(1) $\mathrm{Hom}(E(-mC), \mathcal{O}_C(-1)) = 0$,

(2) $\mathrm{Hom}(\mathcal{O}_C, E(-mC)) = 0$,

(3) $E(-mC)$ *is torsion free outside* C.

A construction of moduli spaces of stable perverse coherent sheaves will be given in the subsequent paper. It can be adapted to a construction of the framed moduli spaces. However our proof also gives an universal family on the space of ζ-stable data, so our result can be also considered as a (different) construction of the fine moduli spaces of framed stable perverse coherent sheaves.

1.3. Morphisms between moduli spaces

Let us consider the affine algebro-geometric quotient $\mu^{-1}(0) /\!\!/ \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$. We have a morphism $\mu^{-1}(0) /\!\!/ \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \to M_0(r, n)$ ($n = \min(\dim V_0, \dim V_1)$) given by

$$[(B_1, B_2, d, i, j)] \longmapsto [(dB_1, dB_2, di, j)] \text{ or } [(B_1 d, B_2 d, i, jd)].$$

The coordinate ring of $\mu^{-1}(0) /\!\!/ \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$ is generated by $\mathrm{tr}(X)$ and $\langle jXi, \chi \rangle$, where X are compositions of B_1, B_2 and d in various orders (so that $X \colon V_\alpha \to V_\alpha$ in the first case, $X \colon V_0 \to V_1$ in the second case), and $\chi \in W^*$. The same statement holds for $M_0(r, n)$ [9]. Therefore the morphism is a closed embedding. We have the inverse map given by $[(B_1', B_2', i', j')] \mapsto [(B_1', B_2', d = \mathrm{id}, i', j')]$ composed with the extension to V_0 or V_1 by 0. Therefore $\mu^{-1}(0) /\!\!/ \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \cong M_0(r, n)$.

Composing the natural projective morphism $\widehat{M}^{\mathrm{s}}_\zeta(r, k, n) \to \mu^{-1}(0) /\!\!/ \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$, we have a morphism

$$\widehat{M}^{\mathrm{s}}_\zeta(r, k, n) \to M_0(r, \min(\dim V_0, \dim V_1)).$$

When the parameter ζ corresponds to the framed moduli space of torsion free sheaves ($^\infty\zeta$ in Figure 1), the morphism is one constructed in [16, Th.3.3]. It factors through the Uhlenbeck (partial) compactification $\widehat{M}_0(r, k, n)$, which corresponds to a parameter on the line $\zeta_0 + \zeta_1 = 0$.

§2. Moduli spaces

In this paper we assume $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2$ satisfies

(2.1) $$\zeta_0 + \zeta_1 < 0, \quad \zeta_0 < 0$$

except in §2.3

2.1.   Smoothness of the moduli space

We show that $\widehat{M^s_\zeta}(r, k, n)$ is smooth in this subsection.

Lemma 2.2. *Under the assumption* (2.1), *ζ-semistability implies the condition* (S2).

Proof. As $\zeta_1 < -\zeta_0$, we have

$$0 \le \zeta_0 \operatorname{codim} T_0 + \zeta_1 \operatorname{codim} T_1 \le \zeta_0(\operatorname{codim} T_0 - \operatorname{codim} T_1).$$

As $\zeta_0 < 0$, this implies $\operatorname{codim} T_0 \le \operatorname{codim} T_1$. If the equality holds, we have $\operatorname{codim} T_1 = 0$ from the second inequality. Therefore $(T_0, T_1) = (V_0, V_1)$.   Q.E.D.

Lemma 2.3. *Suppose that* (B_1, B_2, d, i, j) *satisfies* (S2). *Then* $V_0 = \operatorname{Im} B_1 + \operatorname{Im} B_2 + \operatorname{Im} i$. *The same holds if* (B_1, B_2, d, i, j) *is ζ-semistable thanks to Lemma 2.2.*

Proof. Take $T_0 = \operatorname{Im} B_1 + \operatorname{Im} B_2 + \operatorname{Im} i$, $T_1 = V_1$. Then they satisfy the assumption in (S2). As $\operatorname{codim} T_1 = 0$, we must have $\operatorname{codim} T_0 = 0$, i.e., $T_0 = V_0$.   Q.E.D.

Recall $\mu(B_1, B_2, d, i, j) = B_1 dB_2 - B_2 dB_1 + ij$.

Lemma 2.4. *Suppose* (B_1, B_2, d, i, j) *is ζ-stable.*
(1) *The stabilizer of* (B_1, B_2, d, i, j) *in* $\operatorname{GL}(V_0) \times \operatorname{GL}(V_1)$ *is trivial.*
(2) *The differential $d\mu$ of μ at* (B_1, B_2, d, i, j) *is surjective.*

Proof. (1) Suppose that the pair $(g_0, g_1) \in \operatorname{GL}(V_0) \times \operatorname{GL}(V_1)$ stabilizes (B_1, B_2, d, i, j). Then $S_0 = \operatorname{Im}(g_0 - \operatorname{id}_{V_0})$, $S_1 = \operatorname{Im}(g_1 - \operatorname{id}_{V_1})$ satisfies the assumption in (1) of the ζ-stability condition. Therefore we have $\zeta_0 \dim S_0 + \zeta_1 \dim S_1 \le 0$. Similarly $T_0 = \operatorname{Ker}(g_0 - \operatorname{id}_{V_0})$, $T_1 = \operatorname{Ker}(g_1 - \operatorname{id}_{V_1})$ satisfy $\zeta_0 \operatorname{codim} T_0 + \zeta_1 \operatorname{codim} T_1 \ge 0$. But we have $\operatorname{codim} T_0 = \dim S_0$, $\operatorname{codim} T_1 = \dim S_1$, so both the inequalities must be equalities. Therefore we have $S_0 = 0$, $S_1 = 0$. It means that $g_0 = \operatorname{id}_{V_0}$, $g_1 = \operatorname{id}_{V_1}$.
(2) Suppose $C \in \operatorname{Hom}(V_0, V_1)$ is orthogonal to the image of $d\mu$, with respect to the pairing given by the trace. Then we have

$$CB_\alpha d = dB_\alpha C \ (\alpha = 1, 2), \quad B_1 CB_2 = B_2 CB_1, \quad jC = 0, \quad Ci = 0.$$

Let us consider $S_0 = \operatorname{Im}(B_1 C)$, $S_1 = \operatorname{Im}(CB_1)$, $T_0 = \operatorname{Ker}(B_1 C)$, $T_1 = \operatorname{Ker}(CB_1)$. By the same argument as in (1), the ζ-stability implies $S_0 = S_1 = 0$, that is

$$B_1 C = 0, \quad CB_1 = 0.$$

Exchanging B_1 and B_2, we also have $B_2 C = 0$, $CB_2 = 0$. Then C is equal to 0 on $\operatorname{Im} B_1 + \operatorname{Im} B_2 + \operatorname{Im} i$. Then by Lemma 2.3 we have $C = 0$.   Q.E.D.

By the standard argument from the geometric invariant theory, we have

Theorem 2.5. $\widehat{M}^s_\zeta(r, k, n)$ *is nonsingular of dimension*

$$\dim W(\dim V_0 + \dim V_1) - (\dim V_0 - \dim V_1)^2 = 2nr,$$

provided $\widehat{M}^s_\zeta(r, k, n) \neq \emptyset$. *Moreover,* $\widehat{M}^s_\zeta(r, k, n)$ *is the fine moduli space of* ζ-*stable* $\operatorname{GL}(V_0) \times \operatorname{GL}(V_1)$-*orbits.*

For a later purpose (in the subsequent paper) let us describe the tangent space: it is the middle cohomology of

$$
(2.6) \quad
\begin{matrix}
& & \operatorname{Hom}(V_0, V_1) & & \\
& & \oplus & & \\
\operatorname{Hom}(V_0, V_0) & & \mathbb{C}^2 \otimes \operatorname{Hom}(V_1, V_0) & & \\
\oplus & \xrightarrow{\ \iota\ } & \oplus & \xrightarrow{\ d\mu\ } & \operatorname{Hom}(V_1, V_0), \\
\operatorname{Hom}(V_1, V_1) & & \operatorname{Hom}(W, V_0) & & \\
& & \oplus & & \\
& & \operatorname{Hom}(V_1, W) & &
\end{matrix}
$$

with

$$
\iota \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} d\xi_0 - \xi_1 d \\ B_1 \xi_1 - \xi_0 B_1 \\ B_2 \xi_1 - \xi_0 B_2 \\ \xi_0 i \\ -j\xi_1 \end{bmatrix}, (d\mu) \begin{bmatrix} \widetilde{d} \\ \widetilde{B}_1 \\ \widetilde{B}_2 \\ \widetilde{i} \\ \widetilde{j} \end{bmatrix} = \begin{matrix} B_1 d\widetilde{B}_2 + B_1 \widetilde{d} B_2 + \widetilde{B}_1 dB_2 \\ - B_2 d\widetilde{B}_1 - B_2 \widetilde{d} B_1 - \widetilde{B}_2 dB_1 \\ + \widetilde{i} j + i\widetilde{j}, \end{matrix}
$$

where $d\mu$ is the differential of μ, and ι is the differential of the group action. Remark that $d\mu$ is surjective and ι is injective by the above lemma.

2.2. The ζ-stable point with $W = 0$

For a later purpose we study the case $W = 0$ in this subsection. We assume (2.1) and

$$(2.7) \qquad\qquad \zeta_0 \dim V_0 + \zeta_1 \dim V_1 = 0.$$

Definition 2.8. Suppose $W = 0$ and $\zeta_0 \dim V_0 + \zeta_1 \dim V_1 = 0$. We say $X = (B_1, B_2, d)$ as above is ζ-*semistable* if the condition (1)

in Definition 1.3 holds. (Note that the condition (2) is equivalent to (1) when $W = 0$). And the ζ-*stability* is defined by requiring that the inequality is strict unless $(S_0, S_1) = (0, 0)$ or (V_0, V_1).

We consider the quotient space

$$\left\{ (B_1, B_2, d) \, \middle| \, \begin{array}{l} B_1 dB_2 - B_2 dB_1 = 0 \\ \zeta\text{-stable} \end{array} \right\} \bigg/ \left(\mathrm{GL}(V_0) \times \mathrm{GL}(V_1) / \mathbb{C}^* \right),$$

where $\mathbb{C}^* = \{ (\lambda \, \mathrm{id}_{V_0}, \lambda \, \mathrm{id}_{V_1}) \in \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \mid \lambda \in \mathbb{C}^* \}$. As it acts trivially on the data, the quotient group acts on the space.

Lemma 2.9. *Suppose* (B_1, B_2, d) *is* ζ-*stable. Then* $d = 0$.

Proof. Consider $S_0 = \mathrm{Ker}(B_1 d)$, $S_1 = \mathrm{Ker}(dB_1)$. By the equation $B_1 dB_2 = B_2 dB_1$, they satisfy the assumption in the ζ-semistability condition, hence we have $\zeta_0 \dim S_0 + \zeta_1 \dim S_1 \leq 0$. Similarly we have $\zeta_0 \dim S_0' + \zeta_1 \dim S_1' \leq 0$ for $S_0' = \mathrm{Im}(B_1 d)$, $S_1' = \mathrm{Im}(dB_1)$. But as $\dim S_\alpha + \dim S_\alpha' = \dim V_\alpha$ ($\alpha = 0, 1$), the assumption (2.7) implies that both inequalities must be equalities. Hence $S_0 = 0$, $S_1 = 0$ or $S_0 = V_0$, $S_1 = V_1$. Suppose that the first case occurs. Then both $B_1 d$ and dB_1 are injective. Therefore both B_1, d must be isomorphisms, and hence $\dim V_0 = \dim V_1$. But this is impossible by (2.1) and (2.7). Therefore $B_1 d = 0$, $dB_1 = 0$. Exchanging B_1 and B_2, we have $B_2 d = 0$, $dB_2 = 0$. But as we have $V_0 = \mathrm{Im}\, B_1 + \mathrm{Im}\, B_2$ by the same argument as in Lemma 2.3, we get $d = 0$.   Q.E.D.

Therefore the data $(B_1, B_2, d) = (B_1, B_2, 0)$ can be considered as a representation of the Kronecker quiver. By the same argument in Lemma 2.4(1), it has the trivial stabilizer in $\mathrm{GL}(V_0) \times \mathrm{GL}(V_1) / \mathbb{C}^*$. In particular, it is an indecomposable representation. The classification of indecomposable representations of the Kronecker quiver is well-known (see e.g., [5, 1.8]). If we take suitable bases of V_0 and V_1, the data is written as either of the followings:

(2.10a) $B_1 = \begin{bmatrix} 1_m \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1_m \end{bmatrix} \in \mathrm{Mat}(m + 1, m) \quad (m \geq 0)$,

(2.10b) $B_1 = \begin{bmatrix} 1_m & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1_m \end{bmatrix} \in \mathrm{Mat}(m, m + 1) \quad (m \geq 0)$,

(2.10c) $B_1 = 1_m, \quad B_2 = J_m \quad (m \geq 1)$,

(2.10d) $B_1 = a 1_m + J_m, \quad B_2 = 1_m \quad (a \in \mathbb{C}, \, m \geq 1)$.

Here 1_m is the identity matrix of size m, J_m is the Jordan block of size m with eigenvalue 0, and $\mathrm{Mat}(k, l)$ is the space of all $k \times l$-matrices. By (2.1) and (2.7) we have $\dim V_0 < \dim V_1$. Thus we only have the possibility (2.10b).

Remark 2.11. It is well-known that the derived category of finite dimensional representations of the Kronecker quiver is equivalent to the derived category of coherent sheaves on $\mathbb{P}^1$. The equivalence is given by

$$R\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \bullet) \colon D^b(\operatorname{Coh} \mathbb{P}^1) \to D^b(\operatorname{mod-}A),$$

where A is the path algebra of the Kronecker quiver, or more explicitly, $V_0 = R\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \bullet)$, $V_1 = R\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1), \bullet)$ with homomorphisms given by z_1 and z_2 of the homogeneous coordinate $[z_1 : z_2]$ of $\mathbb{P}^1$. Then the objects corresponding to above indecomposable representations are

a) $\mathcal{O}(m)$ $(m \geq 0)$,
b) $\mathcal{O}(-m-1)[1]$ $(m \geq 0)$,
c) $\mathcal{O}/(z_2/z_1)^m \mathcal{O}$ $(m \geq 1)$,
d) $\mathcal{O}/(z-a)^m \mathcal{O}$ $(z = z_1/z_2, a \in \mathbb{C}, m \geq 1)$.

Lemma 2.12. *The data given in* (2.10b) *is* ζ-*stable.*

Proof. We take $V_0 = \mathbb{C}^m$, $V_1 = \mathbb{C}^{m+1}$ and B_1, B_2 as in b). We have $m\zeta_0 + (m+1)\zeta_1 = 0$ by the assumption (2.7). We further have $\zeta_1 \geq 0$ by (2.1).

The condition $S_0 \subset V_0$, $S_1 \subset V_1$ with $B_\alpha(S_1) \subset S_0$ means that the restriction $(B_1|_{S_0}, B_2|_{S_0})$ to S_0, S_1 is a subrepresentation of (B_1, B_2) as the representation of the Kronecker quiver. By the well-known representation theory of the Kronecker quiver, or by a direct calculation, we know that a quotient representation of B_1, B_2 is a direct sum of indecomposable representations (B_1', B_2'), (B_1'', B_2''),..., all of type b) with size n', n'',.... with $n' + n'' + \cdots \leq m$. Then we have

$$
\begin{aligned}
&\zeta_0 \operatorname{codim} S_0 + \zeta_1 \operatorname{codim} S_1 \\
&= \zeta_0(n' + n'' + \cdots) + \zeta_1((n'+1) + (n''+1) + \cdots) \\
&= (\zeta_0 + \zeta_1)\{(n' + n'' + \cdots) - m\} + \zeta_1(\#\{\text{factors}\} - 1) \geq 0,
\end{aligned}
$$

where we have used $m\zeta_0 + (m+1)\zeta_1 = 0$ in the second equality and $\zeta_0 + \zeta_1 < 0$ and $\zeta_1 \geq 0$ in the last inequality. It is also clear that we have $(S_0, S_1) = (0, 0)$ or (V_0, V_1) if the equality holds.   Q.E.D.

In summary we have

Theorem 2.13. *Assume the condition* (2.1). *The moduli space of* ζ-*stable data* $X = (B_1, B_2, d)$ *is a single point given by the data* b) *above with* $d = 0$ *if* $\dim V_0 = m$, $\dim V_1 = m + 1$ *with* $m\zeta_0 + (m+1)\zeta_1 = 0$. *And it is the empty set otherwise.*

Let $C_m = (B_1, B_2, 0)$ denote the data given in this theorem. In Proposition 5.3 we prove that C_m corresponds to $\mathcal{O}_C(-m-1)$.

2.3. Blowup of the plane as a moduli space

If the condition (2.1) is not satisfied, Theorem 2.13 is no longer true. We consider the limiting case

$$(2.14) \qquad \zeta_0 + \zeta_1 = 0, \qquad \zeta_0 < 0.$$

From (2.7) we have $\dim V_0 = \dim V_1$. The data (B_1, B_2, d) is ζ-stable if and only if it satisfies

> **(S0):** for subspaces $S_0 \subset V_0$, $S_1 \subset V_1$ such that $B_\alpha(S_1) \subset S_0$ ($\alpha = 1, 2$), $d(S_0) \subset S_1$, we have either $\dim S_0 > \dim S_1$, $S_0 = S_1 = 0$, or $S_0 = V_0$, $S_1 = V_1$.

This is the '$W = 0$' version of the conditions (S1), (S2) in Theorem 1.1.

Lemma 2.15. *Suppose* B_1, B_2, d *satisfies* (S0) *and* $B_1 d B_2 = B_2 d B_1$. *Then* $\dim V_0 = \dim V_1 = 1$.

Proof. From the equation $B_1 d B_2 = B_2 d B_1$, $B_1 d$ and $B_2 d$ commute. Let $0 \neq v_0 \in V_0$ be a simultaneous eigenvector, and let $v_1 = d v_0$. Then $(S_0, S_1) = (\mathbb{C} v_0, \mathbb{C} v_1)$ satisfies the assumption in (S0). As we cannot have $(S_0, S_1) = (0, 0)$, we either have $(S_0, S_1) = (V_0, V_1)$ or $v_1 = 0$. We are done in the first case. Therefore we may assume the latter case. Hence $\operatorname{Ker} d \neq 0$.

We take S_0, S_1, S_0', S_1' as in the proof of Lemma 2.9. By the same argument there, we get $(S_0, S_1) = (0, 0)$ or (V_0, V_1). But the first case cannot occur as $\operatorname{Ker} d \neq 0$. Therefore only the latter case can occur, thus $d B_1 = 0$. Exchanging B_1 and B_2, we get $d B_2 = 0$. By the same argument as in Lemma 2.3, the condition (S0) implies $V_0 = \operatorname{Im} B_1 + \operatorname{Im} B_2$. Therefore we have $d = 0$. Therefore $(B_1, B_2, d) = (B_1, B_2, 0)$ can be regarded as a representation of the Kronecker quiver. Moreover, by the same argument as in Lemma 2.4(1), it is indecomposable. Therefore after taking a base, (B_1, B_2) can be written as in (2.10). As $\dim V_0 = \dim V_1$, the cases a),b) cannot happen. In the case c) or d), we take

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then $B_1 v_1$ and $B_2 v_1$ are linearly dependent, and span a 1-dimensional subspace S_0'' in V_0. As the pair $(S_0'', S_1'' = \mathbb{C} v_1)$ violates the inequality in (S0), we must have $S_0'' = V_0$, $S_1'' = V_1$. Thus $\dim V_0 = \dim V_1 = 1$ also in this case. Q.E.D.

Theorem 2.16. *Assume* (2.14) *and* $\dim V_0 = \dim V_1$. *Then the moduli space of* ζ-*stable data* (B_1, B_2, d) *is isomorphic to the blowup of*

the plane at the origin $\widehat{\mathbb{C}}^2$ *if* $\dim V_0 = \dim V_1 = 1$. *And it is the empty set otherwise.*

Proof. By the previous lemma, we may assume $V_0 = V_1 = \mathbb{C}$. Then we have a morphism from the moduli space of ζ-stable data (B_1, B_2, d) to $\widehat{\mathbb{C}}^2$ given by

$$[(B_1, B_2, d)] \mapsto ((B_1 d, B_2 d), [B_1 : B_2]) \in \widehat{\mathbb{C}}^2;$$
$$\text{where } \widehat{\mathbb{C}}^2 = \left\{ ((z_1, z_2), [z : w]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid z_1 w = z_2 z \right\}.$$

As we observed already, we have $V_0 = \operatorname{Im} B_1 + \operatorname{Im} B_2$ if (B_1, B_2, d) satisfies (S0) by the argument in Lemma 2.3. Therefore $B_1 = B_2 = 0$ is not possible. Therefore $[B_1 : B_2] \in \mathbb{P}^1$ is defined, and the above actually defines a morphism from the moduli space to $\widehat{\mathbb{C}}^2$.

Conversely take a point $((z_1, z_2), [z : w]) \in \widehat{\mathbb{C}}^2$. We set $V_0 = V_1 = \mathbb{C}$ and

$$B_1 = z, \quad B_2 = w, \quad d = s = \begin{cases} z_1/z & \text{if } z \neq 0, \\ z_2/w & \text{if } w \neq 0. \end{cases}$$

In this case, the condition (S0) is equivalent to $V_0 = \operatorname{Im} B_1 + \operatorname{Im} B_2$, and hence is satisfied. Moreover, the above is well-defined up to the action of $\operatorname{GL}(V_0) \times \operatorname{GL}(V_1)/\mathbb{C}^* \cong \mathbb{C}^*$. It is also clear that it is the inverse of the previous map. Hence we have the assertion. Q.E.D.

From this theorem, we also see that the moduli space $\widehat{M}_\zeta^{\mathrm{ss}}(0, 0, n)$ of S-equivalence classes of ζ-semistable data is the nth symmetric product of $\widehat{\mathbb{C}}^2$.

It is instructive to consider what happens if ζ_0 becomes positive. (We still assume $\zeta_0 + \zeta_1 = 0$ and $\dim V_0 = \dim V_1 = 1$.) In this case, the ζ-stability is equivalent to $\operatorname{Ker} d = 0$, i.e., d is an isomorphism. Then the moduli space is isomorphic to $\mathbb{C}^2$ under the map

$$[(B_1, B_2, d)] \mapsto (B_1 d, B_2 d) \in \mathbb{C}^2.$$

The converse is given by

$$(z_1, z_2) \mapsto (B_1, B_2, d) = (z_1, z_2, 1).$$

When we cross the wall $\zeta_0 = 0$, the exceptional locus $C = \{[(B_1, B_2, 0)] \mid [B_1 : B_2] \in \mathbb{P}^1\}$ is replaced by the single point $0 = \{[(0, 0, 1)]\}$. This example is a prototype of the picture given in the subsequent paper.

Remark 2.17. It is straightforward to generalize the above arguments to classify all ζ-stable solution with $W = 0$ for any parameter ζ

which satisfies (2.7), but not necessarily (2.1). We leave the proof as an exercise for a reader.

(1) Case $\zeta_0 + \zeta_1 < 0$, $\zeta_0 \geq 0$: The only possibility is $V_0 = \mathbb{C}$, $V_1 = 0$ with $\zeta_0 = 0$.

(2) Case $\zeta_0 + \zeta_1 > 0$, $\zeta_1 > 0$: The data given by (2.10a) with $d = 0$ if $(m+1)\zeta_0 + m\zeta_1 = 0$.

(3) Case $\zeta_0 + \zeta_1 > 0$, $\zeta_1 \leq 0$: The only possibility is $V_1 = \mathbb{C}$, $V_0 = 0$ with $\zeta_1 = 0$.

(4) Case $\zeta_0 + \zeta_1 = 0$, $\zeta_0 > 0$: We have $\dim V_0 = \dim V_1 = 1$ and the moduli space is isomorphic to $\mathbb{C}^2$, as mentioned above.

(5) Case $\zeta_0 = \zeta_1 = 0$: We have $\dim V_0 = \dim V_1 = 1$ and the moduli space is isomorphic to $\mathbb{C}^2 \setminus \{(0,0)\}$.

§3. Resolution of the diagonal

Let us define a rank 2 vector bundle $\mathcal{Q}$ over $\widehat{\mathbb{P}}^2$ as the quotient

$$\mathcal{Q} := \mathcal{O}^{\oplus 3} / \operatorname{Im} \iota, \qquad \text{where } \iota = \begin{bmatrix} z_0 \\ z_2 \\ -z_1 \end{bmatrix} : \mathcal{O}(-\ell_\infty) \to \mathcal{O}^{\oplus 3}.$$

We have $\bigwedge^2 \mathcal{Q} = \mathcal{O}(\ell_\infty)$. This can be also described as the quotient

$$0 \to \mathcal{O}(C - \ell_\infty) \xrightarrow{\begin{bmatrix} z_0 \\ w \\ -z \end{bmatrix}} \mathcal{O}(C) \oplus \mathcal{O}^{\oplus 2} \to \mathcal{Q} \to 0,$$

where the isomorphism is the one induced by $\operatorname{diag}(s, \operatorname{id}, \operatorname{id}) \colon \mathcal{O} \oplus \mathcal{O}^{\oplus 2} \to \mathcal{O}(C) \oplus \mathcal{O}^{\oplus 2}$. We define a section φ of the vector bundle $\mathcal{Q} \boxtimes \mathcal{O}(\ell_\infty)$ over $\widehat{\mathbb{P}}^2 \times \widehat{\mathbb{P}}^2$ as the composite of

$$\varphi \colon \mathcal{O} \boxtimes \mathcal{O} \xrightarrow{\begin{bmatrix} \operatorname{id} \boxtimes z_0 \\ \operatorname{id} \boxtimes z_2 \\ -\operatorname{id} \boxtimes z_1 \end{bmatrix}} \mathcal{O}^{\oplus 3} \boxtimes \mathcal{O}(\ell_\infty) \xrightarrow{p \boxtimes \operatorname{id}_{\mathcal{O}(\ell_\infty)}} \mathcal{Q} \boxtimes \mathcal{O}(\ell_\infty),$$

where the second arrow is the homomorphism induced by the projection $p \colon \mathcal{O}^{\oplus 3} \to \mathcal{Q}$. We also define homomorphisms ψ, χ as composites of

$$\psi \colon \mathcal{O}(-C) \boxtimes \mathcal{O}(C) \xrightarrow{\begin{bmatrix} 0 \\ \operatorname{id} \boxtimes w \\ -\operatorname{id} \boxtimes z \end{bmatrix}} \mathcal{O}(-C)^{\oplus 3} \boxtimes \mathcal{O}(\ell_\infty) \to \mathcal{Q}(-C) \boxtimes \mathcal{O}(\ell_\infty),$$

$$\chi \colon \mathcal{O} \boxtimes \mathcal{O} \xrightarrow{\begin{bmatrix} -\operatorname{id} \boxtimes z_0 \\ 0 \\ 0 \end{bmatrix}} \begin{matrix} \mathcal{O} \boxtimes \mathcal{O}(\ell_\infty) \\ \oplus \\ \mathcal{O}(-C)^{\oplus 2} \boxtimes \mathcal{O}(\ell_\infty) \end{matrix} \to \mathcal{Q}(-C) \boxtimes \mathcal{O}(\ell_\infty)$$

respectively. Here we used the second description of $\mathcal{Q}$ for the definition of χ.

In order to distinguish the coordinates on the first and second factors of $\widehat{\mathbb{P}}^2$, we denote the former by $([z_0', z_1', z_2'], [z' : w'])$ and the latter by $([z_0, z_1, z_2], [z : w])$ hereafter. We define a complex $C^\bullet$ of vector bundles over $\widehat{\mathbb{P}}^2 \times \widehat{\mathbb{P}}^2$ by

$$
(3.1) \quad C^\bullet: \quad
\begin{array}{c}
\mathcal{O} \boxtimes \mathcal{O}(C - 2\ell_\infty) \\
\oplus \\
\mathcal{O}(C - \ell_\infty) \boxtimes \mathcal{O}(-2\ell_\infty)
\end{array}
$$

$$
\xrightarrow{d^{-1}}
\begin{array}{c}
\mathcal{O}^{\oplus 2} \boxtimes \mathcal{O}(-\ell_\infty) \\
\oplus \\
\mathcal{Q}^\vee(C) \boxtimes \mathcal{O}(-\ell_\infty)
\end{array}
\xrightarrow{d^0}
\begin{array}{c}
\mathcal{O} \boxtimes \mathcal{O} \\
\oplus \\
\mathcal{O}(C) \boxtimes \mathcal{O}(-C)
\end{array}
$$

with

$$
d^{-1} = \begin{bmatrix} \mathrm{id} \boxtimes z & z' \boxtimes z_0 \\ \mathrm{id} \boxtimes w & w' \boxtimes z_0 \\ 0 & \varphi_\lrcorner \end{bmatrix}, \qquad
d^0 = \begin{bmatrix} \mathrm{id} \boxtimes z_2 & -\mathrm{id} \boxtimes z_1 & \chi^\vee \\ s' \boxtimes w & -s' \boxtimes z & \psi^\vee \end{bmatrix}.
$$

We assign the degree 0 to the middle term. Let $\Delta \equiv \Delta_{\widehat{\mathbb{P}}^2}$ be the diagonal in $\widehat{\mathbb{P}}^2 \times \widehat{\mathbb{P}}^2$. Let $t \colon C^1 \to \mathcal{O}_\Delta$ be the homomorphism given by restricting to the diagonal and taking the difference of traces after the isomorphism $(\mathcal{O} \boxtimes \mathcal{O} \oplus \mathcal{O}(C) \boxtimes \mathcal{O}(-C))|_\Delta \cong \mathcal{H}om(\mathcal{O}, \mathcal{O}) \oplus \mathcal{H}om(\mathcal{O}(C), \mathcal{O}(C))$.

Over $\widehat{\mathbb{P}}^2 \setminus \ell_\infty = \{([1 : z_1' : z_2'], [z' : w'])\}$, we can identify $\mathcal{Q}$ with $\mathcal{O}^{\oplus 2}$ by

$$
\left(\mathcal{O}^{\oplus 3}\right) / \mathcal{O}(-\ell_\infty) \ni \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bmod \mathcal{O}(-\ell_\infty) \longmapsto \begin{bmatrix} b - a z_2' \\ c + a z_1' \end{bmatrix},
$$

$$
\left(\mathcal{O}(C) \oplus \mathcal{O}^{\oplus 2}\right) / \mathcal{O}(C - \ell_\infty) \ni \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bmod \mathcal{O}(C - \ell_\infty) \longmapsto \begin{bmatrix} b - a w' \\ c + a z' \end{bmatrix}.
$$

The sections φ, ψ, χ are re-written as

$$
\varphi \colon \mathcal{O} \boxtimes \mathcal{O} \xrightarrow{\begin{bmatrix} \mathrm{id} \boxtimes z_2 - z_2' \boxtimes z_0 \\ -\mathrm{id} \boxtimes z_1 + z_1' \boxtimes z_0 \end{bmatrix}} \mathcal{O}^{\oplus 2} \boxtimes \mathcal{O}(\ell_\infty),
$$

$$
\psi \colon \mathcal{O}(-C) \boxtimes \mathcal{O}(C) \xrightarrow{\begin{bmatrix} \mathrm{id} \boxtimes w \\ -\mathrm{id} \boxtimes z \end{bmatrix}} \mathcal{O}(-C)^{\oplus 2} \boxtimes \mathcal{O}(\ell_\infty),
$$

$$
\chi \colon \mathcal{O} \boxtimes \mathcal{O} \xrightarrow{\begin{bmatrix} w' \boxtimes z_0 \\ -z' \boxtimes z_0 \end{bmatrix}} \mathcal{O}(-C)^{\oplus 2} \boxtimes \mathcal{O}(\ell_\infty).
$$

We can directly check $d^0 \circ d^{-1} = 0$, $t \circ d^0 = 0$ over $(\widehat{\mathbb{P}}^2 \setminus \ell_\infty) \times \widehat{\mathbb{P}}^2$. It holds over $\widehat{\mathbb{P}}^2 \times \widehat{\mathbb{P}}^2$ by the continuity.

Proposition 3.2 (cf. [8, §3]). *The complex*

$$0 \to C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{t} \mathcal{O}_{\Delta_{\widehat{\mathbb{P}}^2}} \to 0$$

gives a resolution of $\mathcal{O}_{\Delta_{\widehat{\mathbb{P}}^2}}$.

In the following lemma we consider homomorphisms d^0, d^{-1}, etc, as linear maps between fibers of vector bundles.

Lemma 3.3. (1) φ *vanishes over* $(C \times C) \cup \Delta_{\widehat{\mathbb{P}}^2}$.
(2) *The locus* $\{d^{-1}$ *is not injective*$\}$ *is* $\Delta_{\widehat{\mathbb{P}}^2}$. *And* $\operatorname{Ker} d^{-1}|_{\Delta_{\widehat{\mathbb{P}}^2}} = \{(-z_0 f) \oplus f \mid f \in \mathcal{O}(C - 3\ell_\infty)\} \cong \mathcal{O}(C - 3\ell_\infty) \cong K_{\widehat{\mathbb{P}}^2}$.
(3) d^0 *is surjective outside* $\Delta_{\widehat{\mathbb{P}}^2}$, *and the cokernel of* d^0 *is isomorphic to the one-dimensional space of scalar endomorphisms* $\{\lambda \operatorname{id}_{\mathcal{O}} \oplus (-\lambda) \operatorname{id}_{\mathcal{O}(C)}\}$ *on the diagonal* $\Delta_{\widehat{\mathbb{P}}^2}$ *under the isomorphism* $(\mathcal{O} \boxtimes \mathcal{O} \oplus \mathcal{O}(C) \boxtimes \mathcal{O}(-C))|_{\Delta_{\widehat{\mathbb{P}}^2}} \cong \mathcal{H}om(\mathcal{O}, \mathcal{O}) \oplus \mathcal{H}om(\mathcal{O}(C), \mathcal{O}(C))$.

Remark 3.4. After Proposition 3.2 will be proved, it follows that the cohomology groups of the restriction of the complex $C^\bullet$ to the diagonal are $H^{-1} \cong K_{\widehat{\mathbb{P}}^2}$, $H^0 \cong T^*\widehat{\mathbb{P}}^2$, $H^1 \cong \mathcal{O}$, since $C^\bullet \otimes \mathcal{O}_\Delta$ computes $\operatorname{Tor}_\bullet^{\widehat{\mathbb{P}}^2 \times \widehat{\mathbb{P}}^2}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$. We prove this directly for $H^{\pm 1}$ in the above lemma. It is also possible to prove $H^0 \cong T^*\widehat{\mathbb{P}}^2$ directly.

Proof of Lemma 3.3. (1) From the definition φ is the pull-back of the section defined over $\mathbb{P}^2 \times \mathbb{P}^2$. The latter section vanishes on $\Delta_{\mathbb{P}^2}$. Therefore φ vanishes on $(C \times C) \cup \Delta_{\widehat{\mathbb{P}}^2}$.
(2) Suppose d^{-1} is not injective. Take $a \oplus b \in (\mathcal{O} \boxtimes \mathcal{O}(C - 2\ell_\infty)) \oplus (\mathcal{O}(C - \ell_\infty) \boxtimes \mathcal{O}(-2\ell_\infty))$ from $\operatorname{Ker} d^{-1}$. Then $\varphi \lrcorner b = 0$. If $\varphi \neq 0$, then we have $b = 0$. It implies $za = 0 = wa$, and hence $a = 0$. Therefore φ vanishes. By (1) the point is in $(C \times C) \cup \Delta_{\widehat{\mathbb{P}}^2}$. If it is in $(C \times C)$, then we must have $z_0 \neq 0$. Then looking at the first and second row of d^{-1}, we must have $[z : w] = [z' : w']$. Conversely if the point is in the diagonal, it is clear that d^{-1} is not injective. The second assertion is also obvious from the definition of d^{-1}.
(3) We have

$$(d^0)^\vee = \begin{bmatrix} \operatorname{id} \boxtimes z_2 & s' \boxtimes w \\ -\operatorname{id} \boxtimes z_1 & -s' \boxtimes z \\ \chi & \psi \end{bmatrix} : \begin{array}{c} \mathcal{O} \boxtimes \mathcal{O} \\ \oplus \\ \mathcal{O}(-C) \boxtimes \mathcal{O}(C) \end{array} \longrightarrow \begin{array}{c} \mathcal{O}^{\oplus 2} \boxtimes \mathcal{O}(\ell_\infty) \\ \oplus \\ \mathcal{Q}(-C) \boxtimes \mathcal{O}(\ell_\infty) \end{array}.$$

Suppose that $0 \neq \lambda \oplus \mu \in \mathcal{O} \boxtimes \mathcal{O} \oplus \mathcal{O}(-C) \boxtimes \mathcal{O}(C)$ is in the kernel of $(d^0)^\vee$. We have $\lambda\chi + \mu\psi = 0$, hence

$$(3.5) \qquad (-\lambda z_0, \mu z, \mu w) \in \mathbb{C}(z_0', z', w').$$

On the other hand, we have

$$(3.6) \qquad \lambda s + \mu s' = 0$$

as $z_1 = sz$, $z_2 = sw$.

Substituting (3.6) to (3.5), we get $(\lambda z_0, \lambda z_1, \lambda z_2) \in \mathbb{C}(z_0', z_1', z_2')$. Therefore if $[z_0 : z_1 : z_2] \neq [z_0' : z_1' : z_2']$, we must have $\lambda = 0$. Hence we have $\mu s' = 0$ from (3.6). On the other hand, from (3.5) and $\lambda = 0$, we have either $\mu = 0$ or $z_0' = 0$. As $\{s' = 0\} \cap \{z_0' = 0\} = \emptyset$, we have $\mu = 0$ anyway.

If $[z : w] \neq [z' : w']$, we must have $\mu = 0$ and $\lambda z_0 = 0$ from (3.5). We also have $\lambda s = 0$ from (3.6) and $\mu = 0$. Therefore we must also have $\lambda = 0$ as $\{s = 0\} \cap \{z_0 = 0\} = \emptyset$.

Finally suppose $[z_0 : z_1 : z_2] = [z_0' : z_1' : z_2']$ and $[z : w] = [z' : w']$. Then we have $\lambda + \mu = 0$ from either $z_0 \neq 0$ or $s \neq 0$. $\qquad$ Q.E.D.

Proof of Proposition 3.2. Let us first compute the alternating sum of Chern characters of terms restricted to the slice $\{x\} \times \widehat{\mathbb{P}}^2$. The ranks and the first Chern classes cancel. The alternating sum of the second Chern characters is 1. In particular (from the cancellation of ranks), the complex is exact outside Δ by Lemma 3.3.

Note the following: Suppose A, B are flat over T, $f : A \to B$ satisfies $f_x : A_x \to B_x$ is injective. Then f is injective and B/A is flat over T.

From the above with Lemma 3.3(2), we have $D := C^0/C^{-1}$ is a family of torsion free sheaves on $\widehat{\mathbb{P}}^2$ parametrized by $\widehat{\mathbb{P}}^2$, and D_x is locally free over $\widehat{\mathbb{P}}^2 \setminus \{x\}$. Next by Lemma 3.3(3) $D_x \to C_x^1$ is injective. So $D \to C^1$ is injective and C^1/D is a flat family of sheaves parametrized by $\widehat{\mathbb{P}}^2$. Moreover $D_x \to C_x^1$ is an isomorphism on $\widehat{\mathbb{P}}^2 \setminus \{x\}$. So C^1/D is a family of Artinian sheaves. Finally we consider the induced homomorphism $C^1/D \to \mathcal{O}_\Delta$. By Lemma 3.3(3) $(C^1/D)_x \to \mathbb{C}_x$ is nonzero. On the other hand, from the computation of the Chern classes, we see $(C^1/D)_x \cong \mathbb{C}_x$. Thus $C^1/D \to \mathcal{O}_\Delta$ is an isomorphism. $\qquad$ Q.E.D.

§4. Analysis of stability
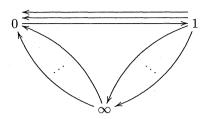
We fix $\zeta = (\zeta_1, \zeta_2)$ as in (2.1).

Fig. 2. New quiver

4.1. Necessary conditions for stability

We shall use the language of the representation theory of a quiver with relation. The data (B_1, B_2, d, i, j) with the underlying vector spaces V_0, V_1, W, as in Theorem 1.1 is a representation of a quiver with relation. But as W plays a different role from V_0, V_1, it is more natural to construct a new quiver following [4].

We fix the vector space W with $\dim W = r$. We define a new quiver with three vertexes 0, 1, ∞. We write two arrows from 1 to 0 corresponding to the data B_1, B_2, and one arrow from 0 to 1 corresponding to the data d. Instead of writing one arrow from ∞ to 0, we write r-arrows. Similarly we write r-arrows from 1 to ∞. And instead of putting W at ∞, we replace it the one dimensional space $\mathbb{C}$ on ∞. We denote it by V_∞. It means that instead of considering the homomorphism i from W to V_0, we take r-homomorphisms $i_1, i_2, \ldots, i_r$ from V_∞ to V_0 by taking a base of W. (See Figure 2.)

We consider the full subcategory of the abelian category of representations of the new quiver with the relation, consisting of representations such that $\dim V_\infty = 0$ or 1. An object can be considered as a representation of the original quiver with $\dim W = 0$, or $\dim W = r$, according to $\dim V_\infty = 0$ or 1. Note that we do not allow a representation of the original quiver with $\dim W \neq r, 0$. For objects X and Y, let $\operatorname{Hom}(X, Y)$ denote the space of morphisms from X to Y.

Recall $C_m = (B_1, B_2, 0)$ denote the data given in Theorem 2.13.

Proposition 4.1. *Suppose* $X = (B_1, B_2, d, i, j)$ *satisfies* $\mu(B_1, B_2, d, i, j) = 0$. *If* X *is* ζ-*semistable, then the following holds:*

(1)   *If* $m\zeta_0 + (m+1)\zeta_1 > 0$, *then* $\operatorname{Hom}(C_m, X) = 0$.
(2)   *If* $m\zeta_0 + (m+1)\zeta_1 < 0$, *then* $\operatorname{Hom}(X, C_m) = 0$.

If X *is* ζ-*stable, then we also have*

(3)   *If* $m\zeta_0 + (m+1)\zeta_1 = 0$, *then* $\operatorname{Hom}(X, C_m) = 0 = \operatorname{Hom}(C_m, X)$.

Proof. (1) Suppose $X = (B_1, B_2, d, i, j)$ is ζ-semistable, and take m with $m\zeta_0 + (m+1)\zeta_1 > 0$ and a homomorphism $\xi \colon C_m \to X$. Consider $S_0 = \operatorname{Im} \xi_0$, $S_1 = \operatorname{Im} \xi_1$. They satisfy the assumption in the stability condition for X in Definition 1.3. By the ζ-semistability of X we have

$$\zeta_0 \dim S_0 + \zeta_1 \dim S_1 \leq 0.$$

On the other hand, $S_0' = \operatorname{Ker} \xi_0$, $S_1' = \operatorname{Ker} \xi_1$ satisfy the assumption in the stability condition for C_m in Definition 1.3. As C_m is $(-1, m/(m+1))$-stable by Lemma 2.12 we have

$$-\dim S_0' + \frac{m}{m+1} \dim S_1' \leq 0,$$

or equivalently

$$-\dim S_0 + \frac{m}{m+1} \dim S_1 \geq 0.$$

But the two inequalities contradict with the assumption $m\zeta_0 + (m+1)\zeta_1 > 0$ unless $\dim S_0 = \dim S_1 = 0$.

The statement (2) can be proved in the same way.

(3) The above proof works until the last stage under $m\zeta_0 + (m+1)\zeta_1 = 0$. The first inequality and the last inequality are in the opposite direction, so we have equalities in all the three inequalities. As X is further ζ-stable, we have $S_0 = 0$, $S_1 = 0$.   Q.E.D.

4.2. Sufficient conditions for stability

The purpose of this section is to show a partial converse to the previous proposition:

Proposition 4.2. *Suppose that $X = (B_1, B_2, d, i, j)$ satisfies $\mu(B_1, B_2, d, i, j) = 0$ and the condition (S2) in Theorem 1.1. If X satisfies (1) and (2) in Proposition 4.1 for a given ζ, then X is ζ-semistable.*

Note that (S2) is a necessary condition for ζ-semistability by Lemma 2.2. Hence X is ζ-semistable if and only if X satisfies (S2), (1) and (2) in Proposition 4.1. A meaning of the condition (S2) will be clarified in Lemma 5.1.

For the proof we use the Harder–Narasimhan and Jordan–Hölder filtration of a representation of a quiver. For this purpose, we slightly need to modify the stability condition, which is suitable for representations for the new quiver. We fix data (B_1, B_2, d, i, j) defined for V_0, V_1, W, and consider it as a representation X of the new quiver as before. We define ζ_∞ by $\zeta_\infty = -\zeta_0 \dim V_0 - \zeta_1 \dim V_1$. For another nonzero representation

Y of the new quiver with the underlying vector space $V_0' \oplus V_1' \oplus V_\infty'$, we define the *slope* by

$$\theta(Y) := \frac{\zeta_0 \dim V_0' + \zeta_1 \dim V_1' + \zeta_\infty \dim V_\infty'}{\dim V_0' + \dim V_1' + \dim V_\infty'}.$$

We only consider the case $\dim V_\infty' = 0$ or 1 as before. We say Y is θ-*semistable* if we have

$$\theta(S) \leq \theta(Y)$$

for any subrepresentation $0 \neq S$ of Y. We say Y is θ-*stable* if the inequality is strict unless $S = Y$. If $(\dim V_0', \dim V_1', \dim V_\infty') = (\dim V_0, \dim V_1, 1)$ (for example, when $Y = X$), then $\theta(Y) = 0$. In this case, θ-(semi)stability is equivalent to ζ-(semi)stability. In fact, if a subrepresentation S has $S_\infty = 0$, then $\theta(S) \leq 0$ is equivalent to $\zeta_0 \dim S_0 + \zeta_1 \dim S_1 \leq 0$. If a subrepresentation T has $T_\infty = \mathbb{C}$, then $\theta(T) \leq 0$ is equivalent to $\zeta_0 \operatorname{codim} T_0 + \zeta_1 \operatorname{codim} T_1 \geq 0$.

When $(\dim V_0', \dim V_1', \dim V_\infty')$ is not equal to $(\dim V_0, \dim V_1, 1)$, we define a new parameter $\zeta' = (\zeta_0', \zeta_1', \zeta_\infty')$ by

(4.3)
$$(\zeta_0', \zeta_1', \zeta_\infty') = (\zeta_0, \zeta_1, \zeta_\infty) - \theta(Y)(1,1,1).$$

Then the θ-(semi)stability of Y is equivalent to (ζ_0', ζ_1')-(semi)stability of Y, as

$$\frac{\zeta_0' \dim S_0 + \zeta_1' \dim S_1 + \zeta_\infty' \dim S_\infty}{\dim S_0 + \dim S_1 + \dim S_\infty} = \theta(S) - \theta(Y).$$

Suppose $\dim V_\infty' = 0$. Then

(4.4)
$$\zeta_0' + \zeta_1' = \frac{1}{\dim V_0' + \dim V_1'}(\zeta_0 - \zeta_1)(\dim V_1' - \dim V_0'),$$

$$\zeta_0' = \frac{\dim V_1'}{\dim V_0' + \dim V_\infty'}(\zeta_0 - \zeta_1).$$

Theorem 4.5. [19] (1) *A representation Y has a Harder–Narasimhan filtration:*

$$Y = Y^0 \supset Y^1 \supset \cdots \supset Y^N \supset Y^{N+1} = 0$$

such that Y^k/Y^{k+1} is θ-semistable for $k = 0, 1, \ldots, N$ and $\theta(Y^0/Y^1) < \theta(Y^1/Y^2) < \cdots < \theta(Y^N/Y^{N+1})$.

(2) *A θ-semistable representation Y has a Jordan–Hölder filtration:*

$$Y = Y^0 \supset Y^1 \supset \cdots \supset Y^N \supset Y^{N+1} = 0$$

such that Y^k/Y^{k+1} is θ-stable for $k = 0, 1, \ldots, N$ and $\theta(Y^0/Y^1) = \theta(Y^1/Y^2) = \cdots = \theta(Y^N/Y^{N+1})$.

Proof of Proposition 4.2. Suppose that X is not ζ-semistable. Since ζ-semistability is equivalent to the θ-semistability in this case, X is not θ-semistable. So we take a Harder–Narasimhan filtration $X = X^0 \supset \cdots \supset X^N \supset X^{N+1} = 0$ as in Theorem 4.5. We have $N > 0$ by the assumption. If $\theta(X^0/X^1) \geq 0$, then $\theta(X^k/X^{k+1}) > 0$ for $k = 1, \dots, N$. It contradicts with $\theta(X) = 0$. Therefore $\theta(X^0/X^1) < 0$. Similarly we must have $\theta(X^N/X^{N+1}) > 0$.

Suppose $(X^0/X^1)_\infty = 0$. Then $T_0 = (X^1)_0$, $T_1 = (X^1)_1$ satisfy the condition in (S2) in Theorem 1.1. As we assume (S2), we have $\operatorname{codim} T_0 < \operatorname{codim} T_1$, i.e., $\dim(X^0/X^1)_0 < \dim(X^0/X^1)_1$. If we define the new parameter $\zeta' = (\zeta'_0, \zeta'_1, \zeta'_\infty)$ by the formula (4.3) for the representation X^0/X^1, we have $\zeta'_0 + \zeta'_1 < 0$, and $\zeta'_0 < 0$ by (4.4) and the assumption (2.1). Therefore (2.1) is satisfied for (ζ'_0, ζ'_1). If X^0/X^1 is θ-stable, or equivalently (ζ'_0, ζ'_1)-stable, then by Theorem 2.13 we have $X^0/X^1 = C_m$ for some m. But $\theta(X^0/X^1) = \theta(C_m) < 0$ means $m\zeta_0 + (m+1)\zeta_1 < 0$, hence $X = X^0$ cannot have C_m as a quotient by our assumption (2) in Proposition 4.1. This is a contradiction. Even if X^0/X^1 is not θ-stable, we take a Jordan–Hölder filtration of X^0/X^1, in particular X has C_m as a quotient with $\theta(C_m) < 0$. This is again a contradiction. Therefore we must have $(X^0/X^1)_\infty = \mathbb{C}$.

Next assume $(X^N/X^{N+1})_\infty = X^N_\infty = 0$. Then we have

$$0 < \theta(X^N) = \frac{\zeta_0 \dim X^N_0 + \zeta_1 \dim X^N_1}{\dim X^N_0 + \dim X^N_1}.$$

By the assumption (2.1) we must have $\dim X^N_0 < \dim X^N_1$. The rest of the argument goes along the same line as above. We must have $C_m \subset X$ with $m\zeta_0 + (m+1)\zeta_1 > 0$, but it contradicts with our assumption (1) in Proposition 4.1. Therefore we must have $(X^N/X^{N+1})_\infty = \mathbb{C}$, but as X_∞ is 1-dimensional, it cannot be possible to have both $(X^0/X^1)_\infty = \mathbb{C}$ and $(X^N/X^{N+1})_\infty = \mathbb{C}$ unless $N = 0$. It means that X is ζ-semistable.
Q.E.D.

4.3. Walls and chambers

Take $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2$ as in (2.1). Suppose ζ-semistability is not equivalent to the ζ-stability. Take data $X = (B_1, B_2, d, i, j) \in \mu^{-1}(0)$ which is ζ-semistable, but not ζ-stable. We consider X as a representation of the new quiver as above. Then by Theorem 4.5 we have a Jordan–Hölder filtration $X = X^0 \supset \cdots \supset X^N \supset X^{N+1} = 0$ as in Theorem 4.5. We have $N > 0$ by the assumption. As $\dim X_\infty = 1$, we have either $\dim(X^0/X^1)_\infty = 0$ or $\dim(X^N/X^{N+1})_\infty = 0$. In particular, we have a ζ-stable representation Y with $Y_\infty = 0$. By Theorem 2.13 then

$Y = C_m$ for some $m \in \mathbb{Z}_{\geq 0}$ and we have $m\zeta_0 + (m+1)\zeta_1 = 0$. Thus this is a possible wall in the space of parameters ζ as in (2.1).

A *wall of type* (r, k, n) is, by definition, a hyperplane $\{m\zeta_0 + (m+1)\zeta_1 = 0\}$ such that there is $X \in \mu^{-1}(0)$ as above with the data r, k, n. For fixed r, k, n, walls of type (r, k, n) is finite. A connected component of the complement of walls of type (r, k, n) is called a *chamber of type* (r, k, n). If there is no fear of confusion, we simply say a chamber. If ζ stays in a chamber, the corresponding stability condition remains equivalent. When we cross a wall, then the stability condition and the moduli space are changed.

§5. Construction: from ADHM data to a sheaf

In this section we construct a coherent sheaf E on $\widehat{\mathbb{P}}^2$ together with a framing Φ such that $(E(-mC), \Phi)$ is stable perverse coherent from a ζ-stable data $X = (B_1, B_2, d, i, j)$.

5.1. Surjectivity of β

Lemma 5.1 ([6, 4.1.3]). $X = (B_1, B_2, d, i, j)$ *satisfies the condition* (S2) *in Theorem 1.1 if and only if* β *in the complex* (1.2) *is surjective at every point in* $\widehat{\mathbb{P}}^2$.

Proof. For the sake of completeness, we give a proof. The proof is essentially the same as the original one, but our use of Theorem 2.16 somewhat clarifies the role of [6, Lemma 4.1.1, Remark 4.1.2].

Let us first show the 'only if' part: Assume β is not surjective at a point $([z_0 : z_1 : z_2], [z : w])$. As β is surjective along $z_0 = 0$, we may assume $z_0 = 1$. Then there exists $v_0 \in V_0^*$, $v_1 \in V_1^*$ such that

$$\text{(5.2)} \qquad z_2 v_0 + w d^* v_1 = 0, \quad z_1 v_0 + z d^* v_1 = 0, \quad B_2^* v_0 + w v_1 = 0,$$
$$B_1^* v_0 + z v_1 = 0, \quad i^* v_0 = 0$$

and $(v_0, v_1) \neq (0, 0)$. Then $(S_0, S_1) := (\mathbb{C} v_0, \mathbb{C} v_1)$ satisfies $B_\alpha^*(S_0) \subset S_1$, $d^*(S_1) \subset S_0$, $i^*(S_0) = 0$. Moreover we have $v_0 \neq 0$, as $v_0 = 0$ implies $v_1 = 0$. Therefore $\dim S_0 = 1$ and $\dim S_1 = 0$ or 1. Hence the annihilators $T_0 := S_0^\perp \subset V_0$, $T_1 := S_1^\perp \subset V_1$ violate the condition (S2).

We prove the 'if' part: Assume $X = (B_1, B_2, d, i, j)$ violates the condition (S2). We consider $(B_1^*, B_2^*, d^*, i^*, j^*)$, then we have a pair of subspaces $S_0 \subset V_0^*$, $S_1 \subset V_1^*$ such that $B_\alpha^*(S_0) \subset S_1$ ($\alpha = 1, 2$), $d^*(S_1) \subset S_0$, $i^*(S_0) = 0$, and $\dim S_1 \leq \dim S_0$ with $(S_0, S_1) \neq (0, 0)$. Using the idea of the proof of the Harder–Narasimhan filtration, we take

(S_0, S_1) among such pairs of subspaces, so that

$$\theta(S_0, S_1) := \frac{\dim S_0 - \dim S_1}{\dim S_0 + \dim S_1}$$

takes the maximum. In particular, $\theta(S_0, S_1) \geq 0$. Then the restriction of (B_1^*, B_2^*, d^*) to (S_0, S_1) is θ-semistable. We then consider a Jordan–Hölder filtration with respect to θ, we may further assume that the restriction of (B_1^*, B_2^*, d^*) to (S_0, S_1) is θ-stable. If we define a new parameter (ζ_0', ζ_1') by

$$(\zeta_0', \zeta_1') = (1, -1) - \theta(S_0, S_1)(1, 1),$$

we have $\zeta_0' \dim S_0 + \zeta_1' \dim S_1 = 0$ and θ-stability of $(B_1^*, B_2^*, d^*)|_{(S_0, S_1)}$ is equivalent to its (ζ_0', ζ_1')-stability in Definition 2.8. We have $\zeta_0' + \zeta_1' \leq 0$ and $\zeta_1' < 0$. Remarking that we change the numbering of the vertexes 0, 1 as we are considering the transposes B_1^*, B_2^*, d^*, the parameter (ζ_0', ζ_1') satisfies either (2.1) or (2.14). Therefore the restriction of (B_1^*, B_2^*, d^*) to (S_0, S_1) is classified as in Theorem 2.13 and Theorem 2.16. In the latter case, there is a corresponding point $(z_1, z_2, [z : w]) \in \widehat{\mathbb{C}}^2$. The surjectivity of β is violated at the point. In the former case, there exists $0 \neq v_0 \in S_0$ such that $(wB_1^* - zB_2^*)v_0 = 0$ as $\dim S_0 > \dim S_1$ for any pair (z, w). Therefore we can find vectors $v_0 \in V_0^*$, $v_1 \in V_1^*$ as in (5.2) for any point on the exceptional curve $C = \{(0, 0, [z : w])\}$. Again the surjectivity of β is violated. Q.E.D.

5.2. The definition of a framed sheaf

Suppose that (B_1, B_2, d, i, j) satisfies the condition (S2). Then β in (1.2) is surjective at every point by Lemma 5.1. Let us consider the restriction of (1.2) to the line ℓ_∞ at infinity. Then we can check that $\alpha|_{\ell_\infty}$ is injective and $(\operatorname{Ker} \beta|_{\ell_\infty})/(\operatorname{Im} \alpha|_{\ell_\infty})$ is isomorphic to $W \otimes \mathcal{O}_{\ell_\infty}$ via the natural inclusion of $W \otimes \mathcal{O}_{\ell_\infty}$ to the middle term of (1.2) restricted to ℓ_∞. Hence we have

(1) α is injective as a sheaf homomorphism, and hence the complex (1.2) is quasi-isomorphic to a coherent sheaf $E = \operatorname{Ker} \beta / \operatorname{Im} \alpha$,

(2) the sheaf E has the canonical framing $\varphi \colon E|_{\ell_\infty} \xrightarrow{\cong} W \otimes \mathcal{O}_{\ell_\infty}$.

Furthermore the injectivity of α fails along a subvariety of $\widehat{\mathbb{P}}^2$, therefore α is injective except possibly along C and finite points in $\widehat{\mathbb{C}}^2 \setminus C$. In particular, E is torsion free outside C (see e.g. the argument in [14, §5(i)]).

5.3. The sheaf corresponding to C_m

For a later purpose we notice that this construction works in the case $W = 0$, provided the assumption (S2) is satisfied. For example, the data C_m in Theorem 2.13 gives a sheaf: C_m satisfies (S2) by Lemma 2.2 as it is $(-1, m/(m+1))$-stable. We can explicitly describe it:

Proposition 5.3. *The sheaf corresponding to C_m is $\mathcal{O}_C(-m-1)$.*

Proof. We use the resolution of the diagonal $C^\bullet$ in (3.1). Our proof is a prototype of the argument used in §6. We consider

$$Rp_{2*}\left(p_1^* \mathcal{O}_C(-m-1) \otimes C^\bullet\right),$$

where p_1, p_2 are the projection to the first and second factors of $\widehat{\mathbb{P}}^2 \times \widehat{\mathbb{P}}^2$ respectively. Since $C^\bullet$ is quasi-isomorphic to the diagonal, this is isomorphic to $\mathcal{O}_C(-m-1)$.

Next let us take the tensor product of $p_1^* \mathcal{O}_C(-m-1)$ with $C^\bullet$ in (3.1) and take derived push-forwards. Note the cohomology groups $H^i(\mathcal{O}_C(-m-1))$ and $H^i(\mathcal{O}_C(-m-1) \otimes \mathcal{O}(C)) = H^i(\mathcal{O}_C(-m-2))$ vanishes for $i = 0$, and have dimensions m and $m+1$ respectively for $i = 1$ thanks to the assumption $m \geq 0$. Let $V_0 = H^1(\mathcal{O}_C(-m-1))$, $V_1 = H^1(\mathcal{O}_C(-m-1) \otimes \mathcal{O}(C))$. We consider the homomorphisms on cohomology groups induced from d_1, d_2 in (3.1). As $z_1' = z_2' = s = 0$ on $C = \operatorname{Supp} \mathcal{O}_C(-m-1)$, the only nontrivial homomorphisms are those $V_1 \to V_0$ induced from z', w'. Under a suitable choice of bases of V_1, V_0, the homomorphisms are given by B_1, B_2 in (2.10b). Then we see that the homomorphisms induced from d_2, d_1 are exactly equal to α, β in (1.2) for the data C_m. Therefore we conclude that $\operatorname{Ker} \beta / \operatorname{Im} \alpha$ for C_m is isomorphic to $\mathcal{O}_C(-m-1)$. Q.E.D.

Remark 5.4. The framing of C_m is implicit as $E|_{\ell_\infty} \cong 0$.

5.4. Homomorphisms

Suppose that two framed coherent sheaves (E, Φ) and (E', Φ') are obtained by the construction in §5.2. Let us denote the data for E by V_0, V_1, B_1, etc, and by V_0', V_1', B_1' for E'. Let us denote the corresponding representations of the quiver (with relations) by X and X' respectively. Let $\operatorname{Hom}(X, X')$ be the space of homomorphisms from X to X', considered as representations of the quiver. Here the quiver is as in Theorem 1.1 and is not the *new quiver*, used in §4. But the difference between the old and new quivers is not relevant, as we are interested only in the cases when one of W or W' is 0 (as in §4), or $W = W'$. In the latter case, we are only interested in homomorphisms which are

identity on $W = W'$. Anyway, we use the same notation Hom as in §4, as there is no fear of confusion.

Proposition 5.5. *Let us consider the space of homomorphisms* $\xi\colon E \to E'$. *It is isomorphic to the space of homomorphisms* $\mathrm{Hom}(X, X')$, *i.e., pairs of linear maps* $\xi_0\colon V_0 \to V_0'$, $\xi_1\colon V_1 \to V_1'$, $b_{WW}\colon W \to W'$ *such that*

$$
\begin{aligned}
&\xi_0 B_\alpha = B_\alpha' \xi_1 \quad (\alpha = 1, 2), \qquad \xi_1 d = d' \xi_0, \\
&\quad \xi_0 i = i' b_{WW}, \qquad b_{WW} j = j' \xi_1.
\end{aligned}
$$

(5.6)

Here we understood $i = 0 = j$ (resp. $i' = 0 = j'$) if $W = 0$ (resp. $W' = 0$).

For the proof, we need a terminology from the theory of monads: A *monad* is a complex of vector bundles

$$
0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0
$$

such that α is injective and β is surjective as sheaf homomorphisms, so that it is quasi-isomorphic to a coherent sheaf $E = \mathrm{Ker}\,\beta / \mathrm{Im}\,\alpha$. A morphism from one monad $M\colon \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}$ to another $M'\colon \mathcal{A}' \xrightarrow{\alpha'} \mathcal{B}' \xrightarrow{\beta'} \mathcal{C}'$ is a morphism of complexes, i.e., the diagram

(5.7)
$$
\begin{array}{ccccc}
\mathcal{A} & \xrightarrow{\alpha} & \mathcal{B} & \xrightarrow{\beta} & \mathcal{C} \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\
\mathcal{A}' & \xrightarrow{\alpha'} & \mathcal{B}' & \xrightarrow{\beta'} & \mathcal{C}'.
\end{array}
$$

It is determined by b, as long as it satisfies $b(\mathrm{Im}\,\alpha) \subset \mathrm{Im}\,\alpha'$, $b(\mathrm{Ker}\,\beta) \subset \mathrm{Ker}\,\beta'$. Then we have an induced map $H(b)\colon E \to E'$. Let $\mathrm{Hom_{mon}}(M, M')$ denote the set of morphisms from M to M'.

Lemma 5.8 ([6, 2.2.1]). *Let M, M' be monads as above, and E, E' be corresponding sheaves. Suppose that*

$$
\mathrm{Hom}(\mathcal{B}, \mathcal{A}'), \quad \mathrm{Hom}(\mathcal{C}, \mathcal{B}'), \quad \mathrm{Ext}^1(\mathcal{C}, \mathcal{B}'), \quad \mathrm{Ext}^1(\mathcal{B}, \mathcal{A}'), \quad \mathrm{Ext}^2(\mathcal{C}, \mathcal{A}')
$$

all vanish. Then the above assignment $H\colon \mathrm{Hom_{mon}}(M, M') \to \mathrm{Hom}(E, E')$ is surjective, and the kernel is isomorphic to $\mathrm{Ext}^1(\mathcal{C}, \mathcal{A}')$.

Proof. We give a proof following [18, Ch. II, Lem. 4.1.3].

Let us suppose a homomorphism $\varphi\colon E \to E'$ is given. We claim that there exists a homomorphism $\psi\colon \mathrm{Ker}\,b \to \mathrm{Ker}\,b'$ completing the

commutative diagram (with exact rows)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A} & \longrightarrow & \operatorname{Ker}\beta & \longrightarrow & E & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \\
0 & \longrightarrow & \mathcal{A}' & \longrightarrow & \operatorname{Ker}\beta' & \longrightarrow & E' & \longrightarrow & 0,
\end{array}
$$

and it is unique up to $\operatorname{Hom}(\operatorname{Ker}\beta, \mathcal{A}')$.

First we prove

(5.9) $\qquad \operatorname{Hom}(\operatorname{Ker}\beta, \mathcal{A}') \cong \operatorname{Ext}^1(\mathcal{C}, \mathcal{A}'), \qquad \operatorname{Ext}^1(\operatorname{Ker}\beta, \mathcal{A}') = 0.$

We apply the derived functor $\mathbf{R}\operatorname{Hom}(\bullet, \mathcal{A}')$ to the exact sequence

(5.10) $\qquad\qquad\qquad 0 \to \operatorname{Ker}\beta \to \mathcal{B} \to \mathcal{C} \to 0.$

Then (5.9) follow from the assumption $\operatorname{Hom}(\mathcal{B}, \mathcal{A}') = 0$, $\operatorname{Ext}^1(\mathcal{B}, \mathcal{A}') = 0$, $\operatorname{Ext}^2(\mathcal{C}, \mathcal{A}') = 0$.

We then apply the functor $\mathbf{R}\operatorname{Hom}(\operatorname{Ker}\beta, \bullet)$ to the exact sequence

$$0 \to \mathcal{A}' \to \operatorname{Ker}\beta' \to E' \to 0$$

to get a short exact sequence

$$0 \to \operatorname{Hom}(\operatorname{Ker}\beta, \mathcal{A}') \to \operatorname{Hom}(\operatorname{Ker}\beta, \operatorname{Ker}\beta') \to \operatorname{Hom}(\operatorname{Ker}\beta, E') \to 0.$$

Therefore the composition of φ with the surjection $\operatorname{Ker}b \twoheadrightarrow E$ can be lifted to a homomorphism $\psi\colon \operatorname{Ker}\beta \to \operatorname{Ker}\beta'$, unique up to $\operatorname{Hom}(\operatorname{Ker}\beta, \mathcal{A}')$. Therefore we get the claim.

Finally we need to prove that there exists a unique homomorphism $b\colon \mathcal{B} \to \mathcal{B}'$ completing the commutative diagram (with exact rows)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}\beta & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\
& & {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker}\beta' & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{C}' & \longrightarrow & 0.
\end{array}
$$

Applying $\mathbf{R}\operatorname{Hom}(\bullet, \mathcal{B}')$ to the exact sequence (5.10), we get

$$\operatorname{Hom}(\mathcal{B}, \mathcal{B}') \cong \operatorname{Hom}(\operatorname{Ker}\beta, \mathcal{B}')$$

from the assumption $\operatorname{Hom}(\mathcal{C}, \mathcal{B}') = 0$, $\operatorname{Ext}^1(\mathcal{C}, \mathcal{B}') = 0$. Therefore the composition of the injection $\operatorname{Ker}\beta' \to \mathcal{B}'$ and ψ can be extended uniquely to a homomorphism $b \in \operatorname{Hom}(\mathcal{B}, \mathcal{B}')$. $\qquad\qquad$ Q.E.D.

If both monads M, M' are given by complexes of forms (1.2) corresponding to data X, X', then the vanishing of five cohomology groups in the assumption of Lemma 5.8 can be easily checked. (We only need to compute cohomology groups of line bundles.) And it is also easy to check $\text{Ext}^1(\mathcal{C}, \mathcal{A}') \cong \text{Hom}(V_1, V_0')$. Moreover if we represent a homomorphism $M \to M'$ with respect to the decomposition in (1.2), we get

Lemma 5.11. *A homomorphism (a, b, c) of monads from M to M' is represented as*

$$
a = \begin{pmatrix} \xi_0 + a_{01}d & a_{01}s \\ 0 & \xi_1 \end{pmatrix}, \quad b = \begin{pmatrix} \text{id}_{\mathbb{C}^2} \otimes (\xi_0 + a_{01}d) & \text{id}_{\mathbb{C}^2} \otimes a_{01} & 0 \\ 0 & \text{id}_{\mathbb{C}^2} \otimes \xi_1 & 0 \\ 0 & 0 & b_{WW} \end{pmatrix},
$$

$$
c = \begin{pmatrix} \xi_0 & a_{01}s \\ 0 & \xi_1 + d'a_{01} \end{pmatrix}
$$

satisfying the relations (5.6), where

$$
\xi_0 \colon V_0 \to V_0', \quad \xi_1 \colon V_1 \to V_1', \quad a_{01} \colon V_1 \to V_0', \quad b_{WW} \colon W \to W'.
$$

The above lemma gives an exact sequence

$$
0 \to \text{Hom}(V_1, V_0') \to \text{Hom}_{\text{mon}}(M, M') \to \text{Hom}(X, X') \to 0.
$$

Moreover any element in $\text{Hom}(V_1, V_0')$ maps $\text{Ker}\,\beta$ to $\text{Im}\,\alpha'$. Therefore $\text{Hom}(V_1, V_0')$ is contained in the kernel of $H \colon \text{Hom}_{\text{mon}}(M, M') \to \text{Hom}(E, E')$, which we identified with $\text{Ext}^1(\mathcal{C}, \mathcal{A}) \cong \text{Hom}(V_1, V_0')$ in Lemma 5.8. Therefore $\text{Hom}(V_1, V_0')$ is exactly the same as the kernel of H. We conclude

$$
\text{Hom}(X, X') \cong \text{Hom}(E, E').
$$

This completes the proof of Proposition 5.5.

Proof of Lemma 5.11. We write a, b, c in matrix forms:

$$
a = \begin{bmatrix} a_{00} & a_{01}s \\ 0 & a_{11} \end{bmatrix}, \quad b = \begin{bmatrix} b_{00} & b_{01} & b_{0W} \\ b_{10} & b_{11} & b_{1W} \\ b_{W0} & b_{W1} & b_{WW} \end{bmatrix}, \quad c = \begin{bmatrix} c_{00} & c_{01}s \\ 0 & c_{11} \end{bmatrix},
$$

where $b_{ij} \in \text{Hom}(\mathbb{C}^2 \otimes V_j, \mathbb{C}^2 \otimes V_i)$, etc. Note a and c are upper triangular as $H^0(\mathcal{O}(-C)) = 0$. We also used $H^0(\mathcal{O}(C)) \cong \mathbb{C}s$.

We restrict the monads M, M' to $\ell_\infty = \{z_0 = 0\}$. Then the commutativity of the diagram implies

$$b_{00} = \mathrm{id}_{\mathbb{C}^2} \otimes a_{00}, \quad b_{10} = 0, \quad b_{W0} = 0, \quad b_{01} = \mathrm{id}_{\mathbb{C}^2} \otimes a_{01},$$
$$b_{11} = \mathrm{id}_{\mathbb{C}^2} \otimes a_{11}, \quad b_{W1} = 0,$$
$$b_{00} = (c_{00} + c_{01}d) \otimes \mathrm{id}_{\mathbb{C}^2}, \quad d'b_{00} = c_{11} \otimes \mathrm{id}_{\mathbb{C}^2}, \quad b_{01} = c_{01} \otimes \mathrm{id}_{\mathbb{C}^2},$$
$$d'b_{01} + b_{11} = c_{11} \otimes \mathrm{id}_{\mathbb{C}^2}, \quad b_{0W} = 0, \quad b_{1W} = 0.$$

So linear maps are determined from c_{00}, a_{11}, a_{01}, b_{WW}. We write the first two by ξ_0, ξ_1.

We then consider the commutativity of the diagram on arbitrary point in $\widehat{\mathbb{P}}^2$ to find that ξ_0, ξ_1 satisfy the relations (5.6). Q.E.D.

5.5. Vanishing of homomorphisms

Recall that we take ζ with (1.4). Then the ζ-stability implies (S2) by Lemma 2.2.

Proposition 5.12. *Suppose that* $X = (B_1, B_2, d, i, j)$ *is* ζ-stable *and let* $E = \mathrm{Ker}\,\beta / \mathrm{Im}\,\alpha$ *as above. Then*

(1) $\mathrm{Hom}(E(-mC), \mathcal{O}_C(-1)) = 0$,
(2) $\mathrm{Hom}(\mathcal{O}_C, E(-mC)) = 0$.

Proof. We have $m\zeta_0 + (m+1)\zeta_1 < 0$ and $(m-1)\zeta_0 + m\zeta_1 > 0$ by the assumption (1.4). Hence if $m > 0$, we have $\mathrm{Hom}(E(-mC), \mathcal{O}_C(-1)) = \mathrm{Hom}(E, \mathcal{O}_C(-m-1)) = \mathrm{Hom}(X, C_m) = 0$ and $\mathrm{Hom}(\mathcal{O}_C, E(-mC)) = \mathrm{Hom}(\mathcal{O}_C(-m), E) = \mathrm{Hom}(C_{m-1}, X) = 0$ by Proposition 4.1. Therefore we only need to show (2) for $m = 0$. (In fact, the following argument only uses that E arises as $\mathrm{Ker}\,\beta / \mathrm{Im}\,\alpha$. But $m > 0$ case is already proved as $\mathrm{Hom}(\mathcal{O}_C(-m), E) = 0$ implies $\mathrm{Hom}(\mathcal{O}_C, E) = 0$.)

Let us take a representation of the quiver in Theorem 1.1 with $V_0 = \mathbb{C}$, $V_1 = W = 0$, $B_1 = B_2 = d = 0$. Let us denote it by C'. If we put it in the complex (1.2), we get

$$\mathcal{O}(C - \ell_\infty) \xrightarrow{\alpha = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]} \mathcal{O}^{\oplus 2} \xrightarrow{\beta = [z_2 \ -z_1]} \mathcal{O}(\ell_\infty).$$

This is exact except that we have $\mathrm{Cok}\,\beta = \mathcal{O}_C$. Therefore the complex is quasi-isomorphic to $\mathcal{O}_C[-1]$, where $[-1]$ denote the shift functor of degree (-1). Let $M \colon \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}$ be this complex with shifted of degree 1 (so that it is quasi-isomorphic to $\mathcal{O}_C$), and $M' \colon \mathcal{A}' \xrightarrow{\alpha'} \mathcal{B}' \xrightarrow{\beta'} \mathcal{C}'$ be the complex (1.2) corresponding to the data X. This M is not a monad in

our sense, but we can define a homomorphism as a commuting diagram

$$
\begin{array}{ccccc}
\mathcal{A} & \xrightarrow{\ \alpha\ } & \mathcal{B} & \xrightarrow{\ \beta\ } & \mathcal{C} \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\
\mathcal{A}' & \xrightarrow{\ \alpha'\ } & \mathcal{B}' & \xrightarrow{\ \beta'\ } & \mathcal{C}'.
\end{array}
$$

Let $\mathrm{Hom}_{\mathrm{mon}}(M, M')$ denote the space of homomorphisms.

Claim. $\mathrm{Hom}_{\mathrm{mon}}(M, M') \cong \mathrm{Hom}(\mathcal{O}_C, E)$.

In fact, it is enough to show that we can complete uniquely the following commutative diagram with exact rows, for any given homomorphism $\varphi \colon \mathcal{O}_C \to E$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Im}\,\beta & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & \mathcal{A}' & \longrightarrow & \mathrm{Ker}\,\beta' & \longrightarrow & E & \longrightarrow & 0.
\end{array}
$$

We consider the long exact sequence for the second row tensored with $\mathcal{C}^*$. Then we have

$$
\mathrm{Hom}(\mathcal{C}, \mathcal{A}') = 0
$$

(this can be checked directly, or follows from $\mathrm{Hom}(\mathcal{C}, \mathcal{B}') = 0$) and

$$
\mathrm{Ext}^1(\mathcal{C}, \mathcal{A}') = 0
$$

(this follows from our choice $V_1 = 0$ for the data corresponding to $\mathcal{O}_C$). Thus we have

$$
\mathrm{Hom}(\mathcal{C}, \mathrm{Ker}\,\beta') \cong \mathrm{Hom}(\mathcal{C}, E).
$$

Now the claim follows.

As $\mathrm{Hom}(\mathcal{C}, \mathcal{B}') = 0 = \mathrm{Hom}(\mathcal{B}, \mathcal{A}')$, we have $\mathrm{Hom}_{\mathrm{mon}}(M, M') = 0$.

$$\text{Q.E.D.}$$

Now we have proved that the sheaf $E(-mC)$ is perverse coherent, so completed the half of the proof of Theorem 1.5.

§6. Inverse construction

We give the inverse construction from a sheaf to ADHM data.

6.1. Two spectral sequences from the resolution of the diagonal

Let p_1, $p_2 \colon \widehat{\mathbb{P}}^2 \times \widehat{\mathbb{P}}^2 \to \widehat{\mathbb{P}}^2$ be the projections to the first and second factors respectively. Let $C^\bullet$ be the resolution of the diagonal in (3.1). Suppose that (E, Φ) is a framed sheaf on (the first factor) $\widehat{\mathbb{P}}^2$ such that $(E(-mC), \Phi)$ is stable perverse coherent. We consider the double complex for the hyper direct image $R^\bullet p_{2*}(p_1^* E(-\ell_\infty) \otimes C^\bullet)$ for which we can consider two spectral sequences as usual. If we take the cohomology of $C^\bullet$ first, we get $E_2^{p,q} = E(-\ell_\infty)$ for $(p, q) = (0, 0)$ and $= 0$ otherwise. Thus the spectral sequence degenerates at E_2-term and converges to $E(-\ell_\infty)$.

On the other hand, if we take the direct image first, we get $R^q p_{2*}(p_1^* E(-\ell_\infty) \otimes C^p)$. As each C^p is of a form $C^p = p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2$, this is equal to $\mathcal{F}_2 \otimes H^q(\widehat{\mathbb{P}}^2, E(-\ell_\infty) \otimes \mathcal{F}_1)$.

6.2. Vanishing of cohomology groups

Lemma 6.1. *Suppose that $(E(-mC), \Phi)$ is a stable perverse coherent framed sheaf with $m \geq 0$. Then*

(1) $H^q(\widehat{\mathbb{P}}^2, E(-k\ell_\infty)) = 0$, $H^q(\widehat{\mathbb{P}}^2, E(C - k\ell_\infty)) = 0$ *for $k = 1, 2$, $q = 0, 2$.*

(2) $H^q(\widehat{\mathbb{P}}^2, E(-\ell_\infty) \otimes \mathcal{Q}^\vee(C)) = 0$ *for $q = 0, 2$.*

Proof.   We take direct images of E and $E(C)$ with respect to $p \colon \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ and deduce the assertion from the vanishing theorem for sheaves on $\mathbb{P}^2$ in [13, Lemma 2.4].

As $E(-mC)$ is stable perverse, we have $0 = \operatorname{Hom}(\mathcal{O}_C, E(-mC)) = \operatorname{Hom}(\mathcal{O}_C(-m), E)$ by (3) in the introducion. Therefore $\operatorname{Hom}(\mathcal{O}_C, E) = 0$ as we have $\operatorname{Hom}(\mathcal{O}_C(-m), \mathcal{O}_C) \otimes \mathcal{O}_C(-m) \to \mathcal{O}_C$ is surjective. Similarly we have $\operatorname{Hom}(\mathcal{O}_C, E(C)) = \operatorname{Hom}(\mathcal{O}_C(1), E) = 0$. Note that the condition (3) is equivalent to the torsion freeness of $p_* E$ at $[1 : 0 : 0]$. In fact, if $\mathbb{C}_0$ is the skyscraper sheaf at $[1 : 0 : 0]$ of $\mathbb{P}^2$, then we have $\operatorname{Hom}(\mathcal{O}_C, E) = \operatorname{Hom}(p^* \mathbb{C}_0, E) = \operatorname{Hom}(\mathbb{C}_0, p_* E)$. Therefore both $p_* E$ and $p_*(E(C))$ are torsion-free in our situation as they are also torsion free on $\widehat{\mathbb{P}}^2 \setminus C \xrightarrow{p}_{\cong} \mathbb{P}^2 \setminus \{[1 : 0 : 0]\}$ by the condition (4)'. Therefore we have $H^0(\widehat{\mathbb{P}}^2, E(-k\ell_\infty)) \cong H^0(\mathbb{P}^2, (p_* E)(-k\ell_\infty)) = 0$ and $H^0(\widehat{\mathbb{P}}^2, E(C - k\ell_\infty)) \cong H^0(\mathbb{P}^2, (p_* E(C))(-k\ell_\infty)) = 0$ for $k = 1, 2$ by [13, Lemma 2.4].

Let us turn to H^2. As $R^1 p_*(E)$ is supported at the origin, it does not contribute to H^2. Therefore we have $H^2(\widehat{\mathbb{P}}^2, E(-k\ell_\infty)) \cong H^2(\mathbb{P}^2, (p_* E)(-k\ell_\infty))$. Hence we have $H^2(\widehat{\mathbb{P}}^2, E(-k\ell_\infty)) = 0$ for $k = 1, 2$ by [13, Lemma 2.4]. The same argument works for $E(C)$.

For (2) we recall that $\mathcal{Q}$ is a pull-back of a vector bundle $\bar{\mathcal{Q}}$ over $\mathbb{P}^2$. Its definition is exactly the same as the vector bundle $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}/\mathcal{O}_{\mathbb{P}^2}(-\ell_\infty)$ appeared in [13, §2.1]. Therefore the assertion follows from [13, Lemma 2.4].
$$\text{Q.E.D.}$$

From this vanishing theorem, the spectral sequence must degenerate at the E_2-term. Furthermore, since the spectral sequence must converges to $E(-\ell_\infty)$, we find that (1) E is isomorphic to $\operatorname{Ker}\beta/\operatorname{Im}\alpha$ where

$$
\begin{array}{ccccc}
V_0 \otimes \mathcal{O}(C - \ell_\infty) & & \mathbb{C}^2 \otimes V_0 \otimes \mathcal{O} & & V_0 \otimes \mathcal{O}(\ell_\infty) \\
\oplus & \xrightarrow{\ \alpha\ } & \oplus & \xrightarrow{\ \beta\ } & \oplus \\
V_1 \otimes \mathcal{O}(-\ell_\infty) & & \widetilde{W} \otimes \mathcal{O} & & V_1 \otimes \mathcal{O}(-C + \ell_\infty)
\end{array},
$$

where

$$V_0 = H^1(E(-\ell_\infty)), \quad V_1 = H^1(E(C - 2\ell_\infty)),$$
$$\widetilde{W} = H^1(E(-\ell_\infty) \otimes \mathcal{Q}^\vee(C)),$$

and (2) α is injective and β is surjective. When we have identified the term of the complex, we have used that the multiplication by z_0 induces isomorphisms $H^1(E(C - 2\ell_\infty)) \cong H^1(E(C - \ell_\infty))$, which is proved as in [13, Lemma 2.4] during the course of the above proof.

6.3. A normal form

We next show that the above complex is of the form in (1.2). The homomorphisms α, β are induced from d^{-1}, d^0 in (3.1). Hence we have

$$
\alpha = \begin{bmatrix} \operatorname{id}\otimes z & B_1 \otimes z_0 \\ \operatorname{id}\otimes w & B_2 \otimes z_0 \\ 0 & \alpha_0 z_0 + \alpha_1 z_1 + \alpha_2 z_2 \end{bmatrix},
$$
$$
\beta = \begin{bmatrix} \operatorname{id}\otimes z_2 & -\operatorname{id}\otimes z_1 & \beta_0 z_0 + \beta_1 z_1 + \beta_2 z_2 \\ d \otimes w & -d \otimes z & \gamma_0 w + \gamma_1 z \end{bmatrix}
$$

for $\alpha_0,\ \alpha_1,\ \alpha_2 \in \operatorname{Hom}(V_1, \widetilde{W})$, $\beta_0,\ \beta_1,\ \beta_2 \in \operatorname{Hom}(\widetilde{W}, V_0)$, $\gamma_0,\ \gamma_1 \in \operatorname{Hom}(\widetilde{W}, V_1)$. Also B_1, B_2, d are induced homomorphisms of z', w', s' respectively.

From the form of d^0 and χ, we have $\beta_1 = \beta_2 = 0$. Now we restrict the complex to ℓ_∞ to get

$$V_1 \otimes \mathcal{O}_{\ell_\infty}(-\ell_\infty) \xrightarrow{\alpha_1 z_1 + \alpha_2 z_2} \widetilde{W} \otimes \mathcal{O}_{\ell_\infty} \xrightarrow{\gamma_0 w + \gamma_1 z} V_1 \otimes \mathcal{O}_{\ell_\infty}(\ell_\infty).$$

Then the argument in [13, §2] shows that the trivialization $E|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}^{\oplus r}$ induces a decomposition $\widetilde{W} \to V_1 \oplus V_1 \oplus W$ such that

$$\alpha_1 = \begin{bmatrix} \mathrm{id}_{V_1} \\ 0 \\ 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 \\ \mathrm{id}_{V_1} \\ 0 \end{bmatrix}, \quad \gamma_0 = \begin{bmatrix} \mathrm{id}_{V_1} & 0 & 0 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 0 & -\mathrm{id}_{V_1} & 0 \end{bmatrix}.$$

From the equation $\beta\alpha = 0$, we then get

$$\alpha_0 = \begin{bmatrix} -dB_1 \\ -dB_2 \\ j \end{bmatrix}, \quad \beta_0 = \begin{bmatrix} B_2 & -B_1 & i \end{bmatrix}$$

for $i\colon W \to V_0$, $j\colon V_1 \to W$.

Thus the complex is of the form in (1.2).

6.4. ζ-stability

By Lemma 5.1 the data $X = (B_1, B_2, d, i, j)$ satisfies the condition (S2) in Theorem 1.1. As $E(-mC)$ is perverse coherent, we have

$$\mathrm{Hom}(X, C_n) \cong \mathrm{Hom}(E(-mC), \mathcal{O}_C(m - n - 1)) = 0,$$
$$\mathrm{Hom}(C_{n'}, X) \cong \mathrm{Hom}(\mathcal{O}_C(m - n' - 1), E(-mC)) = 0$$

for $n' < m \le n$. By Proposition 4.2 X is ζ-stable for ζ satisfying (1.4).

6.5. The last part of the proof

Let us finally comment that maps constructed in the previous section and this section are mutually inverse.

The statement 'sheaf $\to$ ADHM $\to$ sheaf is an identity' is proved already in our construction of the data $X = (B_1, B_2, d, i, j)$ via the resolution of the diagonal.

The statement 'ADHM $\to$ sheaf $\to$ ADHM is an identity' means the uniqueness of X for a given framed sheaf E. But this is also already proved in Proposition 5.5.

Remark 6.2. As a corollary of our proof, we have an equivalence between the derived category $D^b(\mathrm{Coh}_C(\widehat{\mathbb{P}^2}))$ of complexes of coherent sheaves whose homologies are supported on C set-theoretically, and the derived category of finite dimensional nilpotent modules of the quiver

$$\bullet \underset{d}{\overset{B_1, B_2}{\rightleftarrows}} \bullet$$

with relation $B_1 dB_2 = B_2 dB_1$. In fact, the proof is much simpler, as we do not need to study the stability condition.

§7. Distinguished chambers

7.1. Torsion free sheaves on blowup

There are two distinguished chambers in the space of parameters for the stability conditions. The first one is the case when the stability condition is given by (S1)' and (S2). This is the chamber adjacent to the boundary $\zeta_0 + \zeta_1 = 0$ of the region (2.1). Let us take a parameter $^\infty\zeta$ from the chamber as in Figure 1. The corresponding moduli space is the framed moduli space of torsion free sheaves on $\widehat{\mathbb{P}}^2$. This follows from

Proposition 7.1. *Fix r, k, n. If m is sufficiently large, $E(-mC)$ is stable perverse coherent if and only if E is torsion free.*

This, in turn, follows from the following lemma once we observe that X is $^\infty\zeta$-stable if and only if it satisfies the condition (S2) and (S1)', and the sheaf E corresponding to X is torsion free if and only α is injective except finitely many points. We prove Theorem 1.1 simultaneously. For this we use the observation that the sheaf E corresponding to X is locally free if and only α is injective at every point in $\widehat{\mathbb{P}}^2$.

Lemma 7.2. *Let $X = (B_1, B_2, d, i, j) \in \mu^{-1}(0)$. Then*
(1)([6, 4.1.3]) X satisfies the condition (S1) in Theorem 1.1 if and only if α in the complex (1.2) is injective at every point in $\widehat{\mathbb{P}}^2$.
(2) X satisfies the condition (S1)' in §1 if and only if α is injective possibly except finitely many points.

Proof. The proof is taken from that in [14, Prop. 4.1]. In fact, it becomes much simpler.

Let us first show the 'only if' part: Assume α is not injective at a point $([z_0 : z_1 : z_2], [z : w])$. As α is injective along $z_0 = 0$, we may assume $z_0 = 1$. Then there exists $v_0 \in V_0$, $v_1 \in V_1$ such that

$$(z_1 - dB_1)v_1 = 0, \quad (z_2 - dB_2)v_1 = 0,$$
$$B_1 v_1 + z v_0 = 0, \quad B_2 v_1 + w v_0 = 0, \quad j v_1 = 0$$

and $(v_0, v_1) \neq (0, 0)$. From the equation, we have $v_1 \neq 0$, as $v_1 = 0$ implies $v_0 = 0$. Then $(S_0, S_1) = (\mathbb{C}v_0, \mathbb{C}v_1)$ violates the condition (S1). This completes the proof of 'only if' part for (1).

By the condition (S1)' we must have $\dim S_0 = 1$, i.e., $v_0 \neq 0$. We consider X' induced (from X) on the quotient $V_0' := V_0/S_0$, $V_1' := V_1/S_1$.

We claim X' satisfies the condition (S1)'. Let $S_0' \subset V_0'$, $S_1' \subset V_1'$ be subspaces as in the condition (S1)'. Then their inverse images $\widetilde{S}_0'$, $\widetilde{S}_1'$ in V_0, V_1 must satisfy $\dim \widetilde{S}_0' \geq \dim \widetilde{S}_1'$ or $\widetilde{S}_0' = \widetilde{S}_1' = 0$ as X satisfies

the condition (S1)'. As $\dim S_0 = \dim S_1 = 1$, we have $\dim S_0' \geq \dim S_1'$. This completes the proof of the claim.

By the induction on the dimensions of V_0, V_1 we may assume that α defined for X' is injective possibly except for finitely many points.

Let us consider the complex (1.2) for X, X' and (S_0, S_1) with $(B_1, B_2, d) = (z, w, s)$ as in Theorem 2.16. We have a commutative diagram:

$$\begin{array}{ccccc}
C^{-1}(S) & \xrightarrow{\alpha} & C^0(S) & \xrightarrow{\beta} & C^1(S) \\
\downarrow & & \downarrow & & \downarrow \\
C^{-1}(X) & \xrightarrow{\alpha} & C^0(X) & \xrightarrow{\beta} & C^1(X) \\
\downarrow & & \downarrow & & \downarrow \\
C^{-1}(X') & \xrightarrow{\alpha} & C^0(X') & \xrightarrow{\beta} & C^1(X')
\end{array}$$

From Lemma 3.3(2) (or a direct argument) α for S is injective except at the point $([z_0 : z_1 : z_2], [z : w])$. Therefore the injectivity of α for X fails only at this point or a point where α for X' is not injective. So we only have finitely many points.

The proof of the 'if' part can be given exactly as in Lemma 5.1, so is omitted.   Q.E.D.

Theorem 1.1 with Theorem 2.16 implies $\widehat{M}_\zeta^{ss}(r, k, n)$ is the Uhlenbeck compactification of the framed moduli $\widehat{M}_\zeta^s(r, k, n) = \widehat{M}_0^{reg}(r, k, n)$ of locally free sheaves on $\widehat{\mathbb{P}}^2$ if $\zeta_0 + \zeta_1 = 0$, $\zeta_0 < 0$.

7.2. Another distinguished chamber—torsion free sheaves on blow-down

We describe the other distinguished chamber. It is the chamber $\{\zeta_0 < 0, \zeta_1 < 0\}$. Let us denote a parameter from this chamber by ${}^0\zeta$. In this case the ${}^0\zeta$-stability condition just means that a pair of subspaces $T_0 \subset V_0$, $T_1 \subset V_1$ such that $B_\alpha(T_1) \subset T_0$, $d(T_0) \subset T_1$, $\operatorname{Im} i \subset T_0$ must be $T_0 = V_0$, $T_1 = V_1$. Then both (dB_1, dB_2, di, j) and $(B_1 d, B_2 d, i, jd)$ satisfy the stability condition for the framed moduli space of torsion free sheaves on $\mathbb{P}^2$ described in the beginning of §1. Therefore we have the morphisms $\widehat{M}_{0\zeta}(r, k, n) \to M(r, \dim V_0)$ and $\to M(r, \dim V_1)$. This gives a sharp contrast to moduli spaces from other chambers. In general, we only have morphisms into the Uhlenbeck (partial) compactification $M_0(r, \dim V_0)$ or $M_0(r, \dim V_1)$.

Lemma 7.3. *If $[(B_1, B_2, d, i, j)] \in \widehat{M}_{0\zeta}(r, k, n)$, then d is surjective. In particular, $\widehat{M}_{0\zeta}(r, k, n) = \emptyset$ if $k > 0$.*

Proof. We take $T_0 = V_0$, $T_1 = \operatorname{Im} d$. Then all conditions are satisfied, so we must have $T_1 = V_1$, i.e., d must be surjective. Q.E.D.

We now consider the case $k = 0$. Then the situation is surprisingly very simple:

Proposition 7.4. *The morphism $\widehat{M}{}^{\circ}_{\zeta}(r, 0, n) \to M(r, n)$ is an isomorphism.*

Proof. From the assumption $k = 0$ and Lemma 7.3, d must be an isomorphism. Therefore (B_1, B_2, d, i, j) can be recovered from (dB_1, dB_2, di, j). Conversely if $(\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j})$ is stable for $M(r, n)$, then $(\tilde{B}_1, \tilde{B}_2, \operatorname{id}_V, \tilde{i}, \tilde{j})$ gives a ${}^0\zeta$-stable representation. It gives the inverse of $M{}^{\circ}_{\zeta}(r, 0, n) \to M(r, n)$. Q.E.D.

The $c_1 \neq 0$ case will be studied in the subsequent paper.

Remark 7.5. If we take a parameter ζ from the adjacent chamber, i.e., $\zeta_0 < 0$, $0 < \zeta_1 \ll 1$, the ζ-stability implies that a pair of subspaces $T_0 \subset V_0$, $T_1 \subset V_1$ such that $B_\alpha(T_1) \subset T_0$, $d(T_0) \subset T_1$, $\operatorname{Im} i \subset T_0$ must be $T_0 = V_0$. Then $(B_1 d, B_2 d, i, jd)$ satisfy the stability condition for the framed moduli space of torsion free sheaves on $\mathbb{P}^2$. Therefore we have a morphism $\widehat{M}_\zeta(r, k, n) \to M(r, \dim V_0)$.

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