

Quantization of soliton systems and Langlands duality

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Abstract.

We consider the problem of quantization of classical soliton integrable systems, such as the KdV hierarchy, in the framework of a general formalism of Gaudin models associated to affine Kac–Moody algebras. Our experience with the Gaudin models associated to finite-dimensional simple Lie algebras suggests that the common eigenvalues of the mutually commuting quantum Hamiltonians in a model associated to an affine algebra $\widehat{\mathfrak{g}}$ should be encoded by affine opers associated to the Langlands dual affine algebra ${}^L\widehat{\mathfrak{g}}$. This leads us to some concrete predictions for the spectra of the quantum Hamiltonians of the soliton systems. In particular, for the KdV system the corresponding affine opers may be expressed as Schrödinger operators with spectral parameter, and our predictions in this case match those recently made by Bazhanov, Lukyanov and Zamolodchikov. This suggests that this and other recently found examples of the correspondence between quantum integrals of motion and differential operators may be viewed as special cases of the Langlands duality.

§1. Introduction

Soliton equations, such as the celebrated KdV hierarchy, are infinite-dimensional classical integrable systems possessing infinite collections of Poisson commuting Hamiltonians. It is natural to try to *quantize* these systems. This means, in particular, constructing mutually commuting quantum Hamiltonians in the appropriate associative algebra (such as the completed enveloping algebra of the Virasoro algebra in the case of the KdV hierarchy), whose symbols are the classical Hamiltonians, and

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finding the spectra of these quantum Hamiltonians on representations of this algebra. This problem has acquired additional importance after the work of A. Zamolodchikov [Z], in which he showed that these quantum Hamiltonians may be identified with integrals of motion of certain deformations of conformal field theories.

The existence of local commuting quantum Hamiltonians has been established in [FF4, FF5] in the case of the generalized KdV hierarchies and in [FF6] in the case of the generalized AKNS hierarchies. The corresponding non-local Hamiltonians have been constructed in [BLZ1, BHK] for the KdV hierarchies (we will discuss below an extension of this construction to the AKNS hierarchies). A study of the problem of diagonalization of the quantum KdV Hamiltonians was initiated in the important series of works [BLZ1]–[BLZ3] by V. Bazhanov, S. Lukyanov and A. Zamolodchikov.

These works culminated in a remarkable conjecture made in [BLZ5] (building on the earlier works [DT1, BLZ4]), which related the spectra of the *quantum KdV Hamiltonians* to certain one-dimensional *Schrödinger operators*. This connection appears to be rather mysterious. In this paper we suggest that it may be viewed as a special case of a broader picture which fits in the general framework of integrable systems of Gaudin type and the Langlands correspondence.

The first step in our approach is the realization that the quantum soliton systems may be viewed as special cases of *Gaudin models* associated to affine Kac–Moody algebras. At the classical level, this is quite clear. Indeed, it has been known since [RS1, DS] that the algebra of classical Poisson commuting Hamiltonians of the soliton hierarchies may be obtained using the “shift of argument” method, applied to an affine Kac–Moody algebra. Its finite-dimensional counterpart is the Poisson commuting “shift of argument” subalgebra constructed in [MF]. As explained in [FFT], the corresponding Hamiltonian system is the simplest example of a classical Gaudin model with irregular singularities (which is in turn a special case of the Hitchin systems [H, Bea]). Recently, a quantization of this algebra, as well as its generalizations corresponding to other Gaudin models, has been constructed in [R, FFT]. We expect that the soliton integrable systems may be quantized along the same lines, that is, in the framework of the quantization of Gaudin models.

The second step in our approach is the link between the spectra of the quantum *Gaudin Hamiltonians* and differential operators known as *opers*, discovered in our earlier works [FFR, F3, FFT]. We have shown there that the spectra of the Gaudin Hamiltonians associated to a simple finite-dimensional Lie algebra \mathfrak{g} are encoded by ${}^L\mathfrak{g}$ -opers (where ${}^L\mathfrak{g}$ is

the Langlands dual Lie algebra of \mathfrak{g}) on \mathbb{P}^1 with finitely many singular points.

The key point of our analysis [FFR, F3, FFT] of the Gaudin models associated to a simple Lie algebra \mathfrak{g} is the identification of the commutative algebra of quantum Gaudin Hamiltonians with a quotient of the center $Z(\widehat{\mathfrak{g}})$ of the completed enveloping algebra of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ at the critical level. Now we consider the Gaudin systems associated to an affine algebra $\widehat{\mathfrak{g}}$. Therefore it is natural to assume that the corresponding Hamiltonians come from the center $Z(\widehat{\mathfrak{g}})$ of the completed enveloping algebra of the *double affine algebra* $\widehat{\widehat{\mathfrak{g}}}$. At present, we do not have a precise definition of this enveloping algebra or its center. However, we postulate that it exhibits the same salient features as its affine counterpart $Z(\widehat{\mathfrak{g}})$.

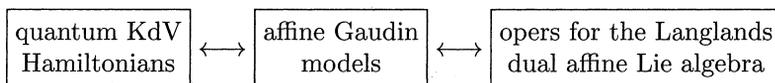
According to a theorem of [FF3, F4], $Z(\widehat{\mathfrak{g}})$ is isomorphic to the algebra of functions on the space $\text{Op}_{L_{\mathfrak{g}}}(D^{\times})$ of $L_{\mathfrak{g}}$ -opers on the (formal) punctured disc D^{\times} . Hence we expect that $Z(\widehat{\mathfrak{g}})$ is related to the space of *affineopers* $\text{Op}_{L_{\widehat{\mathfrak{g}}}}(D^{\times})$ associated to $L_{\widehat{\mathfrak{g}}}$, defined in [F2] (see also [BF]). Here $L_{\widehat{\mathfrak{g}}}$ is the Langlands dual Lie algebra to the affine algebra $\widehat{\mathfrak{g}}$, that is, its Cartan matrix is the transpose of that of $\widehat{\mathfrak{g}}$ (so the dual of an untwisted affine algebra might be a twisted affine algebra). Testing this conjectural relation was in fact one of the main motivations for the present work. Since the structure of the center $Z(\widehat{\mathfrak{g}})$ seems out of reach at the moment, we turn the tables and try to use the available information about the affine Gaudin models and the quantum soliton systems to gain insights into the structure of $Z(\widehat{\mathfrak{g}})$.

The upshot of all this is that the spectra of the quantum $\widehat{\mathfrak{g}}$ -KdV Hamiltonians should be represented by the affine $L_{\widehat{\mathfrak{g}}}$ -opers on \mathbb{P}^1 with prescribed singularities. In the case when $\mathfrak{g} = \mathfrak{sl}_2$ these opers may be represented by Schrödinger operators with a spectral parameter. Thus, we suggest that the connection between the spectra of quantum KdV Hamiltonians and Schrödinger operators conjectured in [BLZ5] comes about as follows:

- The quantum KdV system is interpreted as a generalized Gaudin model associated to the affine Kac–Moody algebra $\widehat{\mathfrak{sl}}_2$ (or $\widehat{\mathfrak{g}}$ for the more general quantum $\widehat{\mathfrak{g}}$ -KdV systems).
- The spectra of the Gaudin Hamiltonians associated to an affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ are encoded by the affine $L_{\widehat{\mathfrak{g}}}$ -opers.
- Since the quantum KdV Hamiltonians may be interpreted as the Gaudin Hamiltonians associated to $\widehat{\mathfrak{sl}}_2$, their spectra

should be encoded by $\widehat{\mathfrak{sl}}_2$ -opers, which are nothing but Schrödinger operators with spectral parameter: $\partial_t^2 - v(t) - \lambda$. More generally, we expect that the spectra of the quantum $\widehat{\mathfrak{g}}$ -KdV Hamiltonians are encoded by affine $L\widehat{\mathfrak{g}}$ -opers. These opers should satisfy an important *no monodromy* property.

Thus, schematically, the correspondence we propose looks as follows:



The appearance of the Langlands dual Lie algebra suggests that this connection (and perhaps the more general ODE/IM correspondence of Dorey, e.a., [DDT1, DMST, DDT2]) should be viewed as a special case of Langlands duality.

In order to test this proposal, we compare our $\widehat{\mathfrak{sl}}_2$ -opers with the Schrödinger operators obtained in [DT1, BLZ4, BLZ5]. We find that they match perfectly, after a simple change of variable. Our formulas easily generalize to an arbitrary affine Lie algebra $\widehat{\mathfrak{g}}$, and again we find that they are consistent with the known formulas for the differential operators associated to the spectra of the quantum Hamiltonians in [DDT1, BHK, DMST, DDT2].

In [DT1, BLZ4, DT2, DDT1, BHK, BLZ5] the connection between the spectra of the quantum KdV Hamiltonians and differential operators is explained by the fact that certain generating functions of the former (the so-called Q -operators) satisfy the same functional relations as connection coefficients (or Stokes multipliers) of the latter. These functional relations are the so-called $T - Q$ relations generalizing Baxter's famous relation. In [DMST, DDT2] (and earlier works referenced there) the functional relations of this type were taken as the basis of an "ODE/IM correspondence" for more general solvable models.

In contrast, in this paper we do not attempt to explain the connection between the spectra of quantum Hamiltonians and affine opers. Rather, our goal is to emphasize that the connection between them fits in the general scheme of quantum Gaudin models and is in fact a manifestation of the Langlands duality. This is based on our analysis [FFR, F3, FFT] in the finite-dimensional case and on the fact that in the affine case our predictions match those obtained in [DT1, BLZ4, DDT1, BHK, BLZ5]. This already has a number of consequences:

- (i) the appearance of the Langlands dual affine Lie algebra (something that would be difficult to see based on the most studied case of $\widehat{\mathfrak{sl}}_n$, which is self-dual);
- (ii) a uniform way of expressing the differential operators encoding the spectra of the quantum Hamiltonians as opers associated to arbitrary affine Lie algebras (again, this is not obvious from the previously known examples);
- (iii) a Bethe Ansatz method for constructing eigenvectors of the quantum Hamiltonians.

In particular, we conjecture explicit Bethe Ansatz formulas (generalizing the Bethe Ansatz in the finite-dimensional Gaudin models) for eigenvectors of the quantum KdV Hamiltonians in the irreducible representations of the Virasoro algebra from the unitary minimal models.

We expect that the appearance of the $T-Q$ relations of [DT1, BLZ4, DT2, DDT1, BHK, BLZ5] may also be explained in the framework of the formalism developed in this paper. More specifically, we believe that it comes from the connection between quantum integrals of motion and the center of a quantum affine algebra at the critical level. We plan to address these issues in a future work.

The paper is organized as follows. In Section 2 we recall the results of [R, FFT] on the quantization of the shift of argument Gaudin algebras in the finite-dimensional case and the identification of their spectra with ${}^L\mathfrak{g}$ -opers. We then consider in Section 3 the affine analogue of this construction which corresponds to the quantization of the Hamiltonians of the soliton hierarchies such as the \mathfrak{g} -AKNS. We discuss in detail the classical Hamiltonians obtained by expansion of the monodromy matrix with respect to the spectral parameter (following [RS1]) and the quantum analogues of both local and non-local Hamiltonians. We then construct eigenvectors of these Hamiltonians using an affine analogue of the Bethe Ansatz method. In order to describe the spectra of the quantum Hamiltonians, we introduce in Section 4 the affine opers, following [F2]. We conjecture that the affine opers that encode the spectra of the quantum Hamiltonians on representations of $\widehat{\mathfrak{g}}$ are ${}^L\widehat{\mathfrak{g}}$ -opers on \mathbb{P}^1 satisfying certain analytic properties (in particular, to integrable representations of $\widehat{\mathfrak{g}}$ correspond ${}^L\widehat{\mathfrak{g}}$ -opers without monodromy).

Next, in Section 5, we apply the same analysis to the generalized KdV hierarchies. We state our conjectures describing the spectra of the quantum KdV Hamiltonians in terms of the affine opers corresponding to the Langlands dual affine Lie algebra ${}^L\widehat{\mathfrak{g}}$. We write down explicit formulas for these opers in the case when $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$ and show that after

a simple change of variables they give rise to the Schrödinger operators constructed in [BLZ5]. We then discuss the generalization of these formulas to the case of $\widehat{\mathfrak{sl}}_n$ and to general affine Lie algebras.

Motivated by this study of the “shift of argument” soliton systems, corresponding to the simplest affine Gaudin models with irregular singularities, we introduce in Section 6 the affine analogues of the Gaudin models with regular singularities. We show that in this case we also have interesting classical integrable systems, which may be quantized by analogy with the finite-dimensional case. Again, we conjecture that the spectra of the corresponding quantum Hamiltonians are described by the affine ${}^L\widehat{\mathfrak{g}}$ -opers on \mathbb{P}^1 with regular singularities at finitely many points. We also construct the eigenvectors of these Hamiltonians using Bethe Ansatz. We show that a particular case of this model is closely related to the GKO coset construction [GKO]. We conjecture that in this case our Bethe vectors are eigenvectors of all quantum KdV Hamiltonians on irreducible modules over the Virasoro algebra from the unitary minimal models.

We summarize our results and conjectures in Section 7.

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§2. The shift of argument Gaudin models in the finite-dimensional case

We start by summarizing what is known about the simplest Gaudin models with irregular singularities, following [R, FFT]. The corresponding classical Hamiltonian system may be viewed as an example of a Hitchin system. The Poisson commutative algebra of classical Hamiltonians is the so-called “shift of argument” subalgebra $\overline{\mathcal{A}}_\chi$ of the symmetric algebra $S(\mathfrak{g})$ of a simple Lie algebra \mathfrak{g} , introduced by A. S. Mishchenko and A. T. Fomenko [MF] (see also [M1]). The corresponding algebra of quantum Hamiltonians is a commutative subalgebra of the universal enveloping algebra $U(\mathfrak{g})$. It has been constructed in [R, FFT] using the center of the completed universal enveloping algebra of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ at the critical level. This allowed us to identify in [FFT] the spectrum of this algebra with the space of ${}^L\mathfrak{g}$ -opers on \mathbb{P}^1 with regular singularity at $0 \in \mathbb{P}^1$ and irregular singularity at $\infty \in \mathbb{P}^1$ (here ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra to \mathfrak{g}).

We also consider here analogous commutative subalgebras in the Hamiltonian reductions of $U(\mathfrak{g})$ known as finite \mathcal{W} -algebras.

In the following sections we will use this construction as a prototype for the description of the quantum “shift of argument” subalgebra when the finite-dimensional Lie algebra \mathfrak{g} is replaced by its affinization $\widehat{\mathfrak{g}}$. It is well-known (see [RS1, DS]) that the corresponding classical systems give rise to hierarchies of soliton equations such as \mathfrak{g} -AKNS and $\widehat{\mathfrak{g}}$ -KdV. In other words, the classical shift of argument subalgebra is generated by the classical Hamiltonians of these hierarchies. Therefore the corresponding quantum Hamiltonians generate the quantum shift of argument subalgebra, and this suggests that their spectra are encoded by affine $L\widehat{\mathfrak{g}}$ -opers. We will find that in the case of the quantum KdV Hamiltonians this matches the predictions of [BLZ5].

2.1. Classical system

The Gaudin models with irregular singularities for finite-dimensional simple Lie algebras have been introduced and studied in [FFT]. Here we will focus on the particular case related to the so-called shift of argument subalgebra $\overline{\mathcal{A}}_\chi \subset S(\mathfrak{g})$ and its quantization $\mathcal{A}_\chi \subset U(\mathfrak{g})$, as it is most closely related to the structures emerging in soliton equations (such as the KdV and n -wave hierarchies) and their quantization, once we switch from \mathfrak{g} to the affine algebra $\widehat{\mathfrak{g}}$. But other Gaudin models with irregular singularities may also have interesting analogues in the affine case.

Our starting point is a classical Hitchin system (see [H, Bea]) in which the phase space is the space of the following \mathfrak{g}^* -valued one-forms on \mathbb{P}^1 :

$$(2.1) \quad \eta = \left(\frac{A}{z} + \chi \right) dz, \quad A \in \mathfrak{g}^*,$$

where $\chi \in \mathfrak{g}^*$ is fixed. This space is therefore isomorphic to \mathfrak{g}^* . A one-form like this has a pole of order 2 at the point ∞ with the leading coefficient $-\chi$, and it also has a pole of order 1 at the point 0.

We define the classical shift of argument algebra $\overline{\mathcal{A}}_\chi$ as the subalgebra of $S(\mathfrak{g}) = \text{Fun } \mathfrak{g}^*$ generated by the coefficients of the polynomials in z^{-1} obtained by evaluating the invariant polynomials $P \in (\text{Fun } \mathfrak{g}^*)^G$, $i = 1, \dots, \ell$, on η . In other words, $\overline{\mathcal{A}}_\chi$ is the subalgebra of $\text{Fun } \mathfrak{g}^*$ generated by the iterated directional derivatives $D_\chi^i P$ of invariant polynomials $P \in (\text{Fun } \mathfrak{g}^*)^G$ in the direction χ , where

$$(2.2) \quad D_\chi P(y) = \left. \frac{d}{du} \right|_{u=0} P(y + u\chi).$$

Equivalently, $\overline{\mathcal{A}}_\chi$ is the subalgebra of $S\mathfrak{g}$ generated by the shifted polynomials

$$(2.3) \quad P_{u\chi}(y) = P(y + u\chi)$$

where P varies in $(\text{Fun } \mathfrak{g}^*)^G$ and $u \in \mathbb{C}$, hence its name.

The algebra $\overline{\mathcal{A}}_\chi$ is Poisson commutative with respect to the Kirillov–Kostant Poisson structure on \mathfrak{g}^* . This follows from the general results on the Hitchin systems with singularities.

The algebra $\overline{\mathcal{A}}_\chi$ was first introduced in [MF]. It has been shown in [MF, Ko, FFT] that for regular χ the algebra $\overline{\mathcal{A}}_\chi$ is a maximal Poisson commutative subalgebra of $S(\mathfrak{g})$ which is a free polynomial algebra in $\dim \mathfrak{b}$ generators

$$(2.4) \quad D_\chi^{n_i} \overline{P}_i, \quad i = 1, \dots, \ell; \quad n_i = 0, \dots, d_i,$$

where \overline{P}_i is a generator of $S(\mathfrak{g})^{\mathfrak{a}}$ of degree $d_i + 1$ and D_χ is the derivative in the direction of χ given by formula (2.2).¹

Note that for any χ the algebra $\overline{\mathcal{A}}_\chi$ belongs to the centralizer $S(\mathfrak{g})^{\mathfrak{a}_\chi}$ of \mathfrak{g}_χ in $S(\mathfrak{g})$, where $\mathfrak{g}_\chi \subset \mathfrak{g}$ is the stabilizer of χ . Let $\psi \in \mathfrak{g}_\chi^*$ be a character of \mathfrak{g}_χ , or, in other words, a one-point coadjoint orbit in \mathfrak{g}_χ^* . Set

$$S_\chi^\psi(\mathfrak{g}) = (S(\mathfrak{g})/(\mathfrak{g}_\chi - \psi(\mathfrak{g}_\chi)))^{\mathfrak{a}_\chi},$$

where $(\mathfrak{g}_\chi - \psi(\mathfrak{g}_\chi))$ is the ideal in $S(\mathfrak{g})$ generated by the elements $A - \psi(A)$, $A \in \mathfrak{g}_\chi$. The algebra $S_\chi^\psi(\mathfrak{g})$ is a Poisson algebra obtained by Hamiltonian reduction of \mathfrak{g}^* with respect to the adjoint action of G_χ and the one-point orbit $\psi \in \mathfrak{g}_\chi^*$. Let $\overline{\mathcal{A}}_\chi^\psi$ be the projection of $\overline{\mathcal{A}}_\chi$ onto $S_\chi^\psi(\mathfrak{g})$. This is a Poisson commutative subalgebra of $S_\chi^\psi(\mathfrak{g})$.

For regular χ the Lie algebra \mathfrak{g}_χ is commutative, and its symmetric algebra $S(\mathfrak{g}_\chi)$ is contained in $\overline{\mathcal{A}}_\chi$. In fact, a basis of \mathfrak{g}_χ may be obtained by taking the maximal non-constant directional derivatives $D_\chi^{d_i} \overline{P}_i$, $i = 1, \dots, \ell$, among the generators (2.4). Therefore for any $\psi \in \mathfrak{g}_\chi^*$ the algebra $\overline{\mathcal{A}}_\chi^\psi$ is a free polynomial algebra generated by the (non-zero) images of

$$D_\chi^{n_i} \overline{P}_i, \quad i = 1, \dots, \ell; \quad n_i = 0, \dots, d_i - 1,$$

in $S_\chi^\psi(\mathfrak{g})$.

¹We recall that $a \in \mathfrak{g}$ is called *regular* if its centralizer in \mathfrak{g} has the smallest possible dimension; namely, ℓ , the rank of \mathfrak{g} . We identify \mathfrak{g}^* with \mathfrak{g} by using a non-degenerate invariant inner product.

For nilpotent elements χ the algebras $S_\chi^\psi(\mathfrak{g})$ are known as the (classical) *finite \mathcal{W} -algebras*, and the above construction produces Poisson commutative subalgebras in these \mathcal{W} -algebras.

For example, suppose that χ corresponds to the maximal root generator $f_\theta \in \mathfrak{n}_- \subset \mathfrak{g}$ under an identification $\mathfrak{g} \simeq \mathfrak{g}^*$ given by a non-degenerate invariant inner product on \mathfrak{g} . Then $\mathfrak{g}_\chi = \mathfrak{n}_-$. Let ψ be a principal character on \mathfrak{n}_- taking non-zero values on the simple root generators f_i . We have $S_\chi^\psi(\mathfrak{g}) = S(\mathfrak{g})^\mathfrak{g}$ in this case, and therefore it is clear that $\overline{\mathcal{A}}_\chi^\psi = S(\mathfrak{g})^\mathfrak{g}$ as well. However, the affine analogue of this construction is non-trivial: the analogue of $S_\chi^\psi(\mathfrak{g})$ is the classical \mathcal{W} -algebra obtained by the Drinfeld–Sokolov reduction of $\widehat{\mathfrak{g}}_1^*$, and the analogue of $\overline{\mathcal{A}}_\chi^\psi$ is its Poisson commutative subalgebra generated by the (local and non-local) Hamiltonians of the $\widehat{\mathfrak{g}}$ -KdV hierarchy (see Section 5).

2.2. Quantization

Let us now discuss the quantization of the algebra $\overline{\mathcal{A}}_\chi$ in the finite-dimensional case. Here by quantization we understand a commutative subalgebra \mathcal{A}_χ of $U(\mathfrak{g})$ such that the associated graded $\text{gr } \mathcal{A}_\chi$ of \mathcal{A}_χ (with respect to the standard filtration on $U(\mathfrak{g})$) is a Poisson commutative subalgebra of $\text{gr } U(\mathfrak{g}) = S(\mathfrak{g})$ which contains $\overline{\mathcal{A}}_\chi$. Such a quantization has been constructed in [R, FFT] in a uniform way for an arbitrary simple Lie algebra \mathfrak{g} (following the procedure that had been used for constructing the algebra of quantum Hamiltonians in the Gaudin models with regular singularities in [FFR, F3], see Section 6.1 below).²

Let $\widehat{\mathfrak{g}}$ be the affine Kac–Moody algebra associated to \mathfrak{g} (see Section 3.1), and

$$(2.5) \quad \mathbb{V}_{\text{crit}} = \text{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}} \mathbb{C}_{-h^\vee}$$

the vacuum $\widehat{\mathfrak{g}}$ -module of critical level $k = -h^\vee$, where h^\vee is the dual Coxeter number of \mathfrak{g} (in this formula $\mathfrak{g}[[t]]$ acts by 0 on the one-dimensional module \mathbb{C}_{-h^\vee} , and $\mathbf{1}$ acts by multiplication by $-h^\vee$). The quantum shift of argument algebra \mathcal{A}_χ was obtained in [R, FFT] as a quotient of the algebra $\text{End}_{\widehat{\mathfrak{g}}} \mathbb{V}_{\text{crit}}$. According to a theorem of [FF3, F4], the latter algebra is isomorphic to the algebra of functions on the space $\text{Op}_{L\mathfrak{g}}(D)$ of $L\mathfrak{g}$ -opers on the (formal) disc. Here $L\mathfrak{g}$ is the Langlands dual Lie algebra to \mathfrak{g} , whose Cartan matrix is the transpose of the Cartan matrix of \mathfrak{g} . The notion of oper was introduced by Beilinson and Drinfeld

²For classical Lie algebras the quantization of $\overline{\mathcal{A}}_\chi$ had been constructed previously in a different way in [NO, T, CT].

[BD1, BD2] following Drinfeld and Sokolov [DS]. We refer the reader to the definition given in [F2, F3], which is well-suited for our present goals. (We will recall this definition below in Section 4.)

Thus, we have an isomorphism

$$(2.6) \quad \text{End}_{\widehat{\mathfrak{g}}} \mathbb{V}_{\text{crit}} \simeq \text{Fun Op}_{L_{\mathfrak{g}}}(D).$$

We note that $\text{End}_{\widehat{\mathfrak{g}}} \mathbb{V}_{\text{crit}}$ is the quotient of the center $Z(\widehat{\mathfrak{g}})$ of the completed enveloping algebra of $\widehat{\mathfrak{g}}$ at the critical level, which is isomorphic to the algebra of functions on the space $\text{Op}_{L_{\mathfrak{g}}}(D^\times)$ of opers on the (formal) punctured disc (see, e.g., [F6], Sect. 4.3).

Using the isomorphism (2.6) and general results on coinvariants from [FB], we have obtained the following description of \mathcal{A}_χ in [FFT]:

Theorem 1. *For regular $\chi \in \mathfrak{g}^*$ the algebra \mathcal{A}_χ is canonically isomorphic to the algebra of functions on the space $\text{Op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)_{\pi(\chi)}$ of $L_{\mathfrak{g}}$ -opers on \mathbb{P}^1 with regular singularity at the point 0 and with irregular singularity of order 2 at ∞ , with the 2-residue $\pi(\chi)$:*

$$(2.7) \quad \mathcal{A}_\chi \simeq \text{Op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)_{\pi(\chi)}.$$

For example, in the case when $\mathfrak{g} = \mathfrak{sl}_2$ and $\chi \in \mathfrak{h}^*$, the space $\text{Op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)_{\pi(\chi)}$ consists of second order differential operators of the form

$$(2.8) \quad \partial_z^2 - \chi^2 + \frac{v_{-1}}{z} + \frac{v_{-2}}{z^2}, \quad v_{-1}, v_{-2} \in \mathbb{C}.$$

Note that they have a pole of order 4 at $z = \infty$ with the leading term $-\chi^2$. For $\mathfrak{g} = \mathfrak{sl}_n$ the role of operators (2.8) is played by the following n th order differential operators:

$$(-\partial_z)^n - v_1(z)(-\partial_z)^{n-2} - \dots + v_{n-2}(z)\partial_z - v_{n-1}(z),$$

where $v_j(z)$ has the form

$$v_j(z) = P_j(\chi) + \sum_{k=1}^{j+1} v_{j,k} z^{-k},$$

and P_j is a particular symmetric polynomial of degree $j + 1$.

For an explicit description of $\text{Op}_{L_{\mathfrak{g}}}(\mathbb{P}^1)_{\pi(\chi)}$ for general simple Lie algebras, see [FFT].

It has been proved in [R, FFT] that $\text{gr } \mathcal{A}_\chi$ contains $\overline{\mathcal{A}}_\chi$, with the equality for regular χ . For general $\chi \in \mathfrak{g}^*$ this equality is a conjecture (see Conjecture 1 of [FFT]).

Let us summarize: the quantization \mathcal{A}_χ of the classical Poisson commutative algebra $\overline{\mathcal{A}}_\chi$ is a commutative subalgebra in a quantum (non-commutative) algebra $U(\mathfrak{g})$, but it turns out to be isomorphic to another classical commutative algebra, namely the algebra of functions on the space $\text{Op}_{L\mathfrak{g}}(\mathbb{P}^1)_{\pi(\chi)}$. A conceptual understanding of this isomorphism is obtained by realizing \mathcal{A}_χ as a quotient of the center of an enveloping algebra of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ and using the identification of the latter with the algebra of functions on opers. The non-triviality of this isomorphism is underscored by the fact that the Lie algebra \mathfrak{g} gets replaced by its Langlands dual Lie algebra ${}^L\mathfrak{g}$ under this isomorphism. In fact, this isomorphism may be viewed as an example of the geometric Langlands correspondence, as explained in [F1, F5, FFT].

The Poisson commutative subalgebras $\overline{\mathcal{A}}_\chi^\psi \subset S_\chi^\psi(\mathfrak{g})$ may also be quantized. Namely, by construction, \mathcal{A}_χ is contained in $U(\mathfrak{g})^{\mathfrak{g}_\chi}$. For a character $\psi : \mathfrak{g}_\chi \rightarrow \mathbb{C}$, let \mathcal{A}_χ^ψ be the projection of \mathcal{A}_χ onto

$$U_\chi^\psi(\mathfrak{g}) = (U(\mathfrak{g})/(\mathfrak{g}_\chi - \psi(\mathfrak{g}_\chi)))^{\mathfrak{g}_\chi},$$

where $(\mathfrak{g}_\chi - \psi(\mathfrak{g}_\chi))$ is the left ideal in $U(\mathfrak{g})$ generated by $A - \psi(A)$, $A \in \mathfrak{g}_\chi$. Thus, $U_\chi^\psi(\mathfrak{g})$ is the result of quantum Hamiltonian (or BRST) reduction of $U(\mathfrak{g})$ with respect to \mathfrak{g}_χ and its character ψ . In particular, if χ is nilpotent, $U_\chi^\psi(\mathfrak{g})$ is a quantum finite \mathcal{W} -algebra, and so \mathcal{A}_χ^ψ gives rise to its commutative subalgebra. For example, if χ corresponds to the maximal root generator $f_\theta \in \mathfrak{n}_- \subset \mathfrak{g}$ and ψ is a principal character on \mathfrak{n}_- (as in Section 2.1), then $\mathcal{A}_\chi^\psi = U_\chi^\psi(\mathfrak{g}) = Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$.

2.3. The spectra of the quantum Hamiltonians

Suppose that $\chi \in \mathfrak{g}^* \simeq \mathfrak{g}$ is regular semi-simple. Without loss of generality we may assume that χ is in a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The algebra \mathcal{A}_χ then contains \mathfrak{h} . It is natural to try to describe the spectrum of \mathcal{A}_χ on a given \mathfrak{g} -module M . Note that since $\mathfrak{h} \subset \mathcal{A}_\chi$, we obtain that the action of \mathcal{A}_χ will preserve the weight decomposition of M . The isomorphism (2.7) implies that the common eigenvalues of \mathcal{A}_χ on M are encoded by ${}^L\widehat{\mathfrak{g}}$ -opers on \mathbb{P}^1 with regular singularity at the point 0 and with singularity of order 2 at ∞ , with the 2-residue $\pi(\chi)$ (which is, by definition, an element of \mathfrak{h}^*/W , see, e.g., [FFT]). In [FFT], Theorem 5.10, we have obtained the following more precise description of the common spectra of \mathcal{A}_χ on the Verma modules and irreducible finite-dimensional representations of \mathfrak{g} .

Theorem 2.

(1) *If $M = M_\nu$, the Verma module with highest weight $\nu \in \mathfrak{h}^*$, then the common eigenvalues of \mathcal{A}_χ on M are encoded by ${}^L\mathfrak{g}$ -opers on*

\mathbb{P}^1 with singularity of order 2 at ∞ , with the 2-residue $\pi(\chi)$ and with regular singularity at the point 0 and the residue (which is, by definition, an element of \mathfrak{h}^*/W , see, e.g., [FFT]) equal to $\varpi(-\nu - \rho)$, where ϖ is the projection $\mathfrak{h}^* \rightarrow \mathfrak{h}^*/W$.

(2) If $M = V_\nu$ is an irreducible finite-dimensional representation of \mathfrak{g} , then the common eigenvalues of \mathcal{A}_χ on M are encoded by the $L\mathfrak{g}$ -opers in part (1) satisfying the additional condition that they have trivial monodromy.

According to Conjecture 2 of [FFT], all opers described in part (2) of this theorem are realized in the spectrum of \mathcal{A}_χ on V_ν .

This is analogous to the description of the spectra of the Hamiltonians in the Gaudin models with regular singularities from [FFR, F3] which we will recall in Section 6.1.

In [FFT] we have constructed eigenvectors of \mathcal{A}_χ by Bethe Ansatz. Suppose that $M = M_\nu$. Then the Bethe eigenvectors have the form

$$(2.9) \quad \phi(w_1^{i_1}, \dots, w_m^{i_m}) = \sum_{\sigma \in S_m} \frac{f_{i_{\sigma(1)}} f_{i_{\sigma(2)}} \cdots f_{i_{\sigma(m)}}}{(w_{\sigma(1)} - w_{\sigma(2)})(w_{\sigma(2)} - w_{\sigma(3)}) \cdots (w_{\sigma(m-1)} - w_{\sigma(m)})w_{\sigma(m)}} v_\nu,$$

where v_ν is the highest weight vector of M_ν , and the sum is over all permutations on m letters. This vector has the weight

$$\nu - \sum_{j=1}^m \alpha_{i_j}.$$

The following result is proved in [FFT], Sect. 6.6 (for the definition of Miura transformation, see [FFT] and Section 4 below).

Theorem 3. *The Bethe vector $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ is an eigenvector of \mathcal{A}_χ if the complex numbers w_j satisfy the Bethe Ansatz equations³*

$$(2.10) \quad \frac{\langle \check{\alpha}_{i_j}, \nu \rangle}{w_j} - \sum_{s \neq j} \frac{\langle \check{\alpha}_{i_j}, \alpha_{i_s} \rangle}{w_j - w_s} + \langle \check{\alpha}_{i_j}, \chi \rangle = 0, \quad j = 1, \dots, m.$$

The corresponding eigenvalues of elements of \mathcal{A}_χ are encoded by the $L\mathfrak{g}$ -opers in $\text{Op}_{L\mathfrak{g}}(\mathbb{P}^1)_{\pi(\chi)}$ obtained by applying the Miura transformation to the connection

$$(2.11) \quad \bar{\nabla} = \partial_z - \chi - \frac{\nu}{z} + \sum_{j=1}^m \frac{\alpha_{i_j}}{z - w_j}$$

³Note that our χ here corresponds to $-\chi$ in [FFT].

on the LH -bundle Ω^p on \mathbb{P}^1

Conjecturally, these Bethe eigenvectors form an eigenbasis of M_ν for generic χ and ν , and a basis of V_ν for a dominant integral ν and generic χ (in this case the above weights should be in the set of weights of V_ν).

Finally, we note that some of the Hamiltonians in \mathcal{A}_χ may be constructed explicitly. Assume again that $\chi \in \mathfrak{h} \subset \mathfrak{g}$ is regular semi-simple. In this case the span of all elements of \mathcal{A}_χ of order less than or equal to 1 (with respect to the standard filtration on $U(\mathfrak{g})$) is the direct sum of the Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_\chi \subset \mathfrak{g} \subset U(\mathfrak{g})$ and the scalars $\mathbb{C} \subset U(\mathfrak{g})$. Next, we have the following quadratic elements:

$$(2.12) \quad T_\gamma(\chi) = \sum_{\alpha \in \Delta_+} \frac{(\alpha, \gamma)}{(\alpha, \chi)} (f_\alpha e_\alpha + e_\alpha f_\alpha), \quad \gamma \in \mathfrak{h}^*,$$

where the elements e_α, f_α generate the \mathfrak{sl}_2 subalgebra \mathfrak{g}_α of \mathfrak{g} corresponding to $\alpha \in \Delta_+$, and they are normalized in such a way that $\kappa_0(e_\alpha, f_\alpha) = 1$, where κ_0 is a non-degenerate invariant inner product on \mathfrak{g} . These are the *DMT Hamiltonians* introduced by C. De Concini (unpublished) and by J. Millson and V. Toledano Laredo [MTL, TL] (see also [FMTV]).⁴ Explicit formulas for higher order Hamiltonians are unknown in general. However, for $\mathfrak{g} = \mathfrak{sl}_n$ explicit formulas have been found in [T, CT].

§3. The shift of argument affine Gaudin models

We now try to generalize the above construction to the situation where a finite-dimensional simple Lie algebra \mathfrak{g} is replaced by the corresponding (extended) affine Kac–Moody algebra $\tilde{\mathfrak{g}}$.

In what follows, we will focus on the case when $\tilde{\mathfrak{g}}$ is an untwisted affine algebra. However, all of our definitions, results and conjectures presented below generalize in a straightforward way to the case of twisted affine algebras.

3.1. Affine Kac–Moody algebras

Let

$$(3.1) \quad 0 \rightarrow \mathbb{C}\mathbf{1} \rightarrow \widehat{\mathfrak{g}} \rightarrow \mathfrak{g}((t)) \rightarrow 0$$

⁴We note that quasi-classical versions of these Hamiltonians have appeared in [Boa] in the study of isomonodromic equations with irregular singularities.

be the Kac–Moody central extension of $\mathfrak{g}((t))$ with the commutation relations

$$\begin{aligned} [A \otimes f(t), B \otimes g(t)] &= [A, B] \otimes fg - \kappa_0(A, B) \operatorname{Res}_{t=0} fdg \cdot \mathbf{1}, \\ [\mathbf{1}, A \otimes f(t)] &= 0, \end{aligned}$$

and κ_0 is the invariant inner product on \mathfrak{g} normalized in such a way that the maximal root has squared length 2. (We will sometimes write (A, B) instead of $\kappa_0(A, B)$.) Let $\tilde{\mathfrak{g}}$ be the corresponding *extended affine Kac–Moody algebra*

$$\tilde{\mathfrak{g}} = \mathbb{C}d \ltimes \hat{\mathfrak{g}},$$

where d acts trivially on $\mathbf{1}$ and as the outer derivation $t\partial_t$ on $\mathfrak{g}((t))$:

$$[d, A \otimes f(t)] = A \otimes t\partial_t f(t).$$

Let us fix a Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

Then the Lie algebra $\tilde{\mathfrak{g}}$ has the following Cartan decomposition:

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+,$$

where

$$\begin{aligned} \tilde{\mathfrak{n}}_+ &= (\mathfrak{n}_+ \otimes \mathbf{1}) \oplus (\mathfrak{g} \otimes t\mathbb{C}[[t]]), \\ \tilde{\mathfrak{n}}_- &= (\mathfrak{n}_- \otimes \mathbf{1}) \oplus (\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]), \\ \tilde{\mathfrak{h}} &= (\mathfrak{h} \otimes \mathbf{1}) \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}d. \end{aligned}$$

The Lie algebra $\tilde{\mathfrak{g}}$ is a symmetrizable Kac–Moody algebra (see [K]). We denote by \tilde{I} the set of vertices of the Dynkin diagram of $\tilde{\mathfrak{g}}$. Thus, $\tilde{I} = I \cup \{0\}$, where $I = \{1, \dots, \ell\}$ is the set of vertices of the Dynkin diagram of \mathfrak{g} . We have the generators e_i of $\tilde{\mathfrak{n}}_+$, f_i of $\tilde{\mathfrak{n}}_-$, $h_i = \tilde{\alpha}_i$ and $\mathbf{1}$ of $\tilde{\mathfrak{h}}$, where in all cases $i \in \tilde{I}$. We also denote by $\alpha_i \in \tilde{\mathfrak{h}}^*$, $i \in \tilde{I}$, the simple roots of $\tilde{\mathfrak{g}}$.

A generalization of the shift of argument method to affine Kac–Moody algebras is well-known to give rise, classically, to Poisson commuting Hamiltonians of a large class of soliton equations. This point of view on soliton equations was explained by A. Reyman and M. Semenov-Tian-Shansky [RS1] (see also related works [M2, RSF]) and has been widely used ever since. In particular, this construction was subsequently generalized by V. Drinfeld and V. Sokolov [DS] to include Hamiltonian reductions (this yields the generalized KdV hierarchies, as we explain

in Section 5 below). In this section and the next we will discuss the affine Gaudin models, both classical and quantum, corresponding to the regular semi-simple shift. They realize the \mathfrak{g} -AKNS soliton models (or n -wave systems). Then in Section 5 we will consider the $\widehat{\mathfrak{g}}$ -KdV models corresponding to the Drinfeld–Sokolov reduction of the shift by the nilpotent maximal root element.

3.2. The classical model

Consider the case when the shift is a regular semi-simple element

$$\chi \in \mathfrak{h} \subset \mathfrak{g} \subset \widetilde{\mathfrak{g}}.$$

The phase space is then an affine analogue of the space of one-forms (2.1), so it consists of the one-forms

$$(3.2) \quad \eta = \left(\frac{A}{z} + \chi \right) dz, \quad A \in \widetilde{\mathfrak{g}}^*.$$

Instead of the full dual space $\widetilde{\mathfrak{g}}^*$ will consider its codimension 2 affine subspace which consists of the functionals taking the value $k \neq 0$ on the central element $\mathbf{1}$ and 0 on the element d . In other words, this is the affine hyperplane $\widehat{\mathfrak{g}}_k^* \subset \widehat{\mathfrak{g}}^*$ which consists of the linear functionals on $\widehat{\mathfrak{g}}$ taking value k on $\mathbf{1}$. It is endowed with the Kirillov–Kostant Poisson structure, and it is well-known that it may be identified with the space of k -connections

$$(3.3) \quad k\partial_t + A(t), \quad A(t) \in \mathfrak{g}^*((t)) \simeq \mathfrak{g}((t))$$

(where the second isomorphism is obtained using the non-degenerate invariant inner product κ_0 on \mathfrak{g}) on the punctured disc, so that the adjoint action of $G((t))$ on $\widehat{\mathfrak{g}}_k^*$ becomes the action of $G((t))$ on k -connections by gauge transformations (see [RS1, FB]).

By rescaling $A(t) \mapsto \frac{1}{k}A(t)$, we may identify $\widehat{\mathfrak{g}}_k^*$ and $\widehat{\mathfrak{g}}_1^*$, so without loss of generality we may, and will, set $k = 1$.

Thus, we replace η in (3.2) by

$$\eta = \left(\frac{\partial_t + A(t)}{z} + \chi \right) dz, \quad A(t) \in \mathfrak{g}((t))$$

and further by the equivalent expression

$$(3.4) \quad L = \partial_t + A(t) + \chi z.$$

We now recognize it as the L -operator of the so-called the \mathfrak{g} -AKNS hierarchy, also known as the generalized n -wave hierarchy (see, e.g., [RS1, DS]).

The Poisson commuting Hamiltonians of the corresponding integrable system are obtained by expanding in z (or z^{-1}) the invariants of the monodromy of the connection. Denote by $M(z) \in G$ (where G is the connected simply-connected Lie group with the Lie algebra \mathfrak{g}) the formal monodromy matrix of the connection operator L given by formula (3.4). More precisely, $M(z)$ is defined as follows: its action on any finite-dimensional representation V of G is obtained by computing the monodromy of the matrix differential operator $\partial_t + A(t) + \chi z$ acting on V .⁵

Now, for any function φ on G which is invariant under the adjoint action of G , the Hamiltonian

$$H_\varphi(z) = \varphi(M(z))$$

is a gauge invariant function of L given by formula (3.4), depending on the spectral parameter z . One shows in the same way as for the Hitchin systems in the finite-dimensional case that

$$(3.5) \quad \{H_\varphi(z), H_\psi(y)\} = 0, \quad \forall \varphi, \psi \in \text{Fun}(G)^G, \quad z, y \in \mathbb{C}^\times.$$

Therefore we may obtain infinite collections of commuting Hamiltonians of the affine Hitchin model by expanding the Hamiltonians $H_\varphi(z)$ around the points 0 and $\infty \in \mathbb{P}^1$.

The asymptotic expansion of $H_\varphi(z)$ at $z = \infty$ gives rise to the local Hamiltonians, in the sense that they may be expressed as integrals of differential polynomials in the matrix coefficients of $A(t)$ (in the adjoint representation). There are standard methods for constructing these Hamiltonians explicitly, see [RS1, DS, FF6]. These Hamiltonians generate the generalized \mathfrak{g} -AKNS (or n -wave) hierarchy. Since the algebra of invariant functions on G is a polynomial algebra with $\ell = \text{rank}(\mathfrak{g})$ generators, we find that there are ℓ local Hamiltonians of each positive spin.

On the other hand, the expansion of $H_\varphi(z)$ around $z = 0$ gives rise to the non-local Hamiltonians. Let us consider the simplest examples of such Hamiltonians obtained by taking the z -linear terms of $H_\varphi(z)$.

The computation of these linear terms is based on the following observation which is well-known in the study of the inverse scattering problem (for a proof, see, e.g., [RS2], Lemma 4.2.4).

⁵Strictly speaking, this monodromy is defined for connections that converge on a small punctured disc around 0 ; however, the coefficients in the expansions that we will consider are also well-defined when $A(t)$ is a formal Laurent power series.

Lemma 1. *Suppose that we have a connection of the form*

$$(3.6) \quad \partial_t + A^{(0)}(t) + \epsilon A^{(1)}(t) + \dots, \quad A^{(i)}(t) \in \mathfrak{g}((t)).$$

Then the ϵ -linear term in the expansion of $\varphi(M(\epsilon))$, where $M(\epsilon) \in G$ is the monodromy of this connection and φ is an invariant function on G , is equal to

$$(3.7) \quad \int (A^{(1)}(t), \Psi_\varphi(t)) dt$$

(the integral is taken over a contour around 0), where $\Psi_\varphi(t)$ is a solution of the equation

$$(3.8) \quad \partial_t \Psi_\varphi(t) + [A^{(0)}(t), \Psi_\varphi(t)] = 0,$$

which is single-valued (i.e., has no monodromy around 0), so that the above contour integral is well-defined.

More concretely, let us write

$$A^{(0)}(t) = \sum_a J_a^{(0)}(t) J^a, \quad A^{(1)}(t) = \sum_a J_a^{(1)}(t) J^a,$$

where $\{J^a\}$ is a basis of \mathfrak{g} . Let $\Psi_b^a(t)$ be the matrix elements of the fundamental solution of the equation (3.8), which may be written explicitly as follows:

$$\partial_t \Psi_b^a(t) + \sum_c J_c^{(0)}(t) \sum_d f_d^{cb} \Psi_b^d(t) = 0,$$

where $\{f_d^{cb}\}$ are the structure constants of \mathfrak{g} . Then the integral in (3.7) is equal to

$$(3.9) \quad \sum_a \int J_a^{(1)}(t) \Psi_\varphi^a(t) dt,$$

where

$$\Psi_\varphi^a(t) = \sum_b c_b(\varphi) J^b \Psi_b^a(t),$$

with the coefficients $c_b(\varphi)$ determined by φ .

Since the algebra of invariant functions on G is isomorphic to the polynomial algebra in $\ell = \text{rank}(\mathfrak{g})$ generators, the elements $\Psi_\varphi^a(t)$, where φ are invariant functions on G , span an ℓ -dimensional subspace in the space of all solutions of (3.8). The corresponding integrals (3.7) give rise to commuting Hamiltonians of an integrable system on the space

of connections of the form (3.6) (in which ϵ is taken as the spectral parameter).

We now apply this in our case, which fits in the statement of Lemma 1 with $\epsilon = z$, $A^{(0)}(t) = A(t)$ and $A^{(1)}(t) = \chi$. Then we obtain that the z -linear term in the expansion of $\varphi(M(z))$, where $\varphi \in (\text{Fun } G)^G$, is given by the formula

$$(3.10) \quad H_\varphi(\chi) = \int \langle \chi, \Psi_\varphi(t) \rangle dt.$$

The Poisson commutativity (3.5) of the Hamiltonians H_φ then implies that

$$\{H_\varphi(\chi), H_\psi(\chi)\} = 0, \quad \varphi, \psi \in (\text{Fun } G)^G.$$

Higher terms in the z -expansion of $\varphi(M(z))$ are given by more complicated formulas, involving multiple integrals of solutions of the equation

$$\partial_t \Psi(t) + [A(t), \Psi(t)] = 0$$

(in the adjoint representation of \mathfrak{g}). The solutions $\Psi(t)$ of this equation, and hence all of these Hamiltonians, are *non-local*, in the sense that they cannot be realized as integrals of differential polynomials in the matrix elements of $A(t)$.

Finally, observe that both local and non-local classical Hamiltonians Poisson commute with the Heisenberg–Poisson algebra generated by the Fourier coefficients of the Cartan components of $A(t)$. Indeed, we obtain our Hamiltonians in effect by differentiating invariant functions on $\widehat{\mathfrak{g}}_1^*$ (namely, φ of the monodromy) in the direction of $\chi \in \mathfrak{h}^* \subset \widehat{\mathfrak{g}}_1^*$. Therefore these Hamiltonians are invariant under the action of the centralizer of χ in $\widehat{\mathfrak{g}}$, which is precisely the (homogeneous) Heisenberg subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$. This is equivalent to the fact that the Hamiltonians Poisson commute with the Heisenberg–Poisson algebra generated by $\widehat{\mathfrak{h}}$ (in the case of local Hamiltonians this commutativity is discussed from a different point of view in [FF6]).

We now wish to quantize these Hamiltonians. The following remark might be useful for understanding what follows.

Recall that the Hamiltonians of the classical shift of argument Gaudin model for a finite-dimensional Lie algebra \mathfrak{g} are obtained by taking the derivatives of invariant polynomials on \mathfrak{g}^* . Each invariant polynomial P on \mathfrak{g}^* has finite degree. Therefore, when we shift the argument by $z\chi$ and expand in powers of z , we obtain a finite polynomial in z , whose coefficients are the derivatives of the P in the direction of χ . In the

affine case the role of \mathfrak{g}^* is played by $\widehat{\mathfrak{g}}_k^*$, which is the space of connections (3.3). The role of invariant polynomials on \mathfrak{g}^* is now played by the spectral invariants $\varphi(M)$ of the monodromy of the connection. As functions on $\widehat{\mathfrak{g}}_k^*$, they have “infinite degree”. When we apply the shift of argument by $z\chi$, the n th coefficient of the z -expansion of the corresponding functions $\varphi(M(z))$ will be equal to the n th derivative of this function in the direction of χ . This way we obtain the non-local classical Hamiltonians. On the other hand, it is instructive to think of the coefficients in the asymptotic z^{-1} -expansion as expressions obtained by differentiating this function $(\infty - n)$ times. These are the local classical Hamiltonians.

Now, the quantization of the derivatives of the invariant functions on \mathfrak{g}^* in the direction of χ gives rise to the quantum Hamiltonians of the shift of argument Gaudin model, as discussed in Section 2. As explained in [R, FFT], to construct these Hamiltonians we really need to use the completed enveloping algebra of $\mathfrak{g}((z))$ (more precisely, of its critical central extension). The quantum Hamiltonians are constructed, roughly speaking, by taking derivatives of central elements of this algebra. Likewise, the quantization of the spectral invariants of the monodromy of the connection gives rise to the quantum Hamiltonians of the affine Gaudin models (quantum soliton systems). These Hamiltonians should also be interpreted as coming from the center of a completed enveloping algebra, but now of the *double affine algebra* $\widehat{\mathfrak{g}}((z))$ (or some extension of it). We expect that this center contains quantum analogues of the spectral invariants of the monodromy of the connection, and the local and non-local quantum Hamiltonians may be obtained by taking the “derivatives” of these central elements (as in the finite-dimensional case), either finitely many times or infinitely many times.

While at present we do not have a precise definition of the completed enveloping algebra of $\widehat{\mathfrak{g}}((z))$ or its center, the quantum Hamiltonians of the affine Gaudin models do exist and may be constructed explicitly, as explained in the next section. We hope that by analyzing these Hamiltonians we may gain insights into the structure of the center of the enveloping algebra of $\widehat{\mathfrak{g}}((z))$.

3.3. Quantum Hamiltonians

The quantization of the local Hamiltonians has been discussed in our paper [FF6], in which we explained how to construct them using the free field (Wakimoto) realization of affine algebras. We refer the reader to [FF6] for more details. According to Conjecture 1 of [FF6], all of the classical local Hamiltonians of the \mathfrak{g} -AKNS hierarchy may be

quantized. Therefore we expect to have ℓ mutually commuting quantum local Hamiltonians of each positive spin.

In addition, for non-zero values of level k these quantum Hamiltonians should commute with the (homogeneous) Heisenberg subalgebra $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$, which is the central extension of $\mathfrak{h}((t))$.

Explicit formulas for the quantum \mathfrak{g} -AKNS Hamiltonians are unknown in general. But we can write down those of spin 1. First of all, we have the following formula, generalizing formula (2.12) for the DMT Hamiltonians (and using the same notation):

$$(3.11) \quad \begin{aligned} \widehat{T}_\gamma(\chi) &= \sum_{\alpha \in \Delta_+} \frac{(\alpha, \gamma)}{(\alpha, \chi)} \int :e_\alpha(w) f_\alpha(w): w dw \\ &= \sum_{\widehat{\alpha} \in \widehat{\Delta}_+^{\text{re}}} \frac{(\widehat{\alpha}, \gamma)}{(\widehat{\alpha}, \chi)} f_{\widehat{\alpha}} e_{\widehat{\alpha}}, \quad \gamma \in \mathfrak{h}^*, \end{aligned}$$

where $\widehat{\Delta}_+^{\text{re}}$ is the set of positive real roots of $\widehat{\mathfrak{g}}$. Here we set

$$e_\alpha(w) = \sum_{n \in \mathbb{Z}} (e_\alpha \otimes t^n) w^{-n-1},$$

and similarly for $f_\alpha(w)$, and assume that χ is regular so that $(\alpha, \chi) \neq 0$ for all $\alpha \in \Delta_+$. The normal ordering is defined as in [FB].

These operators mutually commute, but the drawback is that they do not commute with $\widehat{\mathfrak{h}}$. However, we can modify them slightly to make them commute with $\widehat{\mathfrak{h}}$ provided that the level k is non-zero. Namely, let us set

$$(3.12) \quad \begin{aligned} \widetilde{T}_\gamma(\chi) &= \\ &\sum_{\alpha \in \Delta_+} \frac{(\alpha, \gamma)}{(\alpha, \chi)} \int : \left(e_\alpha(w) f_\alpha(w) + f_\alpha(w) e_\alpha(w) - \frac{1}{k} h_\alpha(w)^2 \right) : w dw, \quad \gamma \in \mathfrak{h}^*, \end{aligned}$$

where $h_\alpha \in \mathfrak{h}$ is the element corresponding to the root $\alpha \in \mathfrak{h}^*$ via the normalized bilinear form.

The following lemma will be proved in the Appendix.⁶

⁶The operators $\widehat{T}_\gamma(\chi)$ may be obtained as certain limits of the connection operators studied in [FMTV], and part (1) of Lemma 2 may be derived from the commutativity of these operators proved in [FMTV] (see Section 4.8 below for more details).

Lemma 2.

(1) For any regular $\chi \in \mathfrak{h}^*$ the Hamiltonians $\widehat{T}_\gamma(\chi), \gamma \in \mathfrak{h}^*$, commute with each other.

(2) For any regular $\chi \in \mathfrak{h}^*$ the Hamiltonians $\widetilde{T}_\gamma(\chi), \gamma \in \mathfrak{h}^*$, commute with each other and with the Heisenberg subalgebra $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$.

Next, we discuss the quantization of the non-local Hamiltonians. In order to do this, we need to construct quantum analogues of the solutions $\Psi(t)$ of the differential equation

$$(3.13) \quad (k\partial_t + A(t))\Psi(t) = 0,$$

taking values in the adjoint representation of \mathfrak{g} . It has been shown by V. Knizhnik and A. Zamolodchikov in [KZ] that these quantum analogues may be obtained from the primary fields of the WZW model corresponding to the adjoint representation. We will now recall how this works for an arbitrary finite-dimensional representations V of \mathfrak{g} . In what follows we will assume that k is a complex number not equal to the critical value $-h^\vee$.

By definition, for given highest weight modules M_1 and M_2 over $\widehat{\mathfrak{g}}$ of the same level k , the primary field corresponding to V is a formal power series

$$\Phi_V(w) = \sum_{n \in \mathbb{Z}} \Phi_{V,(n)} w^{-n} : M_1 \otimes V \rightarrow M_2,$$

which is $\widehat{\mathfrak{g}}$ -invariant, where $\widehat{\mathfrak{g}}$ acts on V by evaluation at w ,

$$J_n^a \stackrel{\text{def}}{=} J^a \otimes t^n \mapsto w^n J^a.$$

This is equivalent to the Fourier coefficients $\Phi_{V,(n)}$ having the following commutation relations with the affine algebra $\widehat{\mathfrak{g}}$:

$$[J_m^a, \Phi_{V,(n)}] = J^a \cdot \Phi_{V,(n+m)}.$$

Here on the left hand side we consider the action of J_n^a on M_1 and M_2 , and on the right hand side the action of J^a on V .

If M_1 and M_2 have highest weights λ_1 and λ_2 and $V = V_\mu$ has highest weight μ , then the corresponding conformal dimensions (the highest weights of the Segal–Sugawara Virasoro algebra) are equal to $\Delta(\lambda_1), \Delta(\lambda_2)$, and $\Delta(\mu)$, respectively, where $\Delta(\nu) = C(\nu)/(k + h^\vee)$, and $C(\nu)$ is the scalar by which the normalized Casimir element of $U(\mathfrak{g})$ acts on V_ν . The true primary field is then

$$\widetilde{\Phi}_V(w) = w^{\Delta(\lambda_2) - \Delta(\lambda_1) - \Delta(\mu)} \Phi_V(w).$$

As explained in [KZ], it satisfies the following differential equation:

$$(3.14) \quad (k + h^\vee)\partial_w \Phi_V(w) - \sum_a :J^a(w)(J_a \cdot \Phi_V)(w): = 0,$$

which may be viewed as a quantization of the differential equation (3.13) written in the representation V (note the “quantum shift” $k \mapsto k + h^\vee$). Therefore we will consider the primary fields $\Psi_V(w)$ as quantizations of solutions of the equation (3.13). Moreover, we see that the monodromy of this $\tilde{\Phi}_V(w)$ is equal to $\exp 2\pi i(\Delta(\lambda_2) - \Delta(\lambda_1) - \Delta(\mu))$.

Suppose now that M_1 and M_2 are Verma modules $M_{\lambda_1, k}$ and $M_{\lambda_2, k}$. It is known that for generic k the space of primary fields of type $V = V_\mu$ acting from $M_{\lambda_1, k}$ to $M_{\lambda_2, k}$ has the dimension equal to the dimension of the component of weight $\lambda_2 - \lambda_1$ in V_μ . Moreover, it is known that this space is isomorphic to the component of weight $\lambda_2 - \lambda_1$ in the irreducible representation of the quantum group $U_q(\mathfrak{g})$ with highest weight μ , where $q = \exp(\pi i/(k + h^\vee))$.

In particular, we find that if $M_1 = M_2 = M_\lambda$, then the space of primary fields $\Phi(w) = \Phi_{\mathfrak{g}}(w)$ of type \mathfrak{g} (the adjoint representation) acting from M_λ to itself is isomorphic to the weight 0 subspace of \mathfrak{g} , which is the Cartan subalgebra of \mathfrak{g} . It therefore has dimension ℓ , the rank of \mathfrak{g} . It is clear that this subspace is precisely the quantization of the space of single-valued solutions of the equation (3.13), spanned by $\Psi_\varphi, \varphi \in (\text{Fun } G)^G$. Let us choose a basis $\Phi_p(w), p = 1, \dots, \ell$, of this space of primary fields.

Then the vector space of the quantum Hamiltonians that quantize the Hamiltonians $H_p(\chi)$ given by formula (3.10) is just the span of the 0th Fourier coefficients of the fields $\Phi_p(w)$,

$$(3.15) \quad H_p(\chi) = \int \Phi_p(w) \frac{dw}{w} = \int \tilde{\Phi}_p(w) w^{h^\vee/(k+h^\vee)} \frac{dw}{w}$$

of the primary fields $\Phi_p(w) : M_\lambda \rightarrow M_\lambda$. Here we use the fact that

$$\Delta(\theta) = h^\vee/(k + h^\vee),$$

where θ is the maximal root, which is the highest weight of the adjoint representation of \mathfrak{g} .

We claim that the Hamiltonians $H_p(\chi), p = 1, \dots, \ell$, commute with each other and with the local Hamiltonians (3.11). This is checked by a direct calculation which will be presented in [FF7].

The quantization of the classical non-local Hamiltonians, corresponding to the higher terms in the z -expansion of the monodromy $\varphi(M(z))$ is expressed by multiple integrals of the adjoint primary fields $\tilde{\Phi}(w) w^{\frac{h^\vee}{k+h^\vee}}$

(including the components acting between different Verma modules). We will now briefly sketch the construction of these integrals. The details will appear in [FF7].

This construction is analogous to the construction of [BLZ1, BHK], where non-local quantum KdV Hamiltonians were constructed using bosonic screening operators corresponding to the simple roots of $\widehat{\mathfrak{g}}$. These operators satisfy the q -Serre relations, and hence, in a certain sense, generate the quantized enveloping algebra $U_q(\widetilde{\mathfrak{n}}_-)$, where $\widetilde{\mathfrak{n}}_-$ is the lower nilpotent Lie subalgebra of $\widehat{\mathfrak{g}}$ (see [BMP, FF5] for the precise definition of what this means). The non-local quantum KdV Hamiltonians are then obtained from elements of $U_q(\widetilde{\mathfrak{n}}_-)$ which correspond to singular vectors in the Verma modules over $U_q(\widehat{\mathfrak{g}})$ of *critical level*.

More precisely, as explained in [FF5], whenever we have a singular vector of weight μ in the Verma module M_λ^q over $U_q(\widehat{\mathfrak{g}})$ of highest weight λ and level 0, we have a well-defined operator $\pi_\lambda \rightarrow \pi_\mu$, where π_λ denotes the Fock representation of a Heisenberg Lie algebra with highest weight λ . This operator is constructed as follows. Let us write the singular vector as Pv_λ , where $v_\lambda \in M_\lambda^q$ is the highest weight vector and $P \in U_q(\widetilde{\mathfrak{n}}_-)$. This operator is obtained by substituting the bosonic screening operators

$$S_i = \int S_i(w) \frac{dw}{w}, \quad i = 0, \dots, \ell,$$

where the $S_i(w) = e^{\beta\phi_i(w)}$ are the bosonic primary fields corresponding to the simple roots in $\widehat{\mathfrak{g}}$, into P instead of the generators $f_i, i = 0, \dots, \ell$. Here $q = \exp(\pi i\beta^2)$. Now, if we replace the screening operator S_0 by the operator

$$(3.16) \quad \widetilde{S}_0 = \int S_0(w) w^{h^\vee\beta^2} \frac{dw}{w},$$

and keep $S_i, i = 1, \dots, \ell$, unchanged, then the same procedure will work, except that we will now assign operators acting on Fock representations to singular vectors in the Verma modules of *critical level* $k = -h^\vee$ instead of level $k = 0$. There are many more of those than at $k = 0$, and one can check that they commute with each other. These are the non-local quantum Hamiltonians of [BLZ1, BHK, FFS].

The non-local Hamiltonians of the affine shift of argument Gaudin model may be constructed in a similar fashion. The most direct way to do it is to replace the Virasoro (or, more generally, \mathcal{W} -algebra) screening operators S_i with the screening operators for the affine algebra $\widehat{\mathfrak{g}}$ arising from the free field (Wakimoto) realization of $\widehat{\mathfrak{g}}$. These screening operators have been defined in [FF6], where it was shown that they satisfy

the same Serre relations of $U_q(\tilde{\mathfrak{n}}_-)$. Hence, substituting them in the formulas discussed above, we construct non-local commuting Hamiltonians of our model.

However, we want to stress that it is not necessary to use the free field realization to define these quantum non-local Hamiltonians. They may be defined intrinsically in terms of the adjoint primary fields of the WZW model associated to $\hat{\mathfrak{g}}$. In order to do that, we need to replace the algebra $U_q(\tilde{\mathfrak{n}}_-)$ by a smaller quantum algebra, which is a quantization of $U(\tilde{\mathfrak{g}}_-)$, where $\tilde{\mathfrak{g}}_- = t^{-1}\mathfrak{g}[t^{-1}]$. To explain this more precisely, let us recall that the algebra $U_q(\tilde{\mathfrak{n}}_-)$ may be constructed using certain rank one local systems on the configuration spaces (see [SV2, V]). Moreover, homology classes with coefficients in these local systems correspond to singular vectors discussed above. These local systems describe the monodromy properties of the above screening currents, and the non-local Hamiltonians may be interpreted as integrals of products of the screening currents over homology cycles of these local systems (see [FF5] for more details).

Likewise, the quantization of $U(\tilde{\mathfrak{g}}_-)$ that we need here, which we denote by $U_q(\tilde{\mathfrak{g}}_-)$, may be constructed using local systems on the configuration spaces that capture the monodromy properties of the adjoint primary fields $\Phi_{\mathfrak{g}}(w)$. These local systems are defined on the trivial vector bundles on the configuration spaces $\mathbb{C}^N \setminus \text{diag}$ with the fibers $\mathfrak{g}^{\otimes N}$ and the connection given by the corresponding KZ operators. The monodromies of these local systems around the diagonals are given by the R -matrices of $U_q(\mathfrak{g})$ in the adjoint representation. Now, the commuting non-local Hamiltonians quantizing the coefficients in the z -expansion of $\varphi(M(z))$ are constructed by integrating products of the primary fields over homology classes with coefficients in these local systems, which in turn correspond to singular vectors in $U_q(\tilde{\mathfrak{g}}_-)$ -modules. Since we want to use singular vectors at the critical level $k = -h^\vee$ (and not $k = 0$), we need to insert the factor $w^{h^\vee/(k+h^\vee)}$ in the integral (3.15). This is analogous to inserting the factor $w^{h^\vee\beta^2}$ in formula (3.16).

The simplest of these non-local Hamiltonians are the operators $H_p(\chi)$ given by formula (3.15). They correspond to the quantization of the z -linear term in the expansion of $\varphi(M(z))$. Higher degree terms correspond to integrals over more complicated homology classes with coefficients in the above local system. Applying the free field realization of $\hat{\mathfrak{g}}$, we may express these Hamiltonians as contour integrals of the $\hat{\mathfrak{g}}$ screening currents and obtain formulas similar to those of [BLZ1, BHK].

3.4. Bethe Ansatz

Next, we discuss the eigenvectors of the quantum Hamiltonians. The Bethe Ansatz described in Section 2.3 has a straightforward generalization to this case. The formula (2.9) for the Bethe vector and the Bethe Ansatz equations (2.10) easily generalize to the affine case. We just need to enlarge the set I to \tilde{I} and include the generator f_0 (corresponding to the simple root α_0 of $\tilde{\mathfrak{g}}$) and view ν as an element of $\tilde{\mathfrak{h}}^*$. Then for any collection of distinct non-zero complex numbers $w_j, j = 1, \dots, m$, and labels $i_1, \dots, i_m \in \tilde{I}$ of the Dynkin diagram of $\tilde{\mathfrak{g}}$, the vector $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ given by formula (2.9) belongs to the Verma module M_ν over $\hat{\mathfrak{g}}$.

We expect that if the Bethe Ansatz equations (2.10) are satisfied, then the vector $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ is an eigenvector of the quantum Hamiltonians discussed above (both local and non-local). Since we expect that these Hamiltonians commute with the Heisenberg Lie algebra $\hat{\mathfrak{h}}$, it is natural to assume that these vectors are also invariant under the Lie subalgebra $\mathfrak{th}[[t]] \subset \hat{\mathfrak{h}}$ (we have checked that this is indeed true in the simplest examples). Thus, we arrive at the following conjecture.

Conjecture 1. *If the Bethe Ansatz equations (2.10) are satisfied, then the vector $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ is a $\mathfrak{th}[[t]]$ -invariant eigenvector of all local and non-local quantum affine Gaudin Hamiltonians, in particular, the Hamiltonians $\tilde{T}_\gamma, \gamma \in \mathfrak{h}^*$, given by formula (3.12), and $H_p(\chi), p = 1, \dots, \ell$, given by formula (3.15).*

Note that $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ is a homogeneous vector of weight $\lambda - \sum_{j=1}^m \alpha_{i_j}$. It is clear from formulas (3.12) and (3.11) that any homogeneous $\mathfrak{th}[[t]]$ -invariant eigenvector of $\tilde{T}_\gamma, \gamma \in \mathfrak{h}^*$, is automatically an eigenvector of $\hat{T}_\gamma, \gamma \in \mathfrak{h}^*$. Therefore Conjecture 1 implies the following statement.

Conjecture 2. *The vector $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ is an eigenvector of the affine DMT Hamiltonians (3.11) if the Bethe Ansatz equations (2.10) are satisfied.*

To prove this conjecture, we cannot use the argument used in [FFT] because we do not know that the affine DMT Hamiltonians arise by the same mechanism as the DMT Hamiltonians in the finite-dimensional case (that is, from the center of an enveloping algebra of the affinized Lie algebra). But this conjecture can probably be derived from the results of [FMTV] (see Section 4.8 below for more details).

The next question is to describe the common eigenvalues of the quantum Hamiltonians that occur on highest weight $\hat{\mathfrak{g}}$ -modules, such as

Verma modules and irreducible integrable representations. In particular, we wish to describe the eigenvalues on the Bethe eigenvectors.

According to Theorem 2, in the finite-dimensional case the common eigenvalues of the quantum shift of argument algebra \mathcal{A}_χ are encoded by $L\mathfrak{g}$ -opers on \mathbb{P}^1 satisfying certain conditions. In particular, the common eigenvalues on the Bethe vectors (2.9) are encoded by the $L\mathfrak{g}$ -opers obtained by applying the Miura transformation to the connection (2.11). We wish to generalize these results to the affine case. This will be done in the next section.

§4. Affine opers

The notion of \mathfrak{g} -opers, where \mathfrak{g} is a finite-dimensional simple Lie algebra, was introduced by Beilinson and Drinfeld [BD1, BD2] following Drinfeld and Sokolov [DS]. Denote by G the Lie group of inner automorphisms of the Lie algebra \mathfrak{g} and B its Borel subgroup. A \mathfrak{g} -oper on a smooth curve X is by definition a triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$, where \mathcal{F} is a principal G -bundle \mathcal{F} on X , ∇ is a connection on \mathcal{F} and \mathcal{F}_B is a B -reduction of \mathcal{F} such that locally on X , if we choose a local coordinate z and trivialize \mathcal{F}_B , the connection acquires the form

$$(4.1) \quad \nabla = \partial_z + \sum_{i=1}^{\ell} \psi_i(z) f_i + \mathbf{v}(z),$$

where each $\psi_i(z)$ is a nowhere vanishing function, and $\mathbf{v}(z)$ is a \mathfrak{b} -valued function.

In this section we generalize the notion of \mathfrak{g} -opers and define $\widehat{\mathfrak{g}}$ -opers, where $\widehat{\mathfrak{g}}$ is an affine Kac–Moody algebra, following [F2]. We then use these affine opers to give a conjectural description of the spectra of the quantum Hamiltonians of the affine Gaudin model introduced in the previous section.

4.1. Definition

We now consider another copy of the affine Kac–Moody algebra $\widetilde{\mathfrak{g}}$, which we will use to describe the spectra of the quantum affine Gaudin Hamiltonians. It is important to realize that this affine Kac–Moody algebra has nothing to do with the affine Kac–Moody algebra used in the construction of the Gaudin models above (in fact, we expect that the two are Langlands dual to each other). To emphasize this difference, we will now use λ as the loop variable. Thus, in this section we set

$$\widetilde{\mathfrak{g}} = \mathbb{C}d \ltimes \widehat{\mathfrak{g}},$$

where

$$0 \rightarrow \mathbf{C1} \rightarrow \widehat{\mathfrak{g}} \rightarrow \mathfrak{g}((\lambda)) \rightarrow 0.$$

It will be convenient to use the following slightly unusual convention for the Cartan decomposition:

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}_+,$$

where

$$\begin{aligned} \widetilde{\mathfrak{n}}_+ &= (\mathfrak{n}_+ \otimes 1) \oplus (\mathfrak{g} \otimes \lambda^{-1}\mathbf{C}[\lambda^{-1}]), \\ \widetilde{\mathfrak{n}}_- &= (\mathfrak{n}_- \otimes 1) \oplus (\mathfrak{g} \otimes \lambda\mathbf{C}[[\lambda]]), \\ \widetilde{\mathfrak{h}} &= (\mathfrak{h} \otimes 1) \oplus \mathbf{C1} \oplus \mathbf{C}d. \end{aligned}$$

Now let \widetilde{G} be the Lie group (more precisely, ind-group scheme) associated to $\widetilde{\mathfrak{g}}$ (see, e.g., [Ka] for the precise definition). We will denote by \overline{G} the quotient of \widetilde{G} by its center. Thus, the Lie algebra of \overline{G} is

$$\overline{\mathfrak{g}} = \mathbf{C}d \ltimes \mathfrak{g}((\lambda)).$$

The group \overline{G} comes with the lower unipotent and Borel subgroups \overline{N}_- and \overline{B}_- (which are proalgebraic groups) corresponding to the Lie subalgebras

$$\overline{\mathfrak{n}}_- = \widetilde{\mathfrak{n}}_- \quad \text{and} \quad \overline{\mathfrak{b}}_- = (\mathfrak{h} \oplus \mathbf{C}d) \oplus \widetilde{\mathfrak{n}}_-$$

of $\overline{\mathfrak{g}}$, respectively, and the upper unipotent and Borel subgroups \overline{N}_+ and \overline{B}_+ (which are group ind-schemes) corresponding to

$$\overline{\mathfrak{n}}_+ = \widetilde{\mathfrak{n}}_+ \quad \text{and} \quad \overline{\mathfrak{b}}_+ = (\mathfrak{h} \oplus \mathbf{C}d) \oplus \widetilde{\mathfrak{n}}_+,$$

respectively.

Let X be a smooth algebraic curve. We will use the notion of principal \overline{G} -bundle on X and a connection on it given in [F2]. Roughly speaking, such a bundle may be trivialized on open subsets of X (in either Zariski or analytic topology). Having chosen such a trivialization and a local coordinate z on an open subset U , a connection is by definition a first order differential operator $\nabla = \partial_z + A(z)$, where $A(z) \in \overline{\mathfrak{g}}$. If we change the trivialization by $g(z) \in \widetilde{G}$, ∇ transforms by the usual formula

$$\nabla \mapsto \partial_z + gA(z)g^{-1} - (\partial_z g)g^{-1}.$$

Under a change of coordinates $z = \varphi(s)$ the operator ∇ also transforms in the usual way:

$$\nabla \mapsto \partial_s + \varphi'(s)A(\varphi(s)).$$

Now, by analogy with the finite-dimensional case [DS, BD1, BD2], a $\widehat{\mathfrak{g}}$ -oper on X may be defined as a triple $(\mathcal{F}, \nabla, \mathcal{F}_{\overline{B}_+})$, where \mathcal{F} is a principal \overline{G} -bundle on X , ∇ is a connection on \mathcal{F} and $\mathcal{F}_{\overline{B}_+}$ is a \overline{B}_+ -reduction of \mathcal{F} , such that locally, with any choice of local coordinate z and trivialization of $\mathcal{F}_{\overline{B}_+}$, the connection operator has the form

$$(4.2) \quad \nabla = \partial_z + \sum_{i=1}^{\ell} \psi_i(z) f_i + \mathbf{v}(z),$$

where each $\psi_i(z)$ is a nowhere vanishing function, and $\mathbf{v}(z)$ is a $\overline{\mathfrak{b}}_+$ -valued function.

We note that another version of affineopers, introduced earlier in [BF], has $\overline{\mathfrak{b}}_+$ and $\overline{\mathfrak{b}}_-$ switched. For comparison of the two definitions and the explanation as to why the definition given here is the one relevant to the Gaudin model, see [F2].

However, this naive definition appears to be inadequate. The reason is that for ind-groups, such as \overline{G} , a connection does not necessarily give rise to a trivialization of \mathcal{F} , even locally analytically. This is because the equation

$$(\partial_z + A(z))\Psi(z) = 0,$$

where $A(z)$ takes values in $\overline{\mathfrak{g}} \simeq \mathbb{C}d \times \mathfrak{g}((\lambda))$, does not *a priori* have local solutions with values in $\overline{G} \simeq \mathbb{C}^\times \times G((\lambda))$ (we may well obtain power series in λ unbounded from below, which are not well-defined as elements of \overline{G}). This is related to the fact that we do not have an exponential map from the Lie algebra $\overline{\mathfrak{g}}$ to the group \overline{G} . This means that the usual correspondence between flat connections on \overline{G} -bundles and \overline{G} -local systems does not exist. In particular, the notion of monodromy of an abstract oper as defined above does not make sense.

Our experience withopers suggests that their monodromy properties are important in the context of Gaudin models (see [F2, F3]). In order to have well-defined monodromy, we will assume that, locally on X , the oper bundle \mathcal{F} admits a reduction to a *proalgebraic* subgroup of \overline{G} , such as $G[[\lambda]]$, which is preserved by the connection ∇ . Then the connection ∇ does give rise to local analytic trivializations of this $G[[\lambda]]$ -subbundle, and hence of the \overline{G} -bundle \mathcal{F} as well, so the monodromy of the oper connection is well-defined. Thus, from now on by a $\widehat{\mathfrak{g}}$ -oper on X we will understand a triple $(\mathcal{F}, \nabla, \mathcal{F}_{\overline{B}_+})$ as above such that, locally on X , \mathcal{F} admits a horizontal reduction to $G[[\lambda]]$ or to another “maximal compact” subgroup of $G((\lambda))$. We denote the set of $\widehat{\mathfrak{g}}$ -opers on X by $\text{Op}_{\widehat{\mathfrak{g}}}(X)$.

Let

$$p_{-1} = \sum_{i \in \tilde{I}} f_i = \sum_{i \in I} f_i + e_\theta \otimes \lambda.$$

We will say that a $\widehat{\mathfrak{g}}$ -oper on $X \setminus x$ has regular singularity at x if the connection operator may be brought to the form

$$(4.3) \quad \nabla = \partial_z + \frac{1}{z} (p_{-1} + \mathbf{v}(z)), \quad \mathbf{v}(z) \in \overline{\mathfrak{b}}_+[[z]],$$

on the punctured disc D_x^\times near x , where z is a local coordinate at x . We define the residue of such an oper as in the finite-dimensional case (see, e.g., [F2]), so that it takes values in $\overline{\mathfrak{h}}/W_{\text{aff}}$. We will denote by $\overline{\omega}$ the projection $\overline{\mathfrak{h}} \rightarrow \overline{\mathfrak{h}}/W_{\text{aff}}$.

Let ν be a dominant integral coweight such that $\nu_i = \langle \nu, \check{\alpha}_i \rangle \in \mathbb{Z}_+$ for all $i \in \tilde{I}$. We will say that a $\widehat{\mathfrak{g}}$ -oper on X is ν -regular at $x \in X$ if it may be brought to the form

$$(4.4) \quad \nabla = \partial_z + \sum_{i \in \tilde{I}} z^{\nu_i} f_i + \mathbf{v}(z),$$

where $\mathbf{v}(z) \in \overline{\mathfrak{b}}_+[[z]]$, on D_x^\times .

We also defineopers with singularity of order 2 and their 2-residue similarly to the finite-dimensional case (see [FFT]).

4.2. Miuraopers and Miura transformation

Let \overline{H} be the subgroup of \overline{G} corresponding to the Lie algebra $\overline{\mathfrak{h}} = \mathbb{C}d \oplus \mathfrak{h}$. Let $\check{\rho}$ be the element of $\overline{\mathfrak{h}}$ such that $\langle \alpha_i, \check{\rho} \rangle = 1$ for all $i \in \tilde{I}$. It defines a homomorphism $\mathbb{C}^\times \rightarrow \overline{H}$. Let $\Omega^{\check{\rho}}$ be the \overline{H} -bundle on X obtained as the push-forward of the \mathbb{C}^\times -bundle on X corresponding to the canonical line bundle on X . Connections on $\Omega^{\check{\rho}}$ are described as follows. If we choose a local coordinate z on an open subset of X , then we trivialize $\Omega^{\check{\rho}}$ and represent the connection as an operator

$$(4.5) \quad \overline{\nabla} = \partial_z + \mathbf{u}(z), \quad \mathbf{u}(z) \in \overline{\mathfrak{h}}.$$

If s is another coordinate such that $z = \varphi(s)$, then this connection will be represented by the operator

$$(4.6) \quad \partial_s + \varphi'(s)\mathbf{u}(\varphi(s)) - \check{\rho} \cdot \frac{\varphi''(s)}{\varphi'(s)}.$$

We will call them *Cartan connections*.

We define the *Miura transformation* from Cartan connections to $\widehat{\mathfrak{g}}$ -opers taking an operator (4.5) to the gauge equivalence class of

$$(4.7) \quad \nabla = \partial_z + p_{-1} + \mathbf{u}(z).$$

The Miura transformation may be understood more conceptually in terms of the *Miuraopers*. By definition [F4], a Miura $\widehat{\mathfrak{g}}$ -oper on X is a quadruple $(\mathcal{F}, \nabla, \mathcal{F}_{\overline{B}_+}, \mathcal{F}_{\overline{B}_-})$, where $(\mathcal{F}, \nabla, \mathcal{F}_{\overline{B}_+})$ is a $\widehat{\mathfrak{g}}$ -oper on X and $\mathcal{F}_{\overline{B}_-}$ is a \overline{B}_- -reduction of \mathcal{F} which is preserved by ∇ . It is called generic if the reductions $\mathcal{F}_{\overline{B}_+}$ and $\mathcal{F}_{\overline{B}_-}$ are in generic relative position (see [F4]). We have a map from Cartan connections to generic Miuraopers taking an operator (4.5) to the quadruple in which \mathcal{F} is the trivial bundle with the connection (4.7) and the tautological reductions $\mathcal{F}_{\overline{B}_+}, \mathcal{F}_{\overline{B}_-}$ satisfying the above conditions. This map sets up a bijection between generic Miuraopers and Cartan connections (see [F4]). The inverse map sends the quadruple as above to the \overline{H} -bundle $\mathcal{F}_{\overline{B}_-}/\overline{N}_-$ with the connection induced by ∇ .

An arbitrary Miura oper on X is always generic away from finitely many points of X . At each of the remaining points the reductions $\mathcal{F}_{\overline{B}_+}$ and $\mathcal{F}_{\overline{B}_-}$ are in relative position labeled by an element $w \neq 1$ of the (affine) Weyl group W_{aff} of \overline{G} . The corresponding Cartan connection has a singularity at such a point with the residue equal to $\check{\rho} - w(\check{\rho})$ (see [F2] for more details).

The Miura transformation is thus defined as the composition of the map from the Cartan connections to generic Miuraopers and the forgetful map from Miuraopers to opers.

4.3. Parabolic Miuraopers

Finally, we introduce the notion of *parabolic Miura oper*. By definition, a parabolic Miura $\widehat{\mathfrak{g}}$ -oper on X is a quadruple $(\mathcal{F}, \nabla, \mathcal{F}_{\overline{B}_+}, \mathcal{F}_{\overline{G}_-})$, where $(\mathcal{F}, \nabla, \mathcal{F}_{\overline{B}_+})$ is a $\widehat{\mathfrak{g}}$ -oper on X and $\mathcal{F}_{\overline{G}_-}$ is a reduction of \mathcal{F} to $\overline{G}_- = \mathbb{C}^\times \ltimes G[[\lambda]]$ which is preserved by ∇ . We will call it generic if $\mathcal{F}_{\overline{B}_+}$ and $\mathcal{F}_{\overline{G}_-}$ are in generic relative position. We then have the following map from the space $\text{Op}_{\mathfrak{g}}(X)$ of \mathfrak{g} -opers on X to the space of generic parabolic Miura $\widehat{\mathfrak{g}}$ -opers: it sends a \mathfrak{g} -oper represented by the connection

$$\overline{\nabla} = \partial_z + \sum_{i \in I} f_i + \mathbf{v}(z), \quad \mathbf{v}(z) \in \mathfrak{b}_+,$$

to the quadruple in which \mathcal{F} is the trivial bundle,

$$\nabla = \overline{\nabla} + f_0,$$

and $\mathcal{F}_{\overline{B}_+}, \mathcal{F}_{\overline{G}_-}$ are the tautological reductions of \mathcal{F} . However, this map is not bijective, because we are missing the one-dimensional subgroup \mathbb{C}^\times of \overline{G}_- corresponding to the element $d \in \overline{\mathfrak{h}}$.

In order to obtain a bijection, consider the Cartesian product of $\text{Op}_{\mathfrak{g}}(X)$ and the space $\text{Conn}_X(\Omega^h)$, where $h = \langle \delta, \check{\rho} \rangle$ is the Coxeter number of \mathfrak{g} . Define a map from $\text{Op}_{\mathfrak{g}}(X) \times \text{Conn}_X(\Omega^h)$ to the space of generic parabolic Miura $\widehat{\mathfrak{g}}$ -opers sending a pair consisting of an oper $\overline{\nabla} \in \text{Op}_{\mathfrak{g}}(X)$ and a connection $\overline{\nabla}' \in \text{Conn}_X(\Omega^h)$ of the form

$$\overline{\nabla}' = \partial_z + \gamma(z),$$

to the connection

$$\nabla = \overline{\nabla} + f_0 + \gamma(z)d.$$

This map is already bijective. To construct the inverse map, observe that

$$\overline{G}_-/G^{(1)} \simeq G \times \mathbb{C}^\times$$

(where $G^{(1)}$ is the first congruence subgroup of $G[[\lambda]]$). To a quadruple $(\mathcal{F}, \nabla, \mathcal{F}_{\overline{B}_+}, \mathcal{F}_{\overline{G}_-})$ as above, we associate a $G \times \mathbb{C}^\times$ -bundle $\mathcal{F}/G^{(1)}$ together with a connection and a B -reduction $\mathcal{F}_{\overline{B}_+}/G^{(1)}$. This bundle is a product of the \mathfrak{g} -oper bundle and the \mathbb{C}^\times -bundle corresponding to Ω^h . Therefore these data give rise to a point in $\text{Op}_{\mathfrak{g}}(X) \times \text{Conn}_X(\Omega^h)$.

Taking the composition with the forgetful map from parabolic Miura operators to operators, we now obtain a map from $\text{Op}_{\mathfrak{g}}(X) \times \text{Conn}_X(\Omega^h)$ to $\widehat{\mathfrak{g}}$ -opers, which is a parabolic version of the Miura transformation. This gives us a convenient way to represent $\widehat{\mathfrak{g}}$ -opers in terms of \mathfrak{g} -opers, which we will exploit in what follows.

Note that any parabolic Miura $\widehat{\mathfrak{g}}$ -oper on a curve X is generic away from finitely many points of X . At each of those points the relative position of $\mathcal{F}_{\overline{B}_+}$ and $\mathcal{F}_{\overline{G}_-}$ is measured by a coset $\overline{w} \in W \setminus W_{\text{aff}}$, and the corresponding \mathfrak{g} -oper has regular singularity with the residue $\varpi(-\overline{w(\check{\rho})})$, where the bar stands for the projection $\overline{\mathfrak{h}} \rightarrow \mathfrak{h}$ along $\mathbb{C}d$ and ϖ is the projection $\mathfrak{h} \rightarrow \mathfrak{h}/W$, as before (note that $\varpi(-\overline{w_1(\check{\rho})}) = \varpi(-\overline{w_2(\check{\rho})})$ if $w_1 \sim w_2$ in $W \setminus W_{\text{aff}}$, so this formula is well-defined).

Remark 1. There is an analogous notion of parabolic Miura oper in the finite-dimensional case. Let P be a parabolic subgroup of G . Then we define a Miura (\mathfrak{g}, P) -oper to be a quadruple $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_P)$, where $(\mathcal{F}, \nabla, \mathcal{F}_B)$ is a \mathfrak{g} -oper (see the beginning of this section) and \mathcal{F}'_P is a P -reduction of \mathcal{F} which is horizontal with respect to ∇ . (In the affine case the role of P is played by \overline{G}_- .) We have the obvious notion of generic Miura (\mathfrak{g}, P) -oper.

Let $L = P/[P, P]$. Then we define, in the same way as above, a bijective map from the space of L -opers to generic Miura (\mathfrak{g}, P) -opers. Composing it with the forgetful map from Miura (\mathfrak{g}, P) -opers to \mathfrak{g} -opers, we obtain a parabolic analogue of the Miura transformation map from L -opers to \mathfrak{g} -opers. The ordinary Miura opers correspond to the case $P = B$.

4.4. Spectra of quantum Gaudin Hamiltonians and affine opers

Now let ${}^L\widehat{\mathfrak{g}}$ be the Langlands dual Lie algebra of $\widehat{\mathfrak{g}}$, whose Cartan matrix is the transpose of the Cartan matrix of $\widehat{\mathfrak{g}}$ (thus, ${}^L\widehat{\mathfrak{g}}$ is a twisted affine Lie algebra if $\widehat{\mathfrak{g}}$ is non-simply laced). We will use the upper index L to denote all of the above groups corresponding to ${}^L\widehat{\mathfrak{g}}$, such as ${}^L\mathfrak{h}$, ${}^L\overline{H}$, etc. In particular, we have a canonical identification ${}^L\mathfrak{h} = \mathfrak{h}^*$, and so we will consider a weight $\nu \in \mathfrak{h}^*$ as an element of ${}^L\mathfrak{h}$. By abusing notation, we will denote by the same symbol its projection onto ${}^L\overline{\mathfrak{h}} = {}^L\overline{\mathfrak{h}}/\mathbb{C}\mathbf{1} = {}^L\mathfrak{h} \oplus \mathbb{C}d$.

As in the finite-dimensional case (see [FFT] and Section 2.3), we attach to each solution of the Bethe Ansatz equation (2.10), where $i_j \in \widetilde{I}$, the Cartan ${}^L\overline{H}$ -connection

$$(4.8) \quad \overline{\nabla} = \partial_z - \chi - \frac{\nu}{z} + \sum_{j=1}^m \frac{\alpha_{i_j}}{z - w_j}.$$

In the same way as in [F2], Lemma 2.10, we prove the following:

Lemma 3. *The Bethe Ansatz equations (2.10) are equivalent to the condition that the ${}^L\widehat{\mathfrak{g}}$ -oper $\nabla = \overline{\nabla} + p_{-1}$ obtained by applying the Miura transformation to the connection (4.8) is regular at the points $w_j, j = 1, \dots, m$, or, equivalently, has no monodromies around these points.*

Note that this ${}^L\widehat{\mathfrak{g}}$ -oper has regular singularity at the point z with the residue and ∞ with the residue determined by ν , and order 2 singularity at ∞ with the 2-residue $\pi(\chi)$.

This suggests that the true parameters for the common eigenvalues of the quantum Gaudin Hamiltonians are not the Cartan connections (6.15), but the corresponding ${}^L\widehat{\mathfrak{g}}$ -opers obtained by applying the Miura transformation. Thus, we propose the following description of the spectra of the quantum affine Gaudin Hamiltonians (including the affine DMT Hamiltonians $\widehat{T}_\gamma(\chi)$ and the non-local Hamiltonians $H_p(\chi)$):

The common eigenvalues of the quantum affine Gaudin Hamiltonians on the subspace of $\mathfrak{th}[[t]]$ -invariant vectors in an irreducible $\widehat{\mathfrak{g}}$ -module with highest weight ν and level $k \neq 0$ over $\widehat{\mathfrak{g}}$ are encoded by ${}^L\widehat{\mathfrak{g}}$ -opers on \mathbb{P}^1 with regular singularity at 0 with residue $\overline{\omega}(-\nu - \rho)$ and singularity of order 2 at ∞ with 2-residue $\pi(\chi)$.

Here it is important to note that the projective line \mathbb{P}^1 , on which these ${}^L\widehat{\mathfrak{g}}$ -opers are defined, has a global coordinate z , which is nothing but the spectral parameter of the affine Gaudin model (in particular, z is the spectral parameter in the L -operator (3.4)).

4.5. The case of $\widehat{\mathfrak{sl}}_2$

What do these ${}^L\widehat{\mathfrak{g}}$ -opers look like? Consider the case when $\widehat{\mathfrak{g}} = {}^L\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$. The Cartan subalgebra ${}^L\widehat{\mathfrak{h}}$ has a basis consisting of

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\mathbf{1}$, and d , and ${}^L\overline{\mathfrak{h}}$ is spanned by h and d (recall that d acts as $\lambda\partial_\lambda$). Without loss of generality we may, and will, consider only highest weights ν such that $\langle \nu, d \rangle = 0$. They are determined by the numbers $k = \langle \nu, \mathbf{1} \rangle$ and $\ell = \frac{1}{2}\langle \nu, h \rangle$. Then the corresponding element of ${}^L\overline{\mathfrak{h}} = \mathbb{C}h \oplus \mathbb{C}d$ is represented by $\ell h + kd$.

We have two simple roots, α_1 and α_0 , which in ${}^L\overline{\mathfrak{h}}$ are represented as $\alpha_1 = h$ and $\alpha_0 = -h$. We therefore divide the set of points $\{w_1, \dots, w_m\}$ into two subsets: $\{w_1^1, \dots, w_{m_1}^1\}$ and $\{w_1^0, \dots, w_{m_0}^0\}$. The connection (4.8) gives rise to the Miura $\widehat{\mathfrak{sl}}_2$ -oper

$$(4.9) \quad \partial_z + \begin{pmatrix} u(z) & \lambda \\ 1 & -u(z) \end{pmatrix} + \frac{k}{z}d,$$

where

$$(4.10) \quad u(z) = -\chi - \frac{\ell}{z} + \sum_{j=1}^{m_1} \frac{1}{z - w_j^1} - \sum_{j=1}^{m_0} \frac{1}{z - w_j^0}.$$

Concretely, the statement of Lemma 3 is realized as follows. The connection in (4.9) has apparent singularities at the points $z = w_j^1$ and $z = w_j^0$ (in addition to $z = 0$ and $z = \infty$), but the condition of Lemma 3 is that these singularities are “fake”; that is, they can be removed by a gauge transformation (by an element of \overline{B}_+). This imposes certain conditions on the points w_j^1 and w_j^0 , which are in fact the Bethe Ansatz equations (2.10).

Let us see more precisely how this works. This is in fact a local question, which involves only the restrictions of the Miura oper (4.9) to the formal discs around these points.

Suppose that we have a Miura $\widehat{\mathfrak{sl}}_2$ -oper of the form

$$(4.11) \quad \partial_z + \begin{pmatrix} u(z) & \lambda \\ 1 & -u(z) \end{pmatrix} + x(z)d$$

on the formal disc around a point w such that

$$u(z) = \sum_{n \geq -1} u_n(z-w)^n, \quad x(z) = \sum_{n \geq 0} x_n(z-w)^n.$$

Suppose first that $u_{-1} = 1$. This is the case when $u(z)$ is given by formula (4.10) and $w = w_j^1$. Let us apply the gauge transformation by

$$\begin{pmatrix} 1 & -\frac{1}{z-w} \\ 0 & 1 \end{pmatrix}.$$

Then we obtain the operator

$$\partial_z + \frac{2}{z-w} \cdot \left(u(z) - \frac{1}{z-w} \right) \Big|_{z=w} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \dots,$$

where the dots stand for the terms regular at $z = w$. Therefore this operator is regular at $z = w$ if and only if the equation

$$(4.12) \quad u_0 = 0$$

is satisfied.

Next, consider the case when $u_{-1} = -1$, which corresponds to $w = w_j^0$ in formula (4.10). Applying the gauge transformation by

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{z-w} \lambda^{-1} & 1 \end{pmatrix},$$

we obtain the operator whose expansion near $z = w$ has the form

$$\partial_z + \frac{2}{z-w} \cdot \left(-u(z) - \frac{1}{z-w} - \frac{x(z)}{2} \right) \Big|_{z=w} \begin{pmatrix} 0 & 0 \\ \lambda^{-1} & 0 \end{pmatrix} + \dots,$$

where the dots stand for terms regular at $z = w$. Therefore this operator is regular at $z = w$ if and only if the equation

$$(4.13) \quad u_0 + \frac{x_0}{2} = 0$$

is satisfied.

Thus, we obtain that the connection (4.9) with $u_{-1} = 1$ (resp., $u_{-1} = -1$) is gauge equivalent to a connection regular at $z = w$ if the equation (4.12) (resp., (4.13)) is satisfied. Conversely, one shows, following the argument of [F2], Lemma 2.10, that if this equation is not satisfied, then the connection (4.9) has non-trivial monodromy around w . Therefore it is impossible to make it regular by a gauge transformation.

Let us apply this in the case of our Miura oper (4.9). Then equations (4.12) for $w = w_j^1$ become

$$(4.14) \quad \frac{\ell}{w_j^1} - \sum_{s \neq j} \frac{1}{w_j^1 - w_s^1} + \sum_{s=1}^{m_0} \frac{1}{w_j^1 - w_s^0} + \chi = 0, \quad j = 1, \dots, m_1,$$

and equations (4.13) for $w = w_j^0$ become

$$(4.15) \quad \frac{\frac{k}{2} - \ell}{w_j^0} + \sum_{s=1}^{m_1} \frac{1}{w_j^0 - w_s^1} - \sum_{s \neq j} \frac{1}{w_j^0 - w_s^0} - \chi = 0, \quad j = 1, \dots, m_0.$$

These are precisely the Bethe Ansatz equations (2.10).

4.6. Second order operator

Now we wish to rewrite (4.9) as a second order differential operator. To this end, we apply the gauge transformation by

$$(4.16) \quad \begin{pmatrix} 1 & -u(z) \\ 0 & 1 \end{pmatrix}.$$

Then we obtain the following operator:

$$(4.17) \quad \partial_z + \begin{pmatrix} 0 & v(z) + \lambda \\ 1 & 0 \end{pmatrix} + \frac{k}{z}d,$$

where

$$(4.18) \quad \begin{aligned} v(z) &= u(z)^2 + \partial_z u(z) \\ &= \chi^2 + \left(2\ell\chi + \sum_{j=1}^{m_0} \frac{k}{w_j^0} \right) \frac{1}{z} + \frac{\ell(\ell+1)}{z^2} \\ &\quad + \sum_{j=1}^{m_0} \frac{2}{(z - w_j^0)^2} + \sum_{j=1}^{m_0} \frac{k}{w_j^0} \frac{1}{z - w_j^0}. \end{aligned}$$

In deriving this formula, we have used the Bethe Ansatz equations (4.14)–(4.15). They imply in particular that $v(z)$ has no singularities at $w_j^1, j = 1, \dots, m_1$.

In addition, the Bethe Ansatz equations (4.15) imply that the coefficients $v_{j,n}$ in the expansion of $v(z)$ at $z = w_j^0$,

$$v(z) \sim \frac{2}{(z - w_j^0)^2} + \sum_{n \geq -1} v_{j,n} (z - w_j^0)^n,$$

satisfy the algebraic equation

$$(4.19) \quad \frac{1}{4} \left(\frac{k}{w_j^0} \right)^3 - \frac{k}{w_j^0} v_{j,0} + v_{j,1} = 0, \quad j = 1, \dots, m_0.$$

Note also that $v_{j,-1} = k/w_j^0$.

For instance, if $k = 0$, then $v_{j,-1} = 0$ and the equation (4.19) reduces to $v_{j,1} = 0$ for all $j = 1, \dots, m_0$. Hence we obtain that in this case the condition is that the coefficients in front of $(z - w_j^0)^{-1}$ and $(z - w_j^0)$ in the expansion of $v(z)$ in Laurent series in $(z - w_j^0)$ should vanish for all $j = 1, \dots, m_0$.

These equations guarantee that on the formal neighborhood of each of the points $w_j^0, j = 1, \dots, m_0$, the $\widehat{\mathfrak{sl}}_2$ -oper (4.17), viewed as a connection on the trivial \overline{G} -bundle on the punctured disc around this point, is gauge equivalent to a connection without singularity. In other words, this $\widehat{\mathfrak{sl}}_2$ -oper has no monodromy around the points $w_j^0, j = 1, \dots, m_0$.

This is in fact a local condition on this oper which involves only its restrictions to the formal discs around these points. Let us analyze this local condition in more detail.

Suppose that we have an $\widehat{\mathfrak{sl}}_2$ -oper of the form

$$(4.20) \quad \partial_z + \begin{pmatrix} 0 & v(z) + \lambda \\ 1 & 0 \end{pmatrix} + x(z)d,$$

on the formal disc around a point w such that

$$(4.21) \quad v(z) = \frac{2}{(z - w)^2} + \sum_{n \geq -1} v_n (z - w)^n, \quad x(z) = \sum_{n \geq 0} x_n (z - w)^n.$$

What is the condition that the connection (4.17) on the trivial \overline{G} -bundle on the punctured disc around the point w (where $\overline{G} = \mathbb{C}^\times \ltimes G((\lambda))$) is gauge equivalent to a connection without singularity?

This connection preserves the trivial \overline{G}_- -bundle, where $\overline{G}_- = \mathbb{C}^\times \ltimes G[[\lambda]]$. Taking the quotient modulo the subgroup $\mathbb{C}^\times \ltimes G^{(1)}$, where $G^{(1)}$ is the first congruence subgroup of $G[[\lambda]]$, we obtain the connection

$$(4.22) \quad \partial_z + \begin{pmatrix} 0 & v(z) \\ 1 & 0 \end{pmatrix}$$

on the trivial G -bundle, which we view as an \mathfrak{sl}_2 -oper. If the connection (4.20) has no monodromy around $z = w$, then the same is true for the connection (4.22). This implies that the following equation on the coefficients in the expansion of $v(z)$ at $z = w$ must hold:

$$(4.23) \quad \frac{1}{4}v_{-1}^3 - v_0v_{-1} + v_1 = 0$$

(see, e.g., [FFT], Sect. 4.4). But this equation is satisfied if and only if $v(z)$ may be represented as the Miura transformation

$$(4.24) \quad v(z) = u(z)^2 - \partial_z u(z),$$

where

$$u(z) = -\frac{1}{z-w} + \sum_{n \geq 0} u_n(z-w)^n.$$

Thus, our connection (4.20) is gauge equivalent, under the gauge transformation (4.16), to a connection of the form (4.11) with $u_{-1} = -1$. As explained in the previous section, the latter is gauge equivalent to a connection without singularity at $z = w$ if and only if equation (4.13) is satisfied. Substituting this into formula (4.24), we find that

$$(4.25) \quad v_{-1} = x_0.$$

As the result, we obtain the equation

$$(4.26) \quad \frac{1}{4}x_0^3 - v_0x_0 + v_1 = 0.$$

Thus, we obtain that the condition for the operator (4.20) to be monodromy free is given by two equations: (4.26) and (4.25). Applying this to the $\widehat{\mathfrak{sl}}_2$ -oper (4.17) on \mathbb{P}^1 with $w = w_j^0$, we obtain the equations (4.19) and $v_{j,-1} = k/w_j^0$.

Recall that we obtained equation (4.19) originally under the assumption that $v(z)$ may be represented globally, on the entire \mathbb{P}^1 , as the Miura transformation of (4.9). But now we see that we do not need to make this assumption. In fact, the local condition of triviality of the monodromy around a point $z = w$ of any $\widehat{\mathfrak{sl}}_2$ -oper (4.20) with $v(z)$ and $x(z)$

given by formula (4.21) is expressed by the equations (4.23) and (4.25), which give rise to the equation (4.19) in the case of the oper (4.17).

Applying the (formal) gauge transformation by z^{kd} to (4.17), we obtain the operator

$$\partial_z + \begin{pmatrix} 0 & v(z) + \lambda z^k \\ 1 & 0 \end{pmatrix},$$

which may be rewritten as the following second order operator with spectral parameter

$$(4.27) \quad \partial_z^2 - \chi^2 - \left(2\ell\chi + \sum_{j=1}^{m_0} \frac{k}{w_j^0} \right) \frac{1}{z} - \frac{\ell(\ell+1)}{z^2} - \sum_{j=1}^{m_0} \frac{2}{(z-w_j^0)^2} - \sum_{j=1}^{m_0} \frac{k}{w_j^0} \frac{1}{z-w_j^0} - \lambda z^k.$$

Now we may treat λ as a complex parameter. The equations (4.19) then imply that the operator (4.27) has no monodromy around the points $w_j^0, j = 1, \dots, m_0$, for all values of λ .

Let us summarize: a connection $\partial_z + u(z)$ given by formula (6.16) with the numbers w_j^0, w_j^1 satisfying the Bethe Ansatz equations (4.14)–(4.15) gives rise to an $\widehat{\mathfrak{sl}}_2$ -oper (4.17) on \mathbb{P}^1 (equivalently, the second order operator (4.27)) such that the coefficients in its expansion at $w_j^0, j = 1, \dots, m_0$, satisfy the equations (4.19). As we have seen above, these equations are the “no monodromy” conditions which may be viewed as analogues of the Bethe Ansatz equations (4.15). Our proposal for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$ is that for generic χ and ν the common eigenvalues of the quantum affine Gaudin Hamiltonians on the Verma module M_ν over $\widehat{\mathfrak{sl}}_2$ are encoded by $\widehat{\mathfrak{sl}}_2$ -opers of this form.

A solution of the Bethe Ansatz equations (4.14)–(4.15) gives us a solution of the equations (4.19), but we do not expect the converse to be true in general unless we impose additional conditions on $v(z)$. For instance, suppose that $v(z)$ is of the form (4.18) and that in addition to having no monodromy around the points w_j^0 , which is ensured by the equations (4.19), the operator (4.17) has monodromy ± 1 around $z = 0$ (this means, in particular, that ℓ is a non-negative half-integer). Then we believe that such $v(z)$ does come from a solution of the Bethe Ansatz equations (4.14)–(4.15). This is a conjecture, which is closely related to Conjecture 3 below.

Note that the operator (4.17) also gives rise to a parabolic Miura $\widehat{\mathfrak{sl}}_2$ -oper corresponding to the \mathfrak{sl}_2 -oper

$$\partial_z + \begin{pmatrix} 0 & v(z) \\ 1 & 0 \end{pmatrix}$$

and the connection $\partial_z + \frac{k}{z}$ in $\text{Conn}_{\mathbb{P}^1}(\Omega^2)$, in the sense of the definition of Section 4.3. The singular points $w_j^0, j = 1, \dots, m_0$, are the points where the two reductions, $\mathcal{F}_{\overline{B}_+}$ and $\mathcal{F}_{\overline{G}_-}$, of this parabolic Miura oper are in the relative position $\overline{s}_0 \in W \setminus W_{\text{aff}}$, where $s_0 \in W_{\text{aff}}$ is the simple reflection corresponding to the simple root α_0 . This is the generic situation. For special values of ν and χ the positions of some of the poles w_j^0 may coalesce and theseopers may degenerate further. In other words, there may appear points on \mathbb{P}^1 at which the relative positions of the two reductions are given by elements of $W \setminus W_{\text{aff}}$ other than \overline{s}_0 . If the relative position at a point w is given by the element $s_0(s_1 s_0)^n \in W \setminus W_{\text{aff}}, n \in \mathbb{Z}$, then the expansion of the function $v(z)$ appearing in (4.18) around w will have the form

$$v(w) \sim \frac{(2n + 1)(2n + 2)}{(z - w)^2} + \dots$$

The coefficients in the expansion of this function at w will also have to satisfy an analogue of equation (4.19), which means that our $\widehat{\mathfrak{sl}}_2$ -oper has no monodromy at w .

In addition, some of the points w_j^0 may tend to 0. This happens when the Verma module M_ν becomes reducible. The common eigenvalues encoded by these degenerations of theopers (4.17) should correspond to the eigenvectors which appear in a submodule of M_ν .

Finally, note that we may rewrite (4.17) as the following partial differential operator:

$$\left(\partial_z + \frac{k}{z} \lambda \partial_\lambda \right)^2 - v(z) - \lambda.$$

This formula underscores the fact that we are dealing here with two independent variables: z and λ . In our analysis we have treated z as a variable, but λ as the parameter of the loop algebra $\mathfrak{g}((\lambda))$. Thus, we have considered an ordinary differential operator (4.17), in z only, but with values in an infinite-dimensional Lie algebra; namely, the Lie algebra $\mathbb{C}d \oplus \mathfrak{g}((\lambda))$. However, considering it as a partial differential operator, in both z and λ , might also be useful in some applications.

4.7. Integrable representations

Now we consider the common eigenvalues of the affine Gaudin Hamiltonians on an integrable representation of $\widehat{\mathfrak{g}}$. Recall that for each integral dominant weight ν , i.e., such that $\langle \nu, \check{\alpha}_i \rangle \in \mathbb{Z}_+$, $i \in \widetilde{I}$, so in particular $k = \langle \nu, \mathbf{1} \rangle \in \mathbb{Z}_+$, we have an irreducible integrable representation V_ν , which is the irreducible quotient of the Verma module M_ν . By analogy with the finite-dimensional case (see [FFT], Theorem 2, (2) and the conjecture after it), we propose the following description of the common eigenvalues of the affine Gaudin Hamiltonians on the space of $\mathfrak{th}[[t]]$ -invariant vectors in integrable representations of $\widehat{\mathfrak{g}}$. Note that this space is a module over the corresponding generalized parafermion algebra, which appears in the space of states of the $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{h}})$ coset model.

Conjecture 3. *The set of common eigenvalues of the affine Gaudin Hamiltonians on the space of $\mathfrak{th}[[t]]$ -invariant vectors in an integrable module V_ν over $\widehat{\mathfrak{g}}$ (counted without multiplicities) is in bijection with the set of ${}^L\widehat{\mathfrak{g}}$ -opers on \mathbb{P}^1 which are ν -regular at 0, and have singularity of order 2 at ∞ with the 2-residue $\pi(\chi)$.*

Note that all of these opers necessarily have *trivial monodromy* on \mathbb{P}^1 .

If the projection of the Bethe vector (6.3) onto V_ν is non-zero, then the corresponding common eigenvalues of the affine Gaudin Hamiltonians are recorded by the ${}^L\widehat{\mathfrak{g}}$ -oper obtained by applying the Miura transformation to the Cartan connection (6.15). It then satisfies the conditions listed in Conjecture 3. This provides the first positive test for this conjecture.

It is natural to ask whether the Bethe Ansatz is complete in this model for generic values of χ . As in the finite-dimensional case (see [FFT]), we expect that it is complete, which gives us hope that the Bethe eigenvectors may furnish an explicit family of bases for all integrable representations of affine Kac–Moody algebras (parameterized by χ).

If the Bethe Ansatz is incomplete, then there are eigenvectors other than the Bethe vectors introduced above. In this case we conjecture that the common eigenvalues of the affine Gaudin Hamiltonians would still be encoded by the ${}^L\widehat{\mathfrak{g}}$ -opers as in Conjecture 3. However, the eigenvalues corresponding to the non-Bethe eigenvectors would be encoded by opers that cannot be expressed as the Miura transformation of a connection of the form (4.8). The structure of these opers in the finite-dimensional case is explained in [F3], Sect. 5.5. We expect that the same pattern holds in the affine case as well. In particular, we expect that the opers corresponding to the non-Bethe eigenvectors may still be expressed as

the Miura transformation, but of Cartan connections of the more general form

$$(4.28) \quad \partial_z - \chi + \frac{\rho - y(\nu + \rho)}{z} + \sum_{j=1}^m \frac{\rho - y_j(\rho)}{z - w_j},$$

where $\rho \in {}^L\bar{\mathfrak{h}}$ is defined as in Section 4.3, and y and $y_j, j = 1, \dots, m$, could be arbitrary elements of the affine Weyl group corresponding to ${}^L\mathfrak{g}$. They measure the relative position between the two reductions of the oper bundle, the oper reduction \mathcal{F}_{B_+} and the horizontal reduction \mathcal{F}_{B_-} at the points $z = 0$ and $z = w_j$, respectively.

Note that if $y = s_i$ is the simple reflection corresponding to the i th simple root, then $\rho - y(\rho) = \alpha_i$. Thus, if $y = 1$ and all the y_j 's are simple reflections, then (4.28) becomes (4.8). We expect that this is the generic situation. However, for special values of parameters some of the common eigenvalues may be encoded by theopers corresponding the Miura transformation of the more general connections (4.28). For instance, for $\mathfrak{g} = \mathfrak{sl}_2$ the weights $\rho - y_j(\rho)$ have the form $\nu = 2n_j$, where n_j is an integer. The corresponding operator (4.27) will then have the leading term $-n_j(n_j + 1)/(t - w_j)^2$ (formula (4.27) corresponds to the case when all $n_j = 1$).

4.8. Generalization to other Kac–Moody algebras

It is natural to ask which of the structures discussed above may be generalized to an arbitrary symmetrizable Kac–Moody algebra \mathfrak{g} . Let us fix $\chi \in \mathfrak{h}^*$ such that

$$(\alpha, \chi) \neq 0, \quad \forall \alpha \in \Delta_+,$$

where Δ_+ is the set of positive roots of \mathfrak{g} . The following analogues of the DMT Hamiltonians (2.12) are well-defined on any highest weight module over \mathfrak{g} :

$$(4.29) \quad T_\gamma(\chi) = \sum_{\alpha \in \Delta_+} \frac{(\alpha, \gamma)}{(\alpha, \chi)} f_\alpha^{(i)} e_\alpha^{(i)}, \quad \gamma \in \mathfrak{h}^*,$$

where $\{e_\alpha^{(i)}\}$ and $\{f_\alpha^{(i)}\}$ form bases of the root subspaces \mathfrak{g}_α and $\mathfrak{g}_{-\alpha}$, respectively, which are dual with respect to a non-degenerate invariant inner product on \mathfrak{g} . These operators coincide with the connection operators of a connection studied in [FMTV]. The flatness of this connection,

established in [FMTV] for an arbitrary symmetrizable Kac–Moody algebra, implies that these operators mutually commute for different γ (and fixed χ).⁷

However, note that for the affine Kac–Moody algebras these Hamiltonians are different from the Hamiltonians (3.11). While in formula (4.29) the summation is over the set of all positive roots of $\widehat{\mathfrak{g}}$, in formula (3.11) it is over the set of positive real roots (and χ, γ are elements of the dual space to the Cartan subalgebra of the finite-dimensional Lie algebra \mathfrak{g} , not the extended Cartan subalgebra of $\widehat{\mathfrak{g}}$). The operators (3.11) may be obtained from (4.29) if we choose γ to be orthogonal to the imaginary root δ and then take the limit when χ becomes orthogonal to δ as well.

Consider the Bethe vectors $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ given by formula (2.9), where $i_j \in I$, the set of vertices of the Dynkin diagram of \mathfrak{g} . We then have an analogue of the statement of Conjecture 2 for the operators (4.29). It may be derived from the results of [FMTV] by considering the asymptotics of the solutions of the dynamical KZ equations constructed in [FMTV] near the critical level, along the lines of [RV].⁸ By taking the limit described in the previous paragraph, one can then prove Conjecture 2.

On the other hand, as explained in [F2], the notions of opers, Cartan connections and Miura transformation generalize to arbitrary symmetrizable Kac–Moody algebras in a straightforward fashion. Moreover, to each solution of the Bethe Ansatz equations (2.10) we may attach a Cartan connection by formula (4.8). Then its Miura transformation is an $L\mathfrak{g}$ -oper which satisfies the same properties as in the finite-dimensional and affine cases. Therefore it is tempting to assume that analogues of our conjectures relating $L\mathfrak{g}$ -opers to the spectra of commuting Hamiltonians (in particular, an analogue of Conjecture 3) also hold in this general context.

However, for Kac–Moody algebras other than finite-dimensional or affine, it is not clear how to construct other commuting quantum Hamiltonians. So at present it is not clear what is the commutative algebra whose spectra are described by these $L\mathfrak{g}$ -opers.

⁷This commutativity may probably also be proved by generalizing the argument used in [TL]. We thank V. Toledano Laredo for a useful discussion of this point.

⁸We thank A. Varchenko for pointing this out.

§5. Quantum KdV system

In this section we consider another example of the shift of argument affine Gaudin model, which corresponds to the $\widehat{\mathfrak{g}}$ -KdV hierarchies. In this case the shift χ is the nilpotent element of $\mathfrak{g} \subset \widehat{\mathfrak{g}}$ corresponding to the maximal root. The corresponding shift of argument algebra is not so interesting by itself, so we look instead at its image in the Hamiltonian reduction with respect to the centralizer of χ , which is $\mathfrak{n}((t))$. Classically, this Hamiltonian reduction has been introduced by Drinfeld and Sokolov [DS], and the corresponding commuting local classical Hamiltonians are the Hamiltonians of the $\widehat{\mathfrak{g}}$ -KdV hierarchy, as explained in [DS]. These Hamiltonians generate a commutative subalgebra in the Poisson algebra obtained by the Drinfeld–Sokolov Hamiltonian reduction from the algebra of functions on $\widehat{\mathfrak{g}}_1^*$. This Poisson algebra is known as the classical \mathcal{W} -algebra. Its quantization is the quantum \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g})$ introduced in [FL] for $\mathfrak{g} = \mathfrak{sl}_n$ and in [FF3] for general \mathfrak{g} (see also [FF2, FF5, FB]). Thus, the quantum Gaudin Hamiltonians in this case are commuting elements of $\mathcal{W}(\mathfrak{g})$. They are known as the local quantum KdV Hamiltonians [FF5].

As in the case of regular semi-simple shift of argument, discussed in the previous section, there are actually two types of Hamiltonians: local and non-local. They are obtained by expanding invariant functions of the monodromy at $z = \infty$ and $z = 0$, respectively. The existence of the local quantum Hamiltonians was established in [FF4, FF5], though explicit formulas for them are unknown in general. The non-local quantum Hamiltonians have been constructed in [BLZ1]. Their spectra on irreducible highest weight modules over the Virasoro algebra have been studied in the series of papers [BLZ1]–[BLZ5]. In particular, in [DT1, BLZ4, BLZ5] a remarkable conjecture was made relating these spectra to certain Schrödinger operators. In this section we show that this conjecture fits in perfectly as a special case of our general proposal that the spectra of quantum Hamiltonians in the affine Gaudin model associated to $\widehat{\mathfrak{g}}$ are given by opers for the Langlands dual affine Lie algebra ${}^L\widehat{\mathfrak{g}}$.

5.1. The classical model

The phase space of the classical model is defined in the same way as in Section 3.2. It consists of the operators⁹

$$(5.1) \quad L = \partial_t + A(t) + \chi z, \quad A(t) \in \mathfrak{g}((t))$$

⁹In what follows, in order to simplify the exposition, we will assume that $\widehat{\mathfrak{g}}$ is untwisted. But our discussion applies to twisted affine algebras as well.

where now

$$\chi = e_\theta \in \mathfrak{n}_+ \subset \mathfrak{g} \subset \mathfrak{g}((t))$$

is a nilpotent element corresponding to the maximal root of \mathfrak{g} . The centralizer of χ is therefore the Lie algebra $\mathfrak{n}_+((t))$. As explained in Section 2.1, instead of constructing a Poisson commutative subalgebra in the Poisson algebra of functions on (5.1), we may construct a commutative subalgebra in its Hamiltonian reduction with respect to $\mathfrak{n}_+((t))$ and its character. We choose the character ψ which corresponds to the element

$$(5.2) \quad \bar{p}_{-1} = \sum_{i=1}^{\ell} f_i \in \mathfrak{n}_- \subset \mathfrak{g}((t)).$$

The resulting Hamiltonian reduction is the *Drinfeld–Sokolov reduction* [DS].

The reduced phase space of the Drinfeld–Sokolov reduction is therefore the quotient of the space $\widetilde{\mathcal{M}}(\widehat{\mathfrak{g}})$ of operators of the form

$$(5.3) \quad L' = \partial_t + p_{-1} + \mathbf{v}(t), \quad \mathbf{v}(t) \in \mathfrak{b}_+((t)),$$

by the gauge action of the group $N_+((t))$. Here

$$p_{-1} = \sum_{i \in \bar{I}} f_i = \sum_{i=1}^{\ell} f_i + e_\theta z$$

is a “principal nilpotent element” of $\mathfrak{g}((z))$. According to [DS], the action of $N_+((t))$ on $\widetilde{\mathcal{M}}(\widehat{\mathfrak{g}})$ is free. The resulting quotient space¹⁰

$$\mathcal{M}(\widehat{\mathfrak{g}}) = \widetilde{\mathcal{M}}(\widehat{\mathfrak{g}})/N_+((t))$$

is isomorphic to the space of \mathfrak{g} -opers on the punctured disc $D^\times = \text{Spec } \mathbb{C}((t))$. The Poisson algebra of local functionals on $\mathcal{M}(\widehat{\mathfrak{g}})$ is known as the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g})$.

For example, $\mathcal{M}(\widehat{\mathfrak{sl}}_2)$ is the quotient of operators of the form

$$\partial_t + \begin{pmatrix} a(t) & b(t) + z \\ 1 & -a(t) \end{pmatrix}, \quad a(t), b(t) \in \mathbb{C}((t)),$$

¹⁰Note that the action of $N_+((t))$, and hence the quotient, do not depend on whether we include $e_\theta z$ in p_{-1} or not.

by the upper triangular gauge transformations depending on t (but not on z). It is easy to see that each orbit contains a unique operator of the form

$$\partial_t + \begin{pmatrix} 0 & v(t) + z \\ 1 & 0 \end{pmatrix}, \quad v(t) \in \mathbb{C}((t)),$$

and hence we may identify $\mathcal{M}(\widehat{\mathfrak{sl}}_2)$ with the space of such operators, or, equivalently, with the space of second order differential operators with spectral parameter

$$\partial_t^2 - v(t) - z, \quad v(t) \in \mathbb{C}((t)).$$

Similarly, the space $\mathcal{M}(\widehat{\mathfrak{sl}}_n)$ may be identified with the space of n th order differential operators with spectral parameter

$$(-\partial_t)^n - v_1(t)(-\partial_t)^{n-2} - \dots + v_{n-2}(t)\partial_t - v_{n-1}(t) - z.$$

For other Lie algebras there also exist canonical representatives for the orbits of the gauge action of $N_+((t))$ on the space of operators of the form (5.3) (see, e.g., [F3]). Recall the element $\bar{p}_{-1} \in \mathfrak{n}_-$ given by formula (5.2). There exists a unique element of \mathfrak{n}_+ of the form

$$\bar{p}_1 = \sum_{i=1}^{\ell} c_i e_i, \quad c_i \in \mathbb{C},$$

such that \bar{p}_1, \bar{p}_{-1} , and $\bar{p}_0 = [\bar{p}_1, \bar{p}_{-1}]$ form an \mathfrak{sl}_2 triple. The element $\frac{1}{2}\bar{p}_0 \in \mathfrak{h}$ then defines the principal grading on \mathfrak{g} such that $\deg \bar{p}_1 = 1, \deg \bar{p}_{-1} = -1$. Let

$$V_{\text{can}} = \bigoplus_{i \in E} V_{\text{can},i}$$

be the space of $\text{ad } \bar{p}_1$ -invariants in \mathfrak{n} , decomposed according to the principal grading. Here

$$E = \{d_1, \dots, d_\ell\}$$

is the set of exponents of \mathfrak{g} . Then \bar{p}_1 spans $V_{\text{can},1}$. Choose a linear generator \bar{p}_j of V_{can,d_j} (if the multiplicity of d_j is greater than one, which happens only in the case $\widehat{\mathfrak{g}} = D_{2n}^{(1)}, d_j = 2n$, then we choose linearly independent vectors in V_{can,d_j}). The following result is due to Drinfeld and Sokolov [DS] (the proof is reproduced in [F2], Lemma 2.1).

Lemma 4. *The gauge action of $N_+((t))$ on the space $\widetilde{\mathcal{M}}(\widehat{\mathfrak{g}})$ is free, and each gauge equivalence class contains a unique operator of the form*

$\partial_t + p_{-1} + \mathbf{v}(t)$, where $\mathbf{v}(t) \in V_{\text{can}}((t))$, so that we can write

$$(5.4) \quad \mathbf{v}(t) = \sum_{j=1}^{\ell} v_j(t) \cdot p_j, \quad v_j(t) \in \mathbb{C}((t)).$$

Thus, each point of the reduced phase space $\mathcal{M}(\widehat{\mathfrak{g}})$ of the Drinfeld–Sokolov reduction is canonically represented by an operator $\partial_t + p_{-1} + \mathbf{v}(t)$, where $\mathbf{v}(t)$ is of the form (5.4).

5.2. Classical Hamiltonians

The Poisson commuting Hamiltonians on the reduced phase space are constructed in the same way as in the case of regular semi-simple χ : by expanding the monodromy of the operators (5.3) with respect to the spectral parameter z . Namely, let $M(z) \in G$ be the monodromy matrix of the operator (5.1). For any invariant function φ on the group G the corresponding function $H_\varphi(z) = \varphi(M(z))$ on $\widetilde{\mathcal{M}}(\widehat{\mathfrak{g}})$ is invariant under the gauge action of $N_+((t))$ and hence gives rise to a well-defined function on the quotient $\mathcal{M}(\widehat{\mathfrak{g}})$. We obtain Poisson commuting Hamiltonians on $\mathcal{M}(\widehat{\mathfrak{g}})$ by expanding $H_\varphi(z)$ in z and z^{-1} .

The asymptotic expansion at $z = \infty$ yields the local Hamiltonians, which generate the $\widehat{\mathfrak{g}}$ -KdV hierarchy of commuting Hamiltonian flows on the Poisson manifold $\mathcal{M}(\widehat{\mathfrak{g}})$. These Hamiltonians have the form

$$H_n = \int P_n(v_j(t), v'_j(t), \dots) dt, \quad n = d_i + Nh, N \in \mathbb{Z}_+,$$

where $d_i \in E$ is an exponent of \mathfrak{g} and h is the Coxeter number. The integrand P_n is a differential polynomial of degree $n + 1$, where we set $\deg v_j^{(m)} = d_j + m + 1$.

On the other hand, the z -expansion at $z = 0$ yields Poisson commuting non-local Hamiltonians. It is instructive to consider the z -linear term in the expansion. For that we apply Lemma 1 with $\epsilon = z$,

$$A^{(0)}(t) = \sum_{i=1}^{\ell} f_i + v(t), \quad v(t) \in \mathfrak{b}_+((t)),$$

and $A^{(1)}(t) = e_\theta$. We obtain that the z -linear term in the expansion of $\varphi(M(z))$, where $\varphi \in (\text{Fun } G)^G$, is given by the formula

$$(5.5) \quad H_\varphi^{(1)} = \int (e_\theta, \Psi_\varphi(t)) dt.$$

This is the lowest weight component of the single-valued solution $\Psi_\varphi(t) \in \mathfrak{g}$ of the equation

$$(5.6) \quad \left(\partial_t + \sum_{i=1}^{\ell} f_i + v(t) \right) \Psi(t) = 0,$$

with values in the adjoint representation of \mathfrak{g} , corresponding to the invariant function φ .

The commutativity (3.5) of the Hamiltonians H_φ then implies that

$$\{H_\varphi^{(1)}, H_\psi^{(1)}\} = 0$$

for all $\varphi, \psi \in (\text{Fun } G)^G$.

Higher order terms in the z -expansion of $\varphi(M(z))$ may be expressed as multiple integrals of the lowest weight components of solutions of the equation (5.6).

For example, if $\mathfrak{g} = \mathfrak{sl}_2$, the equation (5.6) in the adjoint representation is equivalent to the following scalar third order differential equation:

$$(5.7) \quad (\partial_t^3 - 4v\partial_t - 2v')\Psi = 0$$

(its solutions are products of solutions of the second order equation $(\partial_t^2 - v)\psi = 0$). It has a unique (up to a scalar) single-valued solution $\Psi_\varphi(t)$ (here we may choose as φ the trace function on SL_2 , as it generates the algebra of invariant functions on SL_2), and $H_\varphi^{(1)} = \int \Psi_\varphi(t) dt$.

Remark 2. A convenient way to compute the higher order terms in the z -expansion of $\varphi(M(z))$ is to realize the variables of the KdV hierarchy in terms of the variables of the modified KdV (mKdV) hierarchy. This provides a kind of “free field realization”, also known as the Miura transformation, for the commuting Hamiltonians.

Consider the space $\overline{\mathcal{M}}(\widehat{\mathfrak{g}})$ of operators of the form

$$(5.8) \quad \partial_t + p_{-1} + \mathbf{u}(t), \quad \mathbf{u}(t) \in \mathfrak{h}(\!(t)\!).$$

The natural map $\overline{\mathcal{M}}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{M}(\widehat{\mathfrak{g}})$ given by the composition of the inclusion $\overline{\mathcal{M}}(\widehat{\mathfrak{g}}) \rightarrow \widetilde{\mathcal{M}}(\widehat{\mathfrak{g}})$ and the projection $\widetilde{\mathcal{M}}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{M}(\widehat{\mathfrak{g}})$ is the Miura transformation.¹¹ It is a Poisson map with respect to the Heisenberg–Poisson structure on $\overline{\mathcal{M}}(\widehat{\mathfrak{g}})$.

¹¹We have already encountered Miura transformation above, but in a totally different context: namely, it appeared in our study of the eigenvalues of the *quantum* Hamiltonians. Here it appears in the study of the phase space on which the *classical* Hamiltonians are defined.

Because the operator (5.8) has such a simple structure, it is easier to compute the monodromy matrix $M(z)$ for it (and hence for $\varphi(M(z))$) than for the operators of the form (5.3) or (5.4). One can then write down more explicit integral formulas for the coefficients in the z -expansion in $\varphi(M(z))$. For example, in the case when $\mathfrak{g} = \mathfrak{sl}_2$ the operator (5.8) has the form

$$\partial_t + \begin{pmatrix} u(t) & z \\ 1 & -u(t) \end{pmatrix}.$$

The coefficients in the z -expansion of the trace of the monodromy of this operator are written down explicitly in [BLZ1]. They are given by multiple integrals of $\exp(\pm 2\phi(t))$, where $\phi(t)$ is the anti-derivative of $u(t)$, that is, $u(t) = \phi'(t)$. These are the classical analogues of the two screening currents discussed at the end of Section 3.3.

Similar formulas may be obtained for other simple Lie algebras.

5.3. Quantum Hamiltonians

Now we consider the quantization of the Hamiltonians of the $\widehat{\mathfrak{g}}$ -KdV system. The problem of quantization of the local Hamiltonians has been considered, in particular, in [Z, EY, KM, FF4, FF5]. It is closely related to the problem of quantization of the mKdV Hamiltonians and integrals of motion of affine Toda field theories (which are in turn related to deformations of conformal field theories). It was proved in [FF4, FF5] that all classical local $\widehat{\mathfrak{g}}$ -KdV Hamiltonians may be quantized. The quantum local Hamiltonians generate a commutative subalgebra in the quantum \mathcal{W} -algebra $\mathcal{W}_\beta(\mathfrak{g})$, where β is the parameter related to the central charge. This is an associative algebra quantizing the classical (Poisson) \mathcal{W} -algebra of local functionals on $\mathcal{M}(\widehat{\mathfrak{g}})$. It is obtained by the quantum Drinfeld–Sokolov reduction of a completion of the enveloping algebra of $\widehat{\mathfrak{g}}$ and depends on one complex parameter, the central charge c , which is a function of the level of $\widehat{\mathfrak{g}}$ (see [FF3, FB]). For example, $\mathcal{W}_\beta(\mathfrak{sl}_2)$ is nothing but the completed enveloping algebra of the Virasoro algebra. We will discuss the quantum local KdV Hamiltonians in more detail in Section 6.4 below.

The quantum non-local Hamiltonians were introduced in [BLZ1] in the case of $\mathfrak{g} = \mathfrak{sl}_2$ and in [BHK] for $\mathfrak{g} = \mathfrak{sl}_3$ (this construction may be generalized to other \mathfrak{g} [FFS]).

To give an idea as to what these quantum Hamiltonians look like, we discuss the quantization of the Hamiltonians $H_\varphi^{(1)}$ introduced above. Consider first the case when $\mathfrak{g} = \mathfrak{sl}_2$. In this case $H_\varphi^{(1)}$ is given by the integral of a single-valued solutions $\Psi_\varphi(t)$ of the equation (5.7), which is unique up to a scalar. The quantum analogues of solutions of this equations are the primary fields $\Phi_{(1,3)}(w)$ of the Virasoro algebra. There

are actually three primary fields, $\Phi_{(1,3)}^\pm(w)$ and $\Phi_{(1,3)}^0(w)$, acting between Verma modules over the Virasoro algebra, but only one of them, $\Phi_{(1,3)}^0(w)$, acts from a Verma module to itself. This field is the quantization of the unique (up to a scalar) single-valued solution $\Phi_\varphi(t)$ of the equation (5.7). Therefore the quantization of the Hamiltonian $H_\varphi^{(1)}$ is given by the 0th Fourier component of the primary field $\Phi_{(1,3)}^0(w)$. (Here we multiply the primary field by $w^{\Delta_{(1,3)}}$, where $\Delta_{(1,3)}$ is the conformal dimension, so that $\Phi_{(1,3)}^0(w)$ is a formal power series in integer powers of w .)

To construct the quantizations of other coefficients in the z -expansion of $\varphi(M(z))$, we may use the free field realization of the Virasoro algebra in terms of the Heisenberg algebra, quantizing the Miura transformation map between the Poisson algebras of functions on $\mathcal{M}(\widehat{\mathfrak{sl}}_2)$ and $\overline{\mathcal{M}}(\widehat{\mathfrak{sl}}_2)$ described in Remark 2. Then we need to quantize the coefficients in the z -expansion of the monodromy matrix of the operator (5.8). The resulting quantum non-local Hamiltonians will be given by multiple integrals of the screening currents quantizing the classical screening currents $\exp(\pm 2\phi(t))$ introduced in Remark 2. This is the approach taken in [BLZ1].

More precisely, the integrals of the two screening currents satisfy the q -Serre relations of $U_q(\widehat{\mathfrak{sl}}_2)$, where $q = \exp(\pi i \beta^2)$ (see [FF5]). The non-local Hamiltonians correspond to singular vectors in the Verma modules over $U_q(\widehat{\mathfrak{sl}}_2)$ of critical level (see the discussion in Section 3.3). In particular, the simplest of these quantum Hamiltonians, $H_\varphi^{(1)}$, is given by the q -commutator of the two screening charges.

Alternatively, we may use the primary fields $\Phi_{(1,3)}^\pm(w)$ and $\Phi_{(1,3)}^0(w)$, and the fact that their integrals generate a quantum deformation of the algebra $U(\widetilde{\mathfrak{sl}}_{2,-})$, as discussed in Section 3.3.

This construction generalizes to other Lie algebras. For each \mathfrak{g} there are primary fields $\Phi_{(1,\text{adj})}(w)$ for the corresponding quantum \mathcal{W} -algebra $\mathcal{W}_\beta(\mathfrak{g})$, which are the analogues of the primary fields $\Phi_{(1,3)}(w)$. The vector space of this fields is isomorphic to \mathfrak{g} , and it contains an ℓ -dimensional subspace (isomorphic to $\mathfrak{h} \subset \mathfrak{g}$) of those primary fields $\Phi_{(1,\text{adj}),p}(w)$ which act from each Verma module over $\mathcal{W}_\beta(\mathfrak{g})$ to itself. These are the quantum analogues of the single-valued solutions $\Psi_\varphi(t)$ of the equation (5.6). The 0th Fourier coefficients of $\Phi_{(1,\text{adj}),p}(w)$, $p = 1, \dots, \ell$ (multiplied by $w^{\Delta_{(1,\text{adj})}}$) are the quantizations of the classical Hamiltonians $H_\varphi^{(1)}$ given by formula (5.5). Other quantum non-local Hamiltonians may be expressed as multiple integrals of the primary fields $\Phi_{(1,\text{adj})}(w)$.

Alternatively, we may use the free field realization of $\mathcal{W}_\beta(\mathfrak{g})$ and express them as multiple integrals of the corresponding screening currents [BHK, FFS]. Note that the quantum non-local Hamiltonians commute with each other and with the local ones. We will discuss this in more detail in [FF7].

The upshot of the above discussion is that we have a large commutative algebra acting on the Verma modules (and other highest weight modules) over the Virasoro algebra. This algebra is generated by the local and non-local quantum KdV Hamiltonians. We now look for a parameterization of the common eigenvalues of these Hamiltonians. Our experience with the shift of argument models in which the shift χ is regular semi-simple suggests that those should be parameterized by affine $L\widehat{\mathfrak{g}}$ -opers of special kind (discussed in Section 4.4).

In the case at hand we have a nilpotent shift of argument. In order to understand better what kind of opers correspond to this shift, we first consider the analogous question in the finite-dimensional case.

5.4. Nilpotent shift in the finite-dimensional case

Recall from [FFT] that the shift of argument Gaudin algebra \mathcal{A}_χ may be obtained as a quotient of the algebra $\text{End}_{\widehat{\mathfrak{g}}} \mathbb{I}_{1,\chi,\kappa_c}$, where

$$\mathbb{I}_{1,\chi} = \text{Ind}_{t\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}} \mathbb{C}_\chi.$$

Here \mathbb{C}_χ is the one-dimensional module over $t\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}$, on which $t\mathfrak{g}[[t]]$ acts via the character

$$t\mathfrak{g}[[t]] \rightarrow t\mathfrak{g}[[t]]/t^2\mathfrak{g}[[t]] \simeq \mathfrak{g} \xrightarrow{-\chi} \mathbb{C}$$

and the central element $\mathbf{1}$ act as $-h^\vee$. Let us choose χ to be the linear functional on \mathfrak{g} taking a non-zero value on e_θ and zero on all other root generators. The centralizer of this χ is the Lie subalgebra $\mathfrak{n}_- \subset \mathfrak{g}$. We wish to understand the corresponding reduced algebra \mathcal{A}_χ^ψ (see Section 2.2), where ψ is a principal character of \mathfrak{n}_- .

Hence we perform the quantum Drinfeld–Sokolov reduction of the module $\mathbb{I}_{1,\chi}$ with respect to the Lie algebra $\mathfrak{n}_-((t))$ and its character

$$(5.9) \quad f_\alpha \otimes t^n \mapsto \begin{cases} 1, & \alpha = \alpha_i, n = 0, \\ 0, & \text{otherwise} \end{cases}$$

(see [FF1]).

As the result, we obtain a module over the center $Z(\widehat{\mathfrak{g}})$ at the critical level, which we will denote by $H_{\text{DS}}(\mathbb{I}_{1,\chi})$. Recall that $Z(\widehat{\mathfrak{g}})$ is a completion of the polynomial algebra with generators $S_{i,(n)}, i = 1, \dots, \ell; n \in \mathbb{Z}$,

where we use the convention, in which the generating currents are

$$S_i(w) = \sum_{n \in \mathbb{Z}} S_{i,(n)} w^{-n-1}$$

(the first of them, $S_1(z)$ is the unnormalized Segal–Sugawara current), see, e.g., [FFT]. It is easy to see that the elements $S_{i,(n_i)}, i = 1, \dots, \ell - 1; n_i > d_i$, as well as $S_{\ell,(n_\ell)}, n_\ell > d_\ell + 1$, act by 0 on the generating vector of $\mathbb{I}_{1,\chi}$ and hence on $H_{\text{DS}}(\mathbb{I}_{1,\chi})$. But $S_{\ell,(d_\ell+1)}$ contains a monomial which is the product of $f_i \otimes 1, i = 1, \dots, \ell$, and $e_\theta \otimes t$, with a non-zero coefficient. Each factor of this monomial acts by a non-zero scalar on the generating vector; for $f_i \otimes 1, i = 1, \dots, \ell$, this follows from formula (5.9), and for $e_\theta \otimes t$ this follows from the definition of $\mathbb{I}_{1,\chi}$. Therefore the monomial itself, and hence $S_{\ell,(d_\ell+1)}$, act by a non-zero scalar on $\mathbb{I}_{1,\chi}$ and on $H_{\text{DS}}(\mathbb{I}_{1,\chi})$.

Recall (see, e.g., [F3]) that we have canonical representatives for $L\mathfrak{g}$ -opers on the punctured disc D^\times of the form¹²

$$\partial_z + p_{-1} + \mathbf{v}(z),$$

where

$$(5.10) \quad \mathbf{v}(z) = \sum_{j=1}^{\ell} v_j(z) \cdot \bar{p}_j, \quad v_j(z) \in \mathbb{C}(\!(z)\!).$$

For $c \in \mathbb{C}$, let $\text{Op}_{L\mathfrak{g}}^{\text{irr},c}(D)$ be the space of $L\mathfrak{g}$ -opers on the punctured disc of the form (5.10), where $v_j(z) \in z^{-d_j-1}\mathbb{C}[[z]]$ for $j = 1, \dots, \ell - 1$, and

$$v_\ell(z) = cz^{-d_\ell-2} + \dots$$

Thus, for $c = 0$ this is the space of $L\mathfrak{g}$ -opers D^\times with regular singularity, and for $c \neq 0$ this is the space of opers with the mildest possible irregular singularity at the origin.

Lemma 5. *We have an isomorphism*

$$H_{\text{DS}}(\mathbb{I}_{1,\chi}) \simeq \text{Fun Op}_{L\mathfrak{g}}^{\text{irr},c}(D)$$

for some non-zero value of c .

¹²Note that in Section 5.1 have encountered \mathfrak{g} -opers as points of the phase space $\mathcal{M}(\widehat{\mathfrak{g}})$ of the classical $\widehat{\mathfrak{g}}$ -KdV hierarchy. On the other hand, now we are discussing the Hamiltonians in the quantum shift of argument Gaudin model associated to \mathfrak{g} . The spectra of those Hamiltonians are expressed by $L\mathfrak{g}$ -opers. It is important to realize that these are two different spaces of opers! For more on this, see Section 5.6.

The value c depends on the normalization of \bar{p}_ℓ as well as the normalization of χ , but it is inessential, because by rescaling the coordinate z we may identify the spaces $\text{Op}_{L\mathfrak{g}}^{\text{irr},c}(D)$ for all non-zero values of c .

This has the following immediate corollary:

Corollary 1. *For any $c \in \mathbb{C}^\times$ the algebra \mathcal{A}_χ^ψ is isomorphic to the algebra of functions on the space of $L\mathfrak{g}$ -opers on \mathbb{P}^1 with regular singularity at $0 \in \mathbb{P}^1$ and such that their restriction to the punctured disc D_∞^\times at $\infty \in \mathbb{P}^1$ belongs to $\text{Op}_{L\mathfrak{g}}^{\text{irr},c}(D_\infty)$.*

Note that according to the definition of \mathcal{A}_χ^ψ (see Section 2.2), it is isomorphic to the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. In addition, the algebra \mathcal{A}_0 corresponding to $\chi = 0$ is also isomorphic to $Z(\mathfrak{g})$. Thus, we obtain that $\text{Spec } Z(\mathfrak{g})$ is isomorphic to $\text{Op}_{L\mathfrak{g}}^{\text{irr},c}(D)$ for all $c \in \mathbb{C}$.

We are now ready to give a conjectural description of the spectra of the KdV Hamiltonians.

5.5. Spectra of the quantum KdV Hamiltonians

We start with the case of $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$, which corresponds to the KdV system.

Recall from Section 4.4 that in the case of regular semi-simple shift $\chi \in \mathfrak{h}$ we expect, by analogy with the finite-dimensional Gaudin models, that the spectra of the quantum Hamiltonians are encoded by the second order differential operators on \mathbb{P}^1 with global coordinate z , which have regular singularity at $z = 0$ with the leading term $-\ell(\ell + 1)/z^2$ and irregular singularity at ∞ with the leading term $-\chi^2/s^4$, where $s = z^{-1}$ is a local coordinate at $\infty \in \mathbb{P}^1$. These opers are (generically) represented by formula (4.27).

We stress again that this \mathbb{P}^1 is a curve in the *spectral parameter* of the affine Gaudin model.

Now, Corollary 1 suggests that for the KdV system, in which χ is nilpotent, the relevant space of $\widehat{\mathfrak{sl}}_2$ -opers consists of the second order differential operators on \mathbb{P}^1 that have regular singularity at $z = 0$ with the leading term $-\ell(\ell + 1)/z^2$ and irregular singularity at ∞ , but now of order 3, that is, of the form $c/s^3, c \in \mathbb{C}^\times$. By rescaling the coordinate z on \mathbb{P}^1 , we can make $c = 1$, so without loss of generality we will consider operators with singularity $1/s^3$ at ∞ . The corresponding $\widehat{\mathfrak{sl}}_2$ -opers should therefore have the form

$$\partial_z + \begin{pmatrix} 0 & v(z) + \lambda \\ 1 & 0 \end{pmatrix} + \frac{k}{z}d,$$

where

$$(5.11) \quad v(z) = \frac{\ell(\ell+1)}{z^2} + \frac{1}{z} \left(1 - \sum_{j=1}^m \frac{k}{w_j} \right) + \sum_{j=1}^m \frac{2}{(z-w_j)^2} + \sum_{j=1}^m \frac{k}{w_j} \frac{1}{z-w_j},$$

and the coefficients $v_{j,k}$ in the expansion of $v(z)$ in $z-w_j$ satisfy the equations (4.19),

$$(5.12) \quad \frac{1}{4} \left(\frac{k}{w_j} \right)^3 - \frac{k}{w_j} v_{j,0} + v_{j,1} = 0, \quad j = 1, \dots, m.$$

As explained in Section 4.6, this is the condition of “no monodromy” at the points w_j .

As in Section 4.4, applying gauge transformation by z^{kd} , we obtain the operator

$$(5.13) \quad \partial_z + \begin{pmatrix} 0 & v(z) + \lambda z^k \\ 1 & 0 \end{pmatrix},$$

which we rewrite as the following second order differential operator with spectral parameter:

$$(5.14) \quad \partial_z^2 - \frac{1}{z} \left(1 - \sum_{j=1}^m \frac{k}{w_j} \right) - \frac{\ell(\ell+1)}{z^2} - \sum_{j=1}^m \frac{2}{(z-w_j)^2} - \sum_{j=1}^m \frac{k}{w_j} \frac{1}{z-w_j} - \lambda z^k.$$

The equations (5.12) are equivalent to the condition that this operator has no monodromy around $w_j, j = 1, \dots, m$, that is, no monodromy on \mathbb{P}^1 , except around the points 0 and ∞ , for all values of λ .

According to our proposal, the differential operators of this kind should encode the common eigenvalues of the quantum KdV Hamiltonians on irreducible highest weight modules over the Virasoro algebra. Let us recall that the Virasoro algebra obtained by the Drinfeld–Sokolov reduction (with respect to the loop algebra of \mathfrak{n}_-) from the affine Kac–Moody algebra of level k has the central charge

$$(5.15) \quad c_k = 1 - \frac{6(k+1)^2}{k+2}$$

(see [FF1]). Further, the Drinfeld–Sokolov reduction of the irreducible $\widehat{\mathfrak{sl}}_2$ -module with highest weight $\lambda = 2\ell$ and level k is either 0 (which happens if and only if $k - \lambda \in \mathbb{Z}_+$), or is the irreducible module over the

Virasoro algebra with highest weight (by which we mean the eigenvalue of L_0 on the highest weight vector)

$$(5.16) \quad \Delta_{\ell,k} = \frac{(2\ell + 1)^2 - (k + 1)^2}{4(k + 2)}$$

(see [FF1], p. 316).

Let us denote by $L_{\Delta,c}$ the irreducible module over the Virasoro algebra with highest weight Δ and central charge c (this is the irreducible quotient of the Verma module with this highest weight).

Our proposal is then the following:

Conjecture 4. *The generic common eigenvalues of the quantum KdV Hamiltonians on the subspace of $L_{c_k, \Delta_{\ell,k}}$ of conformal dimension $\Delta_{\ell,k} + m$ are in bijection with the set of operators of the form (5.14) satisfying the equations (5.12).*

For special values of ℓ and k the positions of some of the poles w_j coalesce or approach 0, and theseopers may degenerate, as in the case of regular semi-simple χ discussed in Section 4.4. Namely, there may appear points on \mathbb{P}^1 at which the relative positions of the two reductions in the corresponding parabolic Miura oper are given by elements of $W \setminus W_{\text{aff}}$ other than \bar{s}_0 . If the relative position at a point w_j is given by the element $\overline{s_0(s_1 s_0)^n} \in W \setminus W_{\text{aff}}$, then the expansion of the function $v(z)$ appearing in (5.14) around w_j will have the form

$$v(z) \sim \frac{(2n + 1)(2n + 2)}{(z - w_j)^2} + \dots$$

The coefficients in the expansion of this function at w_j will also have to satisfy an analogue of equation (5.12), which means that our second order operator has no monodromy at w_j for all λ .

In addition, when the Verma module $M_{c_k, \Delta_{\ell,k}}$ becomes reducible, some of the points w_j may tend to 0. The common eigenvalues encoded by the corresponding limits of (5.14) should then correspond to the eigenvectors which appear in a submodule of this Verma module generated by singular vectors.

It would be interesting to develop an analogue of the Bethe Ansatz procedure for constructing eigenvectors of the quantum KdV Hamiltonians. It is not obvious how to do this. As we have seen above, the Bethe Ansatz appears when we represent ouropers as the Miura transformation of Cartan connections. This is possible in the Gaudin models associated to regular semi-simple χ , as discussed in Section 4.4. But theopers (5.14) cannot be represented as Miura transformation of Cartan

connections defined globally on \mathbb{P}^1 . (This becomes clear when we look at the restriction of (5.14) to the punctured disc at the point $z = \infty$: the operator has the form $\partial_s^2 + \tilde{v}(s)$, where $\tilde{v}(s) = c/s^3 + \dots$, $c \in \mathbb{C}^\times$, and $s = z^{-1}$ is the local coordinate at ∞ ; this $\tilde{v}(s)$ cannot be represented in the Miura form $u(s)^2 + \partial_s u(z)$, where $u(s)$ is a Laurent power series in s , as easily checked by substitution.)

5.6. Two different spaces of opers

In the study of the KdV system, second order differential operators, which we call $\widehat{\mathfrak{sl}}_2$ -opers, appear in two different ways. First, they appear as points of the phase space $\mathcal{M}(\widehat{\mathfrak{g}})$ of the classical KdV system, as described in Section 5.1. These operators have the form

$$\partial_t^2 - v(t) - z,$$

where t is a coordinate on a circle, or a punctured disc, and $v(t)$ is a function on it. (Despite what the notation may suggest, t is the “space”, not the “time”, coordinate of the KdV equation.) On the other hand, z is the spectral parameter. The classical KdV Hamiltonians are constructed by expanding the monodromy matrix of this operator, considered as function of z , near $z = 0$ and $z = \infty$.

The $\widehat{\mathfrak{sl}}_2$ -opers appear again as the objects encoding the eigenvalues of the quantum KdV Hamiltonians. They have the form

$$\partial_z^2 - v(z) - \lambda z^k,$$

where z is the spectral parameter of the classical KdV system. Thus, z is a coordinate on \mathbb{P}^1 , and $v(z)$ is a meromorphic function on this \mathbb{P}^1 (see formula (5.14)). On the other hand λ , is a new spectral parameter.

It is important to realize that these two spaces of opers are not related to each other. In fact, they correspond to two *dual* theories, one of which is classical and the other is quantum. In the classical theory the affine Kac–Moody algebra $\widehat{\mathfrak{sl}}_2$ is the central extension of $\mathfrak{sl}_2((z))$, where z is the spectral parameter, and the $\widehat{\mathfrak{sl}}_2$ -opers “live” on a circle with coordinate t , which is the space coordinate of the KdV hierarchy. In the quantum setting, $\widehat{\mathfrak{sl}}_2$ is the central extension of $\mathfrak{sl}_2((\lambda))$, and the $\widehat{\mathfrak{sl}}_2$ -opers “live” on a projective line \mathbb{P}^1 , whose coordinate z is the spectral parameter of the classical system.

Even more important is the fact that if we replace the Lie algebra $\widehat{\mathfrak{sl}}_2$ of the classical theory by a more general affine Kac–Moody algebra $\widehat{\mathfrak{g}}$, then the corresponding Lie algebra of the quantum theory will be not $\widehat{\mathfrak{g}}$, but its *Langlands dual* affine Lie algebra ${}^L\widehat{\mathfrak{g}}$, as we will discuss below. This makes it more clear that the classical and quantum theories are really different.

5.7. Comparison with the conjecture of Bazhanov, Lukyanov, and Zamolodchikov

We wish to compare our proposal to the conjectural relation between the eigenvalues of the quantum KdV Hamiltonians and second order differential operators on \mathbb{P}^1 of special kind, proposed by Bazhanov, Lukyanov, and Zamolodchikov in [BLZ5] (generalizing the proposal made earlier in [DT1, BLZ4] for eigenvalues on the highest weight vectors). Understanding the underlying reasons for this relation was in fact one of the main motivations for this paper.

Let us assume that $k \neq -2$ and set

$$\alpha = -\frac{k+1}{k+2}.$$

We will apply the change of variables

$$z = \frac{x^{2\alpha+2}}{(2\alpha+2)^2}$$

to the differential operator (5.14).

The general transformation formula for a second order operator

$$\partial_z^2 - v(z) : \Omega^{-1/2} \rightarrow \Omega^{3/2}$$

under the change of variables $z = \varphi(x)$ is

$$(5.17) \quad v(z) \mapsto v(\varphi(x)) (\varphi')^2 - \frac{1}{2} \{\varphi, x\},$$

where

$$(5.18) \quad \{\varphi, x\} = \frac{\varphi'''}{\varphi'} - \frac{3}{2} \left(\frac{\varphi''}{\varphi'} \right)^2$$

is the Schwarzian derivative of φ .¹³

After a straightforward calculation, we find that the new operator looks as follows:

$$(5.19) \quad \partial_x^2 - \frac{\tilde{\ell}(\tilde{\ell}+1)}{x^2} - x^{2\alpha} + 2 \frac{d^2}{dx^2} \sum_{j=1}^m \log(x^{2\alpha+2} - z_j) + E,$$

¹³Note that it is imperative to consider our second order operators as acting from $\Omega^{-1/2}$ to $\Omega^{3/2}$, for this is the only way to ensure that their property of having the principal symbol 1 and subprincipal symbol 0 is coordinate-independent.

where

$$\begin{aligned} \tilde{\ell}(\tilde{\ell} + 1) &= 4(\alpha + 1)^2 \ell(\ell + 1) + \alpha^2 + 2\alpha + \frac{3}{4} = 4(\alpha + 1)\Delta_{k,\ell} + \alpha^2 - \frac{1}{4}, \\ z_j &= (2\alpha + 2)^2 w_j, \quad E = -(2\alpha + 2)^{\frac{2\alpha}{\alpha+1}} \lambda. \end{aligned}$$

This operator coincides with the Schrödinger operators parameterizing the spectra of the quantum KdV Hamiltonians on $L_{c_k, \Delta_{\ell, k}}$ introduced in [BLZ5] (formula (1)).¹⁴ Moreover, our condition (5.12) means that the operator (5.19) has no monodromy around the points z_j for all E and is therefore equivalent to the algebraic equations of [BLZ5] (formula (3)).

Thus, we see that our proposal reproduces the proposal of [BLZ5] in the case of the quantum KdV system! We view this as a confirmation of the correspondence between the spectra of the Hamiltonians of quantum soliton systems and opers for the Langlands dual affine algebras which we propose in this paper.

5.8. Generalization to the case of $\widehat{\mathfrak{sl}}_n$

It is easy to generalize formula (5.14) to the case of $\widehat{\mathfrak{sl}}_n$. In this case the Langlands dual Lie algebra is also $\widehat{\mathfrak{sl}}_n$. The $\widehat{\mathfrak{sl}}_n$ -opers may be represented as n th order differential operators with spectral parameter. We wish to describe the differential operators of this kind on \mathbb{P}^1 that encode the spectra of the quantum Hamiltonians of the $\widehat{\mathfrak{sl}}_n$ -KdV system acting on the irreducible module over the \mathcal{W} -algebra $\mathcal{W}_k(\widehat{\mathfrak{sl}}_n)$ obtained by the quantum Drinfeld–Sokolov reduction of the irreducible $\widehat{\mathfrak{g}}$ -module $L_{\nu, k}$ of highest weight $\nu \in \mathfrak{h}^*$ and level k . The quantum Drinfeld–Sokolov reduction is described in [FF3, FKW]. In particular, the central charge of $\mathcal{W}_k(\widehat{\mathfrak{sl}}_n)$ is given by the standard formula

$$c = 1 + 2n(n + 1) - n(n + 1) \left((k + n) + \frac{1}{k + n} \right).$$

These operators have the following general form:

$$(5.20) \quad (-\partial_z)^n - v_1(z)(-\partial_z)^{n-2} - \dots + v_{n-2}(z)\partial_z - v_{n-1}(z) - z^k \lambda,$$

and act from $\Omega^{-(n-1)/2}$ to $\Omega^{(n+1)/2}$. This determines their transformation properties under the changes of coordinate z ; in particular, this ensures that the property that their principal symbol is $(-1)^n$ and the subprincipal symbol is 0 is preserved by the changes of coordinate. The

¹⁴Note that $\tilde{\ell}$ corresponds to the ℓ in [BLZ5], but it is not equal to our ℓ .

coefficients $v_r(z)$ will be rational functions on \mathbb{P}^1 with poles at $z = 0, \infty$, and finitely many other points $w_j, j = 1, \dots, m$ (where m is the degree of the corresponding module over $\mathcal{W}_k(\mathfrak{sl}_n)$). These coefficients should satisfy the following properties:

(1) At $z = \infty$ the operator (5.20) has the mildest possible irregular singularity; namely, we have

$$\begin{aligned}\tilde{v}_r(s) &\sim \frac{\tilde{c}_r}{s^{r+1}} + \dots, & r = 1, \dots, n-2; \\ \tilde{v}_{n-1}(s) &\sim \frac{1}{s^{n+1}} + \dots,\end{aligned}$$

where $\tilde{v}_r(s)$ are the coefficients of the operator obtained from (5.20) by the change of variables $z \mapsto s = z^{-1}$. Here, as in the case of \mathfrak{sl}_2 , we normalize the leading coefficient of $\tilde{v}_{n-1}(s)$ to be equal to 1. This may always be achieved by rescaling z .

(2) Near $z = 0$ the operator (5.20) has regular singularity, that is

$$v_r(z) \sim \frac{c_r(\nu)}{z^{r+1}} + \dots,$$

where the coefficients $c_r(\nu)$ are determined by the highest weight ν of the $\widehat{\mathfrak{g}}$ -module $L_{\nu,k}$. Namely, representing ν as (ν_1, \dots, ν_n) , where $\nu_i \in \mathbb{C}$ and $\sum_{i=1}^n \nu_i = 0$, we find the $c_r(\nu)$'s from the following formula:

$$(5.21) \quad (-\partial_z)^n - \sum_{i=1}^{n-1} \frac{c_i(\nu)}{z^{i+1}} (-\partial_z)^{n-i-1} = \left(-\partial_z + \frac{\nu_1}{z}\right) \dots \left(-\partial_z + \frac{\nu_n}{z}\right).$$

(3) At the points w_j the operator (5.20) has regular singularity,

$$v_r(z) \sim \frac{c_r(\theta)}{(z - w_j)^{r+1}} + \dots,$$

where $\theta = (1, 0, \dots, 0, -1)$ is the maximal root of \mathfrak{sl}_n , which is the highest weight of the adjoint representation. There is also the additional requirement that the operator (5.20) has *trivial monodromy* around the point w_j for each $j = 1, \dots, m$, and all λ .

Now, our proposal is that the n th order differential operators of this kind should correspond to the generic common eigenvalues of the quantum $\widehat{\mathfrak{sl}}_n$ -KdV Hamiltonians on the degree $\Delta + m$ subspace of the irreducible module obtained by the quantum Drinfeld–Sokolov reduction of the irreducible $\widehat{\mathfrak{g}}$ -module $L_{\nu,k}$ of highest weight $\nu \in \mathfrak{h}^*$ and level k . In general, some of the poles w_j may coalesce, as in the case of $\widehat{\mathfrak{sl}}_2$.

On the other hand, in [BHK] third order differential operators have been associated to the common eigenvalues of the quantum $\widehat{\mathfrak{sl}}_3$ -KdV (Boussinesq) Hamiltonians on the highest weight vectors in irreducible representations of the \mathcal{W} -algebra associated to $\widehat{\mathfrak{sl}}_3$. Analogous formulas have been proposed in [DDT1, DMST, DDT2] in the case of $\widehat{\mathfrak{sl}}_n$. As in the case of $\widehat{\mathfrak{sl}}_2$ discussed in the previous section, we expect that by applying a change of variables to the above operators when the set of points $w_j, j = 1, \dots, m$, is empty (this corresponds to the highest weight vector), we will obtain the operators of [BHK, DDT1, DMST, DDT2]. For $m > 0$ we obtain a generalization of the proposal of [BHK, DDT1, DMST, DDT2] to non-highest weight vectors.

Explicitly, this change of variables is given by the formula

$$z = \frac{x^{n\alpha+n}}{(n\alpha+n)^n}$$

where

$$\alpha = -\frac{k+n-1}{k+n}.$$

It is easy to see that under this change of variables the irregular term of our differential operator gives rise to the term $x^{n\alpha}$, and the term λx^k gives rise to the new spectral parameter term

$$E = (n\alpha+n)^{\frac{n\alpha}{\alpha+1}} \lambda,$$

which is independent of x . The poles of the new operator will be at the points $x = 0, \infty$ and $z_j, j = 1, \dots, m$, where

$$z_j = (n\alpha+n)^n w_j.$$

If the affine Lie algebra $L\widehat{\mathfrak{g}}$ is of classical type, we may also realize affine opers as scalar (pseudo)differential operators, following [DS]. It is possible to describe those of them that encode the spectra of the quantum Hamiltonians of the $\widehat{\mathfrak{g}}$ -KdV system in a similar way to the case of $\widehat{\mathfrak{sl}}_n$. We expect that the corresponding operators (in the special case when the set of extra points w_j is empty) are related to those proposed in [DDT1, DMST, DDT2] after a change of variables. However, if we wish to describe these affine opers for an arbitrary $L\widehat{\mathfrak{g}}$, we should use instead the canonical form for opers discussed in the next section.

5.9. Generalization to an arbitrary affine algebra $\widehat{\mathfrak{g}}$

It is also easy to generalize formula (5.14) to the case of an arbitrary affine Kac–Moody algebra. For simplicity we will assume here that $L\widehat{\mathfrak{g}}$ is

an untwisted affine Kac–Moody algebra $\widehat{\mathfrak{l}}$, where \mathfrak{l} is a simple Lie algebra, but the general case may be analyzed along the same lines.

Let us recall (see, e.g., [F2]) that \mathfrak{l} -opers have canonical form

$$\nabla = \partial_z + \bar{p}_{-1} + \sum_{r=1}^{\ell} v_r(z) \cdot \bar{p}_r,$$

where $\bar{p}_r, r = 1, \dots, \ell$, are the elements defined in Section 5.1. Here $v_1(z)$ transforms as a projective connection (that is, by formula (5.17)) and $v_r(z), r > 1$, transforms as a $(d_r + 1)$ -differential under the changes of coordinate z . Our $\widehat{\mathfrak{l}}$ -opers on \mathbb{P}^1 will have the form

$$(5.22) \quad \partial_z + \bar{p}_{-1} + \sum_{r=1}^{\ell} v_r(z) \cdot \bar{p}_r + \lambda z^k \bar{p}_{\ell},$$

where each $v_r(z)$ is a rational function on \mathbb{P}^1 with poles at $z = 0, \infty$, and finitely many other points $w_j, j = 1, \dots, m$. These coefficients should satisfy the following properties:

(1) At $z = \infty$ the operator (5.22) has the form

$$\begin{aligned} \tilde{v}_r(s) &\sim \frac{\tilde{c}_r}{s^{d_r+1}} + \dots, & r = 1, \dots, \ell - 1; \\ \tilde{v}_{\ell}(s) &\sim \frac{1}{s^{d_{\ell}+2}} + \dots, \end{aligned}$$

where $\tilde{v}_r(s)$ are the coefficients of the operator obtained from (5.22) by the change of variables $z \mapsto s = z^{-1}$.

(2) Near $z = 0$ the operator (5.22) has regular singularity, that is

$$v_r(z) \sim \frac{c_r(\nu)}{z^{d_r+1}} + \dots,$$

where the coefficients $c_r(\nu)$ are determined by the highest weight ν of the $\widehat{\mathfrak{g}}$ -module $L_{\nu, k}$. They are determined by the following rule: the element

$$\bar{p}_{-1} + \sum_{r=1}^{\ell} \left(c_r(\nu) + \frac{1}{4} \delta_{r,1} \right) \bar{p}_r$$

is the unique element in the Kostant slice of regular elements,

$$\bar{p}_{-1} + \mathbf{v}, \quad \mathbf{v} \in \mathfrak{b},$$

which is conjugate to $\bar{p}_{-1} - \nu$ (see [F6], Sect. 9.1).

(3) At the points w_j the operator (5.22) has regular singularity,

$$v_r(z) \sim \frac{c_r(\theta)}{(z - w_j)^{d_r+1}} + \dots,$$

where θ is the maximal coroot of \mathfrak{l} . In addition, we require that the oper have *trivial monodromy* around the point w_j for each $j = 1, \dots, m$, and all λ .

Our proposal is that the $L\widehat{\mathfrak{g}}$ -opers (5.22) satisfying these conditions encode the generic common eigenvalues of the quantum Hamiltonians of the $\widehat{\mathfrak{g}}$ -KdV system on the irreducible modules of the corresponding \mathcal{W} -algebra obtained by the quantum Drinfeld–Sokolov reduction [FF3, FKW] from the irreducible module over $\widehat{\mathfrak{g}}$ with highest weight ν and level k .

§6. Affine Gaudin models with regular singularities

In the previous section we have discussed the affine Gaudin models corresponding to the shift of argument method. We have seen that the classical Hamiltonians of these models are integrals of motion of familiar soliton hierarchies, such as the KdV or AKNS hierarchies. Hence the quantization of these Gaudin models naturally leads to the corresponding quantum Hamiltonians and their diagonalization. However, in the universe of all Gaudin models (see [FFT]), the shift of argument Gaudin models are very special: they correspond to two points on \mathbb{P}^1 , one with regular singularity and one with irregular singularity of order 2. It is natural to develop a similar theory for more general, multi-point, Gaudin models.

The simplest of these models are the ones with “regular singularities” at finitely many points of \mathbb{P}^1 , in the terminology of [FFT]. In this section we study the corresponding affine Gaudin models. We start out by briefly summarizing the main features of the Gaudin models with regular singularities in the finite-dimensional case. We then discuss analogous patterns in the affine Gaudin models and their potential connection to the quantum KdV system. In particular, we will suggest an explicit construction of eigenvectors of the quantum KdV Hamiltonians on representations of the unitary minimal models via Bethe Ansatz.

6.1. Gaudin model, finite-dimensional case

Let \mathfrak{g} be a simple finite-dimensional Lie algebra over \mathbb{C} . Recall that it carries the invariant inner product κ_0 normalized in such a way that the squared length of the maximal root is equal to 2. Let $\{J^a\}$ be a basis

of \mathfrak{g} and $\{J_a\}$ the dual basis with respect to this inner product. The *Gaudin Hamiltonians* are the elements of $U(\mathfrak{g})^{\otimes N}$ given by the following formula:

$$(6.1) \quad \Xi_i = \sum_{j \neq i} \sum_a \frac{J_a^{(i)} J_a^{(j)}}{z_i - z_j}, \quad i = 1, \dots, N.$$

Here $A^{(i)} = 1 \otimes \dots \otimes A \otimes \dots \otimes 1$, with A in the i th position.

As shown in [FFR, F3], the quadratic Gaudin Hamiltonians are included in a large commutative subalgebra of $U(\mathfrak{g})^{\otimes N}$, called the *Gaudin algebra*. According to [F3], this algebra is isomorphic to the algebra $\text{Fun Op}_{L_{\mathfrak{g}}}^{\text{RS}}(\mathbb{P}^1)_{(z_i), \infty}$ of functions on the space $\text{Op}_{L_{\mathfrak{g}}}^{\text{RS}}(\mathbb{P}^1)_{(z_i), \infty}$ of $L_{\mathfrak{g}}$ -opers on \mathbb{P}^1 with regular singularities at the points z_1, \dots, z_N and ∞ . Here $L_{\mathfrak{g}}$ is the Langlands dual Lie algebra of \mathfrak{g} , whose Cartan matrix is the transpose of the Cartan matrix of \mathfrak{g} .

This isomorphism comes about [FFR, F3] in the same way as in the case of the shift of argument Gaudin models, discussed in Section 2. Namely, using the coinvariants (or conformal blocks) of modules over the affine algebra $\widehat{\mathfrak{g}}$ of critical level, we identify the Gaudin algebra with a quotient of the algebra $\text{End}_{\widehat{\mathfrak{g}}} \mathbb{V}_{\text{crit}}$, where \mathbb{V}_{crit} is the vacuum module of critical level (see formula (2.5)). The latter algebra is isomorphic to the algebra of functions on the space $\text{Op}_{L_{\mathfrak{g}}}(D)$ of $L_{\mathfrak{g}}$ -opers on the (formal) disc [FF3, F4]. This implies that the Gaudin algebra is isomorphic to the algebra $\text{Fun Op}_{L_{\mathfrak{g}}}^{\text{RS}}(\mathbb{P}^1)_{(z_i), \infty}$ of functions on $\text{Op}_{L_{\mathfrak{g}}}^{\text{RS}}(\mathbb{P}^1)_{(z_i), \infty}$.

As before, the appearance of the Langlands dual Lie algebra $L_{\mathfrak{g}}$ here is important. It signifies the fact that the above isomorphism may be viewed as a special case of the geometric Langlands duality (for more on this, see [F1, F5, FFT]).

Thus, we obtain that the common eigenvalues of this Gaudin algebra, including those of the Gaudin Hamiltonians Ξ_i , are encoded by $L_{\mathfrak{g}}$ -opers in $\text{Op}_{L_{\mathfrak{g}}}^{\text{RS}}(\mathbb{P}^1)_{(z_i), \infty}$. For example, in the case of $\mathfrak{g} = \mathfrak{sl}_2$ the corresponding opers are the second order operators of the form

$$(6.2) \quad \partial_z^2 - \sum_{i=1}^N \frac{\Delta_i}{(z - z_i)^2} - \sum_{i=1}^N \frac{c_i}{z - z_i}.$$

In particular, the eigenvalues of the Gaudin algebra on the tensor product of the Verma modules $\bigotimes_{i=1}^N M_{\nu_i}$ are encoded by such differential operators, in which

$$\Delta_i = \frac{1}{4} \nu_i (\nu_i + 2);$$

this is the value of the Casimir operator in $U(\mathfrak{sl}_2)$ on the i th factor M_{ν_i} . The operator (6.2) is then determined by the remaining parameters $c_i, i = 1, \dots, N$, and in fact they are nothing but the eigenvalues of $\Xi_i, i = 1, \dots, N$ (see [F3]).

Thus, in the case of $\mathfrak{g} = \mathfrak{sl}_2$ the Gaudin algebra is generated by the Ξ_i 's and the Casimir operators along each factor in $U(\mathfrak{sl}_2)^{\otimes N}$. But for all other simple Lie algebras there are higher order Gaudin Hamiltonians, whose eigenvalues are recorded by polynomials in the coefficients appearing in the expansions of the $L\mathfrak{g}$ -opers near the points z_i . For instance, for $\mathfrak{g} = \mathfrak{sl}_n$ the role of (6.2) is played by the n th order differential operators with regular singularities:

$$(-\partial_z)^n - v_1(z)(-\partial_z)^{n-2} - \dots + v_{n-2}(z)\partial_z - v_{n-1}(z),$$

where $v_j(z)$ has the form

$$v_j(z) = \sum_{i=1}^N \sum_{k=1}^{j+1} \frac{c_{ijk}}{(z - z_i)^k}.$$

Thus, such an operator is determined by the coefficients c_{ijk} appearing in its expansion around the singular points $z_i, i = 1, \dots, N$. The quadratic Gaudin Hamiltonians correspond to $c_{i,1,1}$, and the coefficients with $j > 1$ correspond to higher Gaudin Hamiltonians of order $j + 1$.

6.2. Bethe Ansatz

Let us choose a collection of weights $\nu_1, \dots, \nu_N \in \mathfrak{h}^*$. Then we have the Verma modules M_{ν_i} with highest weights $\nu_i, i = 1, \dots, \nu_N$. Eigenvectors of the higher Gaudin Hamiltonians on the tensor product $\bigotimes_{i=1}^N M_{\nu_i}$ may be constructed by Bethe Ansatz (see [FFR, F3]).

Namely, for any collection of distinct complex numbers w_1, \dots, w_m such that $w_j \neq z_i$ for all i and j , and a collection $\alpha_{i_1}, \dots, \alpha_{i_m}$ of simple roots of \mathfrak{g} (so that $i_j \in I$, the set of vertices of the Dynkin diagram of \mathfrak{g}) consider the following *Bethe vector*

(6.3)

$$\begin{aligned} &\phi(w_1^{i_1}, \dots, w_m^{i_m}) \\ &= \sum_{p=(P^1, \dots, P^N)} \prod_{k=1}^N \frac{f_{i_1^k}^{(k)} f_{i_2^k}^{(k)} \dots f_{i_m^k}^{(k)}}{(w_{j_1^k} - w_{j_2^k})(w_{j_2^k} - w_{j_3^k}) \dots (w_{j_{\alpha_k}^k} - z_k)} v_{\nu_1} \otimes \dots \otimes v_{\nu_N}. \end{aligned}$$

Here $f_i^{(k)}$ denotes the generator of the Lie algebra $\mathfrak{n}_- \subset \mathfrak{g}$ corresponding to the i th simple root, acting on the k th factor of $\bigotimes_{j=1}^N M_{\nu_j}$, and the

summation is taken over all *ordered* partitions $P^1 \cup P^2 \cup \dots \cup P^N$ of the set $\{1, \dots, m\}$, where $P^k = \{j_1^k, j_2^k, \dots, j_{a_k}^k\}$ and $\{i_1^k, i_2^k, \dots, i_{a_k}^k\}$ are the corresponding labels in I .

It is proved in [FFR, F3] that the vector $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ is a common eigenvector of all higher Gaudin Hamiltonians if the following system of equations is satisfied

$$(6.4) \quad \sum_{i=1}^N \frac{\langle \check{\alpha}_{i_j}, \nu_i \rangle}{w_j - z_i} - \sum_{s \neq j} \frac{\langle \check{\alpha}_{i_j}, \alpha_{i_s} \rangle}{w_j - w_s} = 0, \quad j = 1, \dots, m,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between \mathfrak{h}^* and \mathfrak{h} . These are the *Bethe Ansatz equations* of the Gaudin model.

As shown in [FFR, F3], the common eigenvalues of the higher Gaudin Hamiltonians on the Bethe vector $\phi(w_1^{i_1}, \dots, w_m^{i_m})$ given by formula (6.3) (provided that it is non-zero) are encoded by the ${}^L\mathfrak{g}$ -oper obtained by applying the *Miura transformation* to the Cartan connection on the LH -bundle Ω^ρ over \mathbb{P}^1 ,

$$\bar{\nabla} = \partial_z - \sum_{i=1}^N \frac{\nu_i}{z - z_i} + \sum_{j=1}^m \frac{\alpha_{i_j}}{z - w_j},$$

with regular singularities at the points z_i and w_j . The Miura transformation maps such a connection to an ${}^L\mathfrak{g}$ -oper on \mathbb{P}^1 with regular singularities at these points. This oper has no singularities at the points w_j if and only if the Bethe Ansatz equations (6.4) are satisfied (see [FFR, F3]). Equivalently, we can phrase this as saying that the Bethe Ansatz equations mean that the Miura transformation of the connection $\bar{\nabla}$ have *no monodromy* around $w_j, j = 1, \dots, m$.

There is an additional restriction on the common eigenvalues of the Gaudin Hamiltonians that may occur in the tensor product of finite-dimensional representations, and that is the *condition of no monodromy around the points $z_i, i = 1, \dots, N$, and ∞* . Denote by V_ν the irreducible finite-dimensional \mathfrak{g} -module with highest weight ν , which is dominant and integral. The higher Gaudin Hamiltonians act on $\bigotimes_{i=1}^N V_{\nu_i}$ and commute with the diagonal \mathfrak{g} -action. Let us describe the joint eigenvalues of these Hamiltonians on the isotypical component

$$\left(\bigotimes_{i=1}^N V_{\nu_i} \otimes V_{\nu_\infty} \right)^{\mathfrak{g}},$$

where ν_∞ is another dominant integral weight. According to [F2], Corollary 4.8 (see also [F3], Theorem 2.7,(3)), the set of ${}^L\mathfrak{g}$ -opers representing

them (counted without multiplicities) is a subset of the set of $L\mathfrak{g}$ -opers on \mathbb{P}^1 with regular singularities at $z_i, i = 1, \dots, N$, and ∞ , with fixed residues determined by $\nu_i, i = 1, \dots, N$, and ν_∞ , and with trivial monodromy representation (such opers may equivalently be described as ν_i -regular at z_i and ν_∞ -regular at ∞ , see below). Furthermore, according to Conjecture 1 of [F3], this subset is in fact the entire set of such opers. In other words, all opers of this kind correspond to common eigenvalues of the Gaudin Hamiltonians.

6.3. Generalization to the affine case

Now we would like to generalize the above results on the Gaudin models to the case when a finite-dimensional simple Lie algebra \mathfrak{g} is replaced by an affine Kac–Moody algebra $\tilde{\mathfrak{g}}$, by analogy to the shift of argument models discussed above. We begin by summarizing the known facts about the affine Gaudin model.

Let $\tilde{\Delta}$ be the set of roots of an affine Kac–Moody algebra $\tilde{\mathfrak{g}}$ (see Section 3.1). The Lie algebra $\tilde{\mathfrak{g}}$ has a canonical invariant inner product $\tilde{\kappa}_0$ whose restriction to $\mathfrak{g} \otimes 1$ is equal to κ_0 . Let us choose dual bases $\{J^a\}$ and $\{J_a\}$ of \mathfrak{g} with respect to κ_0 . Then we choose dual bases $\{J^{\tilde{a}}\}, \{J_{\tilde{a}}\}$ of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\kappa}_0$ in the following way. The former consists of $J^a \otimes t^n, n \in \mathbb{Z}, 1$ and d , and the latter consists of $J_a \otimes t^n, n \in \mathbb{Z}, d$ and 1 .

Let z_1, \dots, z_N be a set of N distinct complex numbers. The quadratic Gaudin Hamiltonians associated to $\tilde{\mathfrak{g}}$ are given by the same formula (6.1) as in the finite-dimensional case:

$$\Xi_i = \sum_{j \neq i} \sum_{\tilde{a}} \frac{J_{\tilde{a}}^{(i)} J^{\tilde{a}(j)}}{z_i - z_j}, \quad i = 1, \dots, N$$

(in fact, such Hamiltonians may be defined for any symmetrizable Kac–Moody algebra). The elements Ξ_i belong to a completion of the N -fold tensor power of the universal enveloping algebra $U(\tilde{\mathfrak{g}})$. It gives rise to a well-defined linear operator on the tensor product $\bigotimes_{i=1}^N M_i$, if each M_i is a *smooth* $\tilde{\mathfrak{g}}$ -module. By definition, a $\tilde{\mathfrak{g}}$ -module M is called *smooth* if for any $v \in M$ we have $\mathfrak{g} \otimes t^m \mathbb{C}[[t]]v = 0$ for sufficiently large $m \in \mathbb{Z}_+$. In particular, highest weight modules, or modules from the category \mathcal{O} , are smooth.

It is clear that the Hamiltonians Ξ_i commute with the diagonal action of $\tilde{\mathfrak{g}}$ and that

$$(6.5) \quad \sum_{i=1}^N \Xi_i = 0.$$

For a weight $\nu \in \tilde{\mathfrak{h}}^*$, denote by M_ν the Verma module

$$M_\nu = \text{Ind}_{\tilde{\mathfrak{n}}_+ \oplus \tilde{\mathfrak{h}}}^{\tilde{\mathfrak{g}}} \mathbb{C}_\nu.$$

Consider the problem of diagonalization of the quadratic Gaudin Hamiltonians on the tensor product of Verma modules $\bigotimes_{i=1}^N M_{\nu_i}$. The Bethe Ansatz presented in the previous section generalizes in a straightforward way to the case of affine algebras (and, moreover, to all symmetrizable Kac–Moody algebras).

More precisely, for any collection of distinct complex numbers w_1, \dots, w_m such that $w_j \neq z_i$ for all i and j , and a collection $\alpha_{i_1}, \dots, \alpha_{i_m}$ of simple roots of $\tilde{\mathfrak{g}}$ (so that $i_j \in \tilde{I}$, the set of vertices of the Dynkin diagram of $\tilde{\mathfrak{g}}$) we have the *Bethe vector* given by formula (6.3). Then, according to the results of [FFR], Sect. 5 (based on the computations of [SV1]), this vector is an eigenvector of the quadratic Gaudin Hamiltonians $\Xi_i, i = 1, \dots, N$, if the Bethe Ansatz equations (6.4) are satisfied.¹⁵ This is in complete analogy with the finite-dimensional case.

Thus, so far in the affine case we have the quadratic Gaudin Hamiltonians and the Bethe vectors (6.4). However, by analogy with the finite-dimensional case (and the affine shift of argument Gaudin models), we expect they are included in a large commutative subalgebra of a completion of $U(\tilde{\mathfrak{g}})^{\otimes N}$. This raises the following questions:

- (1) Are there higher order affine Gaudin Hamiltonians?
- (2) What parametrizes the joint eigenvalues of the affine Gaudin Hamiltonians?

In the next section we will take up the first question in the simplest case of two points z_1, z_2 (or, more properly, three points, since we should count ∞ as an extra point). In this case there is only one quadratic Gaudin Hamiltonian, Ξ_1 , because $\Xi_2 = -\Xi_1$ due to the identity (6.5). We will show that Ξ_1 is nothing but the operator L_0 of the *coset Virasoro algebra* acting on the tensor product of two $\hat{\mathfrak{g}}$ -modules $M_1 \otimes M_2$. Therefore all quantum KdV Hamiltonians $H_{2i+1}, i \geq 0$, written with respect to this Virasoro algebra, commute with Ξ_1 (which is equal to $L_0 = H_1$). We will conjecture they these are precisely the higher (local) affine Gaudin Hamiltonians.

In the case of multiple points we do not have an analogous formula for the local Gaudin Hamiltonians, but we will give a procedure for constructing the *non-local* Gaudin Hamiltonians (by analogy with the above construction of the non-local quantum AKNS and KdV Hamiltonians).

¹⁵This may also be proved along the lines of [RV].

We will also consider the quasi-classical limit of this construction, which is an affine analogue of the Hitchin system.

We will then discuss the second question of describing the spectra of the affine Gaudin Hamiltonians. By analogy with the finite-dimensional case, and the affine shift of argument Gaudin models, we will conjecture that the joint eigenvalues of the affine Gaudin Hamiltonians are parameterized by the *affine* $L\widehat{\mathfrak{g}}$ -opers with regular singularities at the points z_1, \dots, z_N and ∞ .

6.4. The case of two points

Let us consider the simplest case of two distinct points z_1, z_2 . Thus, we have two smooth $\widehat{\mathfrak{g}}$ -modules M_1 and M_2 and the quadratic Gaudin Hamiltonian

$$\Xi = \sum_{\tilde{a}} J_{\tilde{a}} \otimes J^{\tilde{a}}$$

acting on $M_1 \otimes M_2$. Note that $\Xi_1 = -\Xi_2 = \Xi/(z_1 - z_2)$. It commutes with the diagonal action of $\tilde{\mathfrak{g}}$ on $M_1 \otimes M_2$. Our question is to construct the higher affine Gaudin Hamiltonians, which commute with the diagonal action of $\widehat{\mathfrak{g}}$ and with Ξ .

We will construct them using the coset construction of [GKO]. Suppose that the levels k_1 and k_2 of the modules M_1 and M_2 are non-critical, i.e., $k_i \neq -h^\vee$ and in addition $k_1 + k_2 \neq -h^\vee$. Then the GKO construction gives us the Virasoro algebra which commutes with the diagonal action of $\widehat{\mathfrak{g}}$ in the completed tensor product $U_{k_1}(\widehat{\mathfrak{g}}) \widehat{\otimes} U_{k_2}(\widehat{\mathfrak{g}})$, where $U_k(\widehat{\mathfrak{g}}) = U(\widehat{\mathfrak{g}})/(1 - k)$. This algebra is generated by $L_n^{\text{GKO}}, n \in \mathbb{Z}$, such that

$$T^{\text{GKO}}(w) = \sum_{n \in \mathbb{Z}} L_n^{\text{GKO}} w^{-n-2} = T_1^{\text{SS}}(w) + T_2^{\text{SS}}(w) - T_{\text{diag}}^{\text{SS}}(w).$$

Here $T^{\text{SS}}(w)$ is the Segal–Sugawara current at level $k \neq -h^\vee$,

$$T^{\text{SS}}(w) = \sum_{n \in \mathbb{Z}} L_n^{\text{SS}} w^{-n-2} = \frac{1}{2(k + h^\vee)} \sum_a :J^a(w)J_a(w):$$

$$J^a(w) = \sum_{n \in \mathbb{Z}} J_n^a w^{-n-1}, \quad J_n^a = J^a \otimes t^n,$$

and the lower indices refer to the first, the second or the diagonal $\widehat{\mathfrak{g}}$ in $U_{k_1}(\widehat{\mathfrak{g}}) \widehat{\otimes} U_{k_2}(\widehat{\mathfrak{g}})$.

Without loss of generality we may, and will, assume that the action of L_0^{SS} on the $\widehat{\mathfrak{g}}$ -modules M_1 and M_2 coincides with the action of $-d \in \widehat{\mathfrak{g}}$.

Then we find that

$$\begin{aligned}
 -(k_1 + k_2 + h^\vee)L_0^{\text{GKO}} &= \sum_a \sum_{n \in \mathbb{Z}} J_{a,n} \otimes J_n^a - k_1 \otimes L_0^{\text{SS}} - L_0^{\text{SS}} \otimes k_2 \\
 &= \sum_{\tilde{a}} J_{\tilde{a}} \otimes J^{\tilde{a}} = \Xi.
 \end{aligned}$$

Thus, up to a non-zero scalar, the quadratic Gaudin Hamiltonian coincides with the generator L_0^{GKO} of the coset Virasoro algebra. Since the entire coset Virasoro algebra commutes with the diagonal action of $\widehat{\mathfrak{g}}$, the first requirement on the higher Gaudin Hamiltonians is satisfied for any element of its completed universal enveloping algebra. Now we search for a commutative subalgebra of this enveloping algebra containing L_0^{GKO} .

Consider the case of $\tilde{\mathfrak{g}} = \tilde{\mathfrak{sl}}_2$. Then a natural candidate for this commutative subalgebra is provided by the algebra of quantum KdV Hamiltonians. Let us recall that the quantum KdV Hamiltonians are certain elements $H_{2i+1}, i \geq 0$, of the completed universal enveloping algebra of the Virasoro algebra whose symbols are the classical KdV Hamiltonians (see Section 5.2). The first of them, H_1 , is nothing but L_0 , and the next one is

$$H_3 = \frac{1}{2}L_0^2 + \sum_{n>0} L_{-n}L_n = \frac{1}{2} \int :T(w)^2: w^3 dw.$$

The other Hamiltonians have the form

$$H_{2i+1} = \int :T(w)^{i+1}: w^{2i+1} dw + \text{lower order terms.}$$

Explicit formulas for the first few of them may be found in [BLZ1]. The existence of these Hamiltonians has been established in [FF4, FF5]. We conjecture that

The quantum KdV Hamiltonians of the coset Virasoro algebra are the higher Gaudin Hamiltonians of the two-point $\widehat{\mathfrak{sl}}_2$ affine Gaudin model with $N = 2$.

More concretely, we conjecture the following:

Conjecture 5. *The affine Bethe vectors (6.3) (for $\widehat{\mathfrak{sl}}_2$ and $N = 2$) are simultaneous eigenvectors of all quantum KdV Hamiltonians of the coset Virasoro algebra, provided that the Bethe Ansatz equations (6.4) are satisfied.*

Consider the special case when the module M_1 is the vacuum module $V_{(0,1)}$ of level $k_1 = 1$ and M_2 is an integrable module $V_{(n,k)}$ of level $k_2 = k \in \mathbb{Z}_+$ and highest weight $n \in \mathbb{Z}_+, n \leq k$, with respect to the one-dimensional Cartan subalgebra $\mathfrak{h} \subset \widehat{\mathfrak{g}}$. Then, according to [GKO], we have the decomposition

$$V_{(0,1)} \otimes V_{(n,k)} \simeq \bigoplus_{m=0}^{k+1} V_{(m,k+1)} \otimes L_{n,m},$$

with respect to the diagonal action of $\widehat{\mathfrak{sl}}_2$, where $L_{n,m}$ is the irreducible module over the Virasoro algebra from the $(k+2, k+3)$ unitary minimal model with the central charge $1 - 6/(k+2)(k+3)$.

If the above conjecture is true, then we obtain an effective method for constructing eigenvectors of the quantum KdV Hamiltonians on the irreducible modules of the Virasoro algebra from the unitary minimal models.

In the case of a general simply-laced Lie algebra \mathfrak{g} , we also expect that the higher affine Gaudin Hamiltonians coincide with the quantum $\widehat{\mathfrak{g}}$ -KdV Hamiltonians, which generate a commutative subalgebra in the coset \mathcal{W} -algebra. If this is true, then the Gaudin Bethe vectors would give us eigenvectors of the quantum $\widehat{\mathfrak{g}}$ -KdV Hamiltonians.

In the above discussion we considered the “local” Gaudin Hamiltonians, that is, those that may be expressed as infinite sums of monomials in the generators of $\widehat{\mathfrak{g}}$ of bounded order (for example, the Hamiltonian Ξ has order 2). However, in addition to these local Hamiltonians there are also non-local ones. In order to explain the appearance of these non-local Hamiltonians, we consider the quasi-classical analogues of the affine Gaudin models: the affine Hitchin systems.

6.5. Affine Hitchin systems

The Gaudin models associated to simple Lie algebras have quasi-classical versions, which may be interpreted as the Beauville–Hitchin systems on \mathbb{P}^1 with ramification (see [H, Bea, Mar, Bot]). In the case of Gaudin models with regular singularities the algebra of quantum Hamiltonians is a commutative subalgebra of $U(\mathfrak{g})^{\otimes N}$. Its quasi-classical counterpart is then a Poisson commutative subalgebra in $S(\mathfrak{g})^{\otimes N} = \text{Fun}((\mathfrak{g}^*)^{\oplus N})$. Let us identify the Poisson manifold $(\mathfrak{g}^*)^{\oplus N}$ (with its Kirillov–Kostant Poisson structure) with the space of one-forms on \mathbb{P}^1 with poles of order one at the points z_1, \dots, z_N , and ∞ ,

$$(6.6) \quad L = \sum_{i=1}^N \frac{A_i}{z - z_i} dz, \quad A_i \in \mathfrak{g}^*.$$

Let $\overline{P}_1, \dots, \overline{P}_\ell$ be generators of the algebra $\text{Inv } \mathfrak{g}^*$ of G -invariant functions on \mathfrak{g}^* , of degrees $d_1 + 1, \dots, d_\ell + 1$, where the d_i 's are the exponents of \mathfrak{g} . Then we have the Hitchin map

$$p : (\mathfrak{g}^*)^{\oplus N} \rightarrow \mathcal{H} = \bigoplus_{i=1}^N H^0 \left(\mathbb{P}^1, (\Omega(z_1 + \dots + z_N + \infty))^{\otimes (d_i+1)} \right),$$

$$L \mapsto (P_1(L), \dots, P_\ell(L)).$$

The pull-backs of polynomial functions on \mathcal{H} to $(\mathfrak{g}^*)^{\oplus N}$ via the map p give rise to Poisson commuting Hamiltonians on $(\mathfrak{g}^*)^{\oplus N}$. They generate a Poisson commutative subalgebra of $S(\mathfrak{g}^*)^{\otimes N}$. According to the results of [F3, FFT], this is precisely the algebra of symbols of the higher Gaudin Hamiltonians.

Now we wish to generalize this construction to the affine case, where we will have commuting quantum Hamiltonians in a completion of $U(\widehat{\mathfrak{g}})^{\otimes N}$. Their symbols should therefore belong to a completion of the algebra of functions on $(\widehat{\mathfrak{g}}^*)^{\oplus N}$. It is natural to identify the latter with the space of one-forms L as in (6.6), but with each A_i now being an element of $\widehat{\mathfrak{g}}^*$. Actually, as in the case of the shift of argument subalgebra, we will consider instead of the i th copy of $\widehat{\mathfrak{g}}^*$, the hyperplane $\widehat{\mathfrak{g}}_{k_i}^*$ in $\widehat{\mathfrak{g}}^*$ which consists of the linear functionals on $\widehat{\mathfrak{g}}$ taking value k_i on the central element $\mathbf{1}$. We will identify this hyperplane with the space of k_i -connections on the punctured disc

$$k_i \partial_t + A_i(t), \quad A_i(t) \in \mathfrak{g}((t)) \overset{k_0}{\simeq} \mathfrak{g}^*((t)),$$

so that the coadjoint action of $G((t))$ becomes the gauge action on the k_i -connections (compare with Section 3.2). This hyperplane carries a canonical Kirillov–Kostant Poisson structure.

Then the L -operator of our affine Hitchin system may be represented in the form

$$(6.7) \quad L = \sum_{i=1}^N \frac{k_i \partial_t + A_i(t)}{z - z_i} dz, \quad A_i \in \mathfrak{g}((t)).$$

In what follows we will assume that $k_i \neq 0$.

Equivalently, we may rewrite this as a connection on the punctured disc, which depends on the spectral parameter $z \in \mathbb{P}^1 \setminus \{z_1, \dots, z_N, \infty\}$:

$$(6.8) \quad L' = \partial_t + \frac{1}{f(z)} \sum_{i=1}^N \frac{A_i(t)}{z - z_i} dz,$$

where

$$(6.9) \quad f(z) = \sum_{i=1}^N \frac{k_i}{z - z_i}.$$

Now we will define commuting Hamiltonians of the affine Hitchin system by evaluating on L' gauge invariant functions on the space of connections $\partial_t + A(t)$.

6.6. Commuting Hamiltonians from the monodromy matrix

As in the case of the shift of argument model (see Section 3.2), the invariant functions are obtained by taking the invariants of the monodromy of the connection with respect to the adjoint action. Denote by $M(z) \in G$ the formal monodromy matrix of the connection operator L' given by formula (6.8). As before, $M(z)$ is defined as follows: its action on any finite-dimensional representation V of G is obtained by computing the monodromy of the corresponding matrix differential operator $\partial_t + A(t)$ acting on V .

For any function φ on G which is invariant under the adjoint action of G , the Hamiltonian

$$H_\varphi(z) = \varphi(M(z))$$

is a gauge invariant function of L' (or of L given by formula (6.7)), depending on the spectral parameter z . One shows in the same way as in the finite-dimensional case that

$$(6.10) \quad \{H_\varphi(z), H_\psi(y)\} = 0, \quad \forall \varphi, \psi \in \text{Fun}(G)^G, \quad z, y \in \mathbb{C} \setminus \{z_1, \dots, z_N\}.$$

Therefore we may obtain families of commuting Hamiltonians of the affine Hitchin model by expanding the Hamiltonians $H_\varphi(z)$ around points of \mathbb{P}^1 .

Let us first consider the expansion around one of the points $z_i, i = 1, \dots, N$. The expansion of the operator $k_i L'$ at z_i reads as follows:

$$k_i \partial_t + A_i(t) + (z - z_i) \left(-A_i(t) \sum_{j \neq i} \frac{k_j}{k_i(z_i - z_j)} + \sum_{j \neq i} \frac{A_j(t)}{z_i - z_j} \right) + \dots,$$

where the dots stand for higher order terms in $(z - z_i)$.

We now apply Lemma 1 in this case, using $(z - z_i)$ as the parameter ϵ . Let us write

$$A_i(t) = \sum_a J_{i,a}(t) J^a,$$

and denote by

$$\Psi_i(t) = \sum_a \Psi_{\varphi,i}^a(t) J^a$$

the single-valued solutions of the equation

$$(6.11) \quad k_i \partial_t \Psi_i(t) + [A_i(t), \Psi_i(t)] = 0,$$

corresponding to $\varphi \in (\text{Fun } G)^G$, as before. Then we find, using Lemma 1, that the $(z - z_i)$ -linear term in the expansion of $\varphi(M(z))$, where $M(z)$ is the monodromy of the operator $k_i L'$, is equal to

$$(6.12) \quad H_{i,\varphi} = \int \left(-J_{i,a}(t) \sum_{j \neq i} \frac{k_j}{k_i(z_i - z_j)} + \sum_{j \neq i} \frac{J_{j,a}(t)}{z_i - z_j} \right) \Psi_{i,\varphi}^a(t) dt$$

(here and below the summation over the repeated index a is understood). The commutativity (6.10) of the Hamiltonians H_φ then implies that

$$\{H_{i,\varphi}, H_{k,\psi}\} = 0, \quad i = 1, \dots, N, \quad \varphi, \psi \in (\text{Fun } G)^G.$$

Higher terms in the expansion of $\varphi(M(z))$ in $(z - z_i)$ are given by more complicated formulas, involving multiple integrals of solutions of (6.11). All of these Hamiltonians are non-local, in the sense that they cannot be realized as integrals of differential polynomials in the $A_i(t)$.

In order to obtain local commuting Hamiltonians, we need to consider the expansion of the operator L' at the points $z = y_1, \dots, y_{n-1}$, which satisfy $f(y_i) = 0$, where $f(z)$ is given by formula (6.9) (this is similar to what happens in the more familiar integrable systems; see, e.g., [DS, RS2]). We expect that for each of these points there is an infinite series of Hamiltonians whose degrees are all positive integers equal to the exponents of $\tilde{\mathfrak{g}}$ modulo the Coxeter number. These Hamiltonians should define interesting hierarchies of commuting flows on our phase space $\prod_{i=1}^N \widehat{\mathfrak{g}}_{k_i}^*$.

We have not seen these particular hierarchies discussed in the literature, but some closely related hierarchies have been studied before. For example, consider the special case when $N = 2$. Then the limit of the L -operator (6.8) when $k_2 \rightarrow 0$ is closely related to the L -operator of the principal chiral field model. We expect that the corresponding limits of our Hamiltonians coincide with the Hamiltonians of that model constructed in [EHMM] (in particular, they have the right degrees). We plan to discuss these Hamiltonians in more detail elsewhere.

6.7. Quantum non-local Hamiltonians

The classical Hamiltonians of the affine Hitchin system may be quantized, giving rise to quantum (non-local) Hamiltonians of the affine Gaudin model.

Let us consider first the case of two points, so that $N = 2$. In this case we have the following classical Hamiltonians:

$$\begin{aligned}
 H_{1,\varphi} &= \int \left(-\frac{k_2}{k_1} J_{1,a}(t) + J_{2,a}(t) \right) \Psi_{1,\varphi}^a(t) dt, \\
 H_{2,\varphi} &= \int \left(-\frac{k_1}{k_2} J_{2,a}(t) + J_{1,a}(t) \right) \Psi_{2,\varphi}^a(t) dt
 \end{aligned}$$

(we have multiplied them by $(z_1 - z_2)$). To quantize these Hamiltonians, we replace $J_{i,a}(t)$ by the current $J_a^{(i)}(z)$ and $\Psi_{i,\varphi}^a(t)$ by the component $\Phi_p^{a(i)}(z)$ of the primary field $\Phi_p^{(i)}(z)$, introduced in Section 3.3,

$$\Phi_p^{(i)}(w) = \sum_a \Phi_p^{a(i)}(w) J_a.$$

Here, as before, the upper index (i) indicates that this operator acts along the i th factor of the tensor product $M_1 \otimes M_2$. We also need to adjust the levels, as in the equation (3.14) (and the Segal–Sugawara formula).

The resulting quantum Hamiltonians read

$$\begin{aligned}
 H_{1,p} &= -\frac{k_2}{k_1 + h^\vee} \sum_{n \in \mathbb{Z}} :J_{a,n} \Phi_{p,(-n)}^a: \otimes 1 + \sum_{n \in \mathbb{Z}} \Phi_{p,(-n)}^a \otimes J_{a,n}, \\
 H_{2,p} &= -\frac{k_1}{k_2 + h^\vee} \sum_{n \in \mathbb{Z}} 1 \otimes :J_{a,n} \Phi_{p,(-n)}^a: + \sum_{n \in \mathbb{Z}} J_{a,n} \otimes \Phi_{p,(-n)}^a.
 \end{aligned}$$

They act on the tensor product $M_{\lambda_1, k_1} \otimes M_{\lambda_2, k_2}$ of Verma modules over $\widehat{\mathfrak{g}}$ (or more general modules $M_1 \otimes M_2$, provided that the primary fields $\Phi_p^{(i)}(w), i = 1, 2$, are well-defined on them).

It is easy to see that they commute with the diagonal action of $\widehat{\mathfrak{g}}$. Indeed, consider again the extended affine algebra $\widetilde{\mathfrak{g}} = \mathbb{C}d \ltimes \widehat{\mathfrak{g}}$. We choose dual bases $\{J^{\bar{a}}\}$ and $\{J_{\bar{a}}\}$ in $\widetilde{\mathfrak{g}}$ as in Section 6.3. Recall that they consist of elements of the form $J^{a,n}$ (resp., $J_{a,n}$), $\mathbf{1}$ and d (resp., d and $\mathbf{1}$). Suppose that we have two homomorphisms of $\widetilde{\mathfrak{g}}$ -modules $\rho_i : \widetilde{\mathfrak{g}} \rightarrow \text{End}_{\mathbb{C}} M_i, i = 1, 2$. Then the endomorphism

$$(6.13) \quad \sum_{\bar{a}} \rho_1(J_{\bar{a}}) \otimes \rho_2(J^{\bar{a}})$$

of $M_1 \otimes M_2$ commutes with the diagonal action of $\tilde{\mathfrak{g}}$.

In our case, we choose as the first homomorphism ρ_1 the action of $\tilde{\mathfrak{g}}$ on M_1 . Under this action $\mathbf{1}$ maps to $k_1 \text{Id}$ and d maps to the Sugawara operator $-L_0$. We define the second homomorphism as follows:

$$\begin{aligned} J^{a,n} &\mapsto \Phi_{p,(n)}^a, \\ d &\mapsto -\frac{1}{k_2 + h^\vee} \sum_{n \in \mathbb{Z}} :J_{a,n} \Phi_{p,(-n)}^a:, \\ \mathbf{1} &\mapsto 0. \end{aligned}$$

One checks, in the same way as for the Segal–Sugawara formula, that this indeed defines a homomorphism of $\tilde{\mathfrak{g}}$ -modules. The resulting operator (6.13) is the Hamiltonian $H_{2,p}$. Hence it commutes with the diagonal action of $\tilde{\mathfrak{g}}$. Since the Hamiltonian $H_{1,p}$ is obtained from $H_{2,p}$ by switching the modules

M_1 and M_2 , we find that it also commutes with $\tilde{\mathfrak{g}}$.

Furthermore, we claim that the Hamiltonians $H_{i,p}$ commute with each other for all $i = 1, 2$ and $p = 1, \dots, \ell$. This may be checked by a direct calculation which will be presented in [FF7].

The above formulas are generalized in a straightforward way to the case of N points. We now have the following commuting Hamiltonians quantizing the classical Hamiltonians (6.12):

$$(6.14) \quad H_{i,p} = \sum_{n \in \mathbb{Z}} : \left(-\frac{J_{a,n}^{(i)}}{k_i + h^\vee} \sum_{j \neq i} \frac{k_j}{z_i - z_j} + \sum_{j \neq i} \frac{J_{a,n}^{(j)}}{z_i - z_j} \right) \Phi_{p,(-n)}^{a(i)} :,$$

where $i = 1, \dots, N, p = 1, \dots, \ell$. They act on the tensor product $\bigotimes_{i=1}^N M_{\lambda_i, k_i}$ of Verma modules, or more general highest weight $\hat{\mathfrak{g}}$ -modules of levels $k_i, i = 1, \dots, N$ (recall that $k_i \neq -h^\vee$ by our assumption). These are the simplest *non-local quantum affine Gaudin Hamiltonians*.

Other non-local quantum Gaudin Hamiltonians may be constructed by generalizing the procedure of [BLZ1, BHK, FFS], as outlined in Section 3.3. We will discuss this in more detail in [FF7].

6.8. Hamiltonians in the limit $z_2 \rightarrow \infty$

It is important to observe that the Hamiltonians of the shift of argument Gaudin model may be obtained from the Hamiltonians of the Gaudin model with regular singularities and $N = 2$ in a certain limit as $z_2 \rightarrow \infty$ (we also set $z_1 = 0$). In the finite-dimensional case this was explained by L. Rybnikov in [R]. The Hamiltonians of the latter model generate a commutative subalgebra in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Consider

the limit $z_2 \rightarrow \infty$, in which we also degenerate the second factor $U(\mathfrak{g})$ (corresponding to the point z_2) to $S(\mathfrak{g})$. Then we obtain a commutative subalgebra in $U(\mathfrak{g}) \otimes S(\mathfrak{g})$. Applying the homomorphism $S(\mathfrak{g}) \rightarrow \mathbb{C}$, corresponding to evaluation at $\chi \in \mathfrak{g}^*$, we then obtain a commutative subalgebra in $U(\mathfrak{g})$. It turns out to be the quantum shift of argument subalgebra \mathcal{A}_χ , as shown in [R].

We may apply the same limiting procedure in the affine case. The above quantum Hamiltonians of the Gaudin model with regular singularities and $N = 2$ generate a commutative subalgebra in the (completed) tensor product $U(\tilde{\mathfrak{g}}) \hat{\otimes} U(\tilde{\mathfrak{g}})$. Taking the limit $z_2 \rightarrow \infty$ and at the same time degenerating $U(\tilde{\mathfrak{g}})$, we obtain a commutative subalgebra in $U(\tilde{\mathfrak{g}}) \hat{\otimes} S(\tilde{\mathfrak{g}})$. We then need to evaluate our Hamiltonians at $\chi \in \tilde{\mathfrak{g}}^*$ along the second factor. For example, let us apply this procedure to the Hamiltonians $H_{1,p}$ and $H_{2,p}$ constructed above. Let us set $k_2 = 0$. Then in our limit the generators $J_{a,n}$ of $\tilde{\mathfrak{g}}$ becomes classical, that is, linear functionals on $\tilde{\mathfrak{g}}^*$. Evaluating them on $\chi \in \mathfrak{h}^* \subset \tilde{\mathfrak{g}}^*$, we obtain precisely the non-local Hamiltonians

$$H_p(\chi) = \sum_a \Phi_{p,(0)}^a(\chi, J_{a,0})$$

introduced in Section 3.3.

On the other hand, if we apply this procedure to $H_{2,p}$, then the primary fields $\Phi_p^a(w)$ will become classical, i.e., will degenerate to $\Psi_\varphi^a(w)$, and we will need to evaluate them on χ . The resulting Hamiltonians will be linear combinations of $J_{a,0}$ that belong to $\mathfrak{h} \subset \tilde{\mathfrak{g}}$, where \mathfrak{h} is the centralizer of χ in \mathfrak{g} . Thus, we obtain local (in fact, linear) Hamiltonians of the shift of argument affine Gaudin model.

Similar formulas will appear in general. The non-local Hamiltonians in the Gaudin model with regular singularities and $N = 2$ have, roughly, the following form:

$$\sum_i A_i \otimes B_i \quad \text{or} \quad \sum_i B_i \otimes A_i,$$

where the A_i are non-local expressions which are quantizations of the derivatives of the “quantum monodromy matrix”, such as $\Phi_{p,(-n)}^a$, and the B_i are local expressions, that is, of finite degree in J_n^a . When we degenerate $U(\tilde{\mathfrak{g}})$ to $S(\tilde{\mathfrak{g}})$ along the second factor, either B_i or A_i become classical, and we may then evaluate them on $\chi \in \tilde{\mathfrak{g}}^*$. In the first case we obtain non-local Hamiltonians of the shift of argument affine Gaudin model (such as the \mathfrak{g} -AKNS model). In the second case, we obtain local Hamiltonians.

A similar limiting procedure may be constructed for the Hamiltonians of the quantum $\widehat{\mathfrak{g}}$ -KdV systems. This suggests a possible way to construct local quantum Hamiltonians of the \mathfrak{g} -AKNS and $\widehat{\mathfrak{g}}$ -KdV systems.

6.9. Spectra of the quantum Hamiltonians and affineopers

In [F2] to each solution of the Bethe Ansatz (6.4) equations for the affine Lie algebra $\widehat{\mathfrak{g}}$ the following connection on the ${}^L\overline{H}$ -bundle Ω^ρ was attached:

$$(6.15) \quad \overline{\nabla} = \partial_z - \sum_{i=1}^N \frac{\nu_i}{z - z_i} + \sum_{j=1}^m \frac{\alpha_{i_j}}{z - w_j}.$$

It was shown in [F2] that the Bethe Ansatz equations are equivalent to the condition that the ${}^L\widehat{\mathfrak{g}}$ -oper $\nabla = \overline{\nabla} + p_{-1}$ obtained by applying the Miura transformation to the connection (6.15) is regular at the points $w_j, j = 1, \dots, m$, or, equivalently, has no monodromy around these points. This oper has regular singularities at the points $z_i, i = 1, \dots, N$ and ∞ with the residues determined by the highest weights $\nu_i, i = 1, \dots, N$.

This suggests that the true parameters of the Bethe vectors are not the Cartan connections (6.15), but the corresponding ${}^L\widehat{\mathfrak{g}}$ -opers obtained by applying the Miura transformation. Thus, we propose the following answer to question (2) of the previous section: what parameterizes common eigenvalues of the affine Gaudin Hamiltonians?

The common eigenvalues of the affine Gaudin Hamiltonians on the tensor product $\bigotimes_{i=1}^N M_{\nu_i}$ of Verma modules over $\widehat{\mathfrak{g}}$ are encoded by ${}^L\widehat{\mathfrak{g}}$ -opers on \mathbb{P}^1 with regular singularities at the points z_1, \dots, z_N and ∞ .

Again, we emphasize that this \mathbb{P}^1 is a curve in the *spectral parameter* z of the affine Gaudin model.

To see what these ${}^L\widehat{\mathfrak{g}}$ -opers look like, consider the case when $\widehat{\mathfrak{g}} = {}^L\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$. We will use the conventions of Section 4.4. We again divide the set of points w_1, \dots, w_m into two subsets $w_1^1, \dots, w_{m_1}^1$, and $w_1^0, \dots, w_{m_0}^0$, corresponding to the simple roots α_1 and α_0 , respectively. The connection (6.15) reads

$$\partial_z + \begin{pmatrix} u(z) & \lambda \\ 1 & -u(z) \end{pmatrix} + \sum_{i=1}^N \frac{k_i}{z - z_i} d,$$

where

$$(6.16) \quad u(z) = -\sum_{i=1}^N \frac{\ell_i}{z - z_i} + \sum_{j=1}^{m_1} \frac{1}{z - w_j^1} - \sum_{j=1}^{m_0} \frac{1}{z - w_j^0}.$$

Applying the gauge transformation by

$$\begin{pmatrix} 1 & -u(z) \\ 0 & 1 \end{pmatrix},$$

we obtain the operator

$$(6.17) \quad \partial_z + \begin{pmatrix} 0 & v(z) + \lambda \\ 1 & 0 \end{pmatrix} + \sum_{i=1}^N \frac{k_i}{z - z_i} d,$$

where

$$(6.18) \quad \begin{aligned} v(z) &= u(z)^2 + \partial_z u(z) \\ &= \sum_{i=1}^N \frac{\ell_i(\ell_i + 1)}{(z - z_i)^2} + \sum_{i=1}^N \frac{c_i}{z - z_i} + \sum_{j=1}^{m_0} \frac{2}{(z - w_j^0)^2} + \sum_{j=1}^{m_0} \frac{v_{j,-1}}{z - w_j^0}, \end{aligned}$$

$$(6.19) \quad c_i = 2\ell_i \left(\sum_{j \neq i} \frac{\ell_j}{z_i - z_j} - \sum_{j=1}^{m_1} \frac{1}{z_i - w_j^1} + \sum_{j=1}^{m_0} \frac{1}{z_i - w_j^0} \right),$$

$$v_{j,-1} = \sum_{i=1}^N \frac{k_i}{w_j^0 - z_i}.$$

These formulas follow from the Bethe Ansatz equations (6.4), which read

$$(6.20) \quad \sum_{i=1}^N \frac{\ell_i}{w_j^1 - z_i} - \sum_{s \neq j} \frac{1}{w_j^1 - w_s^1} + \sum_{s=1}^{m_0} \frac{1}{w_j^1 - w_s^0} = 0, \quad j = 1, \dots, m_1,$$

$$(6.21) \quad \sum_{i=1}^N \frac{\frac{k_i}{2} - \ell_i}{w_j^0 - z_i} + \sum_{s=1}^{m_1} \frac{1}{w_j^0 - w_s^1} - \sum_{s \neq j} \frac{1}{w_j^0 - w_s^0} = 0, \quad j = 1, \dots, m_0.$$

These equations imply, in particular, that $v(z)$ has no singularities at the points $w_j^1, j = 1, \dots, m_1$.

In addition, because $v(z)$ is obtained as the Miura transformation of $u(z)$ given by formula (6.16), the coefficients of its expansion at each point $w_j^0, j = 1, \dots, m_0$,

$$v(z) \sim \frac{2}{(z - w_j^0)^2} + \sum_{n \geq -1} v_{j,n} (z - w_j^0)^n$$

satisfy the equation (4.19). This is the condition that the operator (6.17) has no monodromy around the point w_j^0 . For instance, if all $k_i = 0$, then $v_{j,-1} = 0$ and the equation (4.19) reduces to $v_{j,1} = 0$ for all $j = 1, \dots, m_0$. Hence we obtain that in this case the condition is that the coefficients in front of $(z - w_j^0)^{-1}$ and $(z - w_j^0)$ in the expansion of $v(z)$ in Laurent series in $(z - w_j^0)$ should vanish for all $j = 1, \dots, m_0$.

Thus, a connection $\partial_z + u(z)$ given by formula (6.16) with the numbers w_j^0, w_j^1 satisfying the Bethe Ansatz equations (6.20)–(6.21) gives rise to an operator (6.17) such that its coefficients in the expansion at $w_j^0, j = 1, \dots, m_0$, satisfy the equations (6.19) and (4.19). (However, we do not expect the converse to be true in general.) Our proposal is that the generic common eigenvalues of the affine Gaudin Hamiltonians on the tensor product $\bigotimes_{i=1}^N M_{\nu_i}$ of Verma modules over $\widehat{\mathfrak{sl}}_2$ are encoded by $\widehat{\mathfrak{sl}}_2$ -opers of this kind. In general, some degenerations of these opers will also correspond to some eigenvectors, as in the shift of argument Gaudin model (see Section 4.4).

It is useful to express the operator (6.17) as a second order differential operator (compare with Section 4.4). Applying the (formal) gauge transformation by the element $\prod_{i=1}^N (z - z_i)^{k_i d}$, we obtain

$$\partial_z + \begin{pmatrix} 0 & v(z) + \lambda \prod_{i=1}^N (z - z_i)^{k_i} \\ 1 & 0 \end{pmatrix},$$

which may be rewritten as the following second order operator with spectral parameter

$$\partial_z^2 - v(z) - \lambda \prod_{i=1}^N (z - z_i)^{k_i}.$$

Alternatively, we may rewrite (6.17) as the following partial differential operator:

$$\left(\partial_z + \sum_{i=1}^N \frac{k_i}{z - z_i} \lambda \partial_\lambda \right)^2 - v(z) - \lambda.$$

6.10. Integrable modules

Next, we consider the common eigenvalues of the affine Gaudin Hamiltonians on the tensor product of integrable $\widehat{\mathfrak{g}}$ -modules. Recall that for each integral dominant weight ν we have an integrable module V_ν , which is the irreducible quotient of the Verma module M_ν . By analogy with the finite-dimensional case, we propose the following description of the common eigenvalues of the affine Gaudin Hamiltonians on the tensor product $\bigotimes_{i=1}^N V_{\nu_i}$:

Conjecture 6. *There is a bijection between the set of common eigenvalues of the affine Gaudin Hamiltonians on the tensor product $\bigotimes_{i=1}^N V_{\nu_i}$ of integrable modules over $\widehat{\mathfrak{g}}$ (counted without multiplicities) and the $L\widehat{\mathfrak{g}}$ -opers on \mathbb{P}^1 which are ν_i -regular at z_i , $i = 1, \dots, N$, and have regular singularity at ∞ .*

Note that such opers necessarily have *trivial monodromy* around each of the points z_i , $i = 1, \dots, N$, and hence they have no monodromy on \mathbb{P}^1 .

If the Bethe Ansatz equations (6.4) are satisfied and the projection of the Bethe vector (6.3) onto $\bigotimes_{i=1}^N V_{\nu_i}$ is non-zero, then we obtain an eigenvector of the affine Gaudin Hamiltonians in $\bigotimes_{i=1}^N V_{\nu_i}$. A necessary condition for this is that its weight

$$(6.22) \quad \gamma = \sum_{i=1}^N \nu_i - \sum_{j=1}^m \alpha_{i_j}$$

is dominant. The corresponding eigenvalue should then be encoded by the $L\widehat{\mathfrak{g}}$ -oper obtained by applying the Miura transformation to the Cartan connection (6.15). As shown in [F2], such an oper satisfies the conditions of Conjecture 6. This provides the first consistency check for this conjecture.

Note that we allow the possibility that there exist eigenvectors other than the Bethe vectors in $\bigotimes_{i=1}^N V_{\nu_i}$. In this case we still believe that the common eigenvalues of the affine Gaudin Hamiltonians are encoded by the $L\widehat{\mathfrak{g}}$ -opers as in Conjecture 6. However, the eigenvalues corresponding to the non-Bethe eigenvectors should be encoded by opers that cannot be expressed as the Miura transformation of a connection of the form (6.15), of the kind described at the end of Section 4.4.

Finally, if the weight (6.22) is not dominant, then the corresponding Bethe vector in $\bigotimes_{i=1}^N V_{\nu_i}$ is zero. However, it is still interesting to consider the corresponding space of solutions of the Bethe Ansatz equations (6.4). As shown in [F2], Theorem 5.7, for each fixed weight γ this set may

be identified with a disjoint union of open subsets of a finite-dimensional $L\overline{B}_-$ -orbit (Schubert cell) in the affine flag variety $L\overline{G}/L\overline{B}_-$. We will consider a special case of this correspondence in the next subsection.

6.11. Special case: rational solutions of the KdV hierarchy

Consider the special case of the above construction when $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$, $N = 1$, the highest weight $\nu_1 = 0$, and the level $k_1 = 0$. According to formula (6.18), the corresponding generic $\widehat{\mathfrak{sl}}_2$ -opers may be represented by the following second order differential operators:

$$(6.23) \quad \partial_z^2 - \sum_{j=1}^{m_0} \frac{2}{(z - w_j^0)^2} - \lambda,$$

which satisfy the condition

$$(6.24) \quad \sum_{s \neq j} \frac{1}{(w_s^0 - w_j^0)^3} = 0, \quad j = 1, \dots, m_0.$$

These equations (analogous to the Bethe Ansatz equations (6.4)) mean that the $(z - w_j^0)$ -coefficient $v_{j,1}$ of the expansion of $v(z)$ at $z = w_j^0$ vanishes, which is equivalent to (4.19) in this case. This condition means that the operator (6.23) has *no monodromy* on \mathbb{P}^1 for all values of the spectral parameter λ .

The operators (6.23) were introduced by Airault–McKean–Moser in [AMM] as the generic rational solutions of the KdV hierarchy. They were also encountered by Duistermaat and Grünbaum [DG] in their study of the bispectral problem for second order differential operators. Here we have obtained these operators in a different way; namely, in the framework of the affine Gaudin model in the special case when $N = 1, \nu_1 = 0$ and $k_1 = 0$. Following the argument of [F2], we obtain that the set of these operators may be identified with an open subset of the $SL_2[[\lambda]]$ -orbit

$$SL_2[[\lambda]] \begin{pmatrix} \lambda^{m_0} & 0 \\ 0 & \lambda^{-m_0} \end{pmatrix} SL_2[[\lambda]]$$

in the affine Grassmannian $SL_2((\lambda))/SL_2[[\lambda]]$.

The points outside this locus correspond to the $\widehat{\mathfrak{sl}}_2$ -opers in which some of the points w_j^0 coalesce. They are represented by the more general second order operators of the form

$$(6.25) \quad \partial_z^2 - \sum_{j=1}^{m_0} \frac{\ell_j(\ell_j + 1)}{(z - w_j^0)^2} - \lambda,$$

where $\ell_j \in \mathbb{Z}_{>0}$, which satisfy the no-monodromy condition

$$(6.26) \quad \sum_{s \neq j} \frac{\ell_s(\ell_s + 1)}{(w_s^0 - w_j^0)^{2p_j+1}} = 0, \quad j = 1, \dots, m_0; p_j = 1, \dots, \ell_j.$$

These are precisely the operators constructed in [DG] (see formulas (3.31) and (3.32)), where it was shown that they correspond to the most general rational solutions of the KdV hierarchy decaying at ∞ . The fact that the set of all operators of this form is in bijection with the affine Grassmannian $SL_2((\lambda))/SL_2[[\lambda]]$ is in agreement with the well-known description of the polynomial tau-functions of the KdV hierarchy as points of the affine Grassmannian.¹⁶

The rational solutions of the $\widehat{\mathfrak{g}}$ -KdV hierarchies for other affine Kac–Moody algebras may be interpreted in a similar way.

§7. Concluding remarks

Let us summarize what we have learned so far. Our point of departure was the observation that soliton systems, such as the KdV and AKNS hierarchies, are special cases of the Gaudin models associated to affine Kac–Moody algebras. Therefore the quantization of these models should be understood in this framework. In order to better understand these models, we look at their simplified versions; namely, the Gaudin models associated to finite-dimensional simple Lie algebras. We then use this analogy to describe various elements of the affine Gaudin models, and in particular the quantum soliton systems. In this paper we have made the first steps in this direction.

Let us first discuss the classical soliton systems. These are integrable systems defined on the space of connections

$$\partial_t + A(t), \quad A(t) \in \mathfrak{g}((t)).$$

We apply a shift of argument and consider the space of connections of the form

$$(7.1) \quad \partial_t + A(t) + z\chi, \quad A(t) \in \mathfrak{g}((t)).$$

The shift parameter z plays the role of a new variable, and so these integrable systems are properly understood as integrable systems on the

¹⁶An identification of the space of rational solutions of the KdV hierarchy with the space of affine opers of this kind may also be obtained from the results of [BF].

double loop algebra in variables t and z , as was already noted in [RS1]. The classical commuting Hamiltonians are obtained from the functions $\varphi(M(z))$, where $M(z)$ is the monodromy matrix of (7.1) and φ is a function on G invariant under conjugation. Expanding these functions in z , we obtain local and non-local Hamiltonians. It is important to realize that, when viewed as a function on the entire double loop algebra, $\varphi(M(z))$ is actually central, with respect to the natural Kirillov–Kostant structure.

How can we quantize this system? It is clear that the quantum Hamiltonians should be obtained from quantum versions of the functions $\varphi(M(z))$, which should be in the center of a proper completion of the enveloping algebra of the double loop algebra. Therefore to describe the spectrum of these Hamiltonians we need to understand the structure of this center.

At this point it is instructive to look more closely at the “baby version” of this model; namely, the Gaudin model associated to a finite-dimensional simple Lie algebra \mathfrak{g} . In this case the naive phase space is \mathfrak{g}^* , but because we make the shift of argument, the true phase space is the dual of the loop algebra $\mathfrak{g}((z))$. The classical Hamiltonians appear from the z -expansion of the central functions on $\mathfrak{g}((z))^*$, obtained from invariant polynomials on \mathfrak{g}^* . Naively, we expect that the quantum versions of these functions are central elements in a completed enveloping algebra of $\mathfrak{g}((z))$. But there is an important twist to this story: we need to replace $\mathfrak{g}((z))$ by its central extension; its completed enveloping algebra then has a large center at the critical level.

This center turns out to be isomorphic to the algebra of functions on the space $\text{Op}_{L_{\mathfrak{g}}}(D^{\times})$ of $L_{\mathfrak{g}}$ -opers on the punctured disc, where $L_{\mathfrak{g}}$ is the Langlands dual Lie algebra of \mathfrak{g} [FF3, F6]. This is a non-trivial result, which is closely related to the geometric Langlands correspondence [BD1, F5]. This implies [F3, FFT] that the spectrum of the algebra of quantum Hamiltonians of the Gaudin models associated to \mathfrak{g} is isomorphic to the space of $L_{\mathfrak{g}}$ -opers on \mathbb{P}^1 with prescribed singularities at finitely many points (depending on the particular model).

Now, we conjecture that the algebras of quantum Hamiltonians of the affine Gaudin models exhibit similar features. Namely, their spectra should be expressed in terms of $L_{\widehat{\mathfrak{g}}}$ -opers on \mathbb{P}^1 with prescribed singularities at finitely many points, where $L_{\widehat{\mathfrak{g}}}$ is the Langlands dual Lie algebra of $\widehat{\mathfrak{g}}$. These $L_{\widehat{\mathfrak{g}}}$ -opers may be described more concretely as differential operators. For example, for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$ these are Schrödinger operators with spectral parameter.

We remark that in this picture opers appear in two different contexts, and this may cause some confusion. First of all, the classical

$\widehat{\mathfrak{g}}$ -KdV system has the space of $\widehat{\mathfrak{g}}$ -opers (or \mathfrak{g} -opers) on the punctured disc as the phase space. Then we quantize it and look at the spectra of the corresponding algebra of quantum KdV Hamiltonians. We conjecture that those are encoded by $L\widehat{\mathfrak{g}}$ -opers on \mathbb{P}^1 . So opers appear again, but they have nothing to do with the $\widehat{\mathfrak{g}}$ -opers entering the definition of the classical KdV system! For one thing, they correspond to an *a priori* different Lie algebra; namely, the Langlands dual Lie algebra $L\widehat{\mathfrak{g}}$. Besides, they are defined on \mathbb{P}^1 (on which the spectral parameter z is a global coordinate), not on a punctured disc.

In this paper we have presented explicit conjectures for the $L\widehat{\mathfrak{g}}$ -opers encoding the spectra of the quantum Hamiltonians of the affine Gaudin models associated to $\widehat{\mathfrak{g}}$. We have checked that our proposal coincides with the proposal of [BLZ5] in the case of the quantum KdV Hamiltonians.

What can we learn from these conjectures? First of all, placing the quantum soliton systems in the context of affine Gaudin models is useful, because we may apply the tools of the Gaudin models to investigate the soliton systems. An example of such a tool is Bethe Ansatz, which we use here to produce explicit formulas for the eigenvectors of the quantum Hamiltonians. Second, this points us to a conjectural description of the spectra of the quantum Hamiltonians in terms of opers, which is consistent with and generalizes the proposals obtained by other methods. Finally, we place the quantum soliton systems in the context of Langlands duality. We hope that this will allow us to understand better the true meaning of quantum soliton system and their relations to other areas of mathematics and physics.

§8. Appendix. Proof of Lemma 2

For any simple Lie algebra \mathfrak{g} we have the Segal–Sugawara current

$$S(w) = \frac{1}{2} \sum_{\alpha \in \Delta_+} (:e_\alpha(w)f_\alpha(w): + :f_\alpha(w)e_\alpha(w):) + \frac{1}{2} \sum_{i=1}^{\ell} :h_i(w)h^i(w):,$$

where $\{h^i\}$ is the dual basis to the coroot basis $\{h_i\}$ of \mathfrak{h} . Let us set

$$\widetilde{S}(w) = \sum_{\alpha \in \Delta_+} :e_\alpha(w)f_\alpha(w):$$

and

$$\overline{S}(w) = \frac{1}{2} \sum_{i=1}^{\ell} :h_i(w)h^i(w):.$$

Now denote by

$$S_\alpha(w) = \sum_{n \in \mathbb{Z}} S_{\alpha,n} w^{-n-2}$$

the element $S(w)$ associated to the \mathfrak{sl}_2 subalgebra $\mathfrak{g}_\alpha \subset \mathfrak{g}$ generated by e_α and f_α , and by

$$S_{\alpha\beta}(w) = \sum_{n \in \mathbb{Z}} S_{\alpha\beta,n} w^{-n-2}$$

the element $S(w)$ associated to the rank two Lie subalgebra $\mathfrak{g}_{\alpha\beta} \subset \mathfrak{g}$ whose root system $\Delta_{\alpha\beta} \subset \Delta$ is spanned by the roots $\alpha, \beta \in \Delta_+$. We will use a similar notation for the corresponding currents $\tilde{S}(w)$ and $\bar{S}(w)$.

Adapting the argument proving the commutativity of the DMT Hamiltonians in the finite-dimensional case (see [TL]), we reduce the statement of part (1) of Lemma 2 to checking the following identity:

$$(8.1) \quad [\tilde{S}_{\gamma,0}, \tilde{S}_{\alpha\beta,0}] = 0, \quad \forall \gamma \in \Delta_{\alpha\beta}^+.$$

We have

$$\begin{aligned} \tilde{S}_{\gamma,0} &= S_{\gamma,0} - \bar{S}_{\gamma,0} - \gamma_0, \\ \tilde{S}_{\alpha\beta,0} &= S_{\alpha\beta,0} - \bar{S}_{\alpha\beta,0} - 2\rho_{\alpha\beta,0}. \end{aligned}$$

Here γ (resp., ρ) denotes the element of \mathfrak{h} corresponding to the root γ (resp., the half-sum of positive roots of $\mathfrak{g}_{\alpha\beta}$) in \mathfrak{h}^* under the identification $\mathfrak{h} \simeq \mathfrak{h}^*$ corresponding to κ_0 . We write γ_0 (resp., ρ_0) for $\gamma \otimes 1$ (resp., $\rho \otimes 1$) in $\widehat{\mathfrak{g}}$.

Since all summands in the right hand sides of these formulas commute with $\mathfrak{h} \otimes 1 \subset \widehat{\mathfrak{g}}$, and so in particular with $\rho_{\alpha\beta,0}$, we obtain that (8.1) is equivalent to

$$[\tilde{S}_{\gamma,0}, S_{\alpha\beta,0} - \bar{S}_{\alpha\beta,0}] = 0.$$

Actually, we will show that

$$[\tilde{S}_{\gamma,0}, S_{\alpha\beta,0}] = [\tilde{S}_{\gamma,0}, \bar{S}_{\alpha\beta,0}] = 0.$$

By using the formalism of operator product expansion, it is easy to derive these identities from the following identities:

$$(8.2) \quad S_{\alpha\beta,-1} \tilde{S}_{\gamma,-2} v_k \in \text{Im } T,$$

$$(8.3) \quad \bar{S}_{\alpha\beta,-1} \tilde{S}_{\gamma,-2} v_k \in \text{Im } T$$

in the vertex algebra $V_k(\mathfrak{g})$, where v_k is the vacuum vector and T is the translation operator, in the notation of [FB]. (The sought-after commutation relations will follow after we make the change of variables $w = e^u - 1$ in the corresponding operator product expansion.)

To check (8.1), recall that the Segal–Sugawara operator $S_{\alpha\beta,-1}$ acts on the completed enveloping algebra of $\widehat{\mathfrak{g}}_{\alpha\beta}$ as a multiple of the derivation $T = -t\partial_t$. This implies formula (8.2). To prove formula (8.3), note that $\widetilde{S}_{\gamma,-2}v_k = e_{\gamma,-1}f_{\gamma,-1}v_k$ and $\overline{S}_{\alpha\beta,-1}$ is a linear combination of

$$(8.4) \quad \sum_{n \in \mathbb{Z}} \check{\gamma}_{-1-n}\check{\gamma}_n \quad \text{and} \quad \sum_{n \in \mathbb{Z}} \check{\mu}_{-1-n}\check{\mu}_n,$$

where $\check{\mu} \in \mathfrak{h}$ is orthogonal to $\check{\gamma}$. Therefore the second term, applied to $\widetilde{S}_{\gamma,-2}v_k$, gives 0, and the first term gives

$$\sum_{n \in \mathbb{Z}} \check{\gamma}_{-1-n}\check{\gamma}_n \cdot e_{\gamma,-1}f_{\gamma,-1}v_k = 2\check{\gamma}_{-1}\check{\gamma}_{-2}v_k = T \cdot (\check{\gamma}_{-1})^2 v_k.$$

This implies formula (8.3) and hence proves part (1) of Lemma 2.

The commutativity of the operators \widetilde{T}_γ given by formula (3.12) follows from the corresponding analogue of the identity (8.1), which is proved in the same way as above. In order to complete the proof of part (2) of Lemma 2, it remains to show that the operators \widetilde{T}_γ commute with $\widehat{\mathfrak{h}}$. This, in turn, follows from the formula

$$h_{i,n} \cdot \left(e_{\alpha,-1}f_{\alpha,-1} + f_{\alpha,-1}e_{\alpha,-1} - \frac{1}{k}h_{\alpha,-1}^2 \right) v_k = 0, \quad n \geq 0,$$

which is verified in a straightforward fashion.

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