

## Fermionic formulas for characters of $(1, p)$ logarithmic model in $\Phi_{2,1}$ quasiparticle realisation

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### Abstract.

We give expressions for the characters of  $(1, p)$  logarithmic conformal field models in the Gordon-type form. The formulas are obtained in terms of “quasiparticles” that are Virasoro  $\Phi_{2,1}$  primary fields and generalize the symplectic fermions.

### §1. Introduction

In recent times, logarithmic conformal field theories are investigated from different directions [17, 18, 21, 12, 11, 4, 20, 23]. There exists a class of models that are “extensions” of minimal models [16] by some set of vertex operators [5]. The most popular are the so called  $(1, p)$  models.

The logarithmic  $(1, p)$  models have the central charge

$$(1.1) \quad c = 13 - 6p - \frac{6}{p}.$$

The local chiral algebra of the logarithmic  $(1, p)$  models is the triplet  $W$ -algebra studied in [20, 17]. We let  $\mathcal{W}(p)$  denote this algebra. The algebra  $\mathcal{W}(p)$  bears an action of the  $sl(2)$  algebra that differentiates OPEs. The adjunctive triplet refers to the fact that  $\mathcal{W}(p)$  is generated by the fields  $W^{\pm,0}(z)$ , which are transformed as the spin-1 representation of the  $sl(2)$ . Moreover,  $W^+(z)$  and  $W^-(z)$  are highest and lowest weight vectors of the triplet respectively. The fields  $W^{\pm,0}(z)$  are three solutions of the equation

$$(1.2) \quad \partial^3 \Phi(z) + \text{const}_1 : T(z) \partial \Phi(z) : + \text{const}_2 : \partial T(z) \Phi(z) : = 0$$

for the field  $\Phi_{3,1}(z)$ . The vertex operator algebra  $\mathcal{W}(p)$  is an extension by  $\Phi_{3,1}(z)$  of the Virasoro algebra with the central charge (1.1).

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We let  $Vac_p$  denote the vacuum representation of this Virasoro vertex operator algebra.

The algebra  $\mathcal{W}(p)$  has  $2p$  irreducible representations  $\mathcal{X}_{s,p}^\pm$  ( $1 \leq s \leq p$ ). These representations admit the action of the  $sl(2)$  as well. The representations labeled with the superscript  $+$  decompose into a direct sum of odd dimensional irreducible  $sl(2)$  representations and labeled with the superscript  $-$  into a direct sum of even dimensional. This leads to the factors  $2n + 1$  and  $2n$  in the characters [12, 11]

$$(1.3) \quad \chi_{s,p}^+(q) = \frac{q^{-\frac{1}{24}}}{\prod_{n=1}^{\infty} (1 - q^n)} \sum_{n \in \mathbb{Z}} (2n + 1) q^{p(n + \frac{p-s}{2p})^2},$$

$$(1.4) \quad \chi_{s,p}^-(q) = \frac{q^{-\frac{1}{24}}}{\prod_{n=1}^{\infty} (1 - q^n)} \sum_{n \in \mathbb{Z}} (2n) q^{p(-n + \frac{s}{2p})^2},$$

where  $\chi_{s,p}^\pm(q) = \text{Tr}_{\mathcal{X}_{s,p}^\pm} q^{L_0 - \frac{c}{24}}$ . These expressions for the characters can be considered “bosonic” formulas because they are obtained from a resolution of the irreducible module constructed from some modules, which can be considered  $\mathcal{W}(p)$  Verma modules.

We obtain the “fermionic” formulas for characters in terms of “slightly” bigger algebra  $\mathcal{A}(p)$  that is an extension of the Virasoro vertex operator algebra  $Vac_p$  with two solutions of the equation

$$(1.5) \quad \partial^2 \Phi(z) + \text{const} : T(z) \Phi(z) := 0$$

for the field  $\Phi_{2,1}(z)$  with conformal dimension  $\frac{3p-2}{4}$ . The vertex operator algebra  $\mathcal{A}(p)$  bears the action of  $sl(2)$  and two fields  $a^\pm(z)$  are the highest and the lowest weight vectors of the  $sl(2)$  spin- $\frac{1}{2}$  irreducible representation. The fields  $a^\pm(z)$  are two highest weight vectors of the  $\mathcal{W}(p)$ -module  $\mathcal{X}_{1,p}^-$ . We note that  $\mathcal{A}(p)$  is nonlocal vertex operator algebra, which means that there are exist conformal blocks with  $\mathcal{A}(p)$  fields that have nontrivial monodromy. In the  $p = 2$  case,  $a^\pm(z)$  coincide with derivative of the symplectic fermions [21]. In that case “nonlocality” leads to two sectors in one of which symplectic fermions act with integer and in other with half-integer modes. We have a sequence of extensions of vertex operator algebras

$$(1.6) \quad Vac_p \hookrightarrow \mathcal{W}(p) \hookrightarrow \mathcal{A}(p).$$

The algebra  $\mathcal{A}(p)$  has  $p$  irreducible representations  $\mathcal{X}_{s,p}$  ( $1 \leq s \leq p$ ). Each irreducible  $\mathcal{A}(p)$  module as a  $\mathcal{W}(p)$  module decomposes as  $\mathcal{X}_{s,p} = \mathcal{X}_{s,p}^+ \oplus \mathcal{X}_{s,p}^-$ . We set

$$(1.7) \quad \chi_{s,p}(q) = \text{Tr}_{\mathcal{X}_{s,p}^+ \oplus \mathcal{X}_{s,p}^-} q^{L_0 - \frac{c}{24}} = \chi_{s,p}^+(q) + \chi_{s,p}^-(q).$$

The main result of the paper is formulated as follows.

**Theorem 1.1.** *The characters (1.7) can be written in the form*

$$(1.8) \quad \chi_{s,p}(q) = q^{\frac{s^2-1}{4p} + \frac{1-s}{2} - \frac{c}{24}} \sum_{n_+, n_-, n_1, \dots, n_{p-1} \geq 0} \frac{q^{\frac{1}{2} \mathbf{n} \mathcal{A} \cdot \mathbf{n} + \mathbf{v}_s \cdot \mathbf{n}}{(q)_{n_+} (q)_{n_-} (q)_{n_1} \dots (q)_{n_{p-1}}}},$$

where  $\mathbf{n} = (n_+, n_-, n_1, \dots, n_{p-1})$ ,  $(x)_k = \prod_{i=0}^{k-1} (1 - xq^i)$ ,  $\mathcal{A}$  is the Gordon matrix

$$(1.9) \quad \mathcal{A} = \begin{pmatrix} \frac{p}{2} & \frac{p}{2} & 1 & 2 & 3 & \dots & p-1 \\ \frac{p}{2} & \frac{p}{2} & 1 & 2 & 3 & \dots & p-1 \\ 1 & 1 & 2 & 2 & 2 & \dots & 2 \\ 2 & 2 & 2 & 4 & 4 & \dots & 4 \\ 3 & 3 & 2 & 4 & 6 & \dots & 6 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ p-1 & p-1 & 2 & 4 & 6 & \dots & 2(p-1) \end{pmatrix},$$

and

$$(1.10) \quad \mathbf{v}_s = \left( \frac{p-s}{2}, \frac{p-s}{2}, \underbrace{0, \dots, 0}_{s-1}, \underbrace{1, 2, \dots, p-s}_{p-s} \right).$$

Similar but different fermionic formulas were recently obtained in [13]. We emphasize that the fermionic formulas for characters depend on the chosen set of “particles” in terms of which the formulas are written. The number of particles is equal to the order of the matrix (1.9), i.e.  $p + 1$  in our case. First two rows and columns correspond to  $a^+(z)$  and  $a^-(z)$  and other particles appear in singular terms of the OPE  $a^+(z)a^-(w)$ . We note that the matrix obtained by dropping first two rows and first two columns from (1.9) coincides with the standard Gordon matrix  $2\min(i, j)$ . Our considerations in this paper have many overlaps with the construction for fermionic formulas of minimal models given in [6] in terms of the Virasoro primary field  $\Phi_{2,1}$ . Such a construction is natural in the corner transfer matrix approach to the RSOS models and their connection with the Virasoro minimal models [14, 15]. In this approach one should consider nonlocal vertex operator algebras, which are extensions of the Virasoro algebra by a set of primary fields. However these nonlocal vertex operator algebras have treatable theory of representation.

We now briefly describe the way we prove Theorem 1.1. We construct a degeneration of the chiral algebra  $\mathcal{A}(p)$  to some algebra  $\bar{\mathcal{A}}(p)$  with generators called “particles” that satisfy a set of quadratic defining relations. The structure of these quadratic relations is given by

the Gordon matrix (1.9). Each irreducible representation of  $\mathcal{A}(p)$  has a  $\bar{\mathcal{A}}(p)$  representation counterpart that is the cyclic  $\bar{\mathcal{A}}(p)$  representation with the same character. The characters of  $\bar{\mathcal{A}}(p)$  representations have a natural expression in terms of Gordon-type formulas.

In Sec. 2 we recall some known facts about  $(1, p)$  logarithmic conformal field models and in Sec. 3 give the proof of Thm. 1.1.

**§2. Short description of logarithmic  $(1, p)$  models**

**2.1. Notations**

Throughout the paper we use the standard notation

$$(2.11) \quad \alpha_+ = \sqrt{2p}, \quad \alpha_- = -\sqrt{\frac{2}{p}},$$

$$\alpha_+ \alpha_- = -2, \quad \alpha_0 = \alpha_+ + \alpha_- = \sqrt{\frac{2}{p}}(p-1),$$

where  $p$  is a positive integer. We let  $\mathcal{M}_{r,s;p}$  denote the irreducible module with the highest weight

$$(2.12) \quad \Delta_{r,s} = \frac{p}{4}(r^2 - 1) + \frac{1}{4p}(s^2 - 1) + \frac{1 - rs}{2}, \quad 1 \leq s \leq p, \quad r \in \mathbb{Z}$$

of the Virasoro algebra with the central charge (1.1). We note that  $\mathcal{M}_{r,s;p}$  is the quotient of the Verma module by the submodule generated from one singular vector on the level  $rs$  and such modules exhaust irreducible Virasoro modules that aren't Verma modules.

In terms of the free scalar field  $\varphi$  with the OPE  $\varphi(z)\varphi(w) = \log(z-w)$  the highest weight vector of  $\mathcal{M}_{r,s;p}$  corresponds to the vertex field [16]

$$(2.13) \quad V_{r,s} = :e^{-(\frac{r-1}{2}\alpha_+ + \frac{s-1}{2}\alpha_-)\varphi(z)}:.$$

The generators of the Virasoro algebra are Laurent coefficients of the energy-momentum tensor

$$(2.14) \quad T = \frac{1}{2} : \partial\varphi \partial\varphi : + \frac{\alpha_0}{2} \partial^2\varphi.$$

**2.2. The triplet  $W$ -algebra  $\mathcal{W}(p)$**

The triplet  $W$ -algebra can be described in terms of the lattice vertex operator algebra generated by the vertex operators [11]

$$(2.15) \quad V^\pm(z) = :e^{\pm\alpha_+\varphi(z)}:.$$

The algebra  $\mathcal{W}(p)$  is a subalgebra of this lattice vertex operator algebra. The vacuum representation of  $\mathcal{W}(p)$  is the kernel of the screening operator

$$(2.16) \quad F = \frac{1}{2\pi i} \oint dz :e^{\alpha-\varphi(z)}:$$

acting in the vacuum representation of the lattice VOA. This kernel is generated by the  $sl(2)$ -algebra triplet

$$(2.17) \quad W^- = :e^{-\alpha+\varphi(z)}:, \quad W^0 = [e, W^-], \quad W^+ = [e, W^0],$$

where

$$(2.18) \quad e = \frac{1}{2\pi i} \oint dz :e^{\alpha+\varphi(z)}:$$

is one of the  $sl(2)$  algebra generators. The generator  $f$  in terms of  $\varphi$  is given by a nonlocal expression.  $\mathcal{W}(p)$  contains the energy-momentum tensor (2.14) with the central charge (1.1). The generators  $W^{\pm,0}$  are primary fields of dimension  $2p - 1$ .

### 2.3. Irreducible representations of $\mathcal{W}(p)$

Each irreducible  $\mathcal{W}(p)$  modules  $\mathcal{X}_{s,p}^{\pm}$  can be described in terms of irreducible Virasoro modules  $\mathcal{M}_{r,s;p}$ . Let  $\pi_r$  denote the  $r$ -dimensional irreducible representation of  $sl(2)$ . Then the spaces

$$(2.19) \quad \mathcal{X}_{s,p}^+ = \oplus_{n \in \mathbb{N}} \pi_{2n-1} \otimes \mathcal{M}_{2n-1,s;p},$$

$$(2.20) \quad \mathcal{X}_{s,p}^- = \oplus_{n \in \mathbb{N}} \pi_{2n} \otimes \mathcal{M}_{2n,s;p}$$

admit [4] an action of  $\mathcal{W}(p)$  and are its irreducible modules. These decompositions give formulas (1.3) and (1.4) for the characters.

### 2.4. The algebra $\mathcal{A}(p)$

We consider the “nonlocal” vertex-operator algebra  $\mathcal{A}(p)$  generated by the  $sl(2)$  doublet of fields

$$(2.21) \quad a^+(z) = :e^{-\frac{\alpha}{2}\varphi(z)}:, \quad a^-(z) = [e, a^+(z)] = :D_{p-1}(\partial\varphi(z))e^{\frac{\alpha}{2}\varphi(z)}:,$$

where  $D_{p-1}$  is a degree  $p - 1$  differential polynomial in  $\partial\varphi(z)$ . The conformal dimension of these fields is  $\frac{3p-2}{4}$ . The fields  $a^{\pm}(z)$  have the following OPE

$$(2.22) \quad a^+(z)a^-(w) = (z-w)^{-\frac{3p-2}{2}} \sum_{n \geq 0} (z-w)^n H^n(w)$$

where  $H^n(w)$  are fields with conformal dimension equals to  $n$ . The field  $H^0$  is proportional to the identity field 1,  $H^1 = 0$ ,  $H^2$  is proportional to the energy-momentum tensor  $T$ . About other fields  $H^n$  we can say the following

$$(2.23) \quad H^{2n} = c_{2n} : T^n : + P_{2n}(T), \quad 1 \leq n \leq p-1,$$

$$(2.24) \quad H^{2n+1} = c_{2n+1} \partial : T^n : + P_{2n+1}(T), \quad 1 \leq n \leq p-2,$$

$$(2.25) \quad H^{2p-1} = c_{2p-1} \partial : T^{p-1} : + P_{2p-1}(T) + d_1 W^0,$$

$$(2.26) \quad H^{2p} = c_{2p} : T^p : + P_{2p}(T) + d_2 \partial W^0,$$

where  $: T^n :$  is the normal ordered  $n$ -th power of the energy-momentum tensor,  $P_n(T)$  is a differential polynomial in  $T$  and degree of both  $P_{2n}(T)$  and  $P_{2n+1}(T)$  in  $T$  is equal to  $n-1$ ,  $W^0$  is the field defined in (2.17) and  $c_n, d_1, d_2$  are some nonzero constants. The OPE (2.22) is determined by the conformal weight of fields. The fact that  $c_n, d_1$  and  $d_2$  are not equal to 0 can be checked by a direct calculation using (2.21).

### 2.5. The irreducible representations of $\mathcal{A}(p)$

The vertex operator algebra  $\mathcal{A}(p)$  is graded (by eigenvalues of the zero mode of  $\partial\varphi$ )

$$(2.27) \quad \mathcal{A}(p) = \bigoplus_{\beta \in \frac{\alpha_{\pm}}{2}\mathbb{Z}} \mathcal{A}(p)^{\beta}$$

and  $a^{\pm}(z) \in \mathcal{A}(p)^{\pm \frac{\alpha_{\pm}}{2}}$ . We consider only the graded representations of  $\mathcal{A}(p)$ . For any representation  $\mathcal{X} = \bigoplus_{t \in \mathbb{C}} \mathcal{X}^t$  we have  $a^{\pm}(z) : \mathcal{X}^t \rightarrow \mathcal{X}^{t \pm \frac{\alpha_{\pm}}{2}}$  and  $a^{\pm}(z)$  acting in  $\mathcal{X}^t$  have the decomposition

$$(2.28) \quad a^{\pm}(z) = \sum_{n \in \pm t \frac{\alpha_{\pm}}{2} - \frac{3p-2}{4} + \mathbb{Z}} z^{-n - \frac{3p-2}{4}} a_n^{\pm}.$$

The irreducible representations  $\mathcal{X}_{s,p}$  of  $\mathcal{A}(p)$  are highest-weight modules generated from the vector  $|s, p\rangle \in \mathcal{X}_{s,p}^{\frac{1-s}{2}\alpha_-}$  satisfying

$$(2.29) \quad a_{-\frac{3p-2s}{4}+n}^{\pm} |s, p\rangle = 0, \quad n \in \mathbb{N}, 1 \leq s \leq p.$$

The conformal dimension of  $|s, p\rangle$  is  $\Delta_{1,s} = \frac{s^2-1}{4p} + \frac{1-s}{2}$ . The highest mode of  $a^{\pm}(z)$  that generate new vectors from  $|s, p\rangle$  are

$$(2.30) \quad a_{-\frac{3p-2s}{4}}^{\pm}, \quad 1 \leq s \leq p$$

as it shown in Fig. 1. Proceeding further we obtain the set of extremal

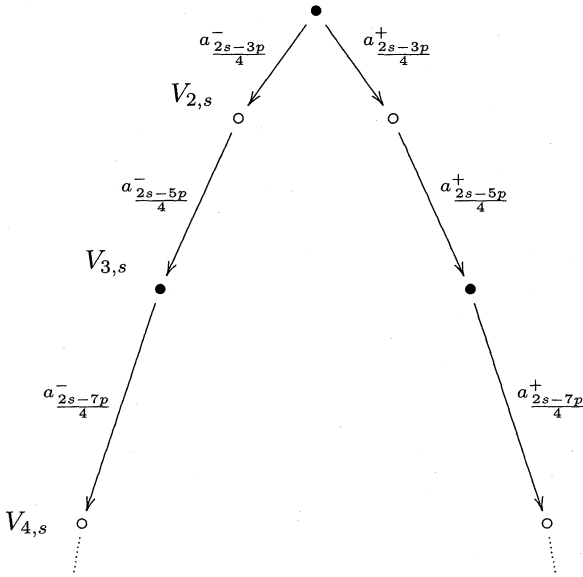


Fig. 1. The irreducible  $\mathcal{A}(p)$  modules. The filled dot on the top is the cyclic vector  $|s, p\rangle$ . The arrows show the action of highest modes of  $a^\pm$  that give nonzero vectors. Filled (open) dots denote vertices belonging to representations  $\mathcal{X}_s^+$  ( $\mathcal{X}_s^-$ ).

vectors shown in Fig. 1.

From (2.19), we immediately obtain that the irreducible representation  $\mathcal{X}_{s,p}$  as a representation of  $sl(2) \oplus \text{Vir}$  decomposes as

$$(2.31) \quad \mathcal{X}_{s,p} = \bigoplus_{n \in \mathbb{N}} \pi_n \otimes \mathcal{M}_{n,s;p}.$$

**Remark 2.1.** In the rest of the paper we use the notation  $\text{ch}V$  for the normalized character of the space  $V$ . Namely the character  $\text{ch}V$  is a Laurent series  $\sum_{i \in \mathbb{Z}} a_i q^i$  such that  $a_i = 0$  for  $i < 0$  and  $a_0 \neq 0$ . For example for  $V = \mathcal{X}_{s,p}$  we have

$$\chi_{s,p}(q) = q^{\frac{s^2-1}{4p} + \frac{1-s}{2} - \frac{c}{24}} \text{ch}\mathcal{X}_{s,p}.$$

The normalization above is natural for us because of the fermionic (particle) approach used in the paper.

§3. Proof of Theorem 1.1

The strategy of the proof is as follows. We introduce a certain filtrations on the algebra  $\mathcal{A}(p)$  such that the adjoint graded algebra  $\bar{\mathcal{A}}(p)$  can be described in terms of generators and quadratic relations. We study highest weight representations of  $\bar{\mathcal{A}}(p)$  and derive fermionic formula for their characters. We show that these characters are equal to  $\chi_{s,p}(q)$ .

3.1. Filtrations and adjoint graded algebras

We introduce a filtration  $F_\bullet$  on  $\mathcal{A}(p)$  by attaching

- degree  $p$  to each mode of  $a^+(z)$ ,
- degree  $p - 1$  to each mode of  $a^-(z)$ ,
- degree 2 to each mode of  $T(z)$ .

We denote the adjoint graded algebra with respect to  $F_\bullet$  by  $\bar{\mathcal{A}}(p)$  and its generators by  $\bar{a}^\pm(z)$  and  $\bar{T}(z)$ .

**Lemma 3.1.** *The following relations hold in  $\bar{\mathcal{A}}(p)$ :*

$$(3.32) \quad \bar{a}^+(z)\bar{a}^+(w) \sim (z - w)^{\frac{p}{2}}, \quad \bar{a}^-(z)\bar{a}^-(w) \sim (z - w)^{\frac{p}{2}},$$

$$(3.33) \quad \bar{T}(z)\bar{a}^\pm(w) \sim z - w,$$

$$(3.34) \quad \bar{a}^+(z)\bar{a}^-(w) \sim (z - w)^{\frac{p}{2}},$$

where  $A(z)B(w) \sim (z - w)^x$  means that the fields  $A(z)$  and  $B(z)$  have the following OPE

$$(3.35) \quad A(z)B(w) = (z - w)^x \sum_{n \geq 0} (z - w)^n C^n(w)$$

with some fields  $C(z)$ . In addition the current  $\bar{T}(z)$  is commutative and satisfy  $\bar{T}(z)^p = 0$ .

*Proof.* From the formula (2.21) we obtain that in  $\bar{\mathcal{A}}(p)$  the following is true:

$$\bar{a}^+(z)\bar{a}^+(w) \sim (z - w)^{\frac{p}{2}}.$$

Therefore the first part of (3.32) holds in  $\mathcal{A}(p)$ . The second part of (3.32) follows from the first part and an equation  $[e, a^-(z)] = 0$ , which follows from the fact that  $a^-(z)$  is a component of the field  $\Phi_{1,2}(z)$ .

To prove (3.33) we use the OPE in  $\mathcal{A}(p)$ :

$$T(z)a^\pm(w) = (z - w)^{-1} \partial a^\pm(w) + :T(z)a^\pm(w): + \dots$$

Now using the relation

$$:T(z)a^\pm(z): + \text{const} \cdot \partial^2 a^\pm(z) = 0$$



(recall that  $a^\pm(z)$  are two components of the field  $\Phi_{2,1}(z)$ ) we obtain (3.33). We now prove (3.34).

The OPE (2.22) gives the following OPE in  $\bar{\mathcal{A}}(p)$ :

$$(3.36) \quad \bar{a}^+(z)\bar{a}^-(w) = (z-w)^{-\frac{3p-2}{2}} \left[ \sum_{i=0}^{2(p-1)} (z-w)^i \bar{H}_i(z) + (z-w)^{2p-1} (c_{2p-1} \partial : \bar{T}^{p-1}(z) : + d_1 \bar{W}^0(z)) + (z-w)^{2p} (c_{2p} : \bar{T}^p(z) : + d_2 \partial \bar{W}^0(z)) \right] + \dots$$

(see (2.23)–(2.26)). We recall that for any  $0 \leq i \leq 2p - 2$  the operator  $H_i(z)$  is a differential polynomial in  $T(z)$  of degree smaller than or equal to  $p - 1$ . Therefore the degree of  $\bar{H}_i$  with respect to our filtration is smaller than  $2p - 1$ . But the degree of  $\bar{a}^+(z)\bar{a}^-(w)$  (which is the left hand side of (3.36)) is exactly  $2p - 1$ . Therefore we can rewrite (3.36) as

$$(3.37) \quad \bar{a}^+(z)\bar{a}^-(w) = (z-w)^{-\frac{3p-2}{2}} \left[ (z-w)^{2p-1} d_1 \bar{W}^0(z) + (z-w)^{2p} (c_{2p} : \bar{T}^p(z) : + d_2 \partial \bar{W}^0(z)) \right] + \dots$$

This proves (3.34).

We now consider the current  $\bar{T}(z)$ . Note that  $\bar{T}(z)$  is commutative, because the degree of each mode of  $\bar{T}(z)$  is equal to 2. We now show that  $\bar{T}^p = 0$ . From (3.37) we obtain that each mode of the current  $\bar{W}^0(z)$  can be expressed as a linear combination of  $\bar{a}_i^+ \bar{a}_j^-$ . The same is true for the modes of  $c_{2p} : \bar{T}^p(z) : + d_2 \partial \bar{W}^0(z)$  and thus for  $\bar{T}^p(z)$ . But the degree of  $\bar{T}^p(z)$  equals  $2p$  and the degree of  $\bar{a}_i^+ \bar{a}_j^-$  is equal to  $2p - 1$ . This gives  $\bar{T}^p(z) = 0$ . Lemma is proved. Q.E.D.

We now want to replace the condition  $\bar{T}^p(z) = 0$  by the set of quadratic relations. We use the standard Lemma (see [10, 6]).

**Lemma 3.2.** *Let  $\mathcal{B}$  be the algebra generated by modes  $J_0, J_1, \dots$  of an abelian current  $J(z)$  with the defining relation  $J(z)^p = 0$ . There exists a filtration  $G_\bullet$  on the algebra  $\mathcal{B}$  such that the adjoint graded algebra is generated by coefficients of series  $J^{[i]}(z)$ , which are images of powers  $J(z)^i$ ,  $1 \leq i < p$ . In addition defining relations in the adjoint graded algebra are given by*

$$(3.38) \quad J^{[n]}(z)J^{[m]}(w) \sim (z-w)^{2\min(n,m)}.$$

We now consider the filtration on  $\bar{\mathcal{A}}(p)$  induced from the filtration  $G_\bullet$  on the algebra generated with the modes of  $\bar{T}(z)$ . We denote the adjoint graded algebra by the same symbol  $\bar{\mathcal{A}}(p)$ . In what follows we use the notation  $\bar{\mathcal{A}}(p)$  to denote the adjoint graded algebra of  $\mathcal{A}(p)$  with respect to the double filtration  $(F_\bullet$  and  $G_\bullet)$ .

We now introduce a new algebra which is quadratic with the defining relations given by (3.38), (3.32), (3.33) and (3.34).

**Definition 3.3.** Let  $\bar{\mathcal{A}}(p)'$  denote an algebra generated with the currents

$$\bar{a}^+(z), \bar{a}^-(z), \bar{T}^{[i]}(z), 1 \leq i < p$$

and defining relations

$$(3.39) \quad \bar{a}^\pm(z)\bar{a}^\pm(w) \sim (z-w)^{\frac{p}{2}},$$

$$(3.40) \quad \bar{a}^\pm(z)\bar{T}^{[n]}(w) \sim (z-w)^n,$$

$$(3.41) \quad \bar{T}^{[n]}(z)\bar{T}^{[m]}(w) \sim (z-w)^{2\min(n,m)}.$$

We note that Lemmas 3.1 and 3.2 gives a surjection

$$(3.42) \quad \bar{\mathcal{A}}(p)' \rightarrow \bar{\mathcal{A}}(p).$$

We define the  $sl(2)$  action on  $\bar{\mathcal{A}}(p)'$  as follows.  $\bar{T}^{[n]}(z)$  are  $sl(2)$  invariants and  $\bar{a}^+(z)$  and  $\bar{a}^-(z)$  are lowest and highest weight vectors of the  $sl(2)$  doublet respectively. This action commutes with the mapping (3.42).

We now study highest weight representations of  $\bar{\mathcal{A}}(p)'$ . Let  $\bar{\mathcal{X}}'_{s,p}$ ,  $1 \leq s \leq p$  denote the cyclic representation of  $\bar{\mathcal{A}}(p)'$  that is generated from the vector  $v_{s,p}$  satisfying the defining relations (The Fourier decomposition of  $\bar{a}^\pm(z)$  is the same as in (2.28).)

$$(3.43) \quad \bar{a}_j^\pm v_{s,p} = 0, \quad j > -\frac{3p-2s}{4},$$

$$(3.44) \quad \bar{T}_j^{[n]} v_{s,p} = 0, \quad j > \begin{cases} -n, & n < s, \\ -2n+s-1, & n \geq s. \end{cases}$$

(3.44) follows from a singular vector in the corresponding Verma module.

We note that because of (3.42) and (2.29) there exists a surjective homomorphism

$$(3.45) \quad \bar{\mathcal{X}}'_{s,p} \rightarrow \bar{\mathcal{X}}_{s,p},$$

where  $\bar{\mathcal{X}}_{s,p}$  is an adjoint graded to  $\mathcal{X}_{s,p}$  with respect to the filtrations induced from  $F_\bullet$  and  $G_\bullet$ . In particular, the characters of  $\bar{\mathcal{X}}_{s,p}$  and  $\mathcal{X}_{s,p}$  coincide and  $\bar{\mathcal{X}}_{s,p} \simeq \mathcal{X}_{s,p}$  as  $sl(2)$  modules.

**Lemma 3.4.** *The character of  $\tilde{X}'_{s,p}$  is given by the right hand side of the formula (1.8).*

*Proof.* We briefly recall the functional realization of the dual space (see for example [9]).

Consider the decomposition

$$\tilde{X}'_{s,p} = \bigoplus_{n_+, n_-, n_1, \dots, n_{p-1} \geq 0} \tilde{X}'_{s,p}(n_+, n_-, n_1, \dots, n_{p-1}),$$

where  $\tilde{X}'_{s,p}(n_+, n_-, n_1, \dots, n_{p-1})$  is the linear span of the vectors of the form

$$\left[ \bar{a}_{i_1^+}^+ \dots \bar{a}_{i_{n_+}^+}^+ \bar{a}_{i_1^-}^- \dots \bar{a}_{i_{n_-}^-}^- \prod_{\alpha=1}^{p-1} \bar{T}_{i_1^\alpha}^{[\alpha]} \dots \bar{T}_{i_{n_\alpha}^\alpha}^{[\alpha]} \right] \cdot v_{s,p}$$

with arbitrary parameters  $i_\beta^\pm, i_\beta^\alpha$ . For  $\theta \in (\tilde{X}'_{s,p}(n_+, n_-, n_1, \dots, n_{p-1}))^*$  and  $\bar{T}^{[\alpha]}$  defined in Def. 3.3, we consider a correlation function

$$F_\theta = \langle \theta | \bar{T}^{[1]}(x_1^+) \dots \bar{T}^{[1]}(x_{n_1}^+) \dots \bar{T}^{[p-1]}(x_1^{p-1}) \dots \bar{T}^{[p-1]}(x_{n_{p-1}}^{p-1}) \bar{a}^+(x_{n_+}^+) \dots \bar{a}^+(x_{n_+}^+) \dots \bar{a}^-(x_{n_-}^-) \dots \bar{a}^-(x_{n_-}^-) | v_{s,p} \rangle.$$

The space of thus obtained functions can be described as the space of functions of the form

$$(f \cdot X)(x_1^+, \dots, x_{n_1}^+, \dots, x_1^{p-1}, \dots, x_{n_{p-1}}^{p-1}, x_1^+, \dots, x_{n_+}^+, x_1^-, \dots, x_{n_-}^-),$$

where  $X$  is a function given by the formula

$$(3.46) \quad \left( \prod_{i=1}^{n_+} x_i^+ \prod_{j=1}^{n_-} x_j^- \right)^{\frac{3p-2s}{4}} \prod_{\alpha=1}^{s-1} \left( \prod_{1 \leq i \leq n_\alpha} x_i^\alpha \right)^\alpha \prod_{\alpha=s}^{p-1} \left( \prod_{1 \leq i \leq n_\alpha} x_i^\alpha \right)^{2\alpha-s+1} \\ \prod_{a=+,-} \prod_{1 \leq i \leq n_a} (x_i^a - x_j^a)^{\frac{p}{2}} \prod_{\substack{1 \leq \alpha \leq p-1 \\ 1 \leq i < j \leq n_\alpha}} (x_i^\alpha - x_j^\alpha)^{2\alpha} \\ \prod_{\substack{1 \leq i \leq n_+ \\ 1 \leq j \leq n_-}} (x_i^+ - x_j^-)^{\frac{p}{2}} \prod_{\substack{b=+,- \\ 1 \leq \alpha \leq p-1}} \prod_{\substack{1 \leq i \leq n_b \\ 1 \leq j \leq n_\alpha}} (x_i^b - x_j^\alpha)^\alpha \prod_{\substack{1 \leq \alpha < \beta \leq p-1 \\ 1 \leq i \leq n_\alpha \\ 1 \leq j \leq n_\beta}} (x_i^\alpha - x_j^\beta)^{2 \min(\alpha, \beta)},$$

and  $f$  is a polynomial symmetric in each group of variables

$$\{x_i^+\}_{i=1}^{n_+}, \{x_i^-\}_{i=1}^{n_-}, \{x_i^\alpha\}_{i=1}^{n_\alpha}, \alpha = 1, \dots, s.$$

The exact form (3.46) of the functions  $F_\theta$  follows from the definition of  $\bar{A}(p)'$  as an algebra with defining quadratic relations and from the

definition of  $\bar{\mathcal{X}}'_{s,p}$ . In particular the factor in the first line of (3.46) comes from the relations (3.43), (3.44) and the the rest factors correspond to (3.39), (3.40), (3.41). Direct computation shows that the character of the space of polynomials (3.46) is given by the right hand side of the formula (1.8). Q.E.D.

This Lemma gives an upper bound for the character of  $\mathcal{X}_{s,p}$ . To prove that (3.45) is an isomorphism, we consider the decomposition

$$(3.47) \quad \bar{\mathcal{X}}'_{s,p} = \bigoplus_{n=1}^{\infty} \pi_n \otimes \bar{\mathcal{X}}'_{s,p}[n],$$

where  $\bar{\mathcal{X}}'_{s,p}[n]$  is a space of multiplicity of  $\pi_n$  in  $\bar{\mathcal{X}}'_{s,p}$ . Our goal is to show that

$$(3.48) \quad \text{ch} \bar{\mathcal{X}}'_{s,p}[r] = \text{ch} \mathcal{M}_{r,s;p}.$$

Because of the surjection (3.45) and formula (2.31) the proof of the equation (3.48) is enough for the proof of the Theorem 1.1.

We divide the proof of (3.48) into 2 parts: we first show that

$$\text{ch} \bar{\mathcal{X}}'_{s,p}[1] = \text{ch} \mathcal{M}_{1,s;p}$$

and then deduce the general  $r$  case.

### 3.2. The proof of $\text{ch} \bar{\mathcal{X}}'_{s,p}[1] = \text{ch} \mathcal{M}_{1,s;p}$

We first let  $p = 1$  and consider the decomposition

$$(3.49) \quad \bar{\mathcal{X}}'_{1,1} = \bigoplus_{n \geq 0} V_n,$$

where  $V_0$  is spanned by the highest weight vector and

$$(3.50) \quad V_{n+1} = \text{span} \langle \bar{a}_i^+ \bar{a}_{j_1}^- \dots \bar{a}_{j_l}^- v, v \in V_n \rangle,$$

with arbitrary  $i, j_1, \dots, j_l$ . Equivalently,

$$V_n = \text{span} \langle \bar{a}_{i_1}^+ \dots \bar{a}_{i_n}^+ \bar{a}_{j_1}^- \dots \bar{a}_{j_l}^- v_{1,1} \rangle,$$

with arbitrary numbers  $i_\alpha, j_\beta$  and  $l$ . We note that the decomposition (3.49) is induced from the grading on  $\bar{\mathcal{A}}(1)'$ , which assigns degree 1 to each mode of  $\bar{a}^+(z)$  and degree 0 to each mode of  $\bar{a}^-(z)$ . We note also that this construction applied to the algebra  $\mathcal{A}(1)$  produces exactly the filtration  $F_\bullet$  (see the beginning of the subsection 3.1).

For any  $M$  with an action of an operator  $h$  and  $l \in \mathbb{Z}$  we set

$$M^l = \{v \in M : hv = lv\}$$

( $h$  is a standard generator of the Cartan subalgebra of  $sl(2)$ ).

**Lemma 3.5.** *Let  $l \geq 0$ . If  $l > n$  then  $V_n^l = 0$ . Otherwise*

$$(3.51) \quad \text{ch}V_n^l = \frac{q^{(n-\frac{1}{2})^2}}{(q)_n(q)_{n-l}}.$$

*Proof.* We recall that  $\bar{a}^+(z)$  and  $\bar{a}^-(z)$  for two-dimensional irreducible representation of  $sl(2)$ . Therefore  $V_n^l$  is the linear span of a set of vectors

$$\bar{a}_{i_1}^+ \dots \bar{a}_{i_n}^+ \bar{a}_{j_1}^- \dots \bar{a}_{j_{n-l}}^- v_{1,1}$$

with arbitrary  $i_\alpha, j_\beta$ . This leads to the description of the dual space  $(V_n^l)^*$  as the space of polynomials in variables  $x_1^+, \dots, x_n^+, x_1^-, \dots, x_{n-l}^-$  of the form

$$\left( \prod_{i=1}^n x_i^+ \prod_{j=1}^{n-l} x_j^- \right)^{\frac{1}{4}} \times \left[ \prod_{1 \leq i < j \leq n} (x_i^+ - x_j^+) \prod_{1 \leq i < j \leq n-l} (x_i^- - x_j^-) \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n-l}} (x_i^+ - x_j^-) \right]^{\frac{1}{2}} \times g,$$

where  $g(x_1^+, \dots, x_n^+, x_1^-, \dots, x_{n-l}^-)$  is an arbitrary polynomial symmetric in each group of variables  $\{x_i^+\}_{i=1}^n$  and  $\{x_j^-\}_{j=1}^{n-l}$ . The degree of the product above is equal to  $(n-l/2)^2 + \text{deg } g$ . Lemma is proved. Q.E.D.

Set  $V_n[1] = V_n \cap \tilde{X}_{1,1}^-[1]$ .

**Proposition 3.6.**

$$(3.52) \quad \text{ch}V_n[1] = \sum_{\substack{n_1, n_2, \dots \geq 0 \\ \sum n_i = n}} \frac{q^{\frac{1}{2} \sum_{i,j \geq 1} 2 \min(i,j) n_i n_j + \sum_{i \geq 1} i n_i}}{(q)_{n_1} (q)_{n_2} \dots}.$$

*Proof.* Using the relation  $\text{ch}V_n[1] = \text{ch}V_n^0 - \text{ch}V_{n+1}^2$  and Lemma above we obtain

$$\text{ch}V_n[1] = \frac{q^{n^2}}{(q)_n^2} - \frac{q^{n^2}}{(q)_{n+1}(q)_{n-1}} = \frac{q^{n^2} q^n (1-q)}{(q)_n (q)_{n+1}}.$$

So we need to show that

$$(3.53) \quad \frac{q^{n^2} q^n (1 - q)}{(q)_n (q)_{n+1}} = \sum_{\substack{n_1, n_2, \dots \geq 0 \\ \sum n_i = n}} q^{\frac{1}{2} \sum_{i,j \geq 1} 2 \min(i,j) n_i n_j + \sum_{i \geq 1} i n_i} \frac{1}{(q)_{n_1} (q)_{n_2} \dots}.$$

Instead we prove a more general relation

$$(3.54) \quad \frac{q^{n^2} u^n}{(q)_n (uq)_n} = \sum_{\substack{n_1, n_2, \dots \geq 0 \\ \sum n_i = n}} \frac{q^{\frac{1}{2} \sum_{i,j \geq 1} 2 \min(i,j) n_i n_j} u^{\sum_{i \geq 1} i n_i}}{(q)_{n_1} (q)_{n_2} \dots},$$

where a new variable  $u$  is introduced (see notation  $(x)_n$  in Thm. 1.1). We note that the relation above reduces to (3.53) after the specialization  $u = q$ .

After the change of variables  $m_i = n_{i+1} + n_{i+2} + \dots, i = 1, 2, \dots$  the equation (3.54) becomes

$$(3.55) \quad \frac{q^{n^2} u^n}{(q)_n (uq)_n} = \sum_{n \geq m_1 \geq m_2 \geq \dots \geq 0} \frac{q^{\sum_{i \geq 1} m_i^2} u^{\sum_{i \geq 1} m_i} q^{n^2} u^n}{(q)_{n-m_1} (q)_{m_1-m_2} \dots},$$

or equivalently

$$(3.56) \quad \frac{1}{(uq)_n} = \sum_{n \geq m_1 \geq m_2 \geq \dots \geq 0} q^{\sum_{i \geq 1} m_i^2} u^{\sum_{i \geq 1} m_i} \binom{n}{m_1}_q \binom{m_1}{m_2}_q \dots,$$

where the notation  $\binom{m}{n}_q = \frac{(q)_m}{(q)_n (q)_{m-n}}$  is used for a  $q$ -binomial coefficient.

We prove (3.56) by induction on  $n$ . The case  $n = 1$  is obvious. For general  $n$  we rewrite the right hand side of (3.56) as

$$(3.57) \quad \sum_{m_1=0}^n q^{m_1^2} \binom{n}{m_1}_q u^{m_1} \sum_{m_1 \geq m_2 \geq \dots \geq 0} q^{\sum_{i \geq 2} m_i^2} u^{\sum_{i \geq 2} m_i} \binom{m_1}{m_2}_q \binom{m_2}{m_3}_q \dots$$

Therefore using the induction assumption it is enough to show that

$$(3.58) \quad \frac{1}{(uq)_n} = \sum_{m=0}^n q^{m^2} u^m \binom{n}{m}_q \frac{1}{(uq)_m}.$$

The left hand side is equal to the  $(u, q)$  character of the space of polynomials in commuting variables  $e_i, 1 \leq i \leq n$ , where  $\deg_u e_i = 1$  and  $\deg_q e_i = i$ . We consider the decomposition

$$\mathbb{C}[e_1, \dots, e_n] = \mathbb{C} \cdot 1 \oplus \bigoplus_{m=1}^n \mathbb{C}[e_1, \dots, e_m] \cdot \text{span}\langle e_{i_1} \dots e_{i_m}, m \leq i_1 \leq \dots \leq i_m \leq n \rangle.$$

The  $(u, q)$  character of the right hand side is equal to the right hand side of (3.58). This finishes the proof of the proposition. Q.E.D.

Consider the space  $\bar{X}'_{s,p} \hookrightarrow \bar{X}'_{s,p}$ , which is generated from the highest weight vector with the modes of  $\bar{a}^\pm(z)$  (but not  $\bar{T}^{[i]}(z)$ ). We have a decomposition

$$\bar{X}'_{s,p} = \bigoplus_{r \geq 1} \pi_r \otimes \bar{X}'_{s,p}[r].$$

**Lemma 3.7.** *For any  $p$  the dual space  $(\bar{X}'_{s,p}[1])^*$  is isomorphic to the direct sum over  $n \geq 0$  of spaces of functions of the form*

$$(3.59) \quad (x_1 \dots x_{2n})^{\frac{3p-2s}{4}} \prod_{1 \leq i < j \leq 2n} (x_i - x_j)^{p/2} g(x_1, \dots, x_{2n}),$$

where  $g(x_1, \dots, x_{2n})$  is a polynomial with values in the space  $(\pi_2^{\otimes 2n})^{sl(2)}$ , which satisfy

$$(3.60) \quad \sigma_{i,j} g(\dots, x_j, \dots, x_i, \dots) = g(\dots, x_i, \dots, x_j, \dots),$$

where  $\sigma_{i,j}$  is a transposition acting on  $\pi_2^{\otimes 2n}$  by permuting  $i$ -th and  $j$ -th factors.

*Proof.* We start with the polynomial realization of the dual space  $(\bar{X}'_{s,p})^*$ . Let  $w_+, w_-$  be the standard basis of the 2-dimensional irreducible representation of  $sl(2)$ . For  $\theta \in (\bar{X}'_{s,p})^*$  we set

$$G_\theta(x_1, \dots, x_k) = \sum_{\alpha_i = \pm} \langle \theta | a^{\alpha_1}(x_1) \dots a^{\alpha_k}(x_k) | v_{s,p} \rangle w_{\alpha_1} \otimes \dots \otimes w_{\alpha_k}.$$

This gives a map from  $(\bar{X}'_{s,p})^*$  to the space of polynomials in variables  $x_1, \dots, x_k$  with values in  $\pi_2^{\otimes k}$ . From (3.39) and (3.43) we obtain that the image of this map coincides with the subspace (3.59) with restriction (3.60).

Note that if  $a_{i_1}^{\alpha_1} \dots a_{i_k}^{\alpha_k} v_{s,p}$  belongs to  $\bar{X}'_{s,p}[1]$  then the number of pluses and minuses in the set  $\{\alpha_i\}_{i=1}^k$  coincide. Therefore to obtain the polynomial realization of the space  $(\bar{X}'_{s,p}[1])^*$  one needs to take an even number  $k = 2n$  of variables and the space of polynomials with values in  $(\pi_2^{\otimes 2n})^{sl(2)}$ . Q.E.D.

We consider the decomposition

$$\bar{X}'_{s,p}[1] = \bigoplus_{n \geq 0} (\bar{X}'_{s,p}[1])_n,$$

where  $(\bar{X}'_{s,p}[1])_n$  is a subspace defined by the formula

$$\text{span} \langle \bar{a}_{i_1}^+ \dots \bar{a}_{i_n}^+ \bar{a}_{j_1}^- \dots \bar{a}_{j_n}^- v_{s,p} \rangle \cap \bar{X}'_{s,p}[1]$$

with arbitrary  $i_\alpha, j_\beta$ . We note that in the case  $p = 1$

$$(3.61) \quad (\bar{X}'_{1,1}[1])_n = V_n[1].$$

**Corollary 3.8.**

$$(3.62) \quad \text{ch}(\bar{X}'_{s,p}[1])_n = \sum_{\substack{n_p, n_{p+1}, \dots \geq 0 \\ \sum n_i = n}} \frac{q^{\frac{1}{2} \sum_{i,j \geq 1} 2 \min(i,j) n_i n_j + \sum_{i \geq p} (i-s+1) n_i}}{(q)_{n_1} (q)_{n_2} \dots}.$$

*Proof.* From the Proposition 3.6 and formula (3.61) we obtain our Corollary in the case  $p = 1$ . We now compare the dual space description (3.59) for general  $(s, p)$  and  $s = p = 1$ . The difference of the degrees is given by the formula

$$\begin{aligned} & \text{deg} \left( \prod_{i=1}^{2n} x_i^{\frac{3p-2s}{4}} \prod_{1 \leq i < j \leq 2n} (x_i - x_j)^{p/2} \right) \\ & - \text{deg} \left( \prod_{i=1}^{2n} x_i^{\frac{1}{4}} \prod_{1 \leq i < j \leq 2n} (x_i - x_j)^{1/2} \right) = n^2(p-1) + n(p-s). \end{aligned}$$

This gives

$$\begin{aligned} & \text{ch}(\bar{X}'_{s,p}[1])_n \\ & = q^{n^2(p-1) + n(p-s)} \sum_{\substack{n_1, n_2, \dots \geq 0 \\ \sum n_i = n}} \frac{q^{\frac{1}{2} \sum_{i,j \geq 1} 2 \min(i,j) n_i n_j + \sum_{i \geq 1} i n_i}}{(q)_{n_1} (q)_{n_2} \dots} \\ & = \sum_{\substack{n_1, n_2, \dots \geq 0 \\ \sum n_i = n}} q^{(p-1)(\sum_{i \geq 1} n_i)^2 + (p-s)(\sum_{i \geq 1} n_i)} \frac{q^{\frac{1}{2} \sum_{i,j \geq 1} 2 \min(i,j) n_i n_j + \sum_{i \geq p} i n_i}}{(q)_{n_1} (q)_{n_2} \dots}. \end{aligned}$$

We now redefine  $n_i \rightarrow n_{p-1+i}$ . Then the formula above gives the right hand side of (3.62). Q.E.D.



**Lemma 3.9.** *The character of  $\mathcal{M}_{1,s;p}$  is given by the Gordon type formula*

$$(3.63) \quad \text{ch}\mathcal{M}_{1,s;p} = \sum_{n_1, n_2, \dots \geq 0} \frac{q^{\frac{1}{2} \sum_{1 \leq i \leq j} 2 \min(i,j) n_i n_j + n_s + 2n_{s+1} + \dots}}{(q)_{n_1} (q)_{n_2} \dots},$$

where the summation goes over all such collections  $n_1, n_2, \dots$  that only finite number of  $n_i \neq 0$ .

*Proof.* We recall that  $\mathcal{M}_{1,s;p}$  is a quotient of the Verma module  $V_{1,s;p}$  by a submodule generated with the a singular vector on the level  $s$  (see [16]). Introduce a filtration  $H_\bullet$  on the Verma module  $V_{1,s;p}$  defined as follows:  $H_0$  is spanned by the highest weight vector and

$$H_{l+1} = \text{span}\{L_n v, v \in H_l, n < 0\} + H_l.$$

In the corresponding adjoint graded space the images of the operators  $L_n$  commute with each other; we denote these operators as  $L_n^{ab}$ . This gives

$$(3.64) \quad \text{ch}\mathcal{M}_{1,s;p} = \text{ch}\mathbb{C}[L_{-1}^{ab}, L_{-2}^{ab}, \dots] / \{p(L_{-i}^{ab})\},$$

where  $\{p(L_{-i}^{ab})\}$  is an ideal generated by some degree  $s$  polynomial  $p(L_{-i}^{ab})$  (we put  $\text{deg } L_i^{ab} = -i$ ). Evidently, the character of this quotient is independent of  $p(L_{-i}^{ab})$  (only the degree  $s$  matters). We fix  $p$  to be equal to  $(L_{-1}^{ab})^s$ .

Let  $T^{ab}(z) = L_{-1}^{ab} + zL_{-2}^{ab} + \dots$ . For  $k \geq s$  let  $R_k$  be a following ring:

$$R_k = \mathbb{C}[L_{-i}^{ab}] / \{T^{ab}(z)^{k+1}, (L_{-1}^{ab})^s\}.$$

Then for the character of  $R_k$  one has a formula

$$\text{ch}R_k = \sum_{n_1, \dots, n_k \geq 0} \frac{q^{\frac{1}{2} \sum_{1 \leq i \leq j \leq k} 2 \min(i,j) n_i n_j + \sum_{i \geq s} (i-s+1) n_i}}{(q)_{n_1} \dots (q)_{n_k}}$$

(see [8]). Obviously

$$\text{ch}\mathcal{M}_{1,s;p} = \text{ch}\mathbb{C}[L_{-1}^{ab}, L_{-2}^{ab}, \dots] / \{(L_{-1}^{ab})^s\} = \lim_{k \rightarrow \infty} \text{ch}R_k.$$

Lemma is proved.

Q.E.D.

**Lemma 3.10.** *The dual space  $(\tilde{X}'_{s,p}[1])^*$  is isomorphic to the direct sum over  $n_1, \dots, n_{p-1}, n \geq 0$  of the  $(\pi_2^{\otimes 2n})^{s\ell(2)}$  valued polynomials*

$$H(x_1^1, \dots, x_{n_1}^1, \dots, x_1^{p-1}, \dots, x_{n_{p-1}}^{p-1}, x_1, \dots, x_{2n})$$

of the form  $Y \cdot g$ , where  $Y$  is a function of the form

$$\begin{aligned}
 (3.65) \quad & \prod_{1 \leq i \leq 2n} x_i^{\frac{3p-2s}{4}} \prod_{1 \leq i < j \leq 2n} (x_i - x_j)^{p/2} \prod_{\substack{1 \leq i \leq 2n \\ 1 \leq \alpha \leq p-1}} \prod_{1 \leq j \leq n_\alpha} (x_i - x_j^\alpha)^{\alpha \times} \\
 & \prod_{\alpha=1}^{s-1} \left( \prod_{1 \leq i \leq n_\alpha} x_i^\alpha \right)^\alpha \prod_{\alpha=s}^{p-1} \left( \prod_{1 \leq i \leq n_\alpha} x_i^\alpha \right)^{2\alpha-s+1} \prod_{1 \leq \alpha < \beta < p-1} \prod_{1 \leq i < j \leq n_\alpha} (x_i^\alpha - x_j^\beta)^{2\alpha} \\
 & \prod_{1 \leq \alpha < \beta \leq p-1} \prod_{\substack{1 \leq i \leq n_\alpha \\ 1 \leq j \leq n_\beta}} (x_i^\alpha - x_j^\beta)^{2 \min(\alpha, \beta)}
 \end{aligned}$$

and  $g$  is  $(\pi_2^{\otimes 2n})^{s\ell(2)}$  valued polynomial symmetric in each group of variables

$$(x_1^\alpha, \dots, x_{n_\alpha}^\alpha), \alpha = 1, \dots, p-1; \quad (x_1, \dots, x_{2n}).$$

In addition  $g$  satisfies the condition (3.60) in variables  $x_1, \dots, x_{2n}$ .

*Proof.* Recall that the currents  $\bar{T}^{[i]}(z)$  commute with the action of  $sl(2)$ . Therefore we obtain

$$(3.66) \quad \bar{X}'_{s,p}[1] = \mathbb{C}[\bar{T}_j^{[i]}] \cdot (\bar{X}'_{s,p}[1]),$$

i.e. the space of invariants  $\bar{X}'_{s,p}[1]$  can be obtained by applying all polynomials in modes of the currents  $\bar{T}^{[i]}(z)$  to vectors of  $\bar{X}'_{s,p}[1]$ . The formula (3.65) is a dual version of (3.66). Namely the first line of (3.65) comes from the polynomial realization of  $(\bar{X}'_{s,p}[1])^*$  (see Lemma 3.7). The second line of (3.65) describes the interaction between  $\bar{a}^\pm(z)$  and  $\bar{T}^{[i]}(z)$  in the algebra  $\bar{A}(p)'$ . Finally the last two lines of (3.65) comes from the polynomial realization of the dual space of the part of  $\bar{X}'_{s,p}[1]$  generated by modes of  $\bar{T}^{[i]}(z)$ . Q.E.D.

**Corollary 3.11.**

$$\text{ch} \bar{X}'_{s,p}[1] = \sum_{n_p, n_{p+1}, \dots \geq 0} \frac{q^{\frac{1}{2} \sum_{i,j \geq 1} 2 \min(i,j) n_i n_j + \sum_{i \geq s} (i-s+1) n_i}}{(q)_{n_1} (q)_{n_2} \dots},$$

where the summation goes over all such collections  $n_p, n_{p+1}, \dots$  that only finite number of  $n_i \neq 0$ .

*Proof.* We note that because of the Corollary 3.8 the character of the space (3.65) is given by a sum over  $n_1, \dots, n_{p-1}, n \geq 0$  of the terms

$$\left[ \sum_{n_p+n_{p+1}+\dots+n} \frac{q^{\frac{1}{2} \sum_{i,j \geq p} 2 \min(i,j) n_i n_j + \sum_{i \geq p} (i-s+1) n_i}}{(q)_{n_1} (q)_{n_2} \dots} \right] q^{\sum_{\alpha=1}^{p-1} 2\alpha n_\alpha} \times \frac{q^{\frac{1}{2} \sum_{1 \leq i,j \leq p-1} 2 \min(i,j) n_i n_j + \sum_{s \leq i \leq p-1} (i-s+1) n_i}}{(q)_{n_1} \dots (q)_{n_{p-1}}},$$

where the first line is the character of the first line of (3.65), the second line is the character of the second line of (3.65) and the third line is the character of the last two lines of (3.65). Rewriting the formula above in terms of  $n_i, i > 0$  we obtain our Corollary. Q.E.D.

**Proposition 3.12.**  $\text{ch} \tilde{\mathcal{X}}'_{s,p}[1] = \text{ch} \mathcal{M}_{1,s;p}$ .

*Proof.* Follows from Lemma 3.9 and Corollary 3.11. Q.E.D.

### 3.3. The general case

We recall the decomposition

$$\mathcal{X}_{s,p} = \bigoplus_{r \geq 1} \pi_r \otimes \mathcal{X}_{s,p}[r].$$

Our goal is to show that

$$(3.67) \quad \mathcal{X}_{s,p}[r] \simeq \tilde{\mathcal{X}}'_{s,p}[r].$$

We first recall the surjection (3.45)

$$\beta_{s,p} : \tilde{\mathcal{X}}'_{s,p} \rightarrow \mathcal{X}_{s,p},$$

which is a homomorphism of  $sl(2)$  modules. This gives a surjection

$$\beta_{s,p}[r] : \tilde{\mathcal{X}}'_{s,p}[r] \rightarrow \mathcal{X}_{s,p}[r].$$

Suppose (3.67) doesn't satisfy. Then  $\beta_{s,p}$  is not an embedding. From the previous section we know that  $\beta_{s,p}[1]$  is an isomorphism. Denote by  $K_r$  the kernel of  $\beta_{s,p}[r]$ . Let  $r$  be a minimal number such that  $K_r$  is not trivial. Fix a vector  $u$  which is a lowest weight vector of some finite-dimensional  $sl(2)$  module  $M \simeq \pi_r \subset K_r$ . We note that for any  $n \in \mathbb{Z}$  the space  $\langle \bar{a}_n^\pm v \rangle$  spanned by  $\bar{a}_n^\pm v$  with  $v \in M$  can be embedded (as  $sl(2)$  module) to  $M \otimes \pi_2$ . In addition  $\langle \bar{a}_n^\pm v \rangle$  is a subspace of  $K_{r+1} \otimes \pi_{r+1} \oplus K_{r-1} \otimes \pi_{r-1}$  because  $sl(2)$  acts on  $\bar{a}_n^\pm$  as on two-dimensional irreducible representation. Because of  $K_{r-1} = 0$ , we obtain that for any  $n \in \mathbb{Z}$

$$(3.68) \quad r \bar{a}_n^- u + \bar{a}_n^+ e u = 0.$$

In fact, the condition  $K_{r-1} = 0$  means that the linear combination  $\alpha \bar{a}_n^- u + \beta \bar{a}_n^+ eu$  vanishes whenever

$$f(\alpha \bar{a}_n^- u + \beta \bar{a}_n^+ eu) = 0.$$

Thus, (3.68) follows from  $fu = 0$  and  $hu = ru$ .

In the following Proposition we show that (3.68) can not be satisfied for all  $n$ .

**Proposition 3.13.** *Let  $u \in \bar{X}'_{s,p}$  be a nonzero vector satisfying  $hu = ru$  with  $r > 0$ . Then there exists  $n \in \mathbb{Z}$  such that  $r\bar{a}_n^- u + \bar{a}_n^+ eu \neq 0$ .*

*Proof.* We use the vertex operator realization of  $\bar{X}'_{s,p}$ . Namely, we consider the space  $\mathfrak{h}$  with a fixed nondegenerate form  $(\cdot, \cdot)$  and an orthogonal basis  $e_1, \dots, e_{p+2}$  such that

$$(e_1, e_1) = \dots = (e_{p+1}, e_{p+1}) = 1, \quad (e_{p+2}, e_{p+2}) = -1.$$

Let  $v_+, v_-, v_1, \dots, v_{p-1} \in \mathbb{R}^{p+2}$  be a set of linearly independent vectors with

$$(v_+, v_+) = (v_+, v_-) = (v_-, v_-) = \frac{p}{2}, \quad (v_\pm, v_i) = i, \quad (v_i, v_j) = 2 \min(i, j).$$

For example, one can fix

$$\begin{aligned} v_i &= \sqrt{2}(e_1 + \dots + e_i), \quad i = 1, \dots, p-1, \\ v_+ &= \frac{1}{\sqrt{2}}(e_1 + \dots + e_{p-1}) + e_p + \frac{1}{\sqrt{2}}e_{p+2}, \\ v_- &= \frac{1}{\sqrt{2}}(e_1 + \dots + e_{p-1}) + e_{p+1} - \frac{1}{\sqrt{2}}e_{p+2}. \end{aligned}$$

Let

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

be the multi-dimensional Heisenberg algebra with the bracket

$$[\alpha \otimes t^i, \beta \otimes t^j] = i\delta_{i,-j}(\alpha, \beta)K, \quad [K, \alpha \otimes t^i] = 0, \quad \alpha, \beta \in \mathfrak{h}.$$

For  $\alpha \in \mathfrak{h}$ , let  $\pi_\alpha$  be the Fock module with highest-weight  $\alpha$ . This module is generated from the highest weight vector  $|\alpha\rangle$  such that

$$(\beta \otimes t^n)|\alpha\rangle = 0, \quad n > 0; \quad (\beta \otimes 1)|\alpha\rangle = (\beta, \alpha)|\alpha\rangle; \quad K|\alpha\rangle = |\alpha\rangle.$$

The  $q$ -degree on  $\pi_\alpha$  is defined by

$$(3.69) \quad \deg_q |\alpha\rangle = \frac{(\alpha, \alpha)}{2}, \quad \deg_q(\beta \otimes t^n) = -n.$$

We also introduce the vertex operators  $\Gamma_\alpha(z) = :e^{\alpha\varphi(z)}:$  acting from  $\pi_\beta$  to  $\pi_{\alpha+\beta}$  with the Fourier decomposition

$$\Gamma_\alpha(z) = \sum_{n \in \mathbb{Z}} \Gamma_\alpha(n) z^{-n - (\alpha, \alpha)/2}.$$

We need two properties of vertex operators:

$$(3.70) \quad [\alpha \otimes t^n, \Gamma_\beta(z)] = (\alpha, \beta) z^n \Gamma_\beta(z),$$

$$(3.71) \quad \Gamma_\alpha(z) \Gamma_\beta(w) \sim (z - w)^{(\alpha, \beta)}.$$

We also need the following statement. There exists an element  $\alpha_s \in \mathfrak{h}$  such that

$$(3.72) \quad \Gamma_{v_\pm}(j) |\alpha_s\rangle = 0, \quad j > -\frac{3p - 2s}{4},$$

$$(3.73) \quad \Gamma_{v_n}(j) |\alpha_s\rangle = 0, \quad j > \begin{cases} -n, & n < s, \\ -2n + s - 1, & n \geq s. \end{cases}$$

We let  $Vert_s$  denote the space generated from the vector  $|\alpha_s\rangle$  with all modes of the vertex operators  $\Gamma_{v_\pm}(z), \Gamma_{v_n}(z)$ . Comparing the definition of  $\bar{X}'_{s,p}$  and formulas (3.72), (3.73), (3.71), we obtain that the proof of the Proposition follows from the Lemma below. Q.E.D.

**Lemma 3.14.** *Let  $u_1, u_2$  be two vectors from some Fock modules. Suppose*

$$(3.74) \quad r\Gamma_{v_-}(n)u_1 + \Gamma_{v_+}(n)u_2 = 0$$

for all  $n$ . Then  $u_1 = u_2 = 0$ .

*Proof.* To prove our lemma we apply an operator  $\alpha \otimes t^i$  to both sides of (3.74). Note that for  $i$  big enough

$$(\alpha \otimes t^i)u_1 = (\alpha \otimes t^i)u_2 = 0.$$

Now using (3.70) we obtain from (3.74) that for all  $n \in \mathbb{Z}$  and all  $\alpha \in \mathfrak{h}$

$$(\alpha, v_-)r\Gamma_{v_-}(n+i)u_1 + (\alpha, v_+)\Gamma_{v_+}(n+i)u_2 = 0.$$

Because of the nondegeneracy of  $(\cdot, \cdot)$  we conclude that for all  $n$

$$\Gamma_{v_-}(n)u_1 = \Gamma_{v_+}(n)u_2 = 0.$$

This gives  $u_1 = u_2 = 0$ . Lemma is proved. Q.E.D.

**Proposition 3.15.**  $\text{ch}\bar{X}_{s,p}[r] = \text{ch}\bar{X}'_{s,p}[r] = \text{ch}\bar{X}'_{s,p}[r]$ .

**Corollary 3.16.**  $\text{ch}\mathcal{X}_{s,p} = \text{ch}\bar{\mathcal{X}}'_{s,p}$  and therefore the statement of the Theorem 1.1 is satisfied.

*Proof.* Follows from Lemma 3.4 and Proposition 3.15. Q.E.D.

#### §4. Conclusion

From the results of the paper we can obtain the following description of  $\mathcal{A}(p)$  irreducible representations  $\mathcal{X}_{s,p}$ . We know that  $\mathcal{X}_{s,p}$  is generated from the vacuum vector  $|s,p\rangle$  satisfying the defining relations (2.29). This means that  $\mathcal{X}_{s,p}$  is induced from trivial representation of the subalgebra generated by  $a_i^\pm$  with  $i = \frac{3p-2s}{4} + n$ ,  $n \in \mathbb{N}$ . In [9], fermionic formulas for irreducible representations of 1-dimensional lattice vertex operator algebras and fermionic formulas for coinvariants in irreducible representations with respect to different subalgebras were obtained. These give graded (or quantum) version of the Verlinde formula for 1-dimensional lattice vertex operator algebras. In this paper generalization of some results of [9] are obtained.

The  $\mathcal{A}(p)$  representation category is equivalent to the representation category  $\mathfrak{C}(p)$  of the small quantum  $sl(2)$  group  $U_q(sl(2))$  with  $q = e^{\frac{i\pi}{p}}$ . This group differs from the quantum group  $\bar{U}_q sl(2)$  from [4] by the relation  $K^p = 1$ . The coinvariants in  $\mathcal{A}(p)$  irreducible representations can be described in terms of  $\mathfrak{C}(p)$ . Therefore the next natural step of investigations can be obtaining of fermionic formulas for coinvariants, which gives  $q$ -versions for multiplicities of a given indecomposable representation in a tensor product of irreducible representations.

The close related to the previous direction of investigations is a monomial basis constructed in terms of  $a^\pm(z)$ -modes like in [7]. These bases allows to establish a contact with some RSOS-like models as in [15].

For applications to percolation type models, a generalization of the results of this paper to  $(p, p')$  models [5] and especially to  $(2, 3)$  model would be very useful.

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