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The third smallest Salem number in automorphisms of K3 surfaces

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Abstract.

We realize the logarithm of the third smallest known Salem number as the topological entropy of a K3 surface automorphism with a Siegel disk and a pointwise fixed curve at the same time. We also show that the logarithm of the Lehmer number, the smallest known Salem number, is not realizable as the topological entropy of any Enriques surface automorphism. These results are entirely inspired by McMullen's works and Mathematica programs.

§1. Introduction

The aim of this note is to remark the following two new phenomena in complex dymanics of automorphisms of compact complex surfaces. These results and their proofs are entirely inspired by impressive works of McMullen [Mc02-1], [Mc02-2], [Mc07], [GM02] and Mathematica programs.

Theorem 1.1. *There is a pair (S, g) of a complex K3 surface S and its automorphism g such that:*

(1) S contains 8 smooth rational curves C_k $(0 \leq k \leq 7)$ whose dual graph forms the Dynkin diagram $E_8(-1)$ and contains no other irreducible complete curve. In particular, S is of algebraic dimension 0;

(2) The topological entropy $h(g)$ is the logarithm of the third smallest known Salem number

$$h(g) = \log 1.200026523...;$$

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(3) The fixed point set S^g consists of one smooth rational curve (in $\cup_{k=0}^{7} C_k$) and 8 isolated points, say Q_i $(1 \leq i \leq 7)$ and Q. The 7 points Q_i are in $\cup_{k=o}^{7} C_k$, but Q is not in $\cup_{k=0}^{7} C_k$;

(4) g has a Siegel disk at Q and g has no Siegel disk at any other point; and

(5) $\mathrm{Aut}\, S = \langle g \rangle \simeq \mathbb{Z}$.

Theorem 1.2. *There does not exist a pair (S, g) of a complex Enriques surface S and its automorphism g such that*

$$h(g) = \log 1.17628081\ldots.$$

Here, the right hand side is the logarithm of the Lehmer number, i.e., the logarithm of the smallest known Salem number.

We shall explain the terms in Theorems (1.1), (1.2) in Section 2. In the rest of the introduction, we shall remark a few differences between our results and some of preceding known results.

In [Mc02-2], McMullen constructed the first examples of surface automorphisms with Siegel disks. They are K3 surface automorphisms arising from certain Salem numbers of degree 22, including the 9-th smallest known one. In his construction, the resulting K3 surfaces are of Picard number 0. So, they have no complete curve, whence, no pointwise fixed curve as well. Theorem (1.1) tells us that it is also possible to have both a Siegel disk and a pointwise fixed curve, necessarily smooth rational, at the same time.

Let S be a rational surface obtained by blowing up at n points on $\mathbb{P}^2$ and g be an automorphism of S. Then, $g^*(K_S) = K_S$ and g naturally acts on the orthogonal complement $K_S^{\perp}$ of the canonical class in $H^2(S, \mathbb{Z})$. The lattice $K_S^{\perp}$ is isomorphic to the lattice $E_n(-1)$, i.e., the lattice represented by the Dynkin diagram with n verices s_k $(0 \leq k \leq n-1)$ of self-intersection -2 such that $n-1$ vertices $s_1, s_2, \cdots, s_{n-1}$ form Dynkin diagram of type $A_{n-1}(-1)$ in this order and the remaining vertex s_0 is joined to only the vertex s_3 by a simple line. (See [Mc07], Section 2, Figure 2.) The lattice $E_n(-1)$ is of signature $(1, n-1)$ when $n \geq 10$. Then, g naturally induces an orthogonal action $g^*|E_n(-1)$ (after fixing a marking). By Nagata [Na61] (see also [Mc07], Theorem (12.4)), $g^*|E_n(-1)$ is an element of the Weyl group $W(E_n(-1))$, i.e., the group generated by the reflections r_k $(0 \leq k \leq n-1)$ corresponding to the vertices s_k. The Weyl group $W(E_n(-1))$ has a special conjugacy class called the *Coxeter class.* It is the conjugacy class of the product (in any order in this case) of the reflections $\Pi_{k=0}^{n-1} r_k$. McMullen ([Mc07], Theorem (1.1)) shows that, when $n \geq 10$, the Coxeter class is realized *geometrically* by a rational surface automorphism. That is, $\Pi_{k=0}^{n-1} r_k =$

$g^*|E_n(-1)$ (under a suitable marking) for an automorphism g of S with suitably chosen n blown up points. When $n = 10$, i.e., for $E_{10}(-1)$, the characteristic polynomial of the Coxeter class is exactly the Lehmer polynimial, i.e., the minimal polynomial of the Lehmer number over $\mathbb{Z}$. In this way, McMullen realized the logarithm of the Lehmer number as the topological entropy of some rational surface automorphisms with $K_S^{\perp} \simeq E_{10}(-1)$. Note that the Lehmer number is the smallest known Salem number. See [FGR99] and the home page quoted there, for the list of the smallest 47 known Salem numbers. Being also based on his preceding result [Mc02-1], Theorem (1.1), McMullen ([Mc07], Theorem (A.1)) also shows that the logarithm of the Lehmer number is in fact the minimal positive entropy of automorphisms of complex surfaces. So, the Lehmer number plays a very special role in automorphisms of compact complex surfaces.

On the other hand, lattice $E_{10}(-1)$ is also isomorphic to the free part of $H^2(S, \mathbb{Z})$ of an Enriques surface S. So, it is natural to ask if the logarithm of the Lehmer number can also be realized as the topological entropy of an Enriques surface automorphism or not. Theorem (1.2) says that it is *not*. This may sound negative. However, I believe that such an impossibility result is also of its own interest.

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§2. Salem numbers in automorphisms of compact Kähler surfaces

In this section, we quickly review the terms in our Theorems (1.1), (1.2). As nothing is new, those who are familiar with these terms should skip this section.

(i) Salem number. Let us start by the definition.

Definition 2.1. A Salem polynomial is a monic irreducible reciprocal polynomial $\varphi(x)$ in $\mathbb{Z}[x]$ such that $\varphi(x) = 0$ has exactly two real roots $\alpha > 1$ and $1/\alpha$ off the unit circle

$$S^1 := \{z \in \mathbb{C} \,|\, |z| = 1\}.$$

It is then of even degree. A Salem number is the unique real root $\alpha > 1$. In other words, a Salem number of degree $2n$ is a real algebraic integer $\alpha > 1$ whose Galois conjugates consist of $1/\alpha$ and $2n-2$ complex numbers on S^1.

Salem numbers of degree 2 are $(m + \sqrt{m^2-4})/2$ $(3 \leq m \in \mathbb{Z})$. For a given integer $n > 0$, there are infinitely many Salem numbers of degree $\leq 2n$ ([GM02], Theorem (1.6)). On the other hand, Salem numbers with bounded degree and bounded (Euclidean) norm are finite. That is, for given $n > 0$ and $N > 0$, Salem numbers α such that $\deg \alpha < 2n$ and $|\alpha| < N$ are finite. In fact, the elementary symmetric functions of the Galois conjugates of α are then bounded, so that the Salem polynomials of such Salem numbers are finite. So, it is in principle possible to list up all the Salem numbers with explicitly boundded norm and degree. In fact, there is a list of all Salem numbers of degree ≤ 40 and norm < 1.3 in the home page quoted by [FGR99], Page 168. The smallest five ones (of degree ≤ 40) are:

$$\alpha_{10} = 1.176280\ldots,\ \alpha_{18} = 1.188368\ldots,$$

$$\alpha_{14} = 1.200026\ldots,\ A_{14} = 1.202616\ldots,$$

$$A_{10} = 1.216391\ldots.$$

Their Salem polynomials are

$$\varphi_{10}(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1,$$

$$\varphi_{18}(x) = x^{18} - x^{17} + x^{16} - x^{15} - x^{12} + x^{11} - x^{10} + x^9 - x^8 + x^7 - x^6 - x^3 + x^2 - x + 1,$$

$$\varphi_{14}(x) = x^{14} - x^{11} - x^{10} + x^7 - x^4 - x^3 + 1,$$

$$\Phi_{14}(x) = x^{14} - x^{12} - x^7 - x^2 + 1,$$

$$\Phi_{10}(x) = x^{10} - x^6 - x^5 - x^4 + 1.$$

The smallest Salem number α_{10} (in this range) is called the *Lehmer number*. This number is discovered by Lehmer [Le33], Page 477. Lehmer stated there that "*We have not made an examination of all* 10*th degree symmetric polynomials but a rather intensive search has failed to reveal*

a better polynomial than $\varphi_{10}(x)$. *All efforts to find a better equation of degree* 12 *and* 14 *have been unsuccessful.*" Since then, it is conjectured that the Lehmer number is the smallest among all the Salem numbers. However, it is neither proved nor disproved so far. Also in this view, the result of McMullen [Mc07], Theorem (A.1) (quoted in the introduction) is very impressive. See also [GH01] and [Mc02-1] for other aspects.

(ii) Topological entropy. Let X be a compact metric space with distance function d. Let g be a continuous self map of X. To make statement simple, we assume that g is *surjective.* The *toplogical entropy* is a measure of "how fast two orbits $\{g^k(x)\}_{k\geq 0}$, $\{g^k(y)\}_{k\geq 0}$ spread out when $k \to \infty$". For the precise definition, we introduce a new kind of distance $d_{g,n}$ for each $n \in \mathbb{Z}_{>0}$ ([KH95], Page 108):

$$d_{g,n}(x,y) := \max\{d(g^k(x), g^k(y)) \,|\, 0 \leq k \leq n-1\}\,.$$

$d_{g,n}(x,y)$ measures a distance between the orbit segments $\{g^k(x)\}_{k=0}^{n-1}$, $\{g^k(y)\}_{k=0}^{n-1}$. Let $B_{g,n}(x,\epsilon)$ be the open ball with center x and of radius ϵ with respect to $d_{g,n}$. We call such a ball (ϵ,n)-ball. Let $S(g,\epsilon,n)$ be the *minimal* number of (ϵ,n)-balls that cover X. Then, "$S(g,\epsilon,n)$ being larger" means that the orbit segments of two close points (uniformly with respect to the original distance d) spread faster in the range $0 \leq k \leq n-1$.

The *topological entropy* of g is the following value ([KH95], Page 108, formula (3.1.10)):

$$h(g) := \lim_{\epsilon\to 0} \operatorname{limsup}_{n\to\infty} \frac{\log S(g,\epsilon,n)}{n}\,.$$

It is shown there that $h(g)$ does not depend on the choice of the distance d giving rise to the same topology on X ([KH95], Page 109, Proposition (3.1.2)). By definition, $h(g) = 0$ if g is an automorphism of finite order.

Let E be an elliptic curve and $A = E \times E$ be the product abelian surface. By definition, $h(t_a) = 0$ for any translation automorphism $t_a(x) = x+a$ $(a \in A)$. Let M be a matrix in $M_2(\mathbb{Z})$ such that $\det M \neq 0$. Then M gives rise to the endmorphism g of A: $g(x) = Mx$. Let α and β be the eigenvalues of M and reorder them so that $|\alpha| \geq |\beta|$. Then, according to the three cases

$$(i)\,|\alpha| \geq |\beta| \geq 1\,,\ (ii)\,|\alpha| \geq 1 \geq |\beta|\,,\ \text{and}\ (iii)\,1 \geq |\alpha| \geq |\beta|\,,$$

we have

$$(i)\,h(g) = \log|\alpha\beta|^2\,,\ (ii)\,h(g) = \log|\alpha|^2\,,\ \text{and}\ (iii)\,h(g) = \log 1 = 0\,.$$

The reason is as follows. First, choose sufficiently small $\epsilon > 0$ and cover A by N mutually disjoint complex 2-dimensional ϵ-cubes (in the product metric) that are "parallel to" the two complex eigenvectors of M (Here we ignore small part of A in covering). Next, divide each of N ϵ-cubes into mutually disjoint (ϵ, n)-cubes with respect to the new distance $d_{g,n}$. Then, according to the cases (i), (ii), (iii), the numbers of the resulting (ϵ, n)-cubes are approximately $N \cdot |\alpha\beta|^{2(n-1)}$, $N \cdot |\alpha|^{2(n-1)}$, and N respectively. This implies the result. (See [KH95], Pages 121–123, for more precise calculations). The values $h(g)$ above coincide with the logarithm of the spectral radius of the action of $g^* | \oplus_{k=0}^{2} H^{2k}(A, \mathbb{Z})$. However, this is not accidental:

Theorem 2.2. *Let X be a compact Kähler manifold of dimension n and let $g : X \longrightarrow X$ be a holomorphic surjective self map of X. Then*

$$h(g) = \log \rho(g^* | \oplus_{k=0}^{n} H^{2k}(X, \mathbb{Z}))\,.$$

Here $\rho(g^* | \oplus_{k=0}^{n} H^{2k}(X, \mathbb{Z}))$ is the spectral radius of the action of g^* on the total cohomology ring of even degree.

This is a fundamental theorem often attributed to Gromov and Yomdin. The explicit statement with full proof (using Yomdin's result) is found in Friedland's paper [Fr95], Theorem (2.1). Note that, in the proof, we only need the estimate by the spectral radius on the cohomology group of even degree. See also [DS04], Pages 315–316, for further discussions. As an immediate consequence, we obtain the following important

Corollary 2.3. *(1) $h(g)$ is the logarithm of an algebraic integer.*
(2) $h(g^n) = nh(g)$ for a positive integer n.

(iii) Toplogogical entropy of a surface automorphism. If $\dim X = 1$, then by Theorem (2.2), $h(g) = \log(\deg g)$ and it is not so informative. Let us consider the case where X is a compact Kähler surface and g is an *automorphism* of X.

The first important fact is the following result due to Cantat ([Ca99], Proposition 1 and its proof):

Theorem 2.4. *Let X be a compact Kähler surface and g be an automorphism of X. Then,*
(1) $h(g) = \log \rho(g^* | H^{1,1}(S, \mathbb{R}))$. Here $\rho(g^* | H^{1,1}(S, \mathbb{R}))$ is the spectral radius of $g^* | H^{1,1}(S, \mathbb{R})$.
(2) Assume that $h(g) > 0$. Then X is isomorphic to either:
(i) a rational surface with $b_2(X) \geq 11$;

(ii) a 2-dimensional complex torus (or its blow up)
(iii) a K3 surface (or its blow up); or
(iv) an Enriques surface (or its blow up).

See eg. [BHPV04], Pages 244–246, for the classification of compact complex surfaces and the definition of surfaces above. Non-minimality in (i) is definitely essential. But non-minimal surfaces in (ii)–(iv) are not so essential. In fact, if X is not minimal in the class (ii)–(iv), then g descends to an automorphism $\overline{g}$ of the minimal model (See eg. [BHPV04], Page 99, Claim). Moreover, $h(g) = h(\overline{g})$. This follows from the fact that the exceptional set forms a negative definite sublattice of $H^2(X, \mathbb{Z})$. So, surfaces having interesting automorphisms (in the view of topological entropy) are non-miminal rational surfaces, 2-dimensional complex tori, K3 surfaces and Enriques surfaces.

Salem numbers naturally appear in the study of automorphisms of complex surfaces:

Theorem 2.5. *Let X be a compact Kähler surface and g be an automorphism of X. Then the characteristic polynomial of $g^*|H^2(X, \mathbb{Z})$ is the product of cyclotomic polynomials and Salem polynomials. In the product, there are at most one Salem factor (counted with multiplicities) and possibly no cyclotomic factor or no Salem factor. In particular, if $h(g) > 0$, then Salem factor appears in the product and $h(g)$ is the logarithm of that Salem number.*

This is due to McMullen [Mc02-2], Theorem (3.2). The argument there is given for K3 surface automorphisms. But it is easily generalized to automorphisms of arbitrary compact Kähler surfaces.

Proof. Consider the real Hodge decomposition of $H^2(X, \mathbb{Z})$:

$$H^2(X, \mathbb{R}) = H^{1,1}_{\mathbb{R}}(X) \oplus V\,.$$

Here V is a vector subspace of $H^2(X, \mathbb{R})$ such that

$$V \otimes \mathbb{C} = H^{2,0}(X) \oplus H^{0,2}(X)\,.$$

The Kähler cone $\mathcal{K}(X)$ forms a strictly convex open cone of $H^{1,1}_{\mathbb{R}}(X)$. Moreover, $H^{1,1}_{\mathbb{R}}(X)$ is of signature $(1, h^{1,1}(X) - 1)$ and V is positive definite (see eg. [BHPV04], Page 143, Theorem (2.14)). As g is an automorphism, we have $g^*(V) = V$, $g^*(H^{1,1}_{\mathbb{R}}(X)) = H^{1,1}_{\mathbb{R}}(X)$ and $g^*(\mathcal{K}(X)) = \mathcal{K}(X)$. As V is positive definite, the eigenvalues of $g^*|V$ are of absolute value 1. As $g^* \in \mathrm{O}(H^2(X, \mathbb{R}))$, it follows that $\det g^*|H^2(X, \mathbb{R}) = \pm 1$. Thus, the product of the eigenvalues of $g^*|H^{1,1}_{\mathbb{R}}(X)$ is of absolute value 1

as well. On the other hand, as $g^*|H^{1,1}_{\mathbb{R}}(X)$ preserves the strictly convex open cone $\mathcal{K}(X)$, the spectral radius of $g^*|H^{1,1}_{\mathbb{R}}(X)$ is given by a real eigenvalue, say $\alpha > 0$, of $g^*|H^{1,1}_{\mathbb{R}}(X)$ with eigenvector, say η, in $\overline{\mathcal{K}(X)}$ (the closure of the Kähler cone). This is due to the (generalized) Perron–Frobenious theorem ([Bi67], Page 274, Theorem). As the product of the eigenvalues is of absolute value 1, it follows that $\alpha \geq 1$. If $\alpha = 1$, then all the eigenvalues of $g^*|H^{1,1}_{\mathbb{R}}(X)$ is also of absolute value 1. Hence, so are the eigenvalues of $g^*|H^2(X,\mathbb{R})$. As $g^*|H^2(X,\mathbb{R})$ is defined over $H^2(X,\mathbb{Z})$, all the eigenvalues are then roots of unity by Kronecker's theorem. Next consider the case where $\alpha > 1$. Consider g^{-1}. Then $1/\alpha < 1$ is an eigenvalue of $(g^{-1})^*|H^{1,1}_{\mathbb{R}}(X)$. Then, again, by the generalized Perron–Frobenius theorem, the spectral radius of $(g^{-1})^*|H^{1,1}_{\mathbb{R}}(X)$ is a real eigenvalue, say β, with eigenvector, say η', in $\overline{\mathcal{K}(X)}$. Since $1/\alpha < 1$, we have $\beta > 1$. Thus, $g^*|H^{1,1}_{\mathbb{R}}(X)$ has an eigenvalue $\alpha' := 1/\beta$ with the same eigenvector η'. Since $\alpha \neq \alpha'$, the linear subspace $H = \mathbb{R}\langle\eta, \eta'\rangle$ is 2-dmensional. From

$$(g^*\eta, g^*\eta) = \alpha^2(\eta,\eta)\ ,\ (g^*\eta', g^*\eta') = (\alpha')^2(\eta',\eta')\,,$$

we obtain $(\eta,\eta) = (\eta',\eta') = 0$. Moreover, we have $(\eta,\eta') > 0$, because both vectors are in the closure of the Kähler cone $\overline{\mathcal{K}(X)}$ (and are linearly independent). Thus, H is of signature $(1,1)$ and the orthogonal complement $V^\perp$ in $H^{1,1}_{\mathbb{R}}(X)$ is negative definite. Thus, the remaining eigenvalues of $g^*|H^{1,1}_{\mathbb{R}}(X)$, that coincide with the eigenvalues of $g^*|H^\perp$, are of absolute value 1. In conclusion, $g^*|H^2(X,\mathbb{R})$ has two real eigenvalues $\alpha > 1$, $0 < \alpha' < 1$ and the other eigenvalues are all of absolute value 1. Also, all these values are algebraic integers. This is because $g^*|H^2(X,\mathbb{R})$ is actually defined over $H^2(X,\mathbb{Z})$. This implies the result. Q.E.D.

Finally we recall the notion of a Siegel disk (for simplicity only 2-dimensional case).

Definition 2.6. (1) Let Δ^2 be a 2-dimensional unit disk with a linear coordinate system (z_1, z_2). A linear automorphism (written under the coordinate action)

$$f^*\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

is called an irrational rotation if $|\rho_1| = |\rho_2| = 1$, and ρ_1 and ρ_2 are multiplicatively independent, in the sense that $(m_1, m_2) = (0,0)$ is the only integer solution to

$$\rho_1^{m_1}\rho_2^{m_2} = 1\,.$$

(2) Let X be a complex analytic surface (not necessarily compact) and g be an automorphism of X. A domain $U \subset X$ is called a Siegel disk of (S, g) if $g(U) = U$ and $(U, g|U)$ is isomorphic to some irrational rotation (Δ^2, f). In other words, g has a Siegel disk if and only if there is a fixed point P at which g can be locally analytically linearlized to an irrational rotation.

The existence of a Siegel disk implies that there is no topologically dense orbit. The first examples of surface automorphisms with Siegel disks were discovered by McMullen ([Mc02-2], Theorem (1.1)) among K3 surfaces. The resulting K3 surfaces X are necessarily of algebraic dimension 0. This follows from the fact that the action on the space of holomorphic 2-forms is finite cyclic if the algebraic dimension $\neq 0$ ([Mc02-2], Theorem (3.5); see also [Og08], Theorem (2.4)). We also note that in this case $NS(X)$ is negative definite (and vice versa), so that X contains at most finitely many irreducible complete curves and they are all smooth rational (if exist). This easily follows from the Riemann–Roch inequality for K3 surfaces (see eg. [BHPV04], Page 312, line 6, formula (2)). Later, McMullen ([Mc07], Theorem (10.1)) also found rational surface automorphisms with Siegel disks. In this case, the resulting surfaces are projective. In fact, they are blowups of $\mathbb{P}^2$.

In general, it is hard to see if a given action is locally analytically linearizable at the fixed point or not. The following criterion, which we only state in dimension 2, is again due to McMullen ([Mc02-2], Theorem (5.1)):

Theorem 2.7. *Let φ be an automorphism of a germ of the origin 0 of $\mathbb{C}^2$ such that $\varphi(0) = 0$ and such that*

$$d\varphi^*(0) = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} .$$

Here $d\varphi^*(0)$ is the action on the cotangent space $\Omega^1_{\mathbb{C}^2}(0)$ at 0 induced by the coordinate action φ^*. (We prefer coordiante action as then everything is covariant.) Assume that:

(1) ρ_1 and ρ_2 are algebraic numbers;

(2) $|\rho_1| = |\rho_2| = 1$; and

(3) ρ_1 and ρ_2 are multiplicatively independent.

Then φ has a Siegel disk at 0, i.e., there is a local coordinate (z_1, z_2) at 0 such that

$$\varphi^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} .$$

This is a highly non-trivial result that involves very deep theorems: the Siegel-Sternberg theorem on analytic linearlization and the Baker-Fel'dman theorem on transcendence of the logarithm of algebraic numbers. See [Mc02-2], Section 5 and the references therein for more details.

§3. Proof of Theorem (1.1)

Let us consider the Salem polynomial

$$\varphi_{14}(x) = x^{14} - x^{11} - x^{10} + x^7 - x^4 - x^3 + 1\,.$$

The third smallest Salem number

$$\alpha_{14} = 1.200026...$$

is the unique real root > 1 of $\varphi_{14}(x) = 0$. The equation $\varphi_{14}(x) = 0$ has one more real root $1/\alpha_{14}$. The other 12 roots, which we denote by

$$\beta_k\ ,\ \overline{\beta_k}\ (1 \leq k \leq 6)\,,$$

are on the unit circle $S^1 = \{z \in \mathbb{C}\,|\,|z| = 1\}$. Among these 12 roots on S^1, we choose two particular ones:

$$\delta := \beta_1 := -(0.990398835230041...) - (0.31823945592693...)i\,,$$

$$\theta := \beta_2 := -(0.371932997164175...) - (0.92825957879273...)i\,.$$

These approximate values are computed by Mathematica program, NSolve.

In what follows, δ and θ always mean these two particular roots.

The following theorem due to Gross and McMullen ([GM02], Theorem 1.3) is essential for our construction:

Theorem 3.1. *Let $\varphi(x) \in \mathbb{Z}[x]$ be an irreducible reciprocal polynomial such that $|\varphi(\pm 1)| = 1$ and p, q be positive integers such that $p \equiv q \,(\mathrm{mod}\, 8)$. Let $\mathbb{R}^{p+q}$ be the real vector space with a symmetric bilinear from of signature (p, q). Assume that $f \in \mathrm{SO}(\mathbb{R}^{p+q})$ and the characteristic polynomial $\Phi_f(x)$ is $\varphi(x)$. Then, there is an even unimodular lattice $L \subset \mathbb{R}^{p+q}$ such that $L_{\mathbb{R}} = \mathbb{R}^{p+q}$ and $f(L) = L$. In other words, f is realized as an automorphism of an even unimodular lattice of signature (p, q).*

We denote the K3 lattice by:

$$\Lambda := \Lambda_{\mathrm{K3}} := E_8(-1)^{\oplus 2} \oplus H^{\oplus 3}\,.$$

Here H is the unique even unimodular lattice of signature $(1,1)$ and $E_8(-1)$ is the unique even unimodular negative definite lattice of rank 8. As well known, the lattice Λ is isomorphic to the second cohomology lattice $(H^2(S,\mathbb{Z}),(*,**))$ of a K3 surface S. Here $(*,**)$ is the cup product on $H^2(S,\mathbb{Z})$. (See eg. [BHPV04], Page 311, Proposition (3.3)(ii).) For a field K, we denote the K-vector space $\Lambda \otimes_{\mathbb{Z}} K$ by Λ_K. A similar abbreviation will be applied to other lattices and vector spaces.

Proposition 3.2. *There are an automorphism F of the K3 lattice Λ and an element σ of $\Lambda_{\mathbb{C}}$ such that:*

(1) $\Phi_F(x) = (x-1)^8 \cdot \varphi_{14}(x)$, where $\Phi_F(x)$ is the characteristic polynomial of F;

(2) $(\sigma,\sigma) = 0$ and $(\sigma,\overline{\sigma}) > 0$; and

(3) $F(\sigma) = \delta\sigma$.

Proof. Our $\varphi_{14}(x)$ is an irreducible reciprocal polynomial in $\mathbb{Z}[x]$ with $|\varphi_{14}(\pm 1)| = 1$. We apply Theorem (3.1) for $\varphi_{14}(x)$, along the line of [GM02], Pages 270–271, Proof of Theorem 2.2.

Let V_k $(0 \le k \le 6)$ be the real vector space $\mathbb{R}^2$. We define symmetric bilinear forms Q_k on V_k $(0 \le k \le 6)$ by

$$Q_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_1 := I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Q_k := -I_2 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \ (k \ge 2).$$

Then (V_0,Q_0) is of signature $(1,1)$, (V_1,Q_1) is positive definite and (V_k,Q_k) $(k \ge 2)$ are negative definite. Noticing $|\beta_k| = 1$, we define $f_k \in \mathrm{SO}(V_k,Q_k)$ $(0 \le k \le 6)$ by

$$f_0 = \begin{pmatrix} \alpha_{14} & 0 \\ 0 & 1/\alpha_{14} \end{pmatrix}, \quad f_k := \begin{pmatrix} \mathrm{Re}\,\beta_k & -\mathrm{Im}\,\beta_k \\ \mathrm{Im}\,\beta_k & \mathrm{Re}\,\beta_k \end{pmatrix} \ (k \ge 1).$$

Here α_{14}, $1/\alpha_{14}$, β_k, $\overline{\beta_k}$ $(1 \le k \le 6)$ are the roots of $\varphi_{14}(x) = 0$. The eigenvalues of f_0 are α_{14} and $1/\alpha_{14}$, and the eigenvalues of f_k $(k \ge 1)$ are β_k and $\overline{\beta_k}$. Set

$$(V,Q) := (\oplus_{i=0}^6 V_i, \oplus_{i=0}^6 Q_i), \ f := \oplus_{k=0}^6 f_k.$$

By construction, $(V,Q) = \mathbb{R}^{3+11}$, $f \in \mathrm{SO}(\mathbb{R}^{3+11})$ and the characteristic polynomial of f is $\varphi_{14}(x)$. Thus, by Theorem (3.1), there is an even unimodular lattice $L \subset \mathbb{R}^{3+11}$ such that $f(L) = L$ and $L_{\mathbb{R}} = \mathbb{R}^{3+11}$. We have an isomorphism

$$L \simeq E_8(-1) \oplus H^{\oplus 3}.$$

This is because the isomorphism class of an even indefinite unimodular lattice is uniquely determined by the signature ([Se73], Page 54, Theorem 5).

We can thus identify

$$\Lambda = E_8(-1) \oplus L\,.$$

Put $F = id_{E_8(-1)} \oplus f$. Then $F \in \mathrm{SO}(\Lambda)$ and the characteristic polynomial of F is

$$\Phi_F(x) = (x-1)^8 \cdot \varphi_{14}(x)\,.$$

It remains to find $\sigma \in \Lambda_{\mathbb{C}}$ that satisfies (2) and (3). Choose an eigenvector $\sigma \in \Lambda_{\mathbb{C}}$ of F with eigenvalue $\delta = \beta_1$. We shall show that this σ satisfies (2) and (3). By definition, we have $F(\sigma) = \delta\sigma$. As F is an automorphism of the *lattice* Λ, it follows that

$$(\sigma, \sigma) = (F(\sigma), F(\sigma)) = \delta^2(\sigma, \sigma)\,.$$

Thus, $(\sigma, \sigma) = 0$ by $\delta^2 \neq 1$. Taking the complex conjugate, we obtain that

$$F(\overline{\sigma}) = \overline{\delta}\overline{\sigma}\,,\ (\overline{\sigma}, \overline{\sigma}) = 0\,.$$

Note that

$$\sigma + \overline{\sigma} \neq 0\,.$$

This is because σ and $\overline{\sigma}$ are eigenvectors corresponding to eigenvalues. On the other hand, by the explicit form of F, we see that

$$\sigma\,,\ \overline{\sigma} \in (V_1)_{\mathbb{C}} \subset L_{\mathbb{C}} \subset \Lambda_{\mathbb{C}}\,.$$

As Q_1 is positive definite on V_1 and $\sigma + \overline{\sigma}$ is a real vector in $V_1 \setminus \{0\}$, it follows that

$$(\sigma + \overline{\sigma}, \sigma + \overline{\sigma}) > 0\,.$$

As $(\sigma, \sigma) = (\overline{\sigma}, \overline{\sigma}) = 0$, this implies $(\sigma, \overline{\sigma}) > 0$. Q.E.D.

Remark 3.3. By changing the symmetric bilinear forms on V_1 by $-I_2$ and on V_2 by I_2, we have an automorphism F' of the K3 lattice Λ and an element $\sigma' \in \Lambda_{\mathbb{C}}$ such that

(1) $\Phi_{F'}(x) = (x-1)^8 \cdot \varphi_{14}(x)$;
(2) $(\sigma', \sigma') = 0$ and $(\sigma', \overline{\sigma'}) > 0$; and
(3) $F'(\sigma') = \theta\sigma'$ (Here we recall that $\theta = \beta_2$).

Theorem 3.4. *There is a pair (S, g) of a K3 surface S and its automorphism g such that*

(1) $g^*\sigma_S = \delta\sigma_S$;
(2) The Néron–Severi lattice $NS(S)$ is isomorphic to $E_8(-1)$; and
(3) $g^*|NS(S) = id_{NS(S)}$.

See eg. [BHPV04], Page 308, line 4 for the definition of the Néron–Severi lattice $NS(S)$.

Proof. Let F and σ be the same as in Proposition (3.2). Then, by Proposition (3.2) (2), the point $[\mathbb{C}\sigma]$ belongs to the period domain of K3 surfaces:

$$\Omega := \{[\mathbb{C}\sigma] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \,|\, (\sigma,\sigma) = 0\,,\ (\sigma,\overline{\sigma}) > 0\}\,.$$

Thus, we can apply the surjectivity of the period mapping for K3 surfaces (see eg. [BHPV04], Page 339, Corollary (14.2)) to get a K3 surface S and an isomorphism $\iota : H^2(S,\mathbb{Z}) \simeq \Lambda$ such that $\iota(\mathbb{C}\sigma_S) = \mathbb{C}\sigma$. Here $H^0(S,\Omega_S^2) = \mathbb{C}\sigma_S$. Define an automorphism f_S of $H^2(S,\mathbb{Z})$ by

$$f_S := \iota^{-1} \circ F \circ \iota\,.$$

We want to find an automorphism g of S such that $f_S = g^*$. According to the global Torelli theorem for K3 surfaces (see eg. [BHPV04], Page 332, Theorem (11.1)), this follows if f_S satisfies the following three properties (i)–(iii):

(i) f_S is an Hodge isometry;

(ii) f_S preserves the positive cone $\mathcal{P}(S)$, i.e., the connected component of

$$\{x \in H^{1,1}(S,\mathbb{R}) \,|\, (x,x) > 0\}$$

containing the Kähler classes of S; and

(iii) f_S preserves the set of classes represented by effective curves in $NS(S)$.

Let us check these properties. By definition of f_S, we have $f_S \in \mathrm{SO}\,(H^2(S,\mathbb{Z}))$ and $f_S(\sigma_S) = \delta\sigma_S$. This shows (i). Recall that the Salem number α_{14} is real and an eigenvalue of F. So, α_{14} is a real eigenvalue of f_S as well. We can then choose a real eigenvector $\eta \in H^2(S,\mathbb{R})$ of f_S with eigenvalue α_{14}. By

$$(\eta,\sigma_S) = (f_S^*(\eta), f_S^*(\sigma_S)) = \alpha_{14}\delta(\eta,\sigma_S)$$

and by $\alpha_{14}\delta \neq 1$, we have $(\eta,\sigma_S) = 0$. As η is real, this implies that $\eta \in H^{1,1}(S,\mathbb{R})$. Moreover, by $\alpha_{14} > 1$ and by

$$(\eta,\eta) = (f_S^*(\eta), f_S^*(\eta)) = \alpha_{14}^2(\eta,\eta)\,,$$

we have $(\eta,\eta) = 0$. Thus, $\eta \in \partial\mathcal{P}(S)$ (the boundary of the positive cone) possibly after replacing η by $-\eta$. As $f_S(\eta) = \alpha_{14}\eta$ with $\eta \neq 0$ and $\alpha_{14} > 0$, this implies (ii).

It remains to check (iii). The $\mathbb{C}$-linear extension of the lattice $L \simeq E_8(-1) \oplus H^{\oplus 3}$ (defined in the proof of Proposition (3.2)) contains σ.

Moreover, as the characteristic polynomial of $F|L = f$, which is $\varphi_{14}(x)$, is irreducible over $\mathbb{Z}$, it follows that the lattice L is the *minimal* primitive lattice of Λ of which the $\mathbb{C}$-linear extension contains σ. Thus, the lattice $\iota^{-1}(L)$ is also the minimal primitive sublattice of $H^2(S, \mathbb{Z})$ of which the $\mathbb{C}$-linear extension contains σ_S. By definition of the transcendental lattice $T(S)$ (See eg. [BHPV04], Page 308, line 5), we have then that

$$T(S) = \iota^{-1}(L) \simeq E_8(-1) \oplus H^{\oplus 3}.$$

Recall that $NS(S) = T(S)^{\perp}$ in $H^2(S, \mathbb{Z})$ and $L^{\perp} = E_8(-1)$ in Λ. Then

$$NS(S) = \iota^{-1}(L^{\perp}) = \iota^{-1}(E_8(-1)) \simeq E_8(-1).$$

Moreover, by $F = id_{E_8(-1)} \oplus f$, it follows that

$$f_S(T(S)) = T(S),\ f_S(NS(S)) = NS(S) \text{ and } f_S|NS(S) = id_{NS(S)}.$$

So, the assertion (3) holds.

Hence there is an automorphism g of S such that $f_S = g^*$. By the proof of (3), our (S, g) also satisfies the assertions (2) and (3) of Theorem (3.4). This completes the proof. Q.E.D.

Remark 3.5. Starting from F' and σ' in Remark (3.3) (instead of F and σ in Proposition (3.2)), we also obtain a pair (S', g') of a K3 surface S' and its automorphism g' such that

(1) $(g')^*\sigma_{S'} = \theta\sigma_{S'}$;
(2) $NS(S') \simeq E_8(-1)$; and
(3) $(g')^*|NS(S') = id_{NS(S')}$.

In the rest, we shall show that the pair (S, g) satisfies the requirement of Theorem (1.1) but the pair (S', g') does not.

Proposition 3.6. *Let S be a K3 surface such that $NS(S) \simeq E_8(-1)$. Then, S contains 8 smooth rational curves C_k $(0 \leq k \leq 7)$ and contains no other irreducible complete curve. Moreover, the dual graph of C_k $(0 \leq k \leq 7)$ is the same as the Dynkin diagram $E_8(-1)$, i.e., $(C_k^2) = -2$, vertices C_1, C_2, $\cdots$, C_7 form Dynkin diagram of Type $A_7(-1)$ in this order and the vertex C_0 is joined to only the vertex C_3 by a simple line. (See* [Mc07], *Section 2, Figure 2. In the figure, set $n = 7$ and replace s_k there by C_k here.)*

Proof. We shall show Proposition (3.6) by dividing into four steps.

Step 1. Let C be an irreducible complete curve on S. Then $C \simeq \mathbb{P}^1$.

Proof. As $NS(S)$ is even, negative definite and S is Kähler (see eg. [BHPV04], Page 144, Theorem (3.1) and Page 310 Proposition (3.3)(i)),

we have that $C \not\equiv 0$ and $(C^2) \leq -2$. Thus, for the arithmetic genus $p_a(C)$, we have

$$0 \leq p_a(C) = (C^2)/2 + 1 \leq 0\,.$$

Hence $p_a(C) = 0$. This implies $C \simeq \mathbb{P}^1$. Q.E.D.

Step 2. $NS(S)$ is generated by the classes of irreducible complete curves. In particular, the number of irreducible complete curves on S is greater than or equal to 8.

Proof. Let e_i $(0 \leq i \leq 7)$ be the basis of $NS(S)$ corresponding to the 8 vertices of $E_8(-1)$. We have $(e_i^2) = -2$. Let $E_i \in \operatorname{Pic} S$ be a representative of e_i. Then by the Riemann–Roch formula and the Serre duality, we have

$$h^0(E_i) + h^0(-E_i) \geq \frac{(E_i^2)}{2} + 2 = 1\,.$$

Thus, for each i, either $|E_i|$ or $|-E_i|$ contains an effective curve. As the class of each irreducible component is also in $NS(S)$, this implies the result. Q.E.D.

Step 3. Let C_k $(0 \leq k \leq m-1)$ be mutually distinct irreducible complete curves on S. Then the classes $[C_k] \in NS(S)$ are linearly independent in $NS(S)$. In particular, the number of irreducible complete curves on S is less than or equal to 8.

Proof. If otherwise, there are subsets I and J of $\{0, 1, \cdots, m-1\}$ such that $I \cap J = \emptyset$ and

$$\sum_{i \in I} a_i [C_i] = \sum_{j \in J} b_j [C_j]\,.$$

Here $a_i \geq 0$ and $b_j \geq 0$ and $a_i \neq 0$ for at least one a_i. As $NS(S)$ is negative definite, it follows that

$$0 > ((\sum_{i \in I} a_i [C_i])^2)\,.$$

On the other hand, we have that

$$((\sum_{i \in I} a_i [C_i])^2) = (\sum_{i \in I} a_i [C_i], \sum_{j \in J} b_j [C_j]) \geq 0\,,$$

a contradiction. This implies the result. Q.E.D.

Step 4. S contains 8 smooth rational curves whose dual graph forms Dynkin diagram $E_8(-1)$ and contains no other irreducible complete curve.

Proof. By Steps 2, 3, S contains exactly 8 irreducible complete curves. We denote them by C_k $(0 \leq k \leq 7)$. Again by Steps 2, 3, $\langle [C_k] \rangle_{k=0}^{7}$ form

a basis of $NS(S)$ over $\mathbb{Z}$. By Step 1, each C_k is also a smooth rational curve. Thus $(C_k^2) = -2$. As $NS(S)$ is negative definite, the dual graph of $\{C_k\}_{k=0}^7$ is then a disjoint union of Dynkin diagrams of type $A_n(-1)$, $D_m(-1)$, $E_6(-1)$, $E_7(-1)$, $E_8(-1)$, with 8 vertices in total. As $NS(S)$ is unimodular and $\langle [C_k] \rangle_{k=0}^7$ forms a basis of $NS(S)$ over $\mathbb{Z}$, the only possible dual graph of $\{C_k\}_{k=0}^7$ is then $E_8(-1)$. In fact, the lattices associated with other Dynkin diagrams are of discriminant ≥ 2. This completes the proof. Q.E.D.

Let us return back to our (S, g) in Theorem (3.4). S has exactly 8 smooth rational curves, say C_k $(0 \leq k \leq 7)$, as described in Proposition (3.6), and no other irreducible complete curve. We set $S^g := \{x \in S \mid g(x) = x\}$.

Lemma 3.7. *(1) $g(C_k) = C_k$ for each C_k.*
(2) Put $\mathcal{F}_C = S^g \cap (\cup_{k=0}^7 C_k)$. Then,

$$\mathcal{F}_C = C_3 \cup \{P_1, P_{12}, P_0, P_{45}, P_{56}, P_{67}, P_7\}.$$

Here P_{ij} is the intersection point of C_i and C_j, and P_i is a point on $C_i \setminus \cup_{j \neq i} C_j$. Moreover, for each $P \in \mathcal{F}_C$, the action $dg^*(P)$ of g^* (the coordinate action of g) on the cotangent space $\Omega_S^1(P)$ is diagonalized as follows:

$$dg^*(P) = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \text{ for } P \in C,$$

$$dg^*(P_{12}) = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta^2 \end{pmatrix},$$

$$dg^*(P_1) = \begin{pmatrix} \delta^{-2} & 0 \\ 0 & \delta^3 \end{pmatrix}, \quad dg^*(P_0) = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta^2 \end{pmatrix},$$

$$dg^*(P_{45}) = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta^2 \end{pmatrix}, \quad dg^*(P_{56}) = \begin{pmatrix} \delta^{-2} & 0 \\ 0 & \delta^3 \end{pmatrix},$$

$$dg^*(P_{67}) = \begin{pmatrix} \delta^{-3} & 0 \\ 0 & \delta^4 \end{pmatrix}, \quad dg^*(P_7) = \begin{pmatrix} \delta^{-4} & 0 \\ 0 & \delta^5 \end{pmatrix}.$$

In particular, at any point $P \in \mathcal{F}_C$, the eigenvalues of $dg^*(P)$ are not multiplicatively independent, so that g has no Siegel disk at $P \in \mathcal{F}_C$.

Proof. As $g^*|NS(S) = id_{NS(S)}$, it follows that $g(C_k) = C_k$ for each k. In particular $g(P_{ij}) = P_{ij}$. Thus $g|C_3 = id_{C_3}$, as $C_3 \simeq \mathbb{P}^1$ and $g|C_3$ fixes the three points $C_3 \cap C_2$, $C_3 \cap C_0$, $C_3 \cap C_4$ on C_3. Then, $dg^*(P)$ is as claimed for $P \in C_3$, by $g^* \sigma_S = \delta \sigma_S$. In particular, $d(g|C_2)^*(P_{23})$

is the multiplication by δ. Hence $d(g|C_2)^*(P_{12})$ is the multiplication by δ^{-1} (and $g|C_2$ has no other fixed point), as $C_2 \simeq \mathbb{P}^1$. Thus, by $g^*\sigma_S = \delta\sigma_S$ again, it follows that $d(g|C_1)^*(P_{12})$ is the multiplication by δ^2. Then, as $C_1 \simeq \mathbb{P}^1$, the automorphism $g|C_1$ has one more fixed point, say, P_1, and $d(g|C_1)^*(P_1)$ is the multiplication by δ^{-2}. Then again by $g^*\sigma_S = \delta\sigma_S$, one can diagonalize the action $dg^*(P_1)$ as claimed. In this way, we figure out the set $\mathcal{F}_C$ and the induced actions on the cotangent spaces as claimed. Note that, by definition, δ^a and δ^b $(a, b \in \mathbb{Z})$ are not multiplicatively independent. From this, the last statement follows. Q.E.D.

Let us define the rational function $\gamma(x) \in \mathbb{Q}(x)$ by

$$\gamma(x) = \frac{1 + x - x^3 - x^4 - x^5 - x^6 - x^7 - x^8 + x^{10} + x^{11}}{1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10}}.$$

We note that the denominator and the numerator are reciprocal of degree 10 and of degree 11. Thus, $\gamma(x)$ is also written in the form

$$\gamma(x) = \frac{(x+1)f_1(x + \frac{1}{x})}{f_2(x + \frac{1}{x})},$$

where $f_1(t)$ and $f_2(t)$ are some polynomials of degree 5 with rational coefficients.

Lemma 3.8. *g has one more fixed point $Q \in S \backslash \cup_{k=0}^7 C_k$. Moreover, the action $dg^*(Q)$ is diagonalized as follows:*

$$dg^*(Q) = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}.$$

Here ϵ_1 and ϵ_2 are the roots of the quadratic equation

$$x^2 - \gamma(\delta)x + \delta = 0.$$

Their approximate values are

$$\epsilon_1 = -(0.8886...) - (0.45858...)i\,,\ \epsilon_2 = -(0.94351...) - (0.33133...)i\,.$$

Proof. As S contains no irreducible complete curve other than $\{C_k\}_{k=0}^7$, the fixed points outside $\cup_{k=0}^7 C_k$ are all isolated and finite (if exist). Let $t \geq 0$ be the number of the fixed points off $\cup_{k=0}^7 C_k$, counted with multiplicities.

Let us determine t first. By the topological Lefschetz fixed point formula (see eg. [GMa93], Theorem (10.3)), we have that

$$\sum_F n(F) = \sum_{k=0}^{4} (-1)^k \mathrm{tr}\,(g^* | H^k(S, \mathbb{Z}))\,.$$

Here the sum on the left hand side runs over all the irreducible components of S^g. For an isolated point P, $n(P)$ is the multiplicity and for a fixed smooth curve C, the number $n(C)$ is the topological Euler number of C if it is smooth, of multiplicity 1 (see ibid.). In our case, each irreducible component of $\mathcal{F}_C$ is smooth, of multiplicity 1 by the explicit description in Lemma (3.7). Thus,

$$\sum_F n(F) = 7 + 2 + t = 9 + t\,.$$

Here 2 is the topological Euler number of the fixed curve $C_3 \simeq \mathbb{P}^1$. On the other hand, using the fact that S is a K3 surface and the fact that the characteristic polynomial of $g^* | H^2(S, \mathbb{Z})$ is $(x-1)^8 \varphi_{14}(x)$, we can calculate the right hand side as follows:

$$\sum_{k=0}^{4} (-1)^k \mathrm{tr}\,(g^* | H^k(S, \mathbb{Z})) = 1 + 1 + \mathrm{tr}\,((x-1)^8 \varphi_{14}(x))$$

$$= 1 + 1 + \mathrm{tr}\,((x-1)^8) + \mathrm{tr}\,(\varphi_{14}(x)) = 1 + 1 + 8 + 0 = 10\,.$$

Thus, $9 + t = 10$ and $t = 1$.

Hence the fixed point outside $\cup_{k=0}^{7} C_k$ is just one point with multiplicity 1. We denote this point by Q. Let ϵ_1 and ϵ_2 be the eigenvalues of $dg^*(Q)$. As Q is an isolated fixed point of multiplicity 1, we have

$$\epsilon_1 \neq 1\,,\ \epsilon_2 \neq 1\,.$$

Let us determine ϵ_1 and ϵ_2. First of all, by $g^* \sigma_S = \delta \sigma_S$, we have

$$\epsilon_1 \epsilon_2 = \delta\,.$$

Next let us compute the sum $\epsilon_1 + \epsilon_2$. For this aim, we want to apply an appropriate form of holomorphic Lefschetz fixed point formula. In our case, g has a fixed curve and g is of infinite order. So, we *can not directly apply* Atiyah–Bott's one [AB68] or Atiyah–Singer's one [AS68]. On the other hand, S^g is smooth and of multiplicity 1 at each irreducible component. Thus, we *can apply* Toledo–Tong's form of the holomorphic

Lefschetz fixed point formula ([TT78], the formula (*) at Page 519 or Theorem (4.10), applied to $E = \mathcal{O}_S$):

$$\sum_F L(F) = \sum_{k=0}^{2} (-1)^k \mathrm{tr}\,(g^* | H^k(\mathcal{O}_S))\,.$$

Here the sum on the left hand side runs over all the irreducible components of S^g. The local contribution terms $L(F)$ are calculated as follows (See ibid.). For an isolated point P,

$$L(P) = \frac{1}{(1-\alpha)(1-\beta)}\,.$$

Here α and β are the eigenvalues of $dg^*(P)$. For a smooth curve F for which $g^*|N^* = \lambda$ on the conormal bundle N^*, we have

$$L(F) = \int_F \mathrm{Td}(F) \cdot \{\mathrm{ch}(\mathcal{O}_F) - \lambda \cdot \mathrm{ch}(N^*)\}^{-1}$$

$$= \int_F (1 + \frac{c_1(F)}{2}) \cdot \frac{1}{1-\lambda} \cdot (1 + \frac{\lambda}{1-\lambda} N^*) = \frac{1 - p_a(F)}{(1-\lambda)} + \frac{\lambda \cdot \mathrm{deg} N^*}{(1-\lambda)^2}\,.$$

For our $C_3 \simeq \mathbb{P}^1$, it is

$$L(C_3) = \frac{1}{(1-\delta)} + \frac{\delta \cdot 2}{(1-\delta)^2} = \frac{1+\delta}{(1-\delta)^2}\,.$$

Thus, the left hand side for our (S, g) is:

$$\sum_F L(F) = \frac{3}{(1-\delta^{-1})(1-\delta^2)} + \frac{2}{(1-\delta^{-2})(1-\delta^3)} + \frac{1}{(1-\delta^{-3})(1-\delta^4)}$$

$$+ \frac{1}{(1-\delta^{-4})(1-\delta^5)} + \frac{1+\delta}{(1-\delta)^2} + \frac{1}{(1-\epsilon_1)(1-\epsilon_2)}\,.$$

Let us compute the right hand side. $g^*|H^2(\mathcal{O}_S)$ is the multiplication by δ^{-1}. This is because $H^2(\mathcal{O}_S)$ is the Serre dual of $H^0(\Omega_S^2) = \mathbb{C}\sigma_S$ and $g^*\sigma_S = \delta\sigma_S$. Thus

$$\sum_{k=0}^{2} (-1)^k \mathrm{tr}\,(g^* | H^k(\mathcal{O}_S)) = 1 + \frac{1}{\delta}\,.$$

Hence

$$1 + \frac{1}{\delta} = \frac{3}{(1-\delta^{-1})(1-\delta^2)} + \frac{2}{(1-\delta^{-2})(1-\delta^3)} + \frac{1}{(1-\delta^{-3})(1-\delta^4)}$$

$$+\frac{1}{(1-\delta^{-4})(1-\delta^{5})}+\frac{1+\delta}{(1-\delta)^{2}}+\frac{1}{(1-\epsilon_1)(1-\epsilon_2)}\,.$$

By transposition, we can rewrite this equation in the following form:

$$\frac{1}{(1-\epsilon_1)(1-\epsilon_2)}=f(\delta)\,.$$

To get an explicit form of $f(\delta)$, *we regard as if* δ *is an indeterminate element* and use Mathematica program, Together. The result is:

$$f(\delta)=\frac{1+\delta-\delta^{3}-\delta^{4}-\delta^{5}-\delta^{6}-\delta^{7}+\delta^{9}+\delta^{10}}{(-1+\delta)^{2}\delta(1+\delta)(1+\delta+\delta^{2})(1+\delta+\delta^{2}+\delta^{3}+\delta^{4})}\,.$$

Note that

$$(1-\epsilon_1)(1-\epsilon_2)=1+\delta-(\epsilon_1+\epsilon_2)\,.$$

Then, by the formula above, we obtain that

$$\epsilon_1+\epsilon_2=1+\delta-\frac{1}{f(\delta)}\,.$$

In order to simplify the right hand side, we again regard δ as an indeterminate element and use Mathematica program, Together. The result is:

$$1+\delta-\frac{1}{f(\delta)}=\gamma(\delta)\,.$$

Here $\gamma(x)$ is the rational function defined just before Lemma (3.8). Thus,

$$\epsilon_1+\epsilon_2=\gamma(\delta)\,.$$

Hence ϵ_1 and ϵ_2 are the roots of the quadratic equation

$$x^{2}-\gamma(\delta)x+\delta=0\,,$$

as claimed. Now using Mathematica again, we can find approximate values of ϵ_1 and ϵ_2 as follows. First, by substituting

$$\delta=(-0.9903988352300419...)-(0.13823945592693967...)i$$

into $\gamma(\delta)$ by Mathematica, we obtain

$$\gamma(\delta)=(0.0548626217729844...)-(0.7899228027367716...)i\,.$$

Our quadratic equation is then

$$x^{2}-((0.0548626217729844...)-(0.7899228027367716...)i)x$$

$$+((-0.9903988352300419...)-(0.13823945592693967...)i)=0\,.$$

Solving this equation by Mathematica program, NSolve, we obtain approximate values of ϵ_1 and ϵ_2 as claimed. They are certainly different and therefore $dg^*(Q)$ can be diagonalized. Q.E.D.

Lemma 3.9. *g has a Siegel disk at Q.*

Note that the diagonalization in Lemma (3.8) is just on the cotangent space level and far from local coordinate level.

Proof. It suffices to check that ϵ_1 and ϵ_2 in Lemma (3.8) satisfy the conditions (1)–(3) in Theorem (2.7).

(1) is clear as ϵ_1 and ϵ_2 are the roots of $x^2-\gamma(\delta)x+\delta=0$, and both δ and $\gamma(\delta)$ are algebraic numbers.

Let us check (2). Mathematica program, Abs applied to ϵ_1 and ϵ_2 certainly indicates the result. However, to conclude that some value x is *exactly* 1, computation based on approximate values of x seems insufficient. Here is a safer argument. Consider

$$e_1:=\frac{\epsilon_1^2}{\delta}\,,\ e_2:=\frac{\epsilon_2^2}{\delta}\,.$$

As $|\delta|=1$, it suffices to show that $|e_1|=|e_2|=1$. We have $e_1e_2=1$ by $\epsilon_1\epsilon_2=\delta$. We also have

$$e_1+e_2=\frac{(\epsilon_1+\epsilon_2)^2-2\epsilon_1\epsilon_2}{\delta}=\frac{\gamma(\delta)^2}{\delta}-2\,.$$

Recall the second expression of $\gamma(\delta)$ given just before Lemma (3.8):

$$\gamma(\delta)=\frac{(\delta+1)f_1(\delta+\frac{1}{\delta})}{f_2(\delta+\frac{1}{\delta})}\,.$$

Then, $e_1+e_2=k(\delta)$, where

$$k(\delta)=\frac{(\gamma(\delta))^2}{\delta}-2=(\delta+\frac{1}{\delta}+2)\{\frac{f_1(\delta+\frac{1}{\delta})}{f_2(\delta+\frac{1}{\delta})}\}^2-2\,,$$

and e_1 and e_2 are the roots of the quadratic equation

$$x^2-k(\delta)x+1=0\,.$$

By the quadratic formula, we have

$$e_1,e_2=\frac{k(\delta)\pm\sqrt{k(\delta)^2-4}}{2}\,.$$

Here, $k(\delta)$ is *real*, as

$$\delta + \frac{1}{\delta} = \delta + \overline{\delta}$$

is real by $|\delta| = 1$. Thus, $|e_1| = |e_2| = 1$ if and only if $|k(\delta)| \leq 2$ from the quadratic formula above. Substituting

$$\delta = -(0.9903...) - (0.3182...)i$$

into $k(\delta)$ by Mathematica, we have

$$k(\delta) = \frac{\gamma(\delta)^2}{\delta} - 2 = -(1.3730...) + (9.4799... \times 10^{-17})i\,.$$

Thus $|k(\delta)| < 2$.

We should remark that there appears an error term

$$(9.4799... \times 10^{-17})i$$

in the above expression of $k(\delta)$. However, *this does not matter*, because it is extremely small compared with the real part and we *know* that $k(\delta)$ is certainly real. Hence the assertion (2) holds for our ϵ_1 and ϵ_2.

It remains to check (3) for our ϵ_1 and ϵ_2. Suppose

$$\epsilon_1^m \epsilon_2^n = 1$$

for $(m, n) \in \mathbb{Z}^2$. Note that $\delta = \beta_1$ and $\theta = \beta_2$ are Galois conjugate, as both are roots of $\varphi_{14}(x) = 0$ (and $\varphi_{14}(x)$ is irreducible). Thus, by taking Galois conjugate, we have

$$(\epsilon_1')^m (\epsilon_2')^n = 1\,.$$

Here ϵ_1' and ϵ_2' are the roots of the quadratic equation

$$x^2 - \gamma(\theta)x + \theta = 0\,.$$

As

$$\theta = -(0.371932997164175...) - (0.92825957879273...)i\,,$$

we have

$$\gamma(\theta) = (1.495690836752066...) - (2.210575209107991...)i\,,$$

by Mathematica. Substituting these values into the quadratic equation above and using Mathematica program, NSolve, we find (up to order) that

$$\epsilon_1' = (0.25262...) - (0.37337...)i, \quad \epsilon_2' = (1.2430...) - (1.837...)i.$$

Clearly $|\epsilon_2'| > 1$. Thus, from

$$1 = |(\epsilon_1')^m(\epsilon_2')^n| = |(\epsilon_1'\epsilon_2')^m(\epsilon_2')^{n-m}| = |(\theta)^m(\epsilon_2')^{n-m}| = |(\epsilon_2')|^{n-m},$$

we conclude $n = m$. Substituting this into $\epsilon_1^n\epsilon_2^m = 1$, we obtain

$$1 = (\epsilon_1\epsilon_2)^n = \delta^n.$$

Here δ is not root of unity. This is because the Salem number $\alpha_{14} > 1$ is a Galois conjugate of δ. Hence $n = 0$, and therefore, $m = n = 0$ by $n = m$. This shows (3). Q.E.D.

Now the following Lemma completes the proof of Theorem (1.1):

Lemma 3.10. $\operatorname{Aut} S = \langle g \rangle \simeq \mathbb{Z}$.

Proof. As δ is not a root of unity, g is of infinite order. So, it suffices to show that $\operatorname{Aut} S$ is generated by g. Let $f \in \operatorname{Aut} S$. As the dual graph of the curves $\{C_k\}_{k=0}^7$ is the Dynkin diagram $E_8(-1)$ and it has no symmetry, we have $f(C_k) = C_k$ $(0 \leq k \leq 7)$. Hence $f^*|NS(S) = id_{NS(S)}$, as $\{C_k\}_{k=0}^7$ generates $NS(S)$. The natural representation of $\operatorname{Aut} S$ on $T(S)$

$$r_T : \operatorname{Aut} S \longrightarrow \mathrm{O}(T_S)$$

is then injective, as so is on $\mathrm{O}(H^2(S, \mathbb{Z}))$ (see eg. [BHPV04], Page 333, Corollary (11.4)). Moreover, as $NS(S) \simeq E_8(-1)$ is negative definite, $\operatorname{Im} r_T$ is isomorphic to $\mathbb{Z}$. This is a special case of [Og08], Theorem (2.4). Hence, $\operatorname{Aut} S$ is isomorphic to $\mathbb{Z}$ as well. Let h be a generator of $\operatorname{Aut} S$. By replacing h by h^{-1} if necessary, we can write $g = h^n$ for some positive integer n. Let $\varphi(x)$ be the characteristic polynomial of $h^*|T(S)$. As $NS(S)$ is negative definite, $\varphi(x)$ is again a Salem polynomial of degree 14 ([Og08], Theorem (3.4)). Let β_{14} be the Salem number of $\varphi(x)$. Then, by $g = h^n$, we have

$$\alpha_{14} = \beta_{14}^n.$$

On the other hand, α_{14} is the *smallest* Salem number of degree 14, as explained in Section 2. Hence $n = 1$, i.e., $g = h$. Q.E.D.

Remark 3.11. Let us consider the pair (S', g') in Remark (3.5). Then, as Lemmas (3.7), (3.8), we have a similar description of the fixed point set:

$$(S')^{g'} = C_3' \cup \{P_1', P_{12}', P_0', P_{45}', P_{56}', P_{67}', P_7'\} \cap \{Q'\}.$$

However, g' has no Siegel disk. In fact, The eigenvalues of $d(g')^*(P')$ $(P' \in C_3')$, $d(g')^*(P_i')$, $d(g')^*(P_{ij}')$ are the same as the eigenvalues of

$dg^*(P)$ $(P \in C_3)$, $dg^*(P_i)$ and $dg^*(P_{ij})$, and they are not multiplicatively independent. The eigenvalues of $d(g')^*(Q')$ are ϵ_1' and ϵ_2'. Here, ϵ_1' and ϵ_2' are the numbers defined at the last part of the proof of Lemma (3.9). Then $|\epsilon_2'| > 1$ as observed there. So, g' has no Siegel disk at Q', either.

Remark 3.12. Recall (from Section 2 (i)) that

$$\Phi_{14}(x) = x^{14} - x^{12} - x^7 - x^2 + 1$$

is the Salem polynomial of the 4-th smallest known Salem number

$$A_{14} = 1.20261\ldots .$$

$|\Phi_{14}(\pm 1)| = 1$, and $\Phi_{14}(x) = 0$ has two particular roots on the unit circle

$$\delta' := -(0.45829\ldots) - (0.88799\ldots)i\,,$$

$$\theta' := -(0.96815\ldots) - (0.25034\ldots)i\,.$$

Then, starting from δ' and arguing exactly in the same way as in Theorem (1.1), we also obtain a K3 surface automorphism of topological entropy $\log A_{14}$, with one pointwise fixed smooth rational curve and a Siegel disk. The resulting action on the Siegel disk is given by $\mathrm{diag}\,(\rho_1, \rho_2)$, where

$$\rho_1 = -(0.29457\ldots) - (0.95562\ldots)i, \quad \rho_2 = (0.98436\ldots) - (0.17614\ldots)i.$$

§4. Proof of Theorem (1.2)

In this section, we shall prove Theorem (1.2). Let S be an Enriques surface and g be an autmorphism of S. Let us denote the free part of $H^2(S, \mathbb{Z})$ by L. Then, by [BHPV04], Page 339, Lemma (15.1) (iii), we have

$$L \simeq H \oplus E_8(-1) \simeq E_{10}(-1)\,.$$

Here the last isomorphism comes from the fact that both $H \oplus E_8(-1)$ and $E_{10}(-1)$ are even unimodular lattices of signature $(1, 9)$. In fact, we can then apply [Se73], Page 54, Theorem 5. We denote by $L(2)$ the lattice such that $L(2) = L$ as $\mathbb{Z}$-module and

$$(x, y)_{L(2)} := 2(x, y)_L$$

for each $x, y \in L(2) = L$. Note that g^* is also an automorphism of the new lattice $L(2)$. Then g^* acts on the discriminant group

$$A_{L(2)} := L(2)^*/L(2) \simeq L/2L \simeq \mathbb{F}_2^{10}\,.$$

Let $\overline{\Phi_{g^*}}(x)$ be the characteristic polynomial of $g^*|A_{L(2)}$. Then $\overline{\Phi_{g^*}}(x) \in \mathbb{F}_2[x]$. More precisely, $\overline{\Phi_{g^*}}(x)$ is the mod 2 reduction of the characteristic polynomial $\Phi_{g^*}(x)$ of $g^*|L = g^*|L(2)$.

Let $\pi : \tilde{S} \longrightarrow S$ be the universal cover of S. Then $\tilde{S}$ is a K3 surface and π is of degree 2 (See eg. [BHPV04], Page 339, Lemma (15.1)(ii)). We denote by ι the covering involution of π. Following [Nm85], Page 203, line 6, we define

$$M := \{x \in H^2(\tilde{S}, \mathbb{Z}) \,|\, \iota^* x = x\}\,, \; N := \{x \in H^2(\tilde{S}, \mathbb{Z}) \,|\, \iota^* x = -x\}\,.$$

By [Nm85], Proposition (2.3), we have

$$M = \pi^*(L) = L(2)\,.$$

Here the last equality is nothing but the following obvious relation

$$(\pi^* x, \pi^* y)_{\tilde{S}} = 2(x, y)_S, \;\; \forall x, y \in L\,.$$

Let $A_M = M^*/M$ and $A_N = N^*/N$ be the discriminant groups of M and N. Note that

$$A_N \simeq A_M = A_{L(2)} \simeq \mathbb{F}_2^{10}\,.$$

Here the first isomorphism is given by the natural surjective morphisms:

$$H^2(\tilde{S}, \mathbb{Z}) \longrightarrow M^*/M\,;\; x \mapsto (x, *) \bmod M\,,$$

$$H^2(\tilde{S}, \mathbb{Z}) \longrightarrow N^*/N\,;\; x \mapsto (x, *) \bmod N.$$

We note that these two morphisms are certainly surjective as $H^2(\tilde{S}, \mathbb{Z})$ is unimodular and both N and M are primitive. Then, it is an easy fact that the both kernels are $N \oplus M$. This is a special case of [Ni80], Corollary (1.5.2).

Let $\tilde{g} \in \operatorname{Aut} \tilde{S}$ be one of the two possible lifts of g on $\tilde{S}$. Then $g \circ \pi = \pi \circ \tilde{g}$ and $\tilde{g} \circ \iota = \iota \circ \tilde{g}$. Thus, $\tilde{g}^*$ preserves both M and N. Hence, $\tilde{g}^*$ induces actions on M and N, and consequently, on the discriminant groups A_M and A_N. Moreover, under the isomorphism of discriminant groups above, we have

$$\tilde{g}^*|A_N = \tilde{g}^*|A_M = g^*|A_{L(2)}\,.$$

Here the last equality follows from $\pi \circ \tilde{g} = g \circ \pi$. Thus, the characteristic polynomial of $g^*|A_N$ is the same as the characteristic polynomial $\overline{\Phi_{g^*}}(x)$ of $g^*|A_{L(2)}$.

Lemma 4.1. *No irreducible component of* $\overline{\Phi_{g^*}}(x) \in \mathbb{F}_2[x]$ *is of degree* 5.

Proof. If $\overline{\Phi_{g^*}}(x)$ would have an irreducible factor of degree 5, then the corresponding eigenvalues of $\tilde{g}^*|A_N$ would be elements of $\mathbb{F}_{32} \setminus \mathbb{F}_2$. Here $32 = 2^5$. As

$$(\mathbb{F}_{32})^\times \simeq \mathbb{Z}/(32-1)\mathbb{Z} = \mathbb{Z}/31\mathbb{Z}\,,$$

the order of $\tilde{g}^*|A_N$ would then be divisible by 31. Thus, the order of $\tilde{g}^*|N$, which is actually finite (Lemma (4.2) below), would be also divisible by 31. However, this is impossible by the next a bit more precise Lemma (4.2). Q.E.D.

Lemma 4.2. *Under the same notations as in the proof of Lemma (4.1), the order of $\tilde{g}^*|N$ is finite, say, d. Let*

$$d = \Pi_{k=1}^n p_k^{m_k}$$

be the prime decomposition of d. Then each primary factor $p_k^{m_k}$ belongs to

$$\{2, 2^2, 2^3, 2^4, 3, 3^2, 5, 7, 11, 13\}\,.$$

Proof. As the lattice $M = \pi^* L$ is of signature $(1,9)$ with pure Hodge type $(1,1)$, the lattice N is of signature $(2,10)$ and N admits the following real Hodge decomposition:

$$N_{\mathbb{R}} = Q \oplus P\,.$$

Here

$$Q := N_{\mathbb{R}} \cap H^{1,1}(\tilde{S})\ ,\ P := \mathbb{R}\langle \mathrm{Re}\,\sigma_{\tilde{S}}, \mathrm{Im}\,\sigma_{\tilde{S}}\rangle,$$

and $\sigma_{\tilde{S}}$ is a nowhere vanishing global holomorphic 2-form on $\tilde{S}$. As $\tilde{g}^*|N$ preserves the Hodge decomposition, we have

$$\tilde{g}^*|N \in \mathrm{O}(P) \times \mathrm{O}(Q)\,.$$

Here P is positive definite and Q is negative definite. Hence $\tilde{g}^*|N$ is diagonalizable and the eigenvalues are of absolute value 1. On the other hand, $\tilde{g}^*|N$ is defined over $\mathbb{Z}$. Thus, all the eigenvalues are roots of unity (Kronecker's theorem). Hence $\tilde{g}^*|N$ is of finite order, say, d. We denote the prime decomposition of d as in the statement. Then $(\tilde{g}^*|N)^e$ with $e = d/p_k^{m_k}$, is of order $p_k^{m_k}$. As $(\tilde{g}^*|N)^e$ is defined over $\mathbb{Z}$, all primitive $p_k^{m_k}$-th roots of unity appear as eigenvalues of $(\tilde{g}^*|N)^e$. As easily seen, their cardinality is exactly $p_k^{m_k-1}(p_k-1)$. As $\mathrm{rank}\, N = 12$, this number can not exceed 12, that is,

$$p_k^{m_k-1}(p_k-1) \leq 12\,.$$

Solving this inequality, we obtain the result. Q.E.D.

Now we are ready to complete the proof of Theorem (1.2). If $h(g)$ would be the logarithm of the Lehmer number, then, as L is of rank 10, the characteristic polynomial $\Phi_{g^*}(x)$ of $g^*|L$ would be the Lehmer polynomial:

$$\varphi_{10}(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1\,.$$

However, this is impossible by Lemma (4.1) and the following:

Lemma 4.3. *Let $\overline{\varphi_{10}}(x) \in \mathbb{F}_2[x]$ be the mod 2 reduction of the Lehmer polynomial. Then the irreducible decomposition of $\overline{\varphi_{10}}(x)$ is:*

$$\overline{\varphi}_{10}(x) = (x^5 + x^3 + x^2 + x + 1)(x^5 + x^4 + x^3 + x^2 + 1)\,.$$

Proof. This immediately follows from Mathematica program, Factor:

$$\text{Factor}\,[x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1, \text{Modulus} \to 2]\,.$$

Q.E.D.

This completes the proof of Theorem (1.2).

Remark 4.4. The second, third, forth smallest known Salem numbers are of degree > 10. So, they can not be realized as the exponential of the topological entropy of an Enriques surface automorphism. Recall (from Section 2 (i)) that

$$\Phi_{10}(x) = x^{10} - x^6 - x^5 - x^4 + 1$$

is the Salem polynomial of the fifth smallest known Salem number

$$A_{10} = 1.216391\ldots\,.$$

Again by Mathematica program, Factor, applied to the mod 2 reduction $\overline{\Phi_{10}}(x)$ of $\Phi_{10}(x)$, we obtain the irreducible factorization:

$$\overline{\Phi_{10}}(x) = (x^5 + x^4 + x^2 + x + 1)(x^5 + x^4 + x^3 + x + 1)\,.$$

Thus, the fifth smallest known Salem number A_{10} can not be realized, either. In conclusion, *none of the smallest five Salem numbers (listed in Section 2 (i)) can be realized as the exponential of the topological entropy of an Enriques surface automorphism.*

We conclude this note with the following natural, probably tractable, open problems relevant to our Theorems (1.1), (1.2) and a few remarks:

Question 4.5. (1) What is the smallest Salem number that can be realized as the exponential of the topological entropy of a K3 surface automorphism?

(2) Is there a surface automorphism having more than one Siegel disks at the same time?

(3) What is the smallest Salem number that can be realized as the exponential of the topological entropy of an Enriques surface automorphism?

(4) What is the smallest Salem number that can be realized as the exponential of the topological entropy of an automorphism of a *generic* Enriques surface?

(5) Are there compact hyperkähler manifolds (of dimension $2n$) having automorphisms with $2n$-dimensional Siegel disks?

Remark 4.6. Here are a few remarks about some of the questions above.

For Question (1). As the Salem numbers in question are of degree ≤ 22, we only need to see the realizability of α_{10} and α_{18}, i.e., the first and second smallest Salem numbers[1].

For Question (4). The automorphism group of a generic Enriques surface is isomorphic to the 2-congruence subgroup $\mathrm{O}^+(E_{10}(-1))(2)$ of $\mathrm{O}^+(E_{10}(-1))$. This is proved by [BP83], Theorem (3.4) and [Nm85], Theorem (5.10). (See also the precise meaning "generic" there.) Thus, this is also a purely group theoretical problem.

For Question (5). In the terminology of [Be83], Page 759, Théorème, a compact hyperkähler manifold of dimension $2n$ is a Ricci flat compact Kähler manifold with $\mathrm{Sp}(n)$ holonomy and a Calabi–Yau manifold of dimension $m \geq 3$ is a Ricci flat compact Kähler manifold with $\mathrm{SU}(m)$ holonomy. Let X be a Calabi–Yau manifold of dimension $m \geq 3$. Then X is projective ([Be83], Page 760, Proposition 1), and therefore, the action of $\mathrm{Aut}\, X$ on the space of holomorphic m-forms is finite cyclic ([Ue75], Page 178, Proposition 14.5). So, Calabi–Yau manifolds of dimension $m \geq 3$ can not admit automorphisms with m-dimensional Siegel disks. For the same reason, compact hyperkähler manifolds having automorphisms with $2n$-dimensional Siegel disks can not be projective as well. A bit more precisely, they are in fact of algebraic dimension 0 ([Og08], Theorem (2.4)). See also [Og09] for the explicit description of

[1]Quite recently, McMullen has shown that the Lehmer number α_{10} is realized as the exponential of the topological entropy of a non-projective K3 surface automorphism.

the topological entropy of automorphisms of compact hyperkähler manifolds and [Zh08] for a more algero-geometric aspect of the topological entropy of automorphisms of higher dimensional manifolds.

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