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Automorphisms of an irregular surface with low slope acting trivially in cohomology

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Abstract.

Let S be a complex minimal nonsingular projective irregular surface of general type with $K_S^2 \leq 4\chi(\mathcal{O}_S)$ and $\chi(\mathcal{O}_S) > 12$. Then the group of automorphisms of S acts faithfully on the cohomology ring $H^*(S, \mathbb{Q})$ with the exceptional case that S is as in [Ca3, Theorem 2.5].

§1. Introduction

Let S be a complex minimal nonsingular projective surface of general type. Let $\mathrm{Aut}_0 S \subset \mathrm{Aut} S$ be the subgroup of automorphisms of S, inducing trivial action on the cohomology ring $H^*(S, \mathbb{Q})$.

It is known that, if the canonical linear system $|K_S|$ of S is base-point-free then $\mathrm{Aut}_0 S$ is trivial, with the possible exceptional case that S satisfies either $K_S^2 = 8\chi(\mathcal{O}_S)$ or $K_S^2 = 9\chi(\mathcal{O}_S)$ [Pet1].

When S has a fibration of genus 2, we have a classification for pairs $(S, \mathrm{Aut}_0 S)$:

Theorem 1. ([Ca2, Theorem 1.1]) *Let S be a complex minimal nonsingular projective surface of general type with a genus 2 fibration $f : S \to C$ and $\chi(\mathcal{O}_S) \geq 5$. Then $|\mathrm{Aut}_0 S| \leq 2$, and if $|\mathrm{Aut}_0 S| = 2$, then the generator of $\mathrm{Aut}_0 S$ is a bi-elliptic involution of f, the canonical map of S factors through f, and S has the following numerical invariants:*

$$K_S^2 = 4\chi(\mathcal{O}_S), \quad q(S) = g(C) = 1.$$

Example 1.1. If S is as in Theorem 1 with $\mathrm{Aut}_0 S$ being non-trivial, then S is birationally equivalent to a double cover of certain elliptic fiber

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bundle. The configuration of the ramification divisor of this covering is determined (see [Ca3, Theorem 2.5] for precise statements). Such a surface can be explicitly constructed (see [Ca2, Example 3.3] for a special case of such a construction).

To the author's knowledge, besides Example 1.1, there are no known examples of S with $p_g(S) \gg 0$ and $\mathrm{Aut}_0 S$ being non-trivial. A natural question is whether it is the only one for minimal surfaces of general type with $K_S^2 \leq 4\chi(\mathcal{O}_S)$.

In this note, we prove it is true for irregular surfaces S. Our main result is the following:

Theorem 2. *Let S be a complex minimal nonsingular projective irregular surface of general type with $\chi(\mathcal{O}_S) > 12$. If $K_S^2 \leq 4\chi(\mathcal{O}_S)$, then $\mathrm{Aut}_0 S$ is trivial with the exceptional case S is as in Example 1.1.*

The sketch of the proof of Theorem 2 is as follows. Thanks to Beauville's and Xiao's results on the canonical map of S [Be; Xi2], the problem reduces to excluding the case that S has a fibration $f\colon S \to C$ of genus 3, and $\mathrm{Aut}_0 S$ is of order two and acts freely on a general fiber of f. In this case, we estimate the number of (-1)-curves on the desingularation $\tilde{T}$ of the quotient $S/\mathrm{Aut}_0 S$, show that the numerical invariants of the minimal model T of $\tilde{T}$ satisfy $K_T^2 < 2\chi(\mathcal{O}_T)$ and $q(T) = 1$, and get a contradiction by a result of of Debarre (cf. [De]).

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Notations. In this paper we denote by $\equiv$ and $\sim$ the linear equivalence and numerical equivalence of two divisors, respectively.

§2. The canonical map is composite with a pencil

Proposition 2.1. *Let S be a complex minimal nonsingular projective surface of general type. Assume that the canonical map ϕ_S of S is composite with a pencil of genus $g \geq 3$. If $K_S^2 < \frac{16}{3}(p_g(S) - 2)$ and $p_g(S) \geq 5$, then $\mathrm{Aut}_0 S$ is trivial.*

Proof. If the moving part $|M|$ of $|K_S|$ has a base point, then $K_S^2 \geq (p_g(S) - 1)^2$ by [K, Lemma 3.3]. So $|M|$ is free from base points, because $(p_g(S) - 1)^2 \geq \frac{16}{3}(p_g(S) - 2)$ when $p_g(S) \geq 5$. By taking the Stein factorization of the canonical map if necessary, we get a fibration $f : S \to B$ of curves of genus $g \geq 3$. By a result of Xiao [Xi1], we have either $q(S) = b = 1$ or $q(S) \leq 2$, $b = 0$, where b denotes the genus of B. The global sections in $H^0(B, f_*\omega_S)$ generate an invertible subsheaf $\mathcal{L}$ of $f_*\omega_S$ satisfying $h^0(B,\mathcal{L}) = h^0(S,\omega_S)$ and $\mathcal{O}_S(M) \simeq f^*\mathcal{L} \sim (\deg\mathcal{L})F$, where F is a general fiber of f. By the Riemann–Roch theorem and the fact that $b \leq 1$, we get $p_g(S) = h^0(B,\mathcal{L}) = \deg\mathcal{L} + 1 - b$. Thus

$$K_S^2 \geq K_S M = (\deg\mathcal{L})K_S F = 2(g-1)\deg\mathcal{L} = 2(g-1)(p_g(S) - 1 + b).$$

Hence $g = 3$ by the assumption. Note also that B is isomorphic to the image of the canonical map of S, because $\mathcal{L}$ is very ample by $\deg\mathcal{L} = p_g(S) - 1 + b \geq 2b + 1$.

Let Z be the fixed part of $|K_S|$, and let H be the horizontal part of Z. We write $H = n_1\Gamma_1 + n_2\Gamma_2 + \cdots$ with $n_1 \geq n_2 \geq \cdots$, where Γ_i $(i = 1, 2, \cdots)$ are the irreducible components of H, with n_i the multiplicity of Γ_i in H. Then $n_1 \leq 4 = ZF = K_S F$. By [K, Lemma 2.1], $(n_1+1)K_S - (\deg\mathcal{L} + 2n_1(b-1))F$ is nef. Considering the intersection number with Z, one gets $K_S Z \geq \frac{4}{(n_1+1)}(\deg\mathcal{L} + 2n_1(b-1))$, and hence

$$K_S^2 = K_S M + K_S Z \geq \frac{4(n_1+2)}{(n_1+1)}(p_g(S) - 1 + b) + \frac{8n_1}{(n_1+1)}(b-1).$$

This gives us $K_S^2 \geq \frac{16}{3}(p_g(S) - 2)$ when $n_1 \leq 2$.

Now we may assume $n_1 \geq 3$. Let $G = \mathrm{Aut}_0 S$. Since $H^0(S,\omega_S)$ is a direct factor of $H^2(S,\mathbb{C})$, G acts trivially on $H^0(S,\omega_S)$. This implies that G acts trivially on $\mathrm{Im}\phi_S$ and there is a homomorphism h of G into $\mathrm{Aut}B$. Since B is isomorphic to $\mathrm{Im}\phi_S$, we have that $\mathrm{Ker}h = G$, i.e., G induces the trivial action on B, and $G \hookrightarrow \mathrm{Aut}F$ for a general fiber F of f.

If $n_1 = 4$, then $H = 4\Gamma_1$, and Γ_1 is a section of f. This implies $F \cap \Gamma_1 \in F$ is a G-fixed point, and hence G is cyclic. Consider the quotient map $\pi \colon F \to F/G$. Since $p_g(S/G) = p_g(S) > 0$, we have $g(F/G) = 1$. Since G is abelian, π has at least two branch points. Using the Hurwitz formula for π, we get $|G| \leq 3$. Now if $|G| = 2$ or 3, then there are at least two G-fixed points on F. Since F is a general fiber of f, this implies that there are G-fixed (multi-)sections. Since any G-fixed curve is contained in the fixed part of $|K_S|$ (see e.g. [Ca1, 1.14]), we get a contradiction. So G must be trivial.

If $n_1 = 3$, then $n_2 = 1$, $H = 3\Gamma_1 + \Gamma_2$, and Γ_1, Γ_2 are sections of f. This implies $p_1 := F \cap \Gamma_1, p_2 := F \cap \Gamma_2 \in F$ are G-fixed points, and hence G is cyclic. Consider the quotient map $\pi \colon F \to F/G$. By the same argument as above, we have $g(F/G) = 1$ and $\deg\pi = 3$. So $K_F \equiv 2p_1 + 2p_2$. On the other hand, from $K_F = (3\Gamma_1 + \Gamma_2 + V)_{|F}$, we get $K_F \equiv 3p_1 + p_2$. This is a contradiction since $p_1 \not\equiv p_2$ on F. Q.E.D.

§3. Proof of Theorem 2

3.1. By Theorem 1 and Proposition 2.1, we may assume that the canonical map ϕ_S of S is generically finite and that S has no pencil of curves of genus 2.

Let $G = \mathrm{Aut}_0 S$. Since $H^0(S, \omega_S)$ is a direct factor of $H^2(S, \mathbb{C})$, it follows that G induces trivial actions on $\mathrm{Im}\phi_S$. So ϕ_S factors through the quotient map

$$\phi_S = \alpha \circ q : S \xrightarrow{\ q\ } S/G \xrightarrow{\ \alpha\ } \Sigma := \mathrm{Im}\phi_S.$$

Thus $\deg \phi_S = |G| \deg\alpha$. Recall that, by [Be, Théorème 3.1], Σ is a canonical surface or satisfies $p_g(\Sigma) = 0$.

If Σ is a canonical surface, then it satisfies the Castelnuovo's inequality $\deg \Sigma \geq 3p_g(S) - 7$ (cf. [Be, 5.6]). We have

$$4\chi(\mathcal{O}_S) \geq K_S^2 \geq (\deg \phi_S) \deg \Sigma \geq |G|(3p_g(S) - 7).$$

This implies that G must be trivial when $\chi(\mathcal{O}_S) \geq 8$.

So we can assume $p_g(\Sigma) = 0$. Then $\deg \alpha \geq 2$. We have

$$4\chi(\mathcal{O}_S) \geq K_S^2 \geq |G| \deg \alpha (p_g(S) - 2).$$

This implies that, when $\chi(\mathcal{O}_S) \geq 7$, G is trivial with one possible exceptional case $|G| = 2$ and $\deg \phi_S = 4$. Note also that in the exceptional case $K_S^2 \geq 4(p_g(S) - 2) > 40$ and $q(S) \leq 3$.

3.2. From now on we assume that the pair (S, G) is as in the exceptional case. By [Xi2, Theorem 1] and its proof, one has that, when $\chi(\mathcal{O}_S) > 12$, S has a fibration $f : S \to C$ of genus 3, and ϕ_S separates fibers of f and maps them onto a pencil of straight lines on Σ. In particular, the degree of the map induced by ϕ_S on the general fiber is four. This implies that the fixed part of $|K_S|$ is vertical with respect to f. Since G induces trivial actions on Σ, and hence on C, $G \hookrightarrow \mathrm{Aut}F$ for a general fiber F of f. Since each G-fixed curve is contained in the fixed part of $|K_S|$ (see [Ca1, 1.14]), we have each G-fixed curve is vertical with respect to f. So G acts freely on F and hence F/G is of genus two. This implies F is hyperelliptic and hence f is an hyperelliptic fibration.

Also, we remark here that any irreducible curve on S with negative self-intersection is G-invariant, since G acts trivially on the cohomology.

3.3. Let σ be the generator of $\mathrm{Aut}_0 S$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\pi}} & \tilde{T} := \tilde{S}/\tilde{\sigma} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle h} \\ S & \xrightarrow{f} & C \end{array}$$

where ρ is the blowup of all isolated fixed points of σ, and $\tilde{\sigma}$ the induced involution on $\tilde{S}$. Then $p_g(\tilde{T}) = p_g(S)$, $q(\tilde{T}) = q(S)$, and $h\colon \tilde{T} \to C$ is a fibration of genus 2. Note also that $\tilde{T}$ is of general type, because the canonical map of $\tilde{T}$ is generically finite by the assumption on ϕ_S and $p_g(\tilde{T}) = p_g(S)$.

Notation 3.4. For any irreducible curve Γ on S, if Γ is vertical w.r.t. f, we denote by m_Γ the multiplicity of Γ in fiber $f^*(f(\Gamma))$.

We have the following simple observations.

Lemma 3.5. (1) *Each (-2)-curve on S is contained in fibers of f.*
(2) *Each (-1)-curve on $\tilde{T}$ is contained in fibers of h.*
(3) *For each (-2)-curve Θ on S, the number of isolated σ-fixed points on Θ is either* 0 *or* 2.
(4) *For each σ-fixed curve D on S, m_D is even.*
(5) *Let Θ be a (-2)-curve on S. If there are no isolated σ-fixed points on Θ, then $m_\Theta \geq 2$.*

Proof. (i) Suppose there is a horizontal (w.r.t. f) (-2)-curve Θ on S. Then $g(C) = 0$ and $d := \Theta F > 0$, where F is a fiber of f. We have $(dK_{S/C} - 4\Theta)F = 0$, where $K_{S/C} = K_S - f^*K_C$ is the relative canonical divisor. Since $F^2 = 0$ and $F \not\sim 0$, by the Hodge index theorem, we have $(dK_{S/C} - 4\Theta)^2 \leq 0$. This implies that $K_{S/C}^2 \leq 48$, and hence $K_S^2 \leq 32$, a contradiction.

(ii) Suppose there is a horizontal (w.r.t. h) (-1)-curve Γ on $\tilde{T}$. Let $h' : T' \to C$ be the relatively minimal model of h. Since $p_g(\tilde{T}) > 0$, Γ does not meet any other (-1)-curve on $\tilde{T}$. So the image of Γ in T' is a (-1)-curve. By the same argument as in (i), we get $K_{T'/C}^2 \leq 8$. Note that, since $h' : T' \to C$ is a relatively minimal fibration of curves of genus 2, one has $K_{T'/C}^2 \geq 2(\chi(\mathcal{O}_{T'}) + 1)$ by the slope inequality. We have $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{T'}) \leq 3$, a contradiction.

(iii) Suppose that σ has precisely one isolated fixed point on Θ. Then $\tilde{\Theta}^2 = -3$, where $\tilde{\Theta}$ be the strict transform of Θ in $\tilde{S}$. On the other hand, from $\tilde{\Theta} = \tilde{\pi}^*D$, where $D = \tilde{\pi}(\tilde{\Theta})$, we get $\tilde{\Theta}^2 = 2D^2$. This is a contradiction.

(iv) By (3.2), $q := f(D)$ is a point. From $(f \circ \rho)^*(q) = \tilde{\pi}^*(h^*(q))$, we have $m_D = \mathrm{mult}_{\tilde{D}}(f \circ \rho)^*(q) = 2\mathrm{mult}_{\bar{D}} h^*(q)$, where $\tilde{D} = \rho^* D$ and $\bar{D} = \tilde{\pi}(\tilde{D})$.

(v) By (iv), we may assume Θ is not σ-fixed. Then Θ meets some σ-fixed curves, say D, D' (maybe $D = D'$) in two points. By (3.2), we have $D, D' < f^*(q)$, where $q = f(\Theta)$.

Let $\bar{D}$ and $\bar{D}'$ be the image of $\rho^* D$ and $\rho^* D'$ in $\tilde{T}$. Let $\tilde{\Theta} = \rho^*\Theta$ and $\Gamma = \tilde{\pi}(\tilde{\Theta})$. Then $\Gamma(\bar{D} + \bar{D}') \geq 2$ ($\Gamma\bar{D} \geq 2$ if $D = D'$). This implies $2 \leq \mathrm{mult}_\Gamma h^*(q) = \mathrm{mult}_{\tilde{\Theta}}(f \circ \rho)^*(q) = m_\Theta$. Q.E.D.

3.6. Let $D_1, \cdots, D_u$ ($u \geq 0$) be the σ-fixed curves and let $\tilde{D}_i = \rho^* D_i$. Let $p_1, \cdots, p_k$ be isolated σ-fixed points, and let $E_i = \rho^* p_i$. We have

$$K_{\tilde{S}} \equiv \tilde{\pi}^* K_{\tilde{T}} + \sum_{i=1}^{u} \tilde{D}_i + \sum_{j=1}^{k} E_j. \tag{1}$$

$$K_{\tilde{S}} \equiv \rho^* K_S + \sum_{j=1}^{k} E_j. \tag{2}$$

Lemma 3.7. *For each (-1)-curve $\tilde{\Gamma}$ on $\tilde{T}$, we have*

(1) $\tilde{\Theta} := \tilde{\pi}^*\tilde{\Gamma}$ *and* $\Theta := \rho_*(\tilde{\Theta})$ *are (-2)-curves.*
(2) *Let Θ be as in (i). Among $D_1, \cdots, D_u$, either there are exactly two curves meet Θ, or there is exactly one curve, which is not a (-2)-curve, meeting Θ in two different points.*

Proof. (i) By (ii) of Lemma 3.5, $q := \tilde{h}(\tilde{\Gamma})$ is a point of C. Let $F' = f^* q$ and $\tilde{F}' = (f \circ \rho)^* q$. We have that $\tilde{\pi}^*\tilde{\Gamma}$ is reduced and irreducible. Indeed, otherwise, we have either $\tilde{\pi}^*\tilde{\Gamma} = \Theta_1 + \Theta_2$ or $\tilde{\pi}^*\tilde{\Gamma} = 2\Theta_3$, where Θ_1, Θ_2 and Θ_3 are curves on $\tilde{S}$. In the former case, $\tilde{\sigma}$ maps Θ_1 to Θ_2, which is absurd since any curve with negative self-intersection is $\tilde{\sigma}$-invariant; In the latter case, from $-2 = \tilde{\pi}^*\tilde{\Gamma}^2 = (2\Theta_3)^2$, we get a contradiction.

Let $\tilde{\Theta} = \tilde{\pi}^*\tilde{\Gamma}$ and $\Theta = \rho_*\tilde{\Theta}$. Since $\tilde{\Theta} < \tilde{F}'$, we have $p_a(\tilde{\Theta}) < 3$. Since $\tilde{\Theta}^2 = -2$, by the adjunction formula, we have $K_{\tilde{S}}\tilde{\Theta} = 0$, 2 or 4.

We show that $K_{\tilde{S}}\tilde{\Theta} = 2$ or 4 does not occur. Otherwise, since $\tilde{\Theta}^2 = -2$, we have that Θ is not a (-2)-curve. Let $m = \mathrm{mult}_\Theta F'$. We have

$$m K_S \Theta \leq K_S F' = 4. \tag{3}$$

Since $\Theta < F'$, we have $\Theta^2 < 0$. This implies that there is at most one isolated σ-fixed point on Θ. So $\tilde{\Theta}\sum_{j=1}^{k} E_j \leq 1$. By (1), we have

$$\tilde{\Theta}\sum_{i=1}^{u}\tilde{D}_i \geq K_{\tilde{S}}\tilde{\Theta}+1. \tag{4}$$

Let I be the subset of $\{1, \cdots, u\}$, such that for each $i \in I$, $D_i < F'$. By Lemma 3.5, we have $2\sum_{i\in I} D_i < F'$. From $\Theta F' = 0$, we get $m\Theta^2 + 2\Theta\sum_{i\in I} D_i \leq 0$. Combining this with (4), (note that $\Theta\sum_{i\in I} D_i = \tilde{\Theta}\sum_{i=1}^{u}\tilde{D}_i$,) we have

$$m\Theta^2 \leq -2K_{\tilde{S}}\tilde{\Theta} - 2 \leq -6. \tag{5}$$

Note that $\Theta^2 = -1$ or -2 and $K_S\Theta \equiv \Theta^2 \bmod 2$, combining (3) with (5), we get a contradiction.

Now we may assume $K_{\tilde{S}}\tilde{\Theta} = 0$. Then $\tilde{\Theta}$ is a (-2)-curve. We have $\tilde{\Theta}E_j = 0$ for each j. (Otherwise, Θ must be (-1)-curve, contrary to the minimality of S.) This implies that there are no isolated σ-fixed points on Θ and Θ is a (-2)-curve.

(ii) Since the intersection number of any two (-2)-curves is less than two, (ii) follows from (i). Q.E.D.

Let $\Gamma_1, \cdots, \Gamma_{n(f)}$ $(n(f) \geq 0)$ be all (-1)-curves on $\tilde{T}$. Since $\tilde{T}$ is of general type, they do not meet each other. Let $\eta : \tilde{T} \to T$ be the map contracting $\Gamma_1, \cdots, \Gamma_{n(f)}$.

Lemma 3.8. *T is a minimal nonsingular surface of general type with $K_T^2 = K_{\tilde{T}}^2 + n(f)$.*

Proof. We prove that T is minimal; the other part is clear. Suppose that there exists a (-1)-curve E on T. Let $\tilde{E} \subset \tilde{T}$ be the strict transform of E. By the definition of η, $\tilde{E}$ is a smooth rational curve with $\tilde{E}^2 \leq -2$, and among $\{\Gamma_1, \cdots, \Gamma_{n(f)}\}$, there is at least one curve, say Γ_1, which meets $\tilde{E}$ with $\Gamma_1\tilde{E} = 1$.

Let $\tilde{\Theta} = \tilde{\pi}^*\Gamma_1$, $\tilde{A} = \tilde{\pi}^*\tilde{E}$, and let $\Theta = \rho_*\tilde{\Theta}$, $A = \rho_*\tilde{A}$. By Lemma 3.7, both $\tilde{\Theta}$ and Θ are (-2)-curves, and Θ meets some σ-fixed curves, say D and D' (maybe $D = D'$) in two points.

We claim that $\tilde{A}$ is irreducible and reduced. Indeed, by the argument as in the proof of Lemma 3.7, we may assume $\tilde{A}_{\mathrm{red}}$ is irreducible. If $\tilde{A} = 2\tilde{A}_1$ for some curve $\tilde{A}_1$, then $\tilde{A}_1$ is $\tilde{\sigma}$-fixed. Since $\Gamma_1\tilde{E} = 1$, we have $\tilde{\Theta}\tilde{A}_1 = 1$. This implies $\tilde{\Theta}$ is $\tilde{\sigma}$-fixed, a contradiction.

Let $\bar{D}$ and $\bar{D}'$ be the image of $\tilde{D}$ and $\tilde{D}'$ (the strict transform of D and D') in T.

If D and D' are (-2)-curves, then both $\bar{D}$ and $\bar{D}'$ are rational with self-intersection not smaller than -3. Let $\eta' : T \to T'$ be the map contracting E. Then $\eta'(\bar{D})$ and $\eta'(\bar{D}')$ are rational with self-intersection not smaller than -2 and they meet at $\eta'(E)$ with the same tangent direction. This is absurd since the induced fibration $T' \to C$ is of genus 2.

Now we may assume one of them, say D, is not a (-2)-curve. Since Θ is a (-2)-curve and $A\Theta = 2$, we have that A is not a (-2)-curve. From $K_S F' = 4$, we have $m_A + m_D \leq m_A K_S A + m_D K_S D \leq 4$. Since m_D is even ((iv) of Lemma 3.5), this implies

$$K_S A = K_S D = 1. \tag{6}$$

Since E, $\bar{D}$ and $\bar{D}'$ pass through $\eta(\Gamma_1)$, we have $\mathrm{mult}_E \hat{h}^*(c) \geq 2$, where $\hat{h} : T \to C$ is the induced fibration and $c = \hat{h}(E)$. Since $\tilde{A}$ is not $\tilde{\sigma}$-fixed, we have $\mathrm{mult}_{\tilde{A}}(f \circ \rho)^*(c) = \mathrm{mult}_{\tilde{E}} h^*(c)$. So $m_A \geq 2$. By (iv) of Lemma 3.5, m_D and $m_{D'}$ are even. From $AF' = 0$, we have $-2m_\Theta + m_D + m_{D'} + 2m_A \leq 0$. So $m_\Theta \geq 4$.

From $AF' = 0$, we have $m_A A^2 + 2m_\Theta = m_A A^2 + m_\Theta A\Theta \leq 0$. So $A^2 \leq -4$. Combining this with (6), by the adjunction formula we get $p_a(A) < 0$, a contradiction. Q.E.D.

Definition 3.1. For an effective divisor A on S, we let $n(A)$ to be the number of (-2)-curves Θ, such that 1) $\Theta < A$, 2) Θ is not σ-fixed, and 3) there are no isolated σ-fixed points on Θ, and we define

$$\delta(A) = n(A) - \sum_D (K_S D - \frac{1}{2} D^2),$$

where the sum $\sum_D$ is taken over all σ-fixed curves contained in A.

By (i) of Lemma 3.5 and Lemma 3.7, we have

$$\sum_{F'} n(F') = n(f), \tag{7}$$

where the sum is taken over all singular fibers of f and $n(f)$ is as in Lemma 3.8.

Lemma 3.9. *For any fiber F' of f, we have $\delta(F') \leq 0$, and $\delta(F') = 0$ holds if and only if F' contains no σ-fixed curves.*

Proof. After suitable re-indexing, we may assume that $D_1, \cdots, D_t$ $(t \geq 0)$ be the σ-fixed curves contained in F', $K_S D_i > 0$ for $i \leq k$ $(0 \leq k \leq t)$ and $D_{k+1}, \cdots, D_t$ are (-2)-curves.

Let $n = n(F')$, and let $\Theta_1, \cdots, \Theta_n$ be (-2)-curves contained in F' such that there are no isolated σ-fixed points on them. After suitable re-indexing, we may assume that $\sum_{i=1}^{k} \Theta_j D_i > 0$ if and only if $j \leq l$ $(0 \leq l \leq n)$.

Let $\mathcal{A}$ be the dual graph of divisor $A := \sum_{i=k+1}^{t} D_i + \sum_{j=l+1}^{n} \Theta_j$. Since A consists of (-2)-curves, we have that every connected component of $\mathcal{A}$ is a tree. By (ii) of Lemma 3.7 and by the definition of A, each boundary vertex (i.e., a vertex connected with other vertices by at most one edge) corresponds to a σ-fixed curve. So we have that, if $A \neq 0$, let $\nu(A)$ be the number of connected components of $\mathcal{A}$, then $m - k \geq n - l + \nu(A)$, and hence

$$\delta(A) = n - l - (t - k) \leq -\nu(A). \tag{8}$$

Let $H = \sum_{i=1}^{k} D_i + \sum_{j=1}^{l} \Theta_j$. Since $m_{D_i} \geq 2$ ((iv) of Lemma 3.5), from

$$2K_S D_1 + \cdots + 2K_S D_k \leq K_S F' = 4, \tag{9}$$

we have $k \leq 2$. So H has at most two connected components.

Case 1. $k = 0$. If $t = 0$, by (3.2) and (ii) of Lemma 3.7, we have $n(F') = 0$ and so $\delta(F') = 0$. If $t > 0$, then $\delta(F') = \delta(A) \leq -1$ by (8).

Case 2. $k = 1$. In this case H is connected. From $D_1 F' = 0$, we get

$$m_{D_1} D_1^2 + 2s \leq m_{D_1} D_1^2 + \sum_{i=1}^{s} m_{\Theta_i} \Theta_i D_1 \leq 0. \tag{10}$$

Case 2.1. $m_{D_1} = 2$. By (10), $\delta(H) \leq -\frac{1}{2} D_1^2 - K_S D_1 < 0$ with the exceptional cases:

(a) $H = D_1 + \Theta_1 + \Theta_2 + \Theta_3$, with $K_S D_1 = 1$, $D_1^2 = -3$ and $\Theta_j D_1 = 1$ for $j = 1, 2, 3$.

(b) $H = D_1 + \Theta_1 + \cdots + \Theta_4$, with $K_S D_1 = 2$, $D_1^2 = -4$ and $\Theta_j D_1 = 1$ for $j = 1, \cdots, 4$.

In each case above, we have $\delta(H) = \frac{1}{2}$, and by (iii) of Lemma 3.5, $A \neq 0$. So by (8), $\delta(F') = \delta(A) + \delta(H) < 0$.

Case 2.2. $m_{D_1} = 4$. We have $K_S D_1 = 1$ and $D_1^2 = -1$ or -3.

If $D_1^2 = -1$, then $\delta(H) < 0$ and so $\delta(F') < 0$, with the exceptional case $H = D_1 + \Theta_1 + \Theta_2$, with $K_S D_1 = 1$, $D_1^2 = -1$ and $\Theta_j D_1 = 1$ for $j = 1, 2$. In the exceptional case, we have $F' = 4D_1 + 2\Theta_1 + 2\Theta_2$. This implies that σ has precisely one isolated fixed point on Θ_j. By (iii) of Lemma 3.5, we get a contradiction.

Now we assume $D_1^2 = -3$. If $\Theta_j D_1 = 1$ for all j, then $s = 6$ and $F' = 4D_1 + \Theta_1 + \cdots + \Theta_6$, with $\Theta_j D_1 = 1$ for all j. We get a contradiction as above.

If $\Theta_j D_1 = 2$ for some j, from $\Theta_j F' = 0$, we have $m_{\Theta_j} \geq 4$. Combining this with (10), we have $\delta(H) < 0$ (and hence $\delta(F') < 0$), with the exceptional case $H = D_1 + \Theta_1 + \Theta_2 + \Theta_3$, with $\Theta_1 D_1 = 2$, and $\Theta_j D_1 = 1$ for $j = 2, 3$. In the exceptional case, we have $\delta(F') < 0$ as in Case 2.1.

Case 3. $k = 2$. By (9), we have $K_S D_i = 1$ and $m_{D_i} = 2$ for $i = 1, 2$. By the adjunction formula, we have $D_i^2 = -1$ or -3.

Since $2H < F'$ ((iv) and (v) of Lemma 3.5), from $D_i F' = 0$, we get

$$2D_i^2 + 2\sum_{j=1}^{s} \Theta_j D_i \leq m_{D_1} D_i^2 + \sum_{j=1}^{s} m_{\Theta_j} \Theta_j D_i \leq 0.$$

So among $\{\Theta_1, \cdots, \Theta_s\}$, there are at most $-D_i^2$ curves meet D_i for $i = 1$ or 2.

If H is connected, then $s \leq -D_1^2 - D_2^2 - 1$, and hence

$$\delta(H) \leq \frac{1}{2}(-D_1^2 - D_2^2) - 3 < 0,$$

with the exceptional case $H = D_1 + D_2 + \Theta_1 + \cdots + \Theta_5$, with $K_S D_i = 1$, $D_i^2 = -3$, $\Theta_1 D_i = 1$ for $i = 1, 2$, and among $\{\Theta_2, \cdots, \Theta_5\}$, there are two curves that meet D_1 and do not meet D_2, and the others meet D_2 and do not meet D_1. In the exceptional case we have $\delta(F') < 0$ as in Case 2.1.

If H is not connected, let H_1, H_2 be connected components of H, by the argument above, we have

$$\delta(H) = \delta(H_1) + \delta(H_2) \leq \frac{1}{2}(-D_1^2 - D_2^2) - 2 < 0,$$

with the exceptional cases:

1) H_1 is of type (a) as in Case 1, and $H_2 = D_1 + \Theta_1$, with $K_S D_1 = 1$, $D_1^2 = -1$ and $\Theta_1 D_1 = 1$.

2) H_i is of type (a) as in Case 1 for $i = 1, 2$.

In case 1), we have $\delta(F') < 0$ as in Case 2.1.

In case 2), by (iii) of Lemma 3.5, the dual graph of A must have at least six boundary points. By the well known facts on the dual graph of connected component consisting (-2)-curves (cf. e.g. [BPV]), we have $\nu(A) \geq 2$. So by (8), $\delta(F') = \delta(A) + \delta(H) < 0$. Q.E.D.

Now by (1) and (2), we have $\rho^* K_S \equiv \tilde{\pi}^* K_{\tilde{T}} + \sum_{i=1}^{u} \rho^* D_i$. So

$$2K_{\tilde{T}}^2 = K_S^2 - \sum_{i=1}^{u} (2K_S D_i - D_i^2). \tag{11}$$

Applying the topological and holomorphic Lefschetz formula to σ (cf. [AS, p. 566]), we have

$$K_S^2 = 8\chi(\mathcal{O}_S) + \sum_{i=1}^{u} D_i^2,$$

where D_i is as in (3.6). The assumption $K_S^2 \leq 4\chi(\mathcal{O}_S)$ implies $u > 0$. By Lemma 3.9, there is a singular fiber F' of f with $\delta(F') < 0$. Combining this with (11), (7), Lemma 3.8, and Lemma 3.9, we have

$$K_T^2 = \frac{1}{2}K_S^2 + \sum_{F'} \delta(F') < \frac{1}{2}K_S^2 \leq 2\chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_T). \tag{12}$$

On the other hand, since T is a minimal irregular surface of general type, by a theorem of Debarre (cf. [De]), one has $K_T^2 \geq 2\chi(\mathcal{O}_T)$, contrary to (12). This finishes the proof of Theorem 2.

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