

Ruan’s conjecture and integral structures in quantum cohomology

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Abstract.

This is an expository article on the recent studies [27, 28, 45, 46, 23] of Ruan’s crepant resolution/flop conjecture [62, 63] and its possible relations to the K -theory integral structure [45, 46, 49] in quantum cohomology.

§1. Introduction

The small quantum cohomology is a family $(H^*(X), \circ_\tau)$ of commutative ring structures on $H^*(X)$ parametrized by $\tau \in H^{1,1}(X)$. In the large radius limit, *i.e.* $-\Re(\int_C \tau) \rightarrow \infty$ for every effective curve $C \subset X$, the quantum product \circ_τ goes to the cup product. For a pair (X_1, X_2) of birational varieties in a “crepant” relationship (such as flops or crepant resolutions), Yongbin Ruan’s conjecture states that the small quantum cohomologies $(H^*(X_1), \circ_{\tau_1})$ and $(H^*(X_2), \circ_{\tau_2})$ are isomorphic under analytic continuation of the parameter τ . The conjectural space where the quantum product \circ_τ is analytically continued is known as a Kähler moduli space \mathcal{M} in physics. (See Figure 1.) In our situation, the space \mathcal{M} has two limit points (which we call cusps) $\mathbf{0}_1, \mathbf{0}_2$ corresponding to the large radius limit points of X_1 and X_2 respectively. A neighborhood V_i of $\mathbf{0}_i$ is identified with an open subset of $H^{1,1}(X_i)$. A weak form of Ruan’s conjecture asserts that *there exists a family (F, \circ_τ) of commutative rings over \mathcal{M} such that its restriction to V_i is isomorphic to the small quantum cohomology of X_i* . In particular, the cohomology rings $H^*(X_1), H^*(X_2)$ are connected via quantum deformations.

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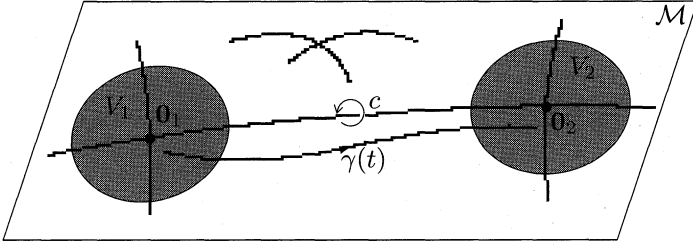


Fig. 1. Kähler moduli space \mathcal{M} containing cusp neighborhoods $V_i \subset H^{1,1}(X_i, \mathbb{C})$, $i = 1, 2$. The *global quantum D-module* over \mathcal{M} develops singularities along thick lines.

In a more precise picture, the family of rings should come from a meromorphic flat connection (F, ∇) over \mathcal{M} , which we call a *global quantum D-module*. The D -module (F, ∇) restricted to V_i is identified with the quantum D -module given by the Dubrovin connection (6):

$$\nabla_\alpha = \frac{\partial}{\partial t^\alpha} + \frac{1}{z} \phi_\alpha \circ \tau, \quad \text{where } z \in \mathbb{C}^* \text{ is a parameter.}$$

It is clear that the operator $z\nabla_\alpha$ recovers the quantum product $\phi_\alpha \circ \tau$ in the limit $z \rightarrow 0$, but the D -module contains much more information than a family of rings. In fact, the global quantum D -module (F, ∇) together with additional data—opposite subspace and dilaton shift—yields a Kyoji Saito’s *flat structure* [66] on the extended Kähler moduli space¹. Moreover, the local monodromy around each cusp determines a canonical choice of the opposite subspace and recovers the flat structure on V_i induced from the identification with the vector space $H^{1,1}(X_i)$. Here, as the example in [27] suggests, the flat structures from the different cusps \mathbf{O}_1 and \mathbf{O}_2 does not necessarily coincide.

In this article, we postulate furthermore that the global quantum D -module is underlain by an integral local system. We also conjecture that, over V_i , the integral local system in question comes from the K -theory of X_i . This has the following physical explanation. Quantum cohomology is part of the A-model topological string theory. A chiral field in the A-model (*i.e.* a section of the quantum D -module) should have a pairing

¹A flat structure is also referred to as a Frobenius manifold structure due to Dubrovin [31]. In the later formulation, the parameter τ will be extended to the total cohomology group $H^*(X_i)$ and \mathcal{M} is also extended accordingly.

with a B-type D-brane (*i.e.* an object of the derived category $D_{\text{coh}}^b(X_i)$) (see *e.g.* [39]). This suggests that a vector bundle on X_i should give a flat section of the quantum D -module. Under mirror symmetry, this corresponds to the fact that a holomorphic n -form has a pairing with a (real) Lagrangian n -cycle by integration. Based on mirror symmetry for toric orbifolds, the author [45, 46] proposed a formula (13) which assigns a flat section of the quantum D -module to an element of the K -group. Katzarkov–Kontsevich–Pantev [49] also found a similar formula for a rational structure independently. The flat sections arising from the K -group define an integral local system over V_i . Via the analytic continuation of K -theory flat sections along a path $\gamma(t)$ connecting V_1 and V_2 (see Figure 1), our picture gives an isomorphism of K -groups:

$$\mathbb{U}_{K,\gamma}: K(X_1) \xrightarrow{\cong} K(X_2).$$

The isomorphism $\mathbb{U}_{K,\gamma}$ here contains complete information on the relationships between the genus zero Gromov–Witten theories (quantum cohomology) of X_1 and X_2 . We hope that $\mathbb{U}_{K,\gamma}$ is given by a certain Fourier–Mukai transformation.

The paper is structured as follows. In Section 2, we review orbifold quantum cohomology/ D -module and introduce the K -theory integral structure on it. In Section 3.1, we formulate a precise picture (Assumption 3.1) of the global quantum D -module sketched above. In Sections 3.2–3.7, we discuss what follows from the picture *without* using integral structures. The main observation here is the fact that *each cusp determines a possibly different flat structure on \mathcal{M}* . The Hard Lefschetz condition in Section 3.7 is a sufficient condition for the Frobenius structures from different cusps to match. These facts were found in [27], but the present article contains a complete proof of the characterization of Frobenius structures at cusps (Theorem 3.13, announced in [27]) and a *generalized* Hard Lefschetz condition (Theorem 3.22). In Sections 3.8, 3.9, we use integral structures to study the crepant resolution conjecture for Calabi–Yau orbifolds and give an explicit prediction (Conjecture 3.31) for the change of co-ordinates in local examples. Readers who want to know a role of integral structures in Ruan’s conjecture can safely skip Sections 3.2–3.7 and go directly to Sections 3.8 or 3.9.

Finally, we remark that we restrict our attentions to the even parity part of the (orbifold) cohomology group throughout this paper.

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§2. K -theory integral structure in quantum cohomology

In this section, we review the orbifold quantum cohomology for smooth Deligne–Mumford stacks and introduce the K -theory integral structure on it. Assuming the convergence of structure constants, quantum cohomology defines a flat connection, called the *Dubrovin connection*, on some cohomology bundle over a neighborhood of the “large radius limit point”. This is called the *quantum D -module*. We will see that the K -group defines an integral lattice in the space of (multi-valued) flat sections of the quantum D -module. The key definition will be given in Definition 2.11. The true origin of this integral structure is yet to be known, but it has a number of good properties:

- This is invariant under every local monodromy around the large radius limit point (Proposition 2.12 (iii)).
- The pairing on quantum cohomology is translated into the Mukai pairing on the K -group (Proposition 2.12 (ii)).
- This gives a real structure which is *pure and polarized* in a neighborhood of the large radius limit point [45, Theorem 3.7]. In particular, we have *tt*-geometry* [18, 37, 47] on quantum cohomology.
- This looks compatible with many computations done in the context of mirror symmetry [43, 7]. In particular, this matches with the integral structure on the Landau–Ginzburg mirror in the case of toric orbifolds [45, 46] (see Example 2.14 (iii)).
- Thus in toric case, this integral structure is compatible also with the *Stokes structure*.

In this article, we will not explain the last three items. See [45, 38, 49] for the properties “pure and polarized” or “compatibility with Stokes structure”.

2.1. Orbifold quantum cohomology

We start with some notation concerning orbifolds. Let \mathcal{X} be a smooth Deligne–Mumford stack with projective coarse moduli space X .

Let $I\mathcal{X}$ be the inertia stack of \mathcal{X} . A point on $I\mathcal{X}$ is given by a pair (x, g) of a point $x \in \mathcal{X}$ and an element g of the automorphism group (local group) $\text{Aut}_{\mathcal{X}}(x)$ at x . The element $g \in \text{Aut}_{\mathcal{X}}(x)$ is also called a stabilizer. Let

$$I\mathcal{X} = \bigsqcup_{v \in \mathbb{T}} \mathcal{X}_v = \mathcal{X}_0 \sqcup \bigsqcup_{v \in \mathbb{T}'} \mathcal{X}_v$$

be the decomposition of $I\mathcal{X}$ into connected components. Here \mathbb{T} is a finite set parametrizing connected components of $I\mathcal{X}$. \mathbb{T} contains a distinguished element $0 \in \mathbb{T}$ which corresponds to the trivial stabilizer $g = 1$ and we set $\mathbb{T} = \{0\} \cup \mathbb{T}'$. Then \mathcal{X}_0 is isomorphic to \mathcal{X} . At each point (x, g) in $I\mathcal{X}$, we can define a rational number $\iota_{(x, g)}$ called *age*. The element g of the automorphism group acts on the tangent space $T_x\mathcal{X}$ and decomposes it into eigenspaces:

$$T_x\mathcal{X} = \bigoplus_{0 \leq f < 1} (T_x\mathcal{X})_f$$

where g acts on $(T_x\mathcal{X})_f$ by $\exp(2\pi i f)$. The age $\iota_{(x, g)}$ is defined to be

$$\iota_{(x, g)} = \sum_{0 \leq f < 1} f \dim_{\mathbb{C}}(T_x\mathcal{X})_f.$$

The age $\iota_{(x, g)}$ is constant along the connected component \mathcal{X}_v of $I\mathcal{X}$, so we denote by ι_v the age $\iota_{(x, g)}$ at any point (x, g) in \mathcal{X}_v . The orbifold or Chen–Ruan cohomology group $H_{\text{CR}}^*(\mathcal{X})$ is a \mathbb{Q} -graded vector space defined by

$$H_{\text{CR}}^p(\mathcal{X}) = \bigoplus_{v \in \mathbb{T}} H^{p-2\iota_v}(\mathcal{X}_v, \mathbb{C}), \quad p \in \mathbb{Q}.$$

This is the same as $H^*(I\mathcal{X}, \mathbb{C})$ as a vector space, but the grading is shifted by the age. In this paper, we only consider the *even parity part* of $H_{\text{CR}}^*(\mathcal{X})$, i.e. the summands satisfying $p - 2\iota_v \equiv 0 \pmod{2}$ in the above decomposition. Unless otherwise stated, we denote by $H_{\text{CR}}^*(\mathcal{X})$ the even parity part. The inertia stack has an involution $\text{inv}: I\mathcal{X} \rightarrow I\mathcal{X}$ which sends (x, g) to (x, g^{-1}) . This induces an involution $\text{inv}: \mathbb{T} \rightarrow \mathbb{T}$ on the index set \mathbb{T} and $\text{inv}^*: H_{\text{CR}}^*(\mathcal{X}) \rightarrow H_{\text{CR}}^*(\mathcal{X})$ on the cohomology. The orbifold Poincaré pairing on $H_{\text{CR}}^*(\mathcal{X})$ is defined by

$$(\alpha, \beta)_{\text{orb}} = \int_{I\mathcal{X}} \alpha \cup \text{inv}^*(\beta) = \sum_{v \in \mathbb{T}} \int_{\mathcal{X}_v} \alpha_v \cup \beta_{\text{inv}(v)},$$

where α_v, β_v are the v -components of α, β . This pairing is symmetric, non-degenerate and of degree $-2 \dim_{\mathbb{C}} \mathcal{X}$.

Gromov–Witten theory for manifolds has been extended to the class of symplectic orbifolds or smooth Deligne–Mumford stacks. This was done by Chen–Ruan [21] in the symplectic category and by Abramovich–Graber–Vistoli [1] in the algebraic category. The formal properties of the genus zero Gromov–Witten theory hold in orbifold theory as well: the genus zero orbifold Gromov–Witten theory defines a cohomological field theory (see *e.g.* [57]) on the metric vector space $(H_{\text{CR}}^*(\mathcal{X}), (\cdot, \cdot)_{\text{orb}})$. In particular, we have the following correlation functions (Gromov–Witten invariants):

$$(1) \quad \langle \cdot, \dots, \cdot \rangle_{0,m,d} : (H_{\text{CR}}^*(\mathcal{X}))^{\otimes m} \rightarrow \mathbb{C}$$

defined for $m \geq 0$ and $d \in H_2(X, \mathbb{Z})$. This is zero when d is not in the semigroup $\text{Eff}_{\mathcal{X}} \subset H_2(X, \mathbb{Z})$ generated by classes of effective curves or $m \leq 2$ and $d = 0$. Also these correlation functions satisfy the so-called *WDVV equation* or the *splitting axiom* (see [1, Theorem 6.4.3]). The genus zero Gromov–Witten invariant is homogeneous with respect to the grading of $H_{\text{CR}}^*(\mathcal{X})$. More precisely, $\langle \alpha_1, \dots, \alpha_m \rangle_{0,m,d} = 0$ unless $p_1 + \dots + p_m = 2(\dim_{\mathbb{C}} \mathcal{X} + \langle c_1(\mathcal{X}), d \rangle + m - 3)$, where $\alpha_i \in H_{\text{CR}}^{p_i}(\mathcal{X})$.

The genus zero Gromov–Witten invariants define a quantum product \bullet_{τ} on $H_{\text{CR}}^*(\mathcal{X})$ parametrized by $\tau \in H_{\text{CR}}^*(\mathcal{X})$:

$$(2) \quad (\alpha \bullet_{\tau} \beta, \gamma)_{\text{orb}} = \sum_{d \in \text{Eff}_{\mathcal{X}}, m \geq 0} \frac{1}{m!} \left\langle \alpha, \beta, \gamma, \overbrace{\tau, \dots, \tau}^{m \text{ times}} \right\rangle_{0,m+3,d} Q^d.$$

Here Q^d denotes the element of the group ring $\mathbb{C}[\text{Eff}_{\mathcal{X}}]$ corresponding to $d \in \text{Eff}_{\mathcal{X}} \subset H_2(X, \mathbb{Z})$. The right-hand side belongs to $\mathbb{C}[[\tau]][[\text{Eff}_{\mathcal{X}}]]$ (a certain completion² of $\mathbb{C}[[\tau]] \otimes \mathbb{C}[\text{Eff}_{\mathcal{X}}]$) and defines the element $\alpha \bullet_{\tau} \beta$ in $H_{\text{CR}}^*(\mathcal{X}) \otimes \mathbb{C}[[\tau]][[\text{Eff}_{\mathcal{X}}]]$ because the orbifold Poincaré pairing is non-degenerate. By extending \bullet_{τ} as a $\mathbb{C}[[\tau]][[\text{Eff}_{\mathcal{X}}]]$ -bilinear map, we have an associative commutative ring $(H_{\text{CR}}^*(\mathcal{X}) \otimes \mathbb{C}[[\tau]][[\text{Eff}_{\mathcal{X}}]], \bullet_{\tau})$. Here the associativity of the product \bullet_{τ} follows from the WDVV equation. This is the *orbifold quantum cohomology* of \mathcal{X} .

Using the *Divisor equation* (see [1, Theorem 8.3.1]), we can write

$$(3) \quad (\alpha \bullet_{\tau} \beta, \gamma)_{\text{orb}} = \sum_{d \in \text{Eff}_{\mathcal{X}}, m \geq 0} \frac{1}{m!} \left\langle \alpha, \beta, \gamma, \overbrace{\tau', \dots, \tau'}^{m \text{ times}} \right\rangle_{0,m+3,d} e^{\langle \tau_0, 2, d \rangle} Q^d,$$

²For example, one completion is given by the additive valuation v on $\mathbb{C}[\text{Eff}_{\mathcal{X}}]$ defined by $v(Q^d) = \int_d \omega$, where ω is a Kähler form on \mathcal{X} .

where we put

$$(4) \quad \tau = \tau_{0,2} + \tau', \quad \tau_{0,2} \in H^2(\mathcal{X}_0), \quad \tau' \in \bigoplus_{p \neq 2} H^p(\mathcal{X}_0) \oplus \bigoplus_{v \in \mathbb{T}'} H^*(\mathcal{X}_v).$$

This shows that the parameters τ and Q in the product \bullet_τ are redundant. In fact \bullet_τ depends only on τ' and $e^{\tau_{0,2}}Q$. We put

$$\circ_\tau := \bullet_\tau|_{Q=1}.$$

The product \circ_τ is a formal power series in τ' and a formal Fourier series in $\tau_{0,2}$. The original product \bullet_τ can be recovered from \circ_τ by substituting $e^{\langle \tau_{0,2}, d \rangle} Q^d$ for $e^{\langle \tau_{0,2}, d \rangle}$. In what follows, we will study \circ_τ instead of \bullet_τ and assume that the product \circ_τ is convergent in some open set U of $H_{\text{CR}}^*(\mathcal{X})$.

Assumption 2.1. *The orbifold quantum product \circ_τ is convergent on a simply-connected open set U containing the following set*

$$\{\tau \in H_{\text{CR}}^*(\mathcal{X}) ; \Re(\langle d, \tau_{0,2} \rangle) < -M, \forall d \in \text{Eff}_{\mathcal{X}} \setminus \{0\}, \|\tau'\| \leq e^{-M}\},$$

where $\tau = \tau_{0,2} + \tau'$ is the decomposition in (4), $M > 0$ is a sufficiently big real number and $\|\cdot\|$ is a suitable norm on $H_{\text{CR}}^*(\mathcal{X})$.

Remark 2.2. Working over a certain formal power series ring, we could discuss the K -theory integral structure without this assumption. However, when considering Ruan's conjecture later, we cannot avoid the convergence problem of quantum cohomology.

The open set U above is considered to be a neighborhood of the "large radius limit point" which is the limit point of the sequence

$$(5) \quad \tau = \tau_{0,2} + \tau' : \Re(\langle d, \tau_{0,2} \rangle) \rightarrow -\infty, \quad \tau' \rightarrow 0.$$

(This notion will be made more precise later.) In this limit, the orbifold quantum product \circ_τ goes to the *Chen–Ruan orbifold cup product* \cup_{CR} . This product \cup_{CR} is the same as the cup product when \mathcal{X} is a manifold, but in the orbifold case, this is different from the cup product on $\mathcal{I}\mathcal{X}$.

2.2. Quantum D -modules with Galois actions

Let $\{\phi_i\}$ be a homogeneous \mathbb{C} -basis of $H_{\text{CR}}^*(\mathcal{X})$ and $\{t^i\}$ be the linear co-ordinate system on $H_{\text{CR}}^*(\mathcal{X})$ dual to the basis $\{\phi_i\}$. Denote by $\tau = \sum_{i=1}^N t^i \phi_i$ a general point on $H_{\text{CR}}^*(\mathcal{X})$. The quantum D -module is a meromorphic flat connection on the trivial $H_{\text{CR}}^*(\mathcal{X})$ -bundle over $U \times \mathbb{C}$. Denote by (τ, z) a general point on the base space $U \times \mathbb{C}$. Let $(-): U \times \mathbb{C} \rightarrow U \times \mathbb{C}$ be the map sending (τ, z) to $(\tau, -z)$.

Definition 2.3. The *quantum D-module* $QDM(\mathcal{X}) = (F, \nabla, (\cdot, \cdot)_F)$ is the trivial holomorphic vector bundle $F := H_{\text{CR}}^*(\mathcal{X}) \times (U \times \mathbb{C}) \rightarrow (U \times \mathbb{C})$ endowed with the meromorphic flat connection ∇ :

$$(6) \quad \begin{aligned} \nabla_i &= \nabla_{\frac{\partial}{\partial t^i}} = \frac{\partial}{\partial t^i} + \frac{1}{z} \phi_i \circ \tau, \\ \nabla_{z\partial_z} &= z \frac{\partial}{\partial z} - \frac{1}{z} E \circ \tau + \mu, \end{aligned}$$

and the ∇ -flat pairing

$$(\cdot, \cdot)_F: (-)^* \mathcal{O}(F) \otimes \mathcal{O}(F) \rightarrow \mathcal{O}_{U \times \mathbb{C}}$$

induced from the orbifold Poincaré pairing $F_{(\tau, -z)} \times F_{(\tau, z)} = H_{\text{CR}}^*(\mathcal{X}) \times H_{\text{CR}}^*(\mathcal{X}) \rightarrow \mathbb{C}$. Here E is the *Euler vector field* on U given by

$$(7) \quad E := c_1(T\mathcal{X}) + \sum_{i=1}^N \left(1 - \frac{1}{2} \deg \phi_i\right) t^i \phi_i$$

and $\mu \in \text{End}(H_{\text{CR}}^*(\mathcal{X}))$ is the *Hodge grading operator* defined by

$$(8) \quad \mu(\phi_i) := \left(\frac{1}{2} \deg \phi_i - \frac{n}{2}\right) \phi_i, \quad n = \dim_{\mathbb{C}} \mathcal{X}.$$

The flat connection ∇ is called the *Dubrovin connection* or *the first structure connection*. Note that ∇_i has a pole of order 1 along $z = 0$ and ∇_{∂_z} has a pole of order 2 along $z = 0$. The flatness of ∇ follows from the WDVV equations and the homogeneity of Gromov–Witten invariants.

Remark 2.4. By *D-module* one means a module over the ring of differential operators. In our case, the ring $\mathcal{O}_{\mathcal{M} \times \mathbb{C}^*} \langle \partial_{t^i}, z\partial_z \rangle$ of differential operators on $\mathcal{M} \times \mathbb{C}^*$ acts on the space of sections of F via the flat connection: $\partial_{t^i} \mapsto \nabla_i$, $z\partial_z \mapsto \nabla_{z\partial_z}$. This explains the name “quantum D-module”.

The quantum D-module admits certain discrete symmetries (Galois actions). Firstly, since $\circ \tau$ depends only on $e^{\tau_{0,2}}$ and τ' , it is clear that $\circ \tau$ is invariant under the following translation:

$$\tau_{0,2} \mapsto \tau_{0,2} - 2\pi i \xi, \quad \xi \in H^2(X, \mathbb{Z}).$$

This is a consequence of the Divisor equations and is familiar in ordinary Gromov–Witten theory. Interestingly, we have a finer symmetry for orbifold theory. Let $H^2(\mathcal{X}, \mathbb{Z})$ be the sheaf cohomology of the constant sheaf \mathbb{Z} on the stack \mathcal{X} (not on the coarse moduli space X). This

group is identified with the set of isomorphism classes of topological orbifold line bundles on \mathcal{X} . Then $H^2(X, \mathbb{Z})$ is identified with the subset of $H^2(\mathcal{X}, \mathbb{Z})$ consisting of line bundles which are pulled back from the coarse moduli space X . For $\xi \in H^2(\mathcal{X}, \mathbb{Z})$, let L_ξ be the corresponding topological orbifold line bundle on \mathcal{X} and $\xi_0 := c_1(L_\xi) \in H^2(X, \mathbb{Q})$ be the first Chern class. For $v \in \mathbb{T}$, define $0 \leq f_v(\xi) < 1$ to be the rational number such that the stabilizer g at $(x, g) \in \mathcal{X}_v$ acts on the fiber $L_{\xi, x}$ by $\exp(2\pi i f_v(\xi))$.

Lemma 2.5 ([46, Proposition 2.3]). *The flat connection ∇ and the pairing $(\cdot, \cdot)_F$ of the quantum D -module $QDM(\mathcal{X}) = (F, \nabla, (\cdot, \cdot)_F)$ are invariant under the following map given for $\xi \in H^2(\mathcal{X}, \mathbb{Z})$:*

$$\begin{aligned} H_{\text{CR}}^*(\mathcal{X}) \times (U \times \mathbb{C}) &\rightarrow H_{\text{CR}}^*(\mathcal{X}) \times (U \times \mathbb{C}) \\ (\phi, \tau, z) &\mapsto (dG(\xi)(\phi), G(\xi)(\tau), z). \end{aligned}$$

Here $G(\xi), dG(\xi): H_{\text{CR}}^*(\mathcal{X}) \rightarrow H_{\text{CR}}^*(\mathcal{X})$ is defined by

$$\begin{aligned} G(\xi)(\tau_0 \oplus \bigoplus_{v \in \mathbb{T}'} \tau_v) &= (\tau_0 - 2\pi i \xi_0) \oplus \bigoplus_{v \in \mathbb{T}'} e^{2\pi i f_v(\xi)} \tau_v \\ dG(\xi)(\tau_0 \oplus \bigoplus_{v \in \mathbb{T}'} \tau_v) &= \tau_0 \oplus \bigoplus_{v \in \mathbb{T}'} e^{2\pi i f_v(\xi)} \tau_v \end{aligned}$$

where we used the decomposition $H_{\text{CR}}^*(\mathcal{X}) = H^*(\mathcal{X}_0) \oplus \bigoplus_{v \in \mathbb{T}'} H^*(\mathcal{X}_v)$ and $\tau_v \in H^*(\mathcal{X}_v)$. (Here we implicitly assume that U is invariant under the map $G(\xi)$, but we can assume this without loss of generality.)

We refer to the symmetry in Lemma 2.5 as a *Galois action* of $H^2(\mathcal{X}, \mathbb{Z})$ or a *local monodromy at the large radius limit*. When $\xi \in H^2(X, \mathbb{Z})$, this is the same as the aforementioned one. Note that the new symmetry can act non-trivially on the fiber of the quantum D -module. The quantum D -module descends to a flat connection on $F/H^2(\mathcal{X}, \mathbb{Z}) \rightarrow (U/H^2(\mathcal{X}, \mathbb{Z})) \times \mathbb{C}$. We call this flat connection on the quotient space also the *quantum D -module*.

We can construct a partial compactification \bar{V} of the quotient $V = U/H^2(\mathcal{X}, \mathbb{Z})$ such that \bar{V} contains the large radius limit point and that the quantum D -module on V extends to a D -module on \bar{V} with a logarithmic singularity along the normal crossing divisor³ $\bar{V} \setminus V$. Choose a \mathbb{Z} -basis p_1, \dots, p_r of $H^2(X, \mathbb{Z})/\text{torsion}$ such that p_a intersects every effective curve $d \in \text{Eff}_{\mathcal{X}}$ non-negatively (i.e. p_a is nef). Then we

³More precisely, $\bar{V} \setminus V$ is a normal crossing divisor on an étale cover of \bar{V} .

have the embedding

$$U/H^2(X, \mathbb{Z}) \hookrightarrow \mathbb{C}^r \times W, \quad \left[\sum_{a=1}^r t^a p_a + \tau' \right] \mapsto (e^{t^1}, \dots, e^{t^r}, \tau')$$

where $W = \bigoplus_{p \neq 2} H^p(\mathcal{X}_0) \oplus \bigoplus_{v \in \mathcal{T}'} H^*(\mathcal{X}_v)$ and $\tau' \in W$. By Assumption 2.1 and the choice of p_a , the image of this embedding contains the open set $((\mathbb{C}^*)^r \times W) \cap \Delta_M$ for a sufficiently big $M > 0$, where

$$\Delta_M = \{(q^1, \dots, q^r, \tau') \in \mathbb{C}^r \times W; |q^a| < e^{-M}, \|\tau'\| < e^{-M}\}.$$

We set $\overline{U/H^2(X, \mathbb{Z})} := (U/H^2(X, \mathbb{Z})) \cup \Delta_M \subset \mathbb{C}^r \times W$. For $\tau_{0,2} = \sum_{a=1}^r t^a p_a$, we have $e^{(\tau_{0,2}, d)} = (e^{t^1})^{\langle p_1, d \rangle} \dots (e^{t^r})^{\langle p_r, d \rangle}$. Therefore by the formula (3), since p_a is nef, the quantum product \circ_τ on $U/H^2(X, \mathbb{Z})$ extends to $\overline{U/H^2(X, \mathbb{Z})}$. The Dubrovin connection on Δ_M in the direction of $q^a = e^{t^a}$ can be written as

$$\nabla_{\frac{\partial}{\partial t^a}} = q^a \frac{\partial}{\partial q^a} + \frac{1}{z} p_a \circ_\tau.$$

Hence it has a logarithmic pole along $q^1 \dots q^r = 0$. We can now define \overline{V} as the quotient space (or stack):

$$\overline{V} := \overline{U/H^2(X, \mathbb{Z})} / (H^2(\mathcal{X}, \mathbb{Z}) / H^2(X, \mathbb{Z})).$$

This contains both $U/H^2(\mathcal{X}, \mathbb{Z})$ and the large radius limit point $q = \tau' = 0$.

Remark 2.6. The partial compactification \overline{V} depends on the choice of a nef basis p_a . A canonical partial compactification is given by the possibly singular quotient stack $[(\text{Spec } \mathbb{C}[\text{Eff}_{\mathcal{X}}] \times W) / (H^2(\mathcal{X}, \mathbb{Z}) / H^2(X, \mathbb{Z}))]$. Note that there exists a natural map from \overline{V} to this stack.

Remark 2.7. Due to the new discrete symmetries, the large radius limit point in \overline{V} can have an orbifold singularity when \mathcal{X} is an orbifold. Also, the quantum D -module $F/H^2(X, \mathbb{Z})$ on the quotient space may not be trivialized in the standard way. In other words, an element of $H_{\text{CR}}^*(\mathcal{X})$ gives a possibly multi-valued section of $F/H^2(\mathcal{X}, \mathbb{Z})$.

2.3. Fundamental solution $L(\tau, z)$ and the space $\mathcal{S}(\mathcal{X})$ of flat sections

We introduce a *fundamental solution* for ∇ -flat sections of the quantum D -module (F, ∇) . Orbifold Gromov–Witten theory also has (gravitational) descendant invariants (as opposed to the primary invariants

(1)) of the form

$$\left\langle \alpha_1 \psi_1^{k_1}, \dots, \alpha_m \psi_m^{k_m} \right\rangle_{0,m,d}$$

where $\alpha_i \in H_{\text{CR}}^*(\mathcal{X})$, $d \in \text{Eff}_{\mathcal{X}}$ and k_i is a non-negative integer. The symbol ψ_i represents the first Chern class of the line bundle on the moduli space of stable maps formed by the cotangent lines at the i -th marked point of the coarse domain curve. As is well-known in manifold Gromov–Witten theory (see *e.g.* [58, Proposition 2]), we can write the fundamental solution to the equation $\nabla s = 0$ by using descendant invariants. Let $\text{pr}: I\mathcal{X} \rightarrow \mathcal{X}$ be the natural projection. For $\tau_0 \in H^*(\mathcal{X}_0)$, we define the action of τ_0 on $H_{\text{CR}}^*(\mathcal{X})$ as

$$\tau_0 \cdot \alpha = \text{pr}^*(\tau_0) \cup \alpha$$

where the right-hand side is the cup product on $H^*(I\mathcal{X})$. (This is known to be the same as the orbifold cup product $\tau_0 \cup_{\text{CR}} \alpha$.) Let $\{\phi_k\}_{k=1}^N$ and $\{\phi^k\}_{k=1}^N$ be bases of $H_{\text{CR}}^*(\mathcal{X})$ dual with respect to the orbifold Poincaré pairing, *i.e.* $(\phi_i, \phi^j)_{\text{orb}} = \delta_i^j$.

Proposition 2.8 (See *e.g.* [46, Proposition 2.4]). *Let $L(\tau, z)$ be the following $\text{End}(H_{\text{CR}}^*(\mathcal{X}))$ -valued function on $U \times \mathbb{C}^*$:*

$$(9) \quad L(\tau, z)\phi = e^{-\tau_0,2/z}\phi - \sum_{\substack{(d,m) \neq (0,0), \\ d \in \text{Eff}_{\mathcal{X}}, 1 \leq k \leq N}} \frac{e^{\langle \tau_0, 2, d \rangle}}{m!} \phi_k \left\langle \phi^k, \tau', \dots, \tau', \frac{e^{-\tau_0,2/z}\phi}{z + \psi_{m+2}} \right\rangle_{0,m+2,d},$$

where $\tau = \tau_{0,2} + \tau'$ is the decomposition in (4) and $1/(z + \psi_{m+2})$ in the correlator should be expanded in the z^{-1} -series $\sum_{k \geq 0} (-1)^k z^{-k-1} \psi_{m+2}^k$. Set $\rho := c_1(\mathcal{X}) \in H^2(\mathcal{X}_0)$ and

$$z^{-\mu} z^\rho := \exp(-\mu \log z) \exp(\rho \log z), \quad \mu \text{ is given in (8)}.$$

Then we have

$$\begin{aligned} \nabla_i(L(\tau, z)z^{-\mu}z^\rho\phi) &= 0, & \nabla_{z\partial_z}(L(\tau, z)z^{-\mu}z^\rho\phi) &= 0, \\ (L(\tau, -z)\phi_i, L(\tau, z)\phi_j)_{\text{orb}} &= (\phi_i, \phi_j)_{\text{orb}}. \end{aligned}$$

In particular, $s_i(\tau, z) = L(\tau, z)z^{-\mu}z^\rho\phi_i$, $1 \leq i \leq N$, form a basis of multi-valued ∇ -flat sections of F satisfying the asymptotic initial condition at the large radius limit (5):

$$s_i(\tau, z) \sim z^{-\mu} z^\rho e^{-\tau_{0,2}} \phi_i.$$

Remark 2.9. The convergence of the fundamental solution $L(\tau, z)$ is not a priori clear. Under the Assumption 2.1, however, we know that $L(\tau, z)$ also converges on $U \times \mathbb{C}^*$ because this is a solution to the linear partial differential equations $\nabla s = 0$.

Definition 2.10. Define $\mathcal{S}(\mathcal{X})$ to be the space of multi-valued ∇ -flat sections of the quantum D -module $QDM(\mathcal{X}) = (F, \nabla, (\cdot, \cdot)_F)$:

$$\mathcal{S}(\mathcal{X}) := \{s(\tau, z) \in \Gamma(U \times \widetilde{\mathbb{C}}^*, \mathcal{O}(F)) ; \nabla s = 0\}.$$

This is a \mathbb{C} -vector space with $\dim_{\mathbb{C}} \mathcal{S}(\mathcal{X}) = \dim_{\mathbb{C}} H_{\text{CR}}^*(\mathcal{X})$. $\mathcal{S}(\mathcal{X})$ is endowed with the pairing $(\cdot, \cdot)_{\mathcal{S}}$:

$$(s_1, s_2)_{\mathcal{S}} := (s_1(\tau, e^{\pi i} z), s_2(\tau, z))_{\text{orb}} \in \mathbb{C},$$

where $s_1(\tau, e^{\pi i} z)$ denotes the parallel translate of $s_1(\tau, z)$ along the counterclockwise path $[0, 1] \ni \theta \mapsto e^{i\pi\theta} z$. Because s_1, s_2 are flat sections, the right-hand side is a complex number independent of (τ, z) . $\mathcal{S}(\mathcal{X})$ is also equipped with the automorphism $G^{\mathcal{S}}(\xi)$ for $\xi \in H^2(\mathcal{X}, \mathbb{Z})$ induced from the Galois action in Lemma 2.5:

$$G^{\mathcal{S}}(\xi): \mathcal{S}(\mathcal{X}) \rightarrow \mathcal{S}(\mathcal{X}), \quad s(\tau, z) \mapsto dG(\xi)(s(G(\xi)^{-1}\tau, z)).$$

In general, $(\cdot, \cdot)_{\mathcal{S}}$ is neither symmetric nor anti-symmetric. When \mathcal{X} is Calabi–Yau, *i.e.* $\rho = c_1(\mathcal{X}) = 0$, $(\cdot, \cdot)_{\mathcal{S}}$ is symmetric when $n = \dim_{\mathbb{C}} \mathcal{X}$ is even and is anti-symmetric when n is odd.

The fundamental solution in Proposition 2.8 gives the *cohomology framing* \mathcal{Z}_{coh} of $\mathcal{S}(\mathcal{X})$:

$$(10) \quad \mathcal{Z}_{\text{coh}}: H_{\text{CR}}^*(\mathcal{X}) \xrightarrow{\cong} \mathcal{S}(\mathcal{X}), \quad \phi \mapsto L(\tau, z)z^{-\mu}z^{\rho}\phi.$$

In terms of this cohomology framing \mathcal{Z}_{coh} , it is easy to check that the pairing and Galois actions on $\mathcal{S}(\mathcal{X})$ can be written as follows:

$$(11) \quad \begin{aligned} (\mathcal{Z}_{\text{coh}}(\alpha), \mathcal{Z}_{\text{coh}}(\beta))_{\mathcal{S}} &= (e^{\pi i \rho} \alpha, e^{\pi i \mu} \beta)_{\text{orb}} \\ G^{\mathcal{S}}(\xi)(\mathcal{Z}_{\text{coh}}(\alpha)) &= \mathcal{Z}_{\text{coh}}\left(\bigoplus_{v \in \mathbb{T}} e^{-2\pi i \xi_0} e^{2\pi i f_v(\xi)}\right) \alpha \end{aligned}$$

where we used the decomposition $H_{\text{CR}}^*(\mathcal{X}) = \bigoplus_{v \in \mathbb{T}} H^*(\mathcal{X}_v)$ in the second line. (See the paragraph before Lemma 2.5 for $\xi_0 \in H^2(\mathcal{X}_0)$ and $f_v(\xi) \in [0, 1)$.)

2.4. K -theory integral lattice of flat sections

We will introduce an integral lattice in the space $\mathcal{S}(\mathcal{X})$ of flat sections using the K -group and the characteristic class called the $\widehat{\Gamma}$ -class. Let $K(\mathcal{X})$ be the Grothendieck group of topological orbifold vector bundles over \mathcal{X} (see *e.g.* [2] for orbifold vector bundles and orbifold K -theory). For simplicity, we assume that \mathcal{X} is isomorphic to a quotient orbifold $[M/G]$ as a topological orbifold, where M is a compact manifold and G is a compact Lie group acting on M with at most finite stabilizers. Under this assumption, $K(\mathcal{X})$ is isomorphic to the G -equivariant K -theory $K_G^0(M)$ and is a finitely generated abelian group [2]. For an orbifold vector bundle V on $I\mathcal{X}$ and a component \mathcal{X}_v of $I\mathcal{X}$, we denote the eigenbundle decomposition of $V|_{\mathcal{X}_v}$ with respect to the stabilizer action as follows:

$$V|_{\mathcal{X}_v} = \bigoplus_{0 \leq f < 1} V_{v,f},$$

where the stabilizer of \mathcal{X}_v acts on $V_{v,f}$ by $\exp(2\pi i f)$. The Chern character $\widetilde{\text{ch}}: K(\mathcal{X}) \rightarrow H^*(I\mathcal{X})$ for orbifold vector bundles is defined as follows:

$$\widetilde{\text{ch}}(V) := \bigoplus_{v \in \mathbb{T}} \sum_{0 \leq f < 1} e^{2\pi i f} \text{ch}((\text{pr}^* V)_{v,f}),$$

where $\text{pr}: I\mathcal{X} \rightarrow \mathcal{X}$ is the natural projection. For an orbifold vector bundle V on \mathcal{X} , let $\delta_{v,f,i}$, $i = 1, \dots, l_{v,f}$ be the Chern roots of the vector bundle $(\text{pr}^* V)_{v,f}$ on \mathcal{X}_v (where $l_{v,f} = \text{rank}(\text{pr}^* V)_{v,f}$). The Todd class $\widetilde{\text{Td}}(V)$ is defined by

$$\widetilde{\text{Td}}(V) := \bigoplus_{v \in \mathbb{T}} \prod_{0 < f < 1, 1 \leq i \leq l_{v,f}} \frac{1}{1 - e^{-2\pi i f} e^{-\delta_{v,f,i}}} \prod_{f=0, 1 \leq i \leq l_{v,0}} \frac{\delta_{v,0,i}}{1 - e^{-\delta_{v,0,i}}}.$$

When the orbifold vector bundle V admits the structure of a holomorphic orbifold vector bundle, the holomorphic Euler characteristic $\chi(V) := \sum_{i=1}^n (-1)^i \dim_{\mathbb{C}} H^i(\mathcal{X}, V)$ is given by the Kawasaki–Riemann–Roch formula [52]:

$$(12) \quad \chi(V) = \int_{I\mathcal{X}} \widetilde{\text{ch}}(V) \cup \widetilde{\text{Td}}(T\mathcal{X}).$$

Note that $\chi(V)$ is an integer by definition. For a (not necessarily holomorphic) topological orbifold vector bundle V on \mathcal{X} , we define $\chi(V)$ to be the right-hand side of the above formula (12). It follows from Kawasaki's index theorem [53] for elliptic operators on orbifolds that $\chi(V)$ is an integer for any V . In fact, the right-hand side of (12) equals

the index of an elliptic operator $\bar{\partial} + \bar{\partial}^* : V \otimes \mathcal{A}_{\mathcal{X}}^{0,\text{even}} \rightarrow V \otimes \mathcal{A}_{\mathcal{X}}^{0,\text{odd}}$, where $\bar{\partial}$ is a not necessarily integrable $(0, 1)$ -connection on V and $\bar{\partial}^*$ is its adjoint with respect to a hermitian metric on V .

Define a multiplicative characteristic class $\widehat{\Gamma} : K(\mathcal{X}) \rightarrow H^*(I\mathcal{X})$ as follows:

$$\widehat{\Gamma}(V) := \bigoplus_{v \in \mathbb{T}} \prod_{0 \leq f < 1} \prod_{i=1}^{l_{v,f}} \Gamma(1 - f + \delta_{v,f,i}).$$

Here $\delta_{v,f,i}$ is the same as above. The Gamma function in the right-hand side should be expanded in Taylor series at $1 - f > 0$. The $\widehat{\Gamma}$ -class can be viewed as a “square root” of the Todd class (more precisely, \widehat{A} -class). In fact, using the Gamma function equality $\Gamma(z)\Gamma(1 - z) = \pi / \sin(\pi z)$, we find

$$\begin{aligned} & [(e^{\pi i \deg / 2} \widehat{\Gamma}(V)) \cup \text{inv}^* \widehat{\Gamma}(V)]_v \cup e^{\pi i (c_1(\text{pr}^* V)|_{\mathcal{X}_v} + \text{age}_v(V))} \\ &= (2\pi i)^{\sum_{f \neq 0} l_{v,f}} [(2\pi i)^{\deg / 2} \widetilde{\text{Td}}(V)]_v, \end{aligned}$$

where $\deg : H^*(I\mathcal{X}) \rightarrow H^*(I\mathcal{X})$ is the ordinary grading operator defined by $\deg = p$ on $H^p(I\mathcal{X})$, $\text{age}_v(V) = \sum_{0 < f < 1} f l_{v,f}$ is the age of V along \mathcal{X}_v , and $[\cdot \cdot \cdot]_v$ is the $H^*(\mathcal{X}_v)$ -component. In this sense⁴, the K -group framing $\mathcal{Z}_K : K(\mathcal{X}) \rightarrow \mathcal{S}(\mathcal{X})$ below can be considered as a “Mukai vector” in quantum cohomology.

Definition 2.11. We define the K -group framing $\mathcal{Z}_K : K(\mathcal{X}) \rightarrow \mathcal{S}(\mathcal{X})$ of the space $\mathcal{S}(\mathcal{X})$ of flat sections by the formula:

$$(13) \quad \begin{aligned} \mathcal{Z}_K(V) &:= \mathcal{Z}_{\text{coh}}(\Psi(V)) = L(\tau, z) z^{-\mu} z^\rho \Psi(V), \\ \text{where } \Psi(V) &:= (2\pi)^{-\frac{n}{2}} \widehat{\Gamma}(T\mathcal{X}) \cup (2\pi i)^{\deg / 2} \text{inv}^* \widetilde{\text{ch}}(V). \end{aligned}$$

Here \mathcal{Z}_{coh} is the cohomology framing (10), $L(\tau, z) z^{-\mu} z^\rho$ is the fundamental solution in Proposition 2.8, $(2\pi i)^{\deg / 2} \in \text{End}(H^*(I\mathcal{X}))$ is defined by $(2\pi i)^{\deg / 2}|_{H^{2p}(I\mathcal{X})} = (2\pi i)^p$ and $\widehat{\Gamma}(T\mathcal{X}) \cup$ is the cup product in $H^*(I\mathcal{X})$. The image $\mathcal{S}(\mathcal{X})_{\mathbb{Z}} := \mathcal{Z}_K(K(\mathcal{X})) \subset \mathcal{S}(\mathcal{X})$ of the K -group framing is called the K -theory integral structure on the quantum cohomology.

The notation \mathcal{Z}_K for the K -group framing is motivated by the *central charge* in physics. Conjecturally, the integral

$$(14) \quad \mathcal{Z}(V) := c(z) \int_{\mathcal{X}} \mathcal{Z}_K(V)(\tau, z) = c(z) (\mathbf{1}, \mathcal{Z}_K(V)(\tau, z))_{\text{orb}}$$

⁴A Mukai vector is given by $\text{ch}(V) \sqrt{\text{Td}(TX)}$ for a vector bundle V on X .

with $c(z) = (2\pi z)^{\frac{3}{2}} / (2\pi i)^n$, $n = \dim \mathcal{X}$ gives the central charge of a B-type D-brane in the class V at the point τ of the (extended) Kähler moduli space. This plays a central role in stability conditions on the derived category $D_{\text{coh}}^b(\mathcal{X})$ [30, 8]. It would be very interesting to find an intrinsic explanation for the formula (13) from this point of view. In the language of quantum D -modules, $Z(V)$ is a coefficient of the unit section **1** expressed in a ∇ -flat frame.

Proposition 2.12 ([46, Proposition 2.10]). (i) *The image $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ of the K -group framing \mathcal{Z}_K is a lattice in $\mathcal{S}(\mathcal{X})$:*

$$\mathcal{S}(\mathcal{X})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \mathcal{S}(\mathcal{X}).$$

(ii) *The pairing $(\cdot, \cdot)_{\mathcal{S}}$ on $\mathcal{S}(\mathcal{X})$ corresponds to the Mukai pairing on $K(\mathcal{X})$ through the K -group framing \mathcal{Z}_K :*

$$(\mathcal{Z}_K(V_1), \mathcal{Z}_K(V_2))_{\mathcal{S}} = \chi(V_1 \otimes V_2^{\vee}).$$

Therefore, we have a \mathbb{Z} -valued pairing $\mathcal{S}(\mathcal{X})_{\mathbb{Z}} \times \mathcal{S}(\mathcal{X})_{\mathbb{Z}} \rightarrow \mathbb{Z}$.

(iii) *For $\xi \in H^2(\mathcal{X}, \mathbb{Z})$, the Galois action $G^{\mathcal{S}}(\xi)$ on $\mathcal{S}(\mathcal{X})$ corresponds to the tensor by the orbifold line bundle L_{ξ}^{\vee} (corresponding to $-\xi$) on $K(\mathcal{X})$:*

$$\mathcal{Z}_K(L_{\xi}^{\vee} \otimes V) = G^{\mathcal{S}}(\xi)(\mathcal{Z}_K(V)).$$

In particular, the lattice $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ is invariant under the Galois action.

The statement (i) follows from the Adem–Ruan decomposition theorem [2, Theorem 5.1], which implies that $\tilde{\text{ch}}: K(\mathcal{X}) \rightarrow H^*(I\mathcal{X})$ is an isomorphism when tensored with \mathbb{C} . The statements (ii) and (iii) follow from straightforward calculations. It is somewhat surprising that many complicated terms finally give the Mukai pairing in (ii) via the Kawasaki–Riemann–Roch formula (12).

Remark 2.13. The formula (13) arose in [45, 46] from the study of mirror symmetry for toric orbifolds. The mirror Landau–Ginzburg model has the natural integral structure and we can shift it to the quantum cohomology. Katzarkov–Kontsevich–Pantev [49] also proposed essentially the same definition (for a rational structure) when \mathcal{X} is a manifold. Closely related results have been observed in the context of mirror symmetry. Calculations and conjectures of Hosono [42], [43, Conjecture 6.3] are compatible with the integral structure above; the works of Horja [40, 41] and Borisov–Horja [7] strongly suggest a relation between K -group and quantum D -module.

Example 2.14. (i) $\mathcal{X} = \mathbb{P}^1$. Let $\omega \in H^2(\mathbb{P}^1, \mathbb{Z})$ be the integral Kähler class. We take $1, \omega$ as a basis of $H^*(\mathbb{P}^1)$. In terms of the cohomology framing $\mathcal{Z}_{\text{coh}}: H^*(\mathbb{P}^1) \cong \mathcal{S}(\mathbb{P}^1)$ in (10), the Galois action and the pairing on $\mathcal{S}(\mathbb{P}^1)$ is represented by the matrices:

$$G^{\mathcal{S}}(\omega) = \begin{bmatrix} 1 & 0 \\ -2\pi i & 1 \end{bmatrix}, \quad (\cdot, \cdot)_{\mathcal{S}} = \begin{bmatrix} 2\pi & i \\ -i & 0 \end{bmatrix}.$$

If an integral lattice L in $H^*(\mathbb{P}^1) \cong \mathcal{S}(\mathbb{P}^1)$ is invariant under $G^{\mathcal{S}}(\omega)$ and if the restriction of $(\cdot, \cdot)_{\mathcal{S}}$ to L gives a perfect pairing $L \times L \rightarrow \mathbb{Z}$, then L must take the following form:

$$L = \mathbb{Z}\sqrt{\frac{n}{2\pi}}(1 + c\omega) \oplus \mathbb{Z}i\sqrt{\frac{2\pi}{n}}\omega$$

for some $n \in \mathbb{Z} \setminus \{0\}$ and $c \in \mathbb{C}$. The K -theory integral structure corresponds to the choice $n = 1$ and $c = -2\gamma$, where $\gamma = 0.57721\dots$ is Euler’s constant (coming from the $\widehat{\Gamma}$ -class $\widehat{\Gamma}(T\mathbb{P}^1) = 1 - 2\gamma\omega$).

(ii) When $\mathcal{X} = X$ is a Calabi–Yau threefold, the $\widehat{\Gamma}$ class is given by

$$\widehat{\Gamma}(TX) = 1 - \frac{\pi^2}{6}c_2(X) - \zeta(3)c_3(X)$$

where $\zeta(3)$ is the special value of Riemann’s zeta function. From this, it follows that the central charges (14) of \mathcal{O}_{pt} , \mathcal{O}_C , \mathcal{O}_S and \mathcal{O} (for any smooth curve C and surface S) restricted to $H^2(X)$ are

$$\begin{aligned} Z(\mathcal{O}_{\text{pt}}) &= 1, \\ Z(\mathcal{O}_C) &= ((1 - g(C)) - \frac{\tau}{2\pi i}) \cap [C], \\ Z(\mathcal{O}_S) &= \frac{[S]^3}{8} + \frac{\chi(S)}{24} + \frac{\tau}{2\pi i} \cap \frac{[S]^2}{2} + \frac{d_{[S]}F_0(\tau)}{(2\pi i)^2}, \\ Z(\mathcal{O}) &= -\frac{\zeta(3)}{(2\pi i)^3}\chi(X) - \frac{\tau}{2\pi i} \cdot \frac{c_2(X)}{24} + \frac{H(\tau)}{(2\pi i)^3}, \end{aligned}$$

where $\tau = \tau_{0,2} \in H^2(X)$, $g(C)$ is the genus of C , and $\chi(X)$ and $\chi(S)$ are the Euler numbers of X and S . $F_0(\tau)$ is the genus zero potential of X

$$F_0(\tau) := \frac{1}{6} \int_X \tau^3 + \sum_{d \in \text{Eff}_X \setminus \{0\}} \langle \rangle_{0,0,d} e^{\langle \tau, d \rangle},$$

$d_{[S]}F_0$ is its derivative in the direction of the Poincaré dual of $[S]$ and $H(\tau) := 2F_0(\tau) - \sum_i t^i \partial_i F_0(\tau)$. The zeta value $\zeta(3)$ also appeared in the quintic mirror calculation of Candelas–de la Ossa–Green–Parkes [16].

(iii) When \mathcal{X} is a weak Fano compact toric orbifold, it is shown in [45], [46, Theorem 4.13] that the central charge of the structure sheaf can be written as an oscillating integral of the mirror Landau–Ginzburg model $W_\tau: (\mathbb{C}^*)^n \rightarrow \mathbb{C}$:

$$Z(\mathcal{O}_{\mathcal{X}})(\tau, z) = \frac{1}{(2\pi i)^n} \int_{\Gamma_{\mathbb{R}} \subset (\mathbb{C}^*)^n} e^{-W_\tau(y)/z} \frac{dy}{y}, \quad n = \dim_{\mathbb{C}} \mathcal{X}.$$

Here dy/y is an invariant holomorphic n -form on $(\mathbb{C}^*)^n$ and $\Gamma_{\mathbb{R}}$ is a non-compact cycle (Lefschetz thimble) in $(\mathbb{C}^*)^n$. (Strictly speaking, we need a “mirror map” between $\tau \in H_{\text{CR}}^2(\mathcal{X})$ in the left-hand side and the parameter τ in the Landau–Ginzburg potential W_τ .) Under mirror symmetry, the unit section $\mathbf{1}$ of the quantum D -module corresponds to the oscillating form $e^{-W_\tau/z}(dy/y)$ in the Landau–Ginzburg model. Thus the formula above shows that the structure sheaf $\mathcal{O}_{\mathcal{X}}$ corresponds to the Lefschetz thimble $\Gamma_{\mathbb{R}}$. Moreover, it turns out that the K -theory integral structure in Definition 2.11 corresponds to the lattice of Lefschetz thimbles under mirror symmetry. See [45, 46] for more details.

(iv) The $\widehat{\Gamma}$ -class contains odd zeta values $\zeta(3), \zeta(5), \dots$ and products of Gamma values. When \mathcal{X} is holomorphic symplectic, however, the $\widehat{\Gamma}$ -class is defined over $\mathbb{Q}(\zeta)[\pi]$ for some root of unity ζ . This could be related to the fact that there is no quantum correction⁵.

Remark 2.15. We can consider the Grothendieck group of algebraic vector bundles or coherent sheaves on \mathcal{X} instead of topological K -groups. In this case, the K -theory integral structure is defined on the algebraic part of the orbifold cohomology $H_{\text{CR}}^*(\mathcal{X})$, *i.e.* cohomology classes on $I\mathcal{X}$ which can be written as linear combinations of Poincaré duals of algebraic cycles with complex coefficients. The algebraic part of orbifold quantum cohomology makes sense due to the algebraic construction of orbifold Gromov–Witten theory [1]. A theoretical difficulty is that we do not know if the orbifold Poincaré pairing is non-degenerate when restricted to the algebraic part of $H_{\text{CR}}^*(\mathcal{X})$: This would be a consequence of the famous Hodge conjecture/Grothendieck standard conjecture. Apart from this point, many discussions in this paper can be equally applied to *algebraic* K -theory integral structures.

2.5. Remark on non-compact case

Even when the space \mathcal{X} is non-compact, we can sometimes define the (orbifold) quantum cohomology. Non-compact local cases are important in the study of Ruan’s conjecture. One standard way is to use

⁵In toric examples [27, 23], the zeta-values appear in the analytic continuation of the J -function, a generating function of Gromov–Witten invariants.

the *torus-equivariant* Gromov–Witten theory. If \mathcal{X} admits a torus action and the fixed point set is compact, we can define torus-equivariant orbifold Gromov–Witten invariants using the Atiyah–Bott style localization on the moduli space of stable maps [35]. In good cases, we can take the non-equivariant limit and obtain the non-equivariant quantum cohomology. In general, we can define Gromov–Witten invariants if the moduli space of stable maps to \mathcal{X} is compact⁶. More generally, even when the moduli space may not be compact, if the evaluation map from the moduli space to the inertia stack $I\mathcal{X}$ is proper, we can define the quantum product by the push-forward by the evaluation map at the “last” marked point. As suggested in [14], this happens for example when X is semi-projective, *i.e.* projective over an affine scheme. In this section, assuming the existence of a well-defined orbifold quantum cohomology for a non-compact space, we describe a possible framework for K -theory integral structures in this case.

Assume that the (non-equivariant) quantum cohomology of \mathcal{X} is well-defined. Quantum cohomology defines the Dubrovin connection and the quantum D -module in the same fashion as in Definition 2.3. The discrete Galois symmetry in Lemma 2.5 is also well-defined. A problem in the non-compact case is that the orbifold Poincaré pairing on $H_{\text{CR}}^*(\mathcal{X})$ is degenerate. However, we have a non-degenerate pairing between $H_{\text{CR}}^*(\mathcal{X})$ and the *compactly supported* orbifold cohomology $H_{\text{CR},c}^*(\mathcal{X})$, which is defined to be the direct sum of compactly supported cohomology groups of the inertia components \mathcal{X}_v (with the same grading shift as before):

$$(\cdot, \cdot)_{\text{orb}}: H_{\text{CR},c}^*(\mathcal{X}) \times H_{\text{CR}}^*(\mathcal{X}) \rightarrow \mathbb{C}.$$

This pairing defines the *dual Dubrovin connection* on the $H_{\text{CR},c}^*(\mathcal{X})$ -bundle $F_c := H_{\text{CR},c}^*(\mathcal{X}) \times (U \times \mathbb{C}) \rightarrow U \times \mathbb{C}$:

$$\begin{aligned} \nabla_i &= \frac{\partial}{\partial t^i} + \frac{1}{z}(\phi_i \circ \tau)^\dagger, \\ \nabla_{z\partial_z} &= z \frac{\partial}{\partial z} - \frac{1}{z}(E \circ \tau)^\dagger + \mu \end{aligned}$$

where $(\phi_i \circ \tau)^\dagger, (E \circ \tau)^\dagger \in \text{End}(H_{\text{CR},c}^*(\mathcal{X}))$ are the adjoint operators with respect to $(\cdot, \cdot)_{\text{orb}}$. We call (F_c, ∇) the *compactly supported quantum*

⁶However, the degree zero moduli space always has a non-compact component, so we indeed need that the evaluation map is proper as stated. This is particularly relevant to the orbifold case where degree zero moduli spaces give a lot of non-trivial Gromov–Witten invariants.

D-module. Note that the dual product $(\phi_i \circ_\tau)^\dagger$ is defined by essentially the same formula as the original product. In fact, $(\alpha \circ_\tau \beta, \gamma)_{\text{orb}} = (\alpha, (\beta \circ_\tau)^\dagger \gamma)_{\text{orb}}$ may be defined by the right-hand side of (2) with $\alpha, \beta \in H_{\text{CR}}^*(\mathcal{X})$ and $\gamma \in H_{\text{CR},c}^*(\mathcal{X})$ (under the assumption that the evaluation map is proper). Tautologically, one has a ∇ -flat pairing:

$$(-)^* \mathcal{O}(F_c) \otimes \mathcal{O}(F) \rightarrow \mathcal{O}_{U \times \mathbb{C}}$$

induced from the orbifold Poincaré pairing, where we recall that $(-): U \times \mathbb{C} \rightarrow U \times \mathbb{C}$ is the map sending (τ, z) to $(\tau, -z)$. One has a natural map

$$(F_c, \nabla) \rightarrow (F, \nabla)$$

induced from $H_{\text{CR},c}^*(\mathcal{X}) \rightarrow H_{\text{CR}}^*(\mathcal{X})$. The fundamental solution in Proposition 2.8 also makes sense. We have two fundamental solutions $\tilde{L}(\tau, z)$, $L(\tau, z)$ taking values in $\text{End}(H_{\text{CR},c}^*(\mathcal{X}))$ and $\text{End}(H_{\text{CR}}^*(\mathcal{X}))$ respectively such that

$$\begin{aligned} \nabla(\tilde{L}(\tau, z)z^{-\mu}z^\rho\varphi) &= 0, & \nabla(L(\tau, z)z^{-\mu}z^\rho\phi) &= 0, \\ (\tilde{L}(\tau, -z)\varphi, L(\tau, z)\phi)_{\text{orb}} &= (\varphi, \phi)_{\text{orb}}, \end{aligned}$$

where $\varphi \in H_{\text{CR},c}^*(\mathcal{X})$ and $\phi \in H_{\text{CR}}^*(\mathcal{X})$. Here again, $\tilde{L}(\tau, z)$ and $L(\tau, z)$ can be defined by the same formula (9), with different domains of definitions⁷. The spaces $\mathcal{S}(\mathcal{X})$, $\mathcal{S}_c(\mathcal{X})$ of multi-valued flat sections of F , F_c are defined in the same way as in Definition 2.10. The symmetries in Lemma 2.5 act on these spaces as automorphisms preserving the pairing:

$$(\cdot, \cdot)_{\mathcal{S}}: \mathcal{S}_c(\mathcal{X}) \times \mathcal{S}(\mathcal{X}) \rightarrow \mathbb{C}, \quad (s_1, s_2) \mapsto (s_1(\tau, e^{\pi i} z), s_2(\tau, z))_{\text{orb}}.$$

Likewise, the formula (13) defines K -group framings

$$\mathcal{Z}_K: K(\mathcal{X}) \rightarrow \mathcal{S}(\mathcal{X}), \quad \mathcal{Z}_{K,c}: K_c(\mathcal{X}) \rightarrow \mathcal{S}_c(\mathcal{X})$$

where $K_c(\mathcal{X})$ is the *compactly supported* K -group. (We need to use $\tilde{L}(\tau, z)$ instead of $L(\tau, z)$ in (13) for the compact support version.) For example, when \mathcal{X} is of the form M/G , one can define $K_c(\mathcal{X})$ as the G -equivariant reduced K -group $\tilde{K}_G^0(M^+)$ of the one-point compactification M^+ of M (as in [69]). One can also use the Grothendieck group $K_{\mathcal{Z}}(\mathcal{X})$ of coherent sheaves on \mathcal{X} supported on a compact set Z . In non-compact

⁷Here one of the dual pairs $\{\phi_k\}, \{\phi^k\}$ in (9) should be taken from $H_{\text{CR},c}^*(\mathcal{X})$ and the other from $H_{\text{CR}}^*(\mathcal{X})$. We take $\phi^k \in H_{\text{CR},c}^*(\mathcal{X})$ when defining $L(\tau, z)$ and $\phi_k \in H_{\text{CR},c}^*(\mathcal{X})$ when defining $\tilde{L}(\tau, z)$.

case, the definition of $K(\mathcal{X})$ may be subject to change *e.g.* we may need to include perfect complexes or infinite dimensional bundles *cf.* [71]. We will not pursue a more precise formulation here. Note that we have a well-defined central charge $Z(V) := c(z) \int_{\mathcal{X}} \mathcal{Z}_{K,c}(V)$ for $V \in K_c(\mathcal{X})$.

Example 2.16 (*cf.* [45, Example 6.5], [46, Section 5.4]). (i) $\mathcal{X} = [\mathbb{C}^2/G]$ where G is a finite subgroup of $SL(2, \mathbb{C})$. The inertia stack $I\mathcal{X}$ is given by

$$I\mathcal{X} = \mathcal{X} \sqcup \bigsqcup_{(g) \neq 1} \mathcal{X}_{(g)}, \quad \mathcal{X}_{(g)} = [\{0\}/C(g)] \quad (g \neq 1),$$

where (g) is a conjugacy class of $g \in G$, and $C(g)$ is the centralizer of g in G . Let $\mathbf{1}$ be the unit class supported on \mathcal{X} and $\mathbf{1}_{(g)} \in H_{\text{CR}}^*(\mathcal{X})$ be the unit class supported on $\mathcal{X}_{(g)}$. The grading is given by

$$\deg \mathbf{1} = 0, \quad \deg \mathbf{1}_{(g)} = 2 \quad (g \neq 1).$$

Since \mathcal{X} is holomorphic symplectic, there is no quantum deformation and \circ_{τ} is trivial: $\mathbf{1} \circ_{\tau} \mathbf{1}_{(g)} = \mathbf{1}_{(g)}$ and all other products are zero. (We can have non-trivial quantum cohomology by considering the equivariant version.) The $\widehat{\Gamma}$ -class is given by

$$\widehat{\Gamma}(T\mathcal{X}) = \mathbf{1} \oplus \bigoplus_{(g) \neq (1)} \frac{\pi}{\sin(\pi f_g)} \mathbf{1}_{(g)} \in H^0(I\mathcal{X})$$

where $0 \leq f_g \leq 1/2$ is the rational number such that the eigenvalues of $g \in SL(2, \mathbb{C})$ are $\exp(\pm 2\pi i f_g)$. Let $\beta, \mathbf{1}_{(g)}$ ($g \neq 1$) be compactly supported cohomology classes on $\mathcal{X}, \mathcal{X}_{(g)}$ such that

$$(\beta, \mathbf{1})_{\text{orb}} = \frac{1}{|G|}, \quad (\mathbf{1}_{(g)}, \mathbf{1}_{(g^{-1})})_{\text{orb}} = \frac{1}{|C(g)|} \quad (g \neq 1).$$

Here $\deg \beta = 4$. We consider the Grothendieck group $K_0^G(\mathbb{C}^2)$ of G -equivariant coherent sheaves on \mathbb{C}^2 supported at the origin. A finite dimensional representation ϱ of G defines a G -equivariant sheaf $\mathcal{O}_0 \otimes \varrho$ on \mathbb{C}^2 . These sheaves generate $K_0^G(\mathbb{C}^2)$ and the Galois action corresponds to the tensor product by a one-dimensional representation. By the equivariant Koszul resolution:

$$0 \rightarrow \mathcal{O}_{\mathbb{C}^2} \otimes \varrho \rightarrow \mathcal{O}_{\mathbb{C}^2} \otimes \varrho \otimes Q^{\vee} \rightarrow \mathcal{O}_{\mathbb{C}^2} \otimes \varrho \rightarrow \mathcal{O}_0 \otimes \varrho \rightarrow 0,$$

where $Q = \mathbb{C}^2$ is the standard G -representation defined by the inclusion $G \subset SL(2, \mathbb{C})$, we compute the Chern character as

$$\widetilde{\text{ch}}(\mathcal{O}_0 \otimes \varrho) = (\dim \varrho) \beta \oplus \bigoplus_{(g) \neq (1)} \text{Tr}(g|_{\varrho} \otimes (\mathbb{C}^2 - Q)) \mathbf{1}_{(g)} \in H_c^*(I\mathcal{X}).$$

Here $\text{Tr}(g|\varrho \otimes (\mathbb{C}^2 - Q))$ is the trace of g on the virtual representation $\varrho \otimes (\mathbb{C}^2 - Q)$. Therefore, using $\tilde{L}(\tau, z) = \exp(-(\tau \circ \tau)^\dagger / z)$, we find

$$(15) \quad Z(\mathcal{O}_0 \otimes \varrho) = e^{-t^0/z} \left(\frac{\dim \varrho}{|G|} + \sum_{(g) \neq 1} \frac{\text{Tr}(g|\varrho) \sin(\pi f_g)}{|C(g)|\pi} t^{(g)} \right),$$

where we put $\tau = t^0 \mathbf{1} + \sum_{(g) \neq 1} t^{(g)} \mathbf{1}_{(g)}$. The simplest central charge is given by the regular representation ϱ_{reg} :

$$Z(\mathcal{O}_0 \otimes \varrho_{\text{reg}}) = e^{-t^0/z}.$$

The vector $[\mathcal{O}_0 \otimes \varrho_{\text{reg}}] \in K_0^G(\mathbb{C}^2)$ is invariant under every Galois action.

(ii) $\mathcal{X} = \mathbb{C}^3/G$ where G is a finite subgroup of $SL(3, \mathbb{C})$. Unlike the previous case (i), \mathcal{X} can have a non-trivial (non-equivariant) quantum cohomology. The inertia stack $I\mathcal{X}$ is given by

$$I\mathcal{X} = \mathcal{X} \sqcup \bigsqcup_{(g) \neq (1)} \mathcal{X}_{(g)}, \quad \mathcal{X}_{(g)} = [(\mathbb{C}^3)^g / C(g)],$$

where $(\mathbb{C}^3)^g \subset \mathbb{C}^3$ is the subspace fixed by g . The ordinary and compactly supported orbifold cohomology are

$$H_{\text{CR}}^*(I\mathcal{X}) = \mathbb{C} \mathbf{1} \oplus \bigoplus_{(g) \neq 1} \mathbb{C} \mathbf{1}_{(g)},$$

$$H_{\text{CR,c}}^*(I\mathcal{X}) = \mathbb{C} \alpha \oplus \bigoplus_{(g): n_g=1} \mathbb{C} \beta_{(g)} \oplus \bigoplus_{(g): n_g=0} \mathbb{C} \mathbf{1}_{(g)},$$

where $n_g = \dim \mathcal{X}_{(g)}$. Here $\mathbf{1}_{(g)}$ is the unit class supported on $\mathcal{X}_{(g)}$ and $\alpha, \beta_{(g)}$ are top classes on $\mathcal{X}, \mathcal{X}_{(g)}$ respectively (with $n_g = 1$) such that

$$(\alpha, \mathbf{1})_{\text{orb}} = \frac{1}{|G|}, \quad (\beta_{(g)}, \mathbf{1}_{(g^{-1})})_{\text{orb}} = (\mathbf{1}_{(g)}, \mathbf{1}_{(g^{-1})})_{\text{orb}} = \frac{1}{|C(g)|}.$$

Note that $\deg \mathbf{1}_{(g)} = 2\iota_{(g)}, \iota_{(g)} = 1$ if $n_g = 1$, $\deg \alpha = 6$ and $\deg \beta_{(g)} = 4$. When $n_g = 1$, let $0 < f_g \leq 1/2$ be a rational number such that $1, e^{\pm 2\pi i f_g}$ are the eigenvalues of $g \in SL(3, \mathbb{C})$. When $n_g = 0$, let $0 < f_{g,1} \leq f_{g,2} \leq f_{g,3} < 1$ be rational numbers such that $e^{2\pi i f_{g,j}}, j = 1, 2, 3$, are the eigenvalues of g . Consider again the Grothendieck group $K_0^G(\mathbb{C}^3)$ of G -equivariant coherent sheaves supported at the origin. A finite dimensional representation ϱ of G gives a class $[\mathcal{O}_0 \otimes \varrho] \in K_0^G(\mathbb{C}^3)$. This yields a dual flat section $\mathcal{Z}_{K,c}(\mathcal{O}_0 \otimes \varrho) = \tilde{L}(\tau, z) z^{-\mu} \Psi(\mathcal{O}_0 \otimes \varrho)$ with $\Psi(\mathcal{O}_0 \otimes \varrho)$ given by

$$(\dim \varrho) \alpha \oplus \bigoplus_{(g): n_g=1} (-1) A_{g^{-1}}^{\varrho} \beta_{(g)} \oplus \bigoplus_{(g): n_g=0} (-1)^{1+\iota_{(g)}} B_{g^{-1}}^{\varrho} \mathbf{1}_{(g)}.$$

Here

$$A_g^e = \text{Tr}(g|\varrho) \frac{\sin(\pi f_g)}{\pi}, \quad B_g^e = \frac{\text{Tr}(g|\varrho)}{\prod_{j=1}^3 \Gamma(1 - f_{g,j})}.$$

The corresponding central charge restricted to $H_{\text{CR}}^2(\mathcal{X})$ is

$$(16) \quad Z(\mathcal{O}_0 \otimes \varrho) = \frac{\dim \varrho}{|G|} + \sum_{(g):n_g=1} \frac{A_g^e}{|C(g)|} t^{(g)} + \sum_{(g):n_g=0} B_g^e F_{0,(g^{-1})}(\tau),$$

where $\tau = \sum_{\iota(g)=1} t^{(g)} \mathbf{1}_{(g)} \in H_{\text{CR}}^2(\mathcal{X})$ and

$$(17) \quad F_{0,(g^{-1})}(\tau) = \begin{cases} t^{(g)}/|C(g)|, & \iota(g) = 1, \\ \sum_{m \geq 2} \frac{1}{m!} \langle \mathbf{1}_{(g^{-1})}, \tau, \dots, \tau \rangle_{0,m+1,0}, & \iota(g) = 2. \end{cases}$$

This follows from $Z(\mathcal{O}_0 \otimes \varrho) = (\tilde{L}(\tau, z)^\dagger \mathbf{1}, z^{-\mu} \Psi(\mathcal{O}_0 \otimes \varrho))_{\text{orb}}$ and the formula for the J -function $J(\tau, -z) = \tilde{L}(\tau, z)^\dagger \mathbf{1}$:

$$J(\tau, -z) = \mathbf{1} - \frac{\tau}{z} + \sum_{\iota(g)=2} F_{0,(g^{-1})}(\tau) |C(g)| \frac{\mathbf{1}_{(g)}}{z^2}.$$

Again the regular representation ρ_{reg} gives the simplest charge 1. The Γ -product $\prod_{j=1}^3 \Gamma(1 - f_{g,j})$ in the central charge may have something to do with the Chowla–Selberg formula [22].

§3. Ruan’s conjecture

We incorporate our K -theory integral structure into the Ruan’s conjecture [62, 63] and discuss what follows from this. We propose the conjecture that an isomorphism between K -theory induces an isomorphism of quantum D -modules via the K -group framing (13).

Ruan’s conjecture can be discussed in many situations. It basically asserts that two birational spaces $\mathcal{X}_1, \mathcal{X}_2$ in a “crepant” relationship have isomorphic (orbifold) quantum cohomology under a suitable identification of quantum parameters. One such relationship is a *crepant resolution*. Let \mathcal{X} be a Gorenstein orbifold without generic stabilizers, *i.e.* the automorphism group at every point x is contained in $SL(T_x \mathcal{X})$. Then the canonical line bundle $K_{\mathcal{X}}$ of \mathcal{X} becomes the pull-back of K_X of the coarse moduli space X . A resolution of singularity $\pi: Y \rightarrow X$ is called *crepant* if $\pi^* K_X \cong K_Y$. We can regard Y and \mathcal{X} as two different crepant resolutions of the same space X :

$$\mathcal{X} \longrightarrow X \longleftarrow Y.$$

In this case, Ruan's conjecture for a pair (\mathcal{X}, Y) is called the *crepant resolution conjecture* and has been studied in many works [15, 59, 14, 27, 11, 5, 13, 23]. Ruan's conjecture have been discussed also for *flops*. Li–Ruan [55] showed that the quantum cohomology is invariant under flops between Calabi–Yau 3-folds. Recently, this was generalized to the case of simple \mathbb{P}^r -flops and Mukai flops [54] in any dimension. The case of certain singular flops between orbifolds are also studied in [19, 20].

More generally, Ruan's conjecture may hold for *K-equivalences*. We say that two smooth Deligne–Mumford stacks $\mathcal{X}_1, \mathcal{X}_2$ are *K-equivalent* if there exist a smooth Deligne–Mumford stack \mathcal{X} and a diagram of projective birational morphisms

$$(18) \quad \mathcal{X}_1 \xleftarrow{p_1} \mathcal{X} \xrightarrow{p_2} \mathcal{X}_2$$

such that $p_1^*K_{\mathcal{X}_1} \cong p_2^*K_{\mathcal{X}_2}$. The most general form of Ruan's conjecture would be the invariance of quantum cohomology under *D-equivalences*, *i.e.* the equivalence of derived categories of coherent sheaves. It is conjectured in [50] that *K-equivalence* is equivalent to *D-equivalence* for smooth birational varieties, but *D-equivalence* does not imply birational equivalence in general. An interesting example is reported [61, 44] where the Gromov–Witten theories of non-birational but *D-equivalent* Calabi–Yau 3-folds have the same mirror family and, in particular, should be equivalent.

One striking feature in Ruan's conjecture is that we need the *analyticity* of the quantum cohomology. In the crepant resolution conjecture, the orbifold quantum cohomology is identified with the *expansion* of the manifold quantum cohomology around a point where the quantum parameter $q = e^{\tau_0, 2}$ is a root of unity. In the flop conjecture, the two quantum cohomology algebras are identified under the transformation $q \mapsto q^{-1}$, where q is the parameter of the exceptional curve.

3.1. A picture of the global quantum *D*-module

Let $\mathcal{X}_1, \mathcal{X}_2$ be a pair of smooth Deligne–Mumford stacks. For a complex analytic space \mathcal{M} , let $\pi: \mathcal{M} \times \mathbb{C} \rightarrow \mathcal{M}$ be the projection to the first factor, z be the co-ordinate on the \mathbb{C} factor and $(-): \mathcal{M} \times \mathbb{C} \rightarrow \mathcal{M} \times \mathbb{C}$ be the map sending (τ, z) to $(\tau, -z)$ as before.

Assumption 3.1 (Global quantum *D*-modules: See Figure 1). *There exists a global quantum *D*-module $(F, \nabla, (\cdot, \cdot)_F, F_{\mathbb{Z}})$ over a global Kähler moduli space \mathcal{M} given by the following data:*

- A connected complex analytic space \mathcal{M} ;
- A holomorphic vector bundle F of rank N over $\mathcal{M} \times \mathbb{C}$;

—A meromorphic flat connection ∇ on F (with poles along $z = 0$):

$$\nabla: \mathcal{O}(F) \rightarrow \mathcal{O}(F)(\mathcal{M} \times \{0\}) \otimes_{\mathcal{O}_{\mathcal{M} \times \mathbb{C}}} (\pi^* \Omega_{\mathcal{M}}^1 \oplus \mathcal{O}_{\mathcal{M} \times \mathbb{C}} \frac{dz}{z})$$

where $\mathcal{O}(F)(\mathcal{M} \times \{0\})$ is the sheaf of meromorphic sections of F with poles of order less than or equal to one only along the divisor $\mathcal{M} \times \{0\}$;

—A non-degenerate, ∇ -flat pairing $(\cdot, \cdot)_F$:

$$(\cdot, \cdot)_F: (-)^* \mathcal{O}(F) \otimes \mathcal{O}(F) \rightarrow \mathcal{O}_{\mathcal{M} \times \mathbb{C}};$$

—An integral local system (\mathbb{Z}^N -subbundle) $F_{\mathbb{Z}} \rightarrow \mathcal{M} \times \mathbb{C}^*$ underlying the flat vector bundle $F|_{\mathcal{M} \times \mathbb{C}^*}$ such that

$$F_{\mathbb{Z}} \subset \text{Ker}(\nabla), \quad F|_{\mathcal{M} \times \mathbb{C}^*} = F_{\mathbb{Z}} \otimes \mathbb{C}, \quad ((-)^* F_{\mathbb{Z}}, F_{\mathbb{Z}})_F \subset \mathbb{Z}.$$

We postulate that the 4-tuple $(F, \nabla, (\cdot, \cdot)_F, F_{\mathbb{Z}})$ satisfies the following.

(i) There exist open subsets $V_i \subset \mathcal{M}$, $i = 1, 2$, such that V_i is identified with the base space of the quantum D -module $QDM(\mathcal{X}_i)$:

$$V_i \cong U_i/H^2(\mathcal{X}_i, \mathbb{Z}),$$

and that the restriction of $(F, \nabla, (\cdot, \cdot)_F)$ to $V_i \times \mathbb{C}$ is isomorphic to $QDM(\mathcal{X}_i)$:

$$(F, \nabla, (\cdot, \cdot)_F)|_{V_i \times \mathbb{C}} \cong QDM(\mathcal{X}_i), \quad i = 1, 2.$$

Here $U_i \subset H_{\text{CR}}^*(\mathcal{X}_i)$ is the convergence domain of the quantum product in Assumption 2.1 and $U_i/H^2(\mathcal{X}_i, \mathbb{Z})$ is the quotient by the Galois action. Moreover, this isomorphism matches the integral local system $F_{\mathbb{Z}}$ with the K -theory integral structure of $QDM(\mathcal{X}_i)$ in Definition 2.11.

(ii) Assume that \mathcal{X}_1 and \mathcal{X}_2 are K -equivalent (18) and also related by a birational correspondence

$$(19) \quad \mathcal{X}_1 \xrightarrow{\pi_1} Z \xleftarrow{\pi_2} \mathcal{X}_2$$

such that $\pi_1 \circ p_1 = \pi_2 \circ p_2$. Take base points $x_i \in V_i$. For a line bundle L on Z , denote by $l_i(L) \in \pi_1(V_i, x_i)$ the homotopy class of a loop given by the class $[\pi_i^*(L)] \in H^2(\mathcal{X}_i, \mathbb{Z})$. (Recall that $V_i \cong U_i/H^2(\mathcal{X}_i, \mathbb{Z})$.) There exists a path $\gamma: [0, 1] \rightarrow \mathcal{M}$ from $\gamma(0) = x_1$ to $\gamma(1) = x_2$ such that $\gamma_*(l_1(L)) = l_2(L)$ for any line bundle L on Z . Here γ is independent of L .

As far as the author knows, all the concrete examples of global quantum D -modules arise from mirror symmetry. For example, in the case of toric flops or toric crepant resolutions (and complete intersections

in them), we can construct a global quantum D -module using the mirror Landau–Ginzburg model and \mathcal{M} is identified with the complex moduli space of the mirror [27, 24, 23]. The space of stability conditions on the derived category $D_{\text{coh}}^b(\mathcal{X}_i)$ due to Douglas and Bridgeland [30, 8] gives a candidate for the universal cover of \mathcal{M} . Another candidate (though being only infinitesimal at present) is the space of A_∞ -deformations of the derived Fukaya category of \mathcal{X}_i .

We assume the existence of a global quantum D -module F connecting $QDM(\mathcal{X}_1)$ and $QDM(\mathcal{X}_2)$. Choosing a path $\gamma: [0, 1] \rightarrow \mathcal{M}$ from a point $x_1 \in V_1$ to a point $x_2 \in V_2$, we have an analytic continuation map P_γ of flat sections

$$(20) \quad P_\gamma: \mathcal{S}(\mathcal{X}_1) \rightarrow \mathcal{S}(\mathcal{X}_2)$$

along the path $\hat{\gamma} = (\gamma, 1): [0, 1] \rightarrow \mathcal{M} \times \mathbb{C}^*$. Here by (i), we identified the space of flat sections of F over $V_i \times \mathbb{C}^*$ with $\mathcal{S}(\mathcal{X}_i)$. This preserves the K -theory integral structures $P_\gamma(\mathcal{S}(\mathcal{X}_1)_\mathbb{Z}) = \mathcal{S}(\mathcal{X}_2)_\mathbb{Z}$ and the pairing $(\cdot, \cdot)_\mathcal{S}$. Then it would be natural to conjecture the following.

Conjecture 3.2. *For each path γ , there exists an isomorphism of K -groups*

$$(21) \quad \mathbb{U}_{K,\gamma}: K(\mathcal{X}_1) \rightarrow K(\mathcal{X}_2)$$

which induces the analytic continuation map P_γ in (20) through the K -group framing (13). $\mathbb{U}_{K,\gamma}$ preserves the Mukai pairing $\chi(\mathbb{U}_{K,\gamma}(V_1) \otimes \mathbb{U}_{K,\gamma}(V_2)^\vee) = \chi(V_1 \otimes V_2^\vee)$. Here $\mathbb{U}_{K,\gamma}$ gives the full relationships between $QDM(\mathcal{X}_1)$ and $QDM(\mathcal{X}_2)$, i.e. we know the quantum cohomology of \mathcal{X}_2 once we know the isomorphism $\mathbb{U}_{K,\gamma}$ and the complete information on the analytic continuation of the quantum cohomology of \mathcal{X}_1 .

We expect that the K -group isomorphisms $\mathbb{U}_{K,\gamma}$ are given by geometric correspondences such as Fourier–Mukai transformations [9, 51]. This conjecture is compatible with Borisov–Horja’s result [7], where they identified the K -group of toric Calabi–Yau orbifold with the space of solutions to the GKZ system and also identified the analytic continuation of GKZ solutions with the Fourier–Mukai transformations between K -groups. If the path γ is the same as what appeared in (ii) of Assumption 3.1, we also expect that $\mathbb{U}_{K,\gamma}$ commutes with the actions of line bundles pulled back from Z , i.e. $\mathbb{U}_{K,\gamma}(\pi_1^*(L) \otimes V) = \pi_2^*(L) \otimes \mathbb{U}_{K,\gamma}(V)$ for a line bundle L on Z . This is compatible with (ii) in Assumption 3.1 and the fact that tensoring by the π_i^*L on $K(\mathcal{X}_i)$ corresponds to the monodromy (Galois) action on $\mathcal{S}(\mathcal{X}_i)$ along the loop $l_i(L)$.

Remark 3.3. (i) Unlike the original quantum D -module, the *global* quantum D -module F is not a priori trivialized in the standard way. This is an important point in this formulation. In fact, for the crepant resolution of $\mathbb{C}^3/\mathbb{Z}_3$ (or its compactification $\mathbb{P}(1, 1, 1, 3)$), F has different trivializations over V_1 and V_2 [3, 27]. Here different trivializations correspond to different Frobenius/flat structures on the base \mathcal{M} .

(ii) The flat connection can have poles along $z = 0$. For a local section s of F around $z = 0$, $\nabla_X s$ has a pole of order ≤ 1 along $z = 0$ for $X \in T\mathcal{M}$ and $\nabla_{\partial_z} s$ has a pole of order ≤ 2 along $z = 0$.

(iii) The K -theory isomorphism (21) depends on the choice of a path γ . It would be very interesting to study the global monodromy of $(F, \nabla, (\cdot, \cdot)_F, F_{\mathbb{Z}})$.

Remark 3.4. In the context of Ruan’s conjecture, the picture of the global quantum D -module has been proposed in [27], [28] in terms of the Givental formalism. An integral structure was incorporated in this picture in [45, 46]. The structure analogous to the global quantum D -module $(F, \nabla, (\cdot, \cdot)_F, F_{\mathbb{Z}})$ first emerged in singularity theory [66] and has been studied under various names: *Frobenius manifolds* [31]; *semi-infinite Hodge structures* [4]; *TE(R)P structures* [37, 38]; *twistor structures* [70, 65]; *non-commutative Hodge structures* [49] etc.

3.2. Family of algebras: isomorphism of F -manifolds

We explain that Assumption 3.1 implies the deformation equivalence of quantum cohomology. Choosing a local trivialization of F , we can write the connection operator ∇_X with $X \in T\mathcal{M}$ in the form

$$\nabla_X = X + \frac{1}{z} \mathcal{A}_X(\tau, z).$$

The residual part $\mathcal{A}_X(\tau, 0) = [z\nabla_X]|_{z=0}$ defines a well-defined endomorphism of $F|_{\mathcal{M} \times \{0\}}$. The flatness of the connection ∇ implies the commutativity of these operators $[\mathcal{A}_X(\tau, 0), \mathcal{A}_Y(\tau, 0)] = 0$. Note that on $V_i \subset \mathcal{M}$, $\mathcal{A}_X(\tau, 0)$ is identified with the quantum product $X \circ_{\tau}$. (Here we identify the tangent vector X with an element of $H_{\text{CR}}^*(\mathcal{X}_i)$.) We say that (F, ∇) is *miniversal* at a point $\tau \in \mathcal{M}$ if there exists a vector $v \in F_{(\tau, 0)}$ such that the map

$$(22) \quad T_{\tau} \mathcal{M} \rightarrow F_{(\tau, 0)}, \quad X \mapsto \mathcal{A}_X(\tau, 0)v$$

is an isomorphism. This property clearly holds at $\tau \in V_i$ since we can choose v to be the unit $\mathbf{1} \in H_{\text{CR}}^*(\mathcal{X})$. The miniversality may fail along a complex analytic subvariety of \mathcal{M} . In the sequel, by deleting such loci if

necessary, we assume that (F, ∇) is miniversal everywhere on \mathcal{M} . Then we can define the product \circ_τ on the tangent space $T_\tau\mathcal{M}$ by the formula:

$$\mathcal{A}_{X \circ_\tau Y}(\tau, 0)v = \mathcal{A}_X(\tau, 0)(\mathcal{A}_Y(\tau, 0)v),$$

where $v \in F_{(\tau, 0)}$ is a vector which makes the map (22) an isomorphism. The unit vector $e \in T_\tau\mathcal{M}$ is defined by

$$\mathcal{A}_e(\tau, 0)v = v.$$

Then $(T_\tau\mathcal{M}, \circ_\tau, e)$ becomes an associative commutative ring by the commutativity of $\mathcal{A}_X(\tau, 0)$. This definition does not depend on the choice of v . In fact, the inclusion

$$T_\tau\mathcal{M} \hookrightarrow \text{End}(F_{(\tau, 0)}), \quad X \mapsto \mathcal{A}_X(\tau, 0)$$

becomes a homomorphism of rings. This product \circ_τ endows the base space \mathcal{M} with the structure of an F -manifold [36]. Here an F -manifold is a complex manifold \mathcal{M} endowed with an associative and commutative multiplication $T\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} T\mathcal{M} \rightarrow T\mathcal{M}$ satisfying a certain axiom (see [36]). This is a weak version of the Frobenius manifold structure explained below in Section 3.4.

The F -manifold \mathcal{M} here admits an *Euler vector field*. In a local frame of F , we can write the connection in the z -direction as

$$(23) \quad \nabla_{z\partial_z} = z\partial_z - \frac{1}{z}\mathcal{U}(\tau) + \mathcal{V}(\tau, z), \quad \mathcal{V}(\tau, z) \text{ is regular at } z = 0.$$

The residual part $\mathcal{U}(\tau) = [z^2\nabla_{\partial_z}]|_{z=0}$ again defines a well-defined endomorphism of the bundle $F|_{\mathcal{M} \times \{0\}}$. The flatness of ∇ implies that the endomorphism $\mathcal{U}(\tau)$ commutes with $\mathcal{A}_X(\tau, 0)$ for every $X \in T\mathcal{M}$. From this (and miniversality) it follows that there exists a unique vector field $E \in \Gamma(\mathcal{M}, T\mathcal{M})$ such that

$$\mathcal{U}(\tau) = \mathcal{A}_E(\tau, 0).$$

This satisfies the axiom of the Euler vector field:

$$(24) \quad [E, X \circ_\tau Y] = [E, X] \circ_\tau Y + X \circ_\tau [E, Y] + X \circ_\tau Y.$$

Proposition 3.5. *Under Assumption 3.1, the quantum cohomologies of \mathcal{X}_1 and \mathcal{X}_2 underlie the same F -manifold \mathcal{M} with the Euler vector field E .*

3.3. Semi-infinite variation of Hodge structures

The deformation equivalence explained in the previous section is a rather weak relationship. The global quantum D -module F has much more information than just a family of algebras. We consider the *semi-infinite variation of Hodge structures* or $\frac{\infty}{2}$ VHS associated to F . This notion was introduced by Barannikov [4]. The information of $\frac{\infty}{2}$ VHS is in fact equivalent to that of the meromorphic flat connection $(F, \nabla, (\cdot, \cdot)_F)$, but the analogy with the ordinary Hodge theory is clearer in this language.

We will work over the universal cover $\widetilde{\mathcal{M}}$ of \mathcal{M} . Let \mathcal{H} be the space of flat sections of F over $\widetilde{\mathcal{M}} \times \mathbb{C}^*$:

$$\mathcal{H} := \{s \in \Gamma(\widetilde{\mathcal{M}} \times \mathbb{C}^*, \mathcal{O}(F)) ; \nabla_X s = 0, \forall X \in T\mathcal{M}\}.$$

Note that $s \in \mathcal{H}$ is flat only in the direction of \mathcal{M} and not necessarily flat in the z direction. This is infinite dimensional over \mathbb{C} . For $\tau \in \widetilde{\mathcal{M}}$, every section $s(\tau, \cdot) \in \Gamma(\{\tau\} \times \mathbb{C}^*, F)$ can be uniquely extended to a section s over $\widetilde{\mathcal{M}} \times \mathbb{C}^*$ which is flat in the \mathcal{M} -direction. Therefore \mathcal{H} is isomorphic to $\Gamma(\{\tau\} \times \mathbb{C}^*, F)$ and is a free $\mathcal{O}(\mathbb{C}^*)$ -module of rank N , where $\mathcal{O}(\mathbb{C}^*)$ is the space of holomorphic functions on \mathbb{C}^* and N is the rank of F . The pairing on \mathcal{H} is defined by

$$(s_1, s_2)_{\mathcal{H}} := (s_1(\tau, -z), s_2(\tau, z))_F \in \mathcal{O}(\mathbb{C}^*).$$

Note that the right-hand side does not depend on τ since s_1, s_2 are flat in the \mathcal{M} -direction. This pairing satisfies $(s_2, s_1)_{\mathcal{H}} = (-)^*(s_1, s_2)_{\mathcal{H}}$. For $\tau \in \widetilde{\mathcal{M}}$, the space of sections of F over $\{\tau\} \times \mathbb{C}$ is naturally embedded into \mathcal{H} (via the \mathcal{M} -flat extension as above)

$$\Gamma(\{\tau\} \times \mathbb{C}, F) \hookrightarrow \mathcal{H}.$$

We denote by \mathbb{F}_{τ} the image of this embedding. Recall that the image of $\Gamma(\{\tau\} \times \mathbb{C}^*, F)$ gives the whole space \mathcal{H} . \mathbb{F}_{τ} consists of flat sections $s \in \mathcal{H}$ such that $s(\tau, \cdot)$ is regular at $z = 0$. We call \mathbb{F}_{τ} the *semi-infinite Hodge structure*. \mathbb{F}_{τ} is a free $\mathcal{O}(\mathbb{C})$ -submodule of \mathcal{H} and can be regarded as a point on the Segal–Wilson Grassmannian [60] of \mathcal{H} as follows: Fix an $\mathcal{O}(\mathbb{C}^*)$ -basis e_1, \dots, e_N of \mathcal{H} . An $\mathcal{O}(\mathbb{C})$ -basis s_1, \dots, s_N of \mathbb{F}_{τ} can be written as $s_j = \sum_{i=1}^N e_i c_{ij}(\tau, z)$. By restricting z to lie on S^1 , the $N \times N$ matrix $(c_{ij}(\tau, z))$ defines an element of the loop group $LGL(N, \mathbb{C})$. A change of the basis s_j changes the matrix (c_{ij}) by the left multiplication by an element of the positive loop group $LGL^+(N, \mathbb{C})$ (whose entries are holomorphic functions on \mathbb{C}). Thus the subspace \mathbb{F}_{τ} is identified with

an element $[(c_{ij}(\tau, z))]$ of $LGL(N, \mathbb{C})/LGL^+(N, \mathbb{C}) =: \text{Gr}_{\frac{\infty}{2}}(\mathcal{H})$. We call the map

$$\widetilde{\mathcal{M}} \ni \tau \mapsto \mathbb{F}_\tau \in \text{Gr}_{\frac{\infty}{2}}(\mathcal{H})$$

the *semi-infinite period map*.

Proposition 3.6 ([27, Proposition 2.9]). *The semi-infinite period map $\tau \mapsto \mathbb{F}_\tau$ satisfies the following:*

$$\begin{aligned} X\mathbb{F}_\tau &\subset z^{-1}\mathbb{F}_\tau, & X &\in T_\tau\mathcal{M}, \\ (\mathbb{F}_\tau, \mathbb{F}_\tau)_\mathcal{H} &\subset \mathcal{O}(\mathbb{C}), \\ (\nabla_{z\partial_z} + E)\mathbb{F}_\tau &\subset \mathbb{F}_\tau, \end{aligned}$$

where we used the fact that $\nabla_{z\partial_z}$ acts on \mathcal{H} as a \mathbb{C} -endomorphism. The first property is an analogue of Griffiths transversality and the second is the Hodge–Riemann bilinear relation.

3.4. Opposite subspace and Frobenius manifolds

As we remarked, the global quantum D -module is not a priori trivialized. A good trivialization is given by the choice of an *opposite subspace* to the $\frac{\infty}{2}$ VHS. The choice of an opposite subspace and a dilaton shift defines a Frobenius structure on the universal cover of \mathcal{M} . The Frobenius/flat structure was discovered by K. Saito [66] as a structure on a miniversal deformation of isolated hypersurface singularities and the use of opposite subspaces goes back to M. Saito's work [67] in that context. Let $\mathcal{O}(\mathbb{P}^1 \setminus \{0\})$ be the space of holomorphic functions on $\mathbb{P}^1 \setminus \{0\}$. This is contained in $\mathcal{O}(\mathbb{C}^*)$.

Definition 3.7. An *opposite subspace* \mathcal{H}_- at $\tau \in \widetilde{\mathcal{M}}$ is a free $\mathcal{O}(\mathbb{P}^1 \setminus \{0\})$ -submodule of \mathcal{H} such that the natural map

$$(25) \quad \mathcal{H}_- \oplus \mathbb{F}_\tau \rightarrow \mathcal{H}$$

is an isomorphism. \mathcal{H}_- is said to be *homogeneous* if

$$\nabla_{z\partial_z}\mathcal{H}_- \subset \mathcal{H}_-$$

and *isotropic* if

$$(\mathcal{H}_-, \mathcal{H}_-)_\mathcal{H} \subset z^{-2}\mathcal{O}(\mathbb{P}^1 \setminus \{0\}).$$

In terms of the loop Grassmannian $LGL(N, \mathbb{C})/LGL^+(N, \mathbb{C})$, \mathcal{H}_- is opposite at τ if \mathbb{F}_τ lies on the “big cell”: an open orbit of $LGL^-(N, \mathbb{C})$. Therefore, the opposite property ((25) is an isomorphism) is an open condition: If \mathcal{H}_- is opposite at τ , then it is opposite in a neighborhood of

τ . Given an opposite subspace \mathcal{H}_- at some point, the opposite property may fail along a complex analytic subvariety of $\widetilde{\mathcal{M}}$.

We explain in the lemma below that a homogeneous opposite subspace corresponds to an extension of (F, ∇) across $z = \infty$ such that the connection ∇ has a logarithmic singularity along $z = \infty$.

Lemma 3.8. *For a point $\tau \in \widetilde{\mathcal{M}}$, the following are equivalent:*

- (i) \mathcal{H}_- is a homogeneous opposite subspace at τ .
- (ii) \mathcal{H}_- is homogeneous and at least one of the following projections

$$z\mathcal{H}_-/\mathcal{H}_- \longleftarrow z\mathcal{H}_- \cap \mathbb{F}_\tau \longrightarrow \mathbb{F}_\tau/z\mathbb{F}_\tau$$

is an isomorphism of finite dimensional \mathbb{C} -vector spaces.

(iii) Define an extension $\widehat{F}_\tau \rightarrow \{\tau\} \times \mathbb{P}^1$ of the vector bundle $F|_{\{\tau\} \times \mathbb{C}}$ to $\{\tau\} \times \mathbb{P}^1$ as follows: A section $s \in \Gamma(\{\tau\} \times \mathbb{C}^*, F)$ extends to a regular section of \widehat{F}_τ over $\{\tau\} \times (\mathbb{P}^1 \setminus \{0\})$ if the image of s in \mathcal{H} lies in $z\mathcal{H}_-$. The extension $(\widehat{F}_\tau, \nabla)$ is a trivial vector bundle over \mathbb{P}^1 and ∇ has a logarithmic singularity at $z = \infty$.

Proof. (i) \Rightarrow (ii). The injectivity of the maps in (ii) is obvious. For $[v] \in z\mathcal{H}_-/\mathcal{H}_-$ with $v \in z\mathcal{H}_-$, write $v = v_0 + v_-$ where $v_0 \in \mathbb{F}_\tau$ and $v_- \in \mathcal{H}_-$. Then $v_0 = v - v_- \in z\mathcal{H}_- \cap \mathbb{F}_\tau$ and $[v] = [v_0]$. This shows the surjectivity of $z\mathcal{H}_- \cap \mathbb{F}_\tau \rightarrow z\mathcal{H}_-/\mathcal{H}_-$. For $[v] \in \mathbb{F}_\tau/z\mathbb{F}_\tau$ with $v \in \mathbb{F}_\tau$, write $z^{-1}v = v_- + v_0$, where $v_- \in \mathcal{H}_-$ and $v_0 \in \mathbb{F}_\tau$. Then $zv_- = v - zv_0 \in \mathbb{F}_\tau \cap z\mathcal{H}_-$ and $[v] = [zv_-]$. This shows the surjectivity of $z\mathcal{H}_- \cap \mathbb{F}_\tau \rightarrow \mathbb{F}_\tau/z\mathbb{F}_\tau$.

(ii) \Rightarrow (iii). Consider the extension $\widehat{F}_\tau \rightarrow \{\tau\} \times \mathbb{P}^1$ in (iii). We can identify $z\mathcal{H}_-/\mathcal{H}_-$ with the fiber $\widehat{F}_{(\tau, \infty)}$, $z\mathcal{H}_- \cap \mathbb{F}_\tau$ with the space of global sections $\Gamma(\mathbb{P}^1, \widehat{F}_\tau)$ and $\mathbb{F}_\tau/z\mathbb{F}_\tau$ with the fiber $\widehat{F}_{(\tau, 0)}$. Since the maps in (ii) are induced from the restrictions, that one of them is an isomorphism implies that \widehat{F}_τ is a trivial holomorphic vector bundle. For a local coordinate $w = z^{-1}$ around $z = \infty$, we have $\nabla_{w\partial_w} = -\nabla_{z\partial_z}$. Hence the homogeneity implies $\nabla_{w\partial_w}(z\mathcal{H}_-) \subset (z\mathcal{H}_-)$, so ∇ has a logarithmic singularity at $w = 0$.

(iii) \Rightarrow (i). Note that \mathcal{H} is identified with the space of sections of \widehat{F}_τ over $\{\tau\} \times \mathbb{C}^*$. Because \widehat{F}_τ is trivial, that (25) is an isomorphism follows from the decomposition

$$\mathcal{O}(\mathbb{C}^*) = z^{-1}\mathcal{O}(\mathbb{P}^1 \setminus \{0\}) \oplus \mathcal{O}(\mathbb{C}).$$

The logarithmic singularity of ∇ implies the homogeneity of \mathcal{H}_- . Q.E.D.

By the isomorphism in (ii) of Lemma 3.8, a homogeneous opposite subspace \mathcal{H}_- gives a local trivialization of F . In fact, since $F|_{\{\tau\} \times \mathbb{C}}$

extends to a trivial vector bundle \widehat{F}_τ over $\{\tau\} \times \mathbb{P}^1$, we have

$$(26) \quad F_{(\tau,z)} \cong \Gamma(\{\tau\} \times \mathbb{P}^1, \widehat{F}_\tau) \cong z\mathcal{H}_- \cap \mathbb{F}_\tau \cong z\mathcal{H}_-/\mathcal{H}_-$$

The finite dimensional vector space $z\mathcal{H}_-/\mathcal{H}_-$ does not depend on τ , so this defines a trivialization of F over an open subset of $\widetilde{\mathcal{M}}$. Under this trivialization, the flat connection ∇ can be written as follows (see e.g. [27, Proposition 2.11])

$$(27) \quad \begin{aligned} \nabla_X &= X + \frac{1}{z} \mathcal{A}_X(\tau), \quad X \in T\mathcal{M}, \\ \nabla_{z\partial_z} &= z\partial_z - \frac{1}{z} \mathcal{U}(\tau) + \mathcal{V}, \end{aligned}$$

where $\mathcal{A}(\tau)$ is an $\text{End}(z\mathcal{H}_-/\mathcal{H}_-)$ -valued 1-form, $\mathcal{U}(\tau)$ is an $\text{End}(z\mathcal{H}_-/\mathcal{H}_-)$ -valued function, and \mathcal{V} is a constant operator in $\text{End}(z\mathcal{H}_-/\mathcal{H}_-)$. Here $\mathcal{A}(\tau), \mathcal{U}(\tau)$ are independent of z and defined on an open subset of $\widetilde{\mathcal{M}}$. Note that $\mathcal{U}(\tau) = \mathcal{A}_E(\tau)$ by the definition of the Euler vector field E .

In order to have a Frobenius structure on $\widetilde{\mathcal{M}}$, in addition to \mathcal{H}_- , we need to choose⁸ an eigenvector $v_0 \in z\mathcal{H}_-/\mathcal{H}_-$ of \mathcal{V} satisfying the miniversality condition:

$$(28) \quad T_\tau \widetilde{\mathcal{M}} \rightarrow z\mathcal{H}_-/\mathcal{H}_-, \quad X \mapsto \mathcal{A}_X(\tau)v_0 \quad \text{is an isomorphism.}$$

We call v_0 a *dilaton shift*. The isomorphism $T_\tau \widetilde{\mathcal{M}} \cong z\mathcal{H}_-/\mathcal{H}_-$ above defines an affine flat structure⁹ on $\widetilde{\mathcal{M}}$. A vector field X is defined to be flat if $\mathcal{A}_X(\tau)v_0$ is a constant element in $z\mathcal{H}_-/\mathcal{H}_-$. A flat co-ordinate system on $\widetilde{\mathcal{M}}$ is constructed as follows. Let $\hat{v}_0 + \psi(\tau)$ be the unique intersection point of \mathbb{F}_τ and the affine subspace $\hat{v}_0 + \mathcal{H}_-$, where $\hat{v}_0 \in z\mathcal{H}_-$ is an (arbitrarily fixed) lift of v_0 and $\psi(\tau) \in \mathcal{H}_-$. See Figure 2. Then the map

$$\widetilde{\mathcal{M}} \ni \tau \mapsto [\psi(\tau)] \in \mathcal{H}_-/z^{-1}\mathcal{H}_-$$

is a local isomorphism and gives a flat co-ordinate system. In fact, the differential of this map is identified with (28). Varying τ , the intersection point $\hat{v}_0 + \psi(\tau) \in \mathbb{F}_\tau$ gives a section s_0 of F which corresponds to $v_0 \in z\mathcal{H}_-/\mathcal{H}_-$ in the trivialization (26). (Note that $\hat{v}_0 + \psi(\tau) \in z\mathcal{H}_- \cap \mathbb{F}_\tau$.) This section s_0 is called a *primitive section*. In Gromov–Witten theory, the corresponding vector $\hat{v}_0 + \psi(\tau) \in \mathcal{H}$ is called the *J-function*.

⁸The author does not claim that v_0 always exists. The action of \mathcal{V} on $z\mathcal{H}_-/\mathcal{H}_-$ is induced from that of $\nabla_{z\partial_z}$ on $z\mathcal{H}_-$.

⁹An affine flat manifold is a manifold with a torsion free flat connection on the tangent bundle.

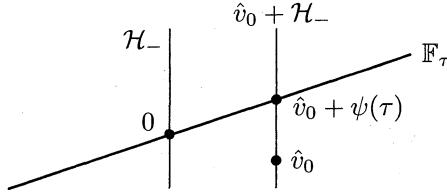


Fig. 2. J -function $\hat{v}_0 + \psi(\tau)$ and flat co-ordinates $[\psi(\tau)] \in \mathcal{H}_-/z^{-1}\mathcal{H}_-$.

For a flat vector field X , we have $\mathcal{V}(\mathcal{A}_X v_0) = \mathcal{A}_{(\alpha+1)X - [X, E]} v_0$ where α is the eigenvalue of v_0 with respect to \mathcal{V} .

When \mathcal{H}_- is isotropic, the pairing $(\cdot, \cdot)_{\mathcal{H}}$ on \mathcal{H} restricts to a symmetric bilinear \mathbb{C} -valued pairing on $z\mathcal{H}_- \cap \mathbb{F}_\tau \cong z\mathcal{H}_-/\mathcal{H}_-$. By pulling back this pairing on $z\mathcal{H}_-/\mathcal{H}_-$ to $T_\tau \widetilde{\mathcal{M}}$ by the map (28), we obtain a \mathbb{C} -bilinear metric $g: T_\tau \widetilde{\mathcal{M}} \times T_\tau \widetilde{\mathcal{M}} \rightarrow \mathbb{C}$. The metric tensor of g is constant in the flat co-ordinates above, so the metric g is flat.

Proposition 3.9 ([27, Proposition 2.12]). *Take an isotropic homogeneous opposite subspace \mathcal{H}_- and a dilaton shift $v_0 \in z\mathcal{H}_-/\mathcal{H}_-$ satisfying (28) at some point τ . Then the F -manifold structure (\circ_τ, e, E) in Proposition 3.5 can be lifted to the Frobenius manifold structure (\circ_τ, e, E, g) on the complement of a complex analytic subvariety in $\widetilde{\mathcal{M}}$. These data satisfy:*

- (i) the Levi-Civita connection ∇^{LC} of g is flat;
- (ii) $(T_\tau \widetilde{\mathcal{M}}, \circ_\tau, g)$ is a commutative Frobenius algebra;
- (iii) the pencil of flat connections $\nabla_X^\lambda = \nabla_X^{\text{LC}} + \lambda X \circ_\tau$ is flat;
- (iv) the unit vector field e is flat;
- (v) the Euler vector field E satisfies (24), $(\nabla^{\text{LC}})^2 E = 0$ and

$$Eg(X, Y) = g([E, X], Y) + g(X, [E, Y]) + 2(\alpha + 1)g(X, Y),$$

where $\alpha \in \mathbb{C}$ is the eigenvalue of v_0 : $\mathcal{V}v_0 = \alpha v_0$.

3.5. Opposite subspaces at cusps

We call the large radius limit point of \mathcal{X}_i a *cusp* of the global Kähler moduli space \mathcal{M} and V_i a neighborhood of a cusp. Since the base space of $QDM(\mathcal{X}_i)$ is a quotient of a vector space, V_i is equipped with the standard Frobenius/flat structure as described in [57, 31]. We will show that, under certain conditions, the Frobenius structure (or the corresponding opposite subspace) of V_i can be uniquely characterized by the monodromy invariance and the compatibility with the Deligne extension. This means that *there is a canonical choice of the Frobenius manifold*

structure at each cusp from a purely D -module theoretic viewpoint. The characterization here was shown in the case $\mathcal{X} = \mathbb{P}(1, 1, 1, 3)$ in [27].

Henceforth we study the global quantum D -module restricted to V_i i.e. $QDM(\mathcal{X}_i)$. We omit the subscript i and write V, \mathcal{X} for V_i, \mathcal{X}_i etc. The open set $U \subset H_{\text{CR}}^*(\mathcal{X})$ in Assumption 2.1 is identified with the universal cover of $V \cong U/H^2(\mathcal{X}, \mathbb{Z})$.

Definition 3.10 (Givental space [26, 33]). The *Givental symplectic space* $\mathcal{H}^{\mathcal{X}}$ is defined to be the free $\mathcal{O}(\mathbb{C}^*)$ -module

$$\mathcal{H}^{\mathcal{X}} := H_{\text{CR}}^*(\mathcal{X}) \otimes \mathcal{O}(\mathbb{C}^*),$$

endowed with an $\mathcal{O}(\mathbb{C}^*)$ -valued pairing $(\cdot, \cdot)_{\mathcal{H}}$:

$$(f(z), g(z))_{\mathcal{H}} = (f(-z), g(z))_{\text{orb}}.$$

As an infinite dimensional vector space over \mathbb{C} , $\mathcal{H}^{\mathcal{X}}$ has the following symplectic form:

$$(29) \quad \Omega(f, g) = \text{Res}_{z=0}(f(-z), g(z))_{\text{orb}} dz.$$

We identify the Givental space $\mathcal{H}^{\mathcal{X}}$ with the space \mathcal{H} of flat sections of $QDM(\mathcal{X})$ over U through the fundamental solution in Proposition 2.8:

$$\mathcal{H}^{\mathcal{X}} \cong \mathcal{H}, \quad \phi(z) \mapsto L(\tau, z)\phi(z).$$

This identification preserves the pairing.

In terms of the Givental space, the semi-infinite Hodge structure \mathbb{F}_{τ} is identified with the Lagrangian subspace:

$$(30) \quad \mathbb{F}_{\tau} = L(\tau, z)^{-1}(H_{\text{CR}}^*(\mathcal{X}) \otimes \mathcal{O}(\mathbb{C})) \subset \mathcal{H}^{\mathcal{X}}, \quad \tau \in U.$$

The Givental space has a standard opposite subspace $\mathcal{H}_{-}^{\mathcal{X}}$:

$$\mathcal{H}_{-}^{\mathcal{X}} := z^{-1}H_{\text{CR}}^*(\mathcal{X}) \otimes \mathcal{O}(\mathbb{P}^1 \setminus \{0\}) \subset \mathcal{H}^{\mathcal{X}}.$$

In fact, this is opposite to \mathbb{F}_{τ} (i.e. $\mathcal{H}_{-}^{\mathcal{X}} \oplus \mathbb{F}_{\tau} = \mathcal{H}^{\mathcal{X}}$) for every $\tau \in U$ because $L(\tau, z)$ is regular at $z = \infty$ and $L(\tau, z) = \text{id} + \mathcal{O}(z^{-1})$.

Proposition 3.11. *The standard opposite subspace $\mathcal{H}_{-}^{\mathcal{X}}$ is homogeneous and isotropic. This $\mathcal{H}_{-}^{\mathcal{X}}$ and the standard dilaton shift $v_0 = \mathbf{1} \in z\mathcal{H}_{-}^{\mathcal{X}}/\mathcal{H}_{-}^{\mathcal{X}}$ endow the base space $V \cong U/H^2(\mathcal{X}, \mathbb{Z})$ of the quantum D -module with the standard Frobenius manifold structure coming from the linear structure on $U \subset H_{\text{CR}}^*(\mathcal{X})$ and the orbifold Poincaré pairing on $T_{\tau}U \cong H_{\text{CR}}^*(\mathcal{X})$. (See Proposition 3.9 for the construction of Frobenius manifolds.)*

Proof. It follows from Proposition 2.8 that $L(\tau, z)$ satisfies the differential equation $\nabla_{z\partial_z} L(\tau, z)\phi = L(\tau, z)(\mu - \rho/z)\phi$ for $\phi \in H_{\text{CR}}^*(\mathcal{X})$. This shows that the action of $\nabla_{z\partial_z}$ on the Givental space is given by

$$(31) \quad \nabla_{z\partial_z} = z\partial_z + \mu - \frac{\rho}{z} \quad \text{on } \mathcal{H}^{\mathcal{X}}.$$

Therefore the standard opposite subspace is homogeneous $\nabla_{z\partial_z} \mathcal{H}_-^{\mathcal{X}} \subset \mathcal{H}_-^{\mathcal{X}}$. It is obvious that $\mathcal{H}_-^{\mathcal{X}}$ is isotropic. Because $L(\tau, z)^{-1}\phi = \phi + O(z^{-1})$ for $\phi \in H_{\text{CR}}^*(\mathcal{X})$, we have $L(\tau, z)^{-1}\phi \in z\mathcal{H}_-^{\mathcal{X}} \cap \mathbb{F}_{\tau}$. Therefore, the constant section ϕ of $QDM(\mathcal{X})$ corresponds to (again) the constant element $\phi \in z\mathcal{H}_-^{\mathcal{X}}/\mathcal{H}_-^{\mathcal{X}}$ under the trivialization (26). This means that $\mathcal{H}_-^{\mathcal{X}}$ yields exactly the *given* trivialization of $QDM(\mathcal{X})$. In particular, the connection operators $\mathcal{A}_{\mathcal{X}}$, \mathcal{U} , \mathcal{V} in (27) are identified with $X_{\circ\tau}$, $E_{\circ\tau}$, μ and $\mathbf{1} \in H_{\text{CR}}^*(\mathcal{X})$ is the eigenvector of $\mathcal{V} = \mu$ of eigenvalue $-\dim_{\mathbb{C}} \mathcal{X}/2$. Now we only need to check that the corresponding flat metric g is the orbifold Poincaré pairing. But this is obvious from $(L(\tau, -z)^{-1}\phi_1, L(\tau, z)^{-1}\phi_2)_{\text{orb}} = (\phi_1, \phi_2)_{\text{orb}}$. Q.E.D.

Here we describe the two characteristic properties of $\mathcal{H}_-^{\mathcal{X}}$: the *monodromy invariance* and the *compatibility with the Deligne extension*.

The monodromy invariance of $\mathcal{H}_-^{\mathcal{X}}$ — We see that $\mathcal{H}_-^{\mathcal{X}}$ is invariant under the local monodromy (or Galois actions) around the large radius limit. The Galois action in Lemma 2.5 acts on the Givental space $\mathcal{H}^{\mathcal{X}}$ by $G^{\mathcal{H}}(\xi)$:

$$G^{\mathcal{H}}(\xi) = \bigoplus_{v \in \mathbb{T}} e^{-2\pi i \xi_0/z} e^{2\pi i f_v(\xi)}, \quad \xi \in H^2(\mathcal{X}, \mathbb{Z}),$$

where we used the decomposition $\mathcal{H}^{\mathcal{X}} = \bigoplus_{v \in \mathbb{T}} H^*(\mathcal{X}_v) \otimes \mathcal{O}(\mathbb{C}^*)$. Since $G^{\mathcal{H}}(\xi)$ contains only negative powers in z , we have

$$(32) \quad G^{\mathcal{H}}(\xi)\mathcal{H}_-^{\mathcal{X}} \subset \mathcal{H}_-^{\mathcal{X}}.$$

The semi-infinite Hodge structures are monodromy-equivariant: $G^{\mathcal{H}}(\xi)\mathbb{F}_{\tau} = \mathbb{F}_{G(\xi)\tau}$. The monodromy-invariance of $\mathcal{H}_-^{\mathcal{X}}$ corresponds to the fact that the corresponding Frobenius manifold structure is well-defined¹⁰ on the quotient $V \cong U/H^2(\mathcal{X}, \mathbb{Z})$. The induced action of $G^{\mathcal{H}}(\xi)$ on $z\mathcal{H}_-^{\mathcal{X}}/\mathcal{H}_-^{\mathcal{X}}$ is given by $\bigoplus_{v \in \mathbb{T}} e^{2\pi i f_v(\xi)}$. Because $f_v(\xi)$ is a rational number, there exists a positive integer $k_0 > 0$ such that

$$(33) \quad (G^{\mathcal{H}}(\xi))^{k_0} = \text{id} \quad \text{on } z\mathcal{H}_-^{\mathcal{X}}/\mathcal{H}_-^{\mathcal{X}}.$$

¹⁰More precisely, we also need the fact that the vector $\mathbf{1} \in z\mathcal{H}_-^{\mathcal{X}}/\mathcal{H}_-^{\mathcal{X}}$ is invariant under the Galois action.

This corresponds to the fact that the monodromy of the Levi-Civita connection ∇^{LC} of the flat metric g (or the monodromy of the trivialization (26)) becomes trivial on a k_0 -fold cover of V . In fact, one can see that the monodromy of ∇^{LC} is trivial on the cover $U/H^2(X, \mathbb{Z}) \rightarrow U/H^2(\mathcal{X}, \mathbb{Z}) \cong V$, where X is the coarse moduli space of \mathcal{X} .

Compatibility with Deligne's canonical extension —As we did at the end of Section 2.2, we can extend the quantum D -module on the cover $U/H^2(X, \mathbb{Z})$ to a connection on $\overline{U/H^2(X, \mathbb{Z})}$ with a logarithmic pole along $q^1 \cdots q^r = 0$ by choosing a nef basis p_1, \dots, p_r of $H^2(X, \mathbb{Z})/\text{tors}$. This is a Deligne's canonical extension [29] of ∇ for a fixed $z \in \mathbb{C}^*$. In order to have a Deligne's extension, we need to choose a logarithm of the monodromy $M_a := G^{\mathcal{H}}(p_a) = e^{-2\pi i p_a/z}$ around the axis $q^a = 0$. In our case, M_a has the standard logarithm $\text{Log}(M_a) = -2\pi i p_a/z$ since M_a is unipotent. We can define the Deligne extension here as follows. A section $s(\tau, z)$ of F over $(U/H^2(X, \mathbb{Z})) \times \mathbb{C}^*$ is defined to be extendible to $\overline{U/H^2(X, \mathbb{Z})} \times \mathbb{C}^*$ if the image $\iota_\tau(s) \in \mathcal{H}^\mathcal{X}$ of $s(\tau, \cdot) \in \Gamma(\{\tau\} \times \mathbb{C}^*, F)$ satisfies the following: *the family of elements in $\mathcal{H}^\mathcal{X}$*

$$U/H^2(X, \mathbb{Z}) \ni [\tau] \mapsto \tilde{s}_\tau := \exp \left(\sum_{a=1}^r \frac{\log q^a}{2\pi i} \text{Log}(M_a) \right) \iota_\tau(s) \in \mathcal{H}^\mathcal{X}$$

extends holomorphically to $\overline{U/H^2(X, \mathbb{Z})}$, where we put $\tau = \tau_{0,2} + \tau'$ as in (4) and $\tau_{0,2} = \sum_{a=1}^r p_a \log q^a$. Note that \tilde{s}_τ is single-valued on $U/H^2(X, \mathbb{Z})$ since the exponential factor offsets the monodromy. Moreover, the limit of $s(\tau, z)$ at $q = \tau' = 0$ is regular at $z = 0$ if $\tilde{s}_\tau|_{q=\tau'=0}$ lies in the *limiting Hodge structure* \mathbb{F}_{lim} :

$$\mathbb{F}_{\text{lim}} := \lim_{\substack{q \rightarrow 0 \\ \tau' \rightarrow 0}} \exp \left(\sum_{a=1}^r \frac{\log q^a}{2\pi i} \text{Log}(M_a) \right) \mathbb{F}_\tau,$$

where we put $\tau = \sum_{a=1}^r p_a \log q^a + \tau'$ as in (4). By using (30) and the definition (9) of $L(\tau, z)$, one can check that \mathbb{F}_{lim} exists and

$$(34) \quad \mathbb{F}_{\text{lim}} = H_{\text{CR}}^*(\mathcal{X}) \otimes \mathcal{O}(\mathbb{C}) \subset \mathcal{H}^\mathcal{X}.$$

The existence of \mathbb{F}_{lim} is an analogue of the *nilpotent orbit theorem* [68] in quantum cohomology. This means that the semi-infinite Hodge structure \mathbb{F}_τ is approximated by the nilpotent orbit $e^{-\sum_{a=1}^r \log q^a \text{Log}(M_a)/(2\pi i)} \mathbb{F}_{\text{lim}}$ as $q, \tau' \rightarrow 0$. The standard opposite subspace is opposite to \mathbb{F}_{lim} :

$$(35) \quad \mathcal{H}_-^\mathcal{X} \oplus \mathbb{F}_{\text{lim}} = \mathcal{H}^\mathcal{X}.$$

This corresponds to the fact that the trivialization induced from $\mathcal{H}_-^\mathcal{X}$ is compatible with the Deligne extension at $q = 0$, *i.e.* a section which

is constant in the trivialization (26) is extendible across $q = 0$ in the Deligne extension. Note that this is a stronger condition than that \mathcal{H}_- is opposite to \mathbb{F}_τ for every $\tau \in U$.

For a multiplicative character $\alpha: H^2(\mathcal{X}, \mathbb{Z}) \rightarrow \mathbb{C}^*$, we put

$$\mathbb{T}_\alpha := \{v \in \mathbb{T} ; \exp(2\pi i f_v(\xi)) = \alpha(\xi), \forall \xi \in H^2(\mathcal{X}, \mathbb{Z})\}.$$

Because $e^{2\pi i \iota_v} = \alpha([-K_{\mathcal{X}}])$ for $v \in \mathbb{T}_\alpha$, the age ι_v for $v \in \mathbb{T}_\alpha$ have the common fractional part for each α . Consider the following two conditions.

$$(36) \quad \forall \alpha, \exists n_\alpha \in \mathbb{Q} \text{ such that } \forall v \in \mathbb{T}_\alpha \ (n_v + 2\iota_v = n_\alpha \text{ or } n_\alpha + 1).$$

$$(37) \quad \iota_v = 0, v \neq 0 \implies \exists \xi \in H^2(\mathcal{X}, \mathbb{Z}) \text{ such that } f_v(\xi) > 0.$$

Here $n_v := \dim_{\mathbb{C}} \mathcal{X}_v$. The first condition is a weaker version of the Hard Lefschetz condition we will see later¹¹. (There we have $n_v + 2\iota_v = \dim_{\mathbb{C}} \mathcal{X}$ for all v .) When (36) is satisfied, we put

$$(38) \quad \mathbb{T}_{\alpha, j} = \{v \in \mathbb{T}_\alpha ; n_v + 2\iota_v = n_\alpha + j\}, \quad \mathbb{T}_\alpha = \mathbb{T}_{\alpha, 0} \sqcup \mathbb{T}_{\alpha, 1}.$$

Example 3.12. If \mathcal{X} is isomorphic to a quotient $[M/G]$ of a manifold M by an *abelian* Lie group G as a topological orbifold, the conditions (36), (37) are satisfied since every \mathbb{T}_α consists of one element. In fact, there are sufficiently many line bundles on $[M/G]$ arising from characters of G which “separate” different inertia components. In particular, these hold for toric orbifolds.

Theorem 3.13. *Assume that the coarse moduli space X of \mathcal{X} is projective and that the quantum cohomology of \mathcal{X} is convergent (Assumption 2.1). The standard opposite subspace $\mathcal{H}_- = \mathcal{H}_-^{\mathcal{X}}$ and the standard dilation shift $v_0 = \mathbf{1}$ of the quantum D -module $QDM(\mathcal{X})$ are characterized as follows.*

(i) *Under the condition (36), there exists a unique homogeneous opposite subspace satisfying the monodromy invariance (32), (33) and the compatibility with the Deligne extension (35).*

(ii) *Under the condition (37), there exists a unique vector $v_0 \in z\mathcal{H}_-^{\mathcal{X}}/\mathcal{H}_-^{\mathcal{X}}$ (up to a scalar multiple) such that v_0 is an eigenvector of $\mu = \mathcal{V} = [\nabla_{z\partial_z}]$ of the smallest eigenvalue $-\dim_{\mathbb{C}} \mathcal{X}/2$ and invariant under every Galois action on $z\mathcal{H}_-^{\mathcal{X}}/\mathcal{H}_-^{\mathcal{X}}$.*

Thus under (36) and (37), the above conditions determine a canonical Frobenius structure at the cusp up to a constant multiple of the flat metric.

¹¹The condition (36) says that $V_\alpha = \bigoplus_{v \in \mathbb{T}_\alpha} H^{*-2\iota_v}(\mathcal{X}_v)$ is bicentric HL in the sense of Definition 3.20. See also Remark 3.21.

Proof. Let $\mathcal{H}_- \subset \mathcal{H}^\mathcal{X}$ be any homogeneous opposite subspace satisfying (32), (33) and (35). We decompose the Galois action as

$$G^\mathcal{H}(\xi) = e^{-2\pi i \xi_0/z} \circ G_0^\mathcal{H}(\xi), \quad G_0^\mathcal{H}(\xi) = \bigoplus_{v \in \mathbb{T}} e^{2\pi i f_v(\xi)}.$$

Claim: \mathcal{H}_- satisfies the following:

$$\xi_0 \cdot \mathcal{H}_- \subset \mathcal{H}_-, \quad G_0^\mathcal{H}(\xi)\mathcal{H}_- \subset \mathcal{H}_-, \quad (z\partial_z + \mu)\mathcal{H}_- \subset \mathcal{H}_-.$$

Take a sufficiently big $k_0 > 0$ such that $(G_0^\mathcal{H}(\xi))^{k_0} = \text{id}$ and (33) hold. Then $(G^\mathcal{H}(\xi))^{k_0} = e^{-k_0 2\pi i \xi_0/z}$ preserves \mathcal{H}_- and acts trivially on $z\mathcal{H}_-/\mathcal{H}_-$. Then $\text{Log}((G^\mathcal{H}(\xi))^{k_0}) = -k_0 2\pi i \xi_0/z$ sends $z\mathcal{H}_-$ to \mathcal{H}_- . This implies the first equation. The second equation follows from $G_0^\mathcal{H}(\xi) = e^{2\pi i \xi_0/z} \circ G^\mathcal{H}(\xi)$ and (32). The third equation follows from $\nabla_{z\partial_z} \mathcal{H}_- \subset \mathcal{H}_-$, the formula (31) for $\nabla_{z\partial_z}$ and $(\rho/z)\mathcal{H}_- \subset \rho\mathcal{H}_- \subset \mathcal{H}_-$.

The third equation in the claim means that \mathcal{H}_- is homogeneous with respect to the usual grading on $H_{\text{CR}}^*(\mathcal{X})$ together with $\text{deg } z = 2$. The opposite property (35) and the formula (34) for \mathbb{F}_{lim} imply that

$$(39) \quad z\mathcal{H}_- \cap \mathbb{F}_{\text{lim}} \cong \mathbb{F}_{\text{lim}}/z\mathbb{F}_{\text{lim}} = H_{\text{CR}}^*(\mathcal{X}).$$

Since $z\partial_z + \mu$ preserves $z\mathcal{H}_- \cap \mathbb{F}_{\text{lim}}$, this is an isomorphism of graded vector spaces. Also $G_0^\mathcal{H}(\xi)$ preserves $z\mathcal{H}_- \cap \mathbb{F}_{\text{lim}}$ and (39) is equivariant with respect to the action of $G_0^\mathcal{H}(\xi)$. Therefore (39) is decomposed into the sum of simultaneous eigenspaces of the commuting operators $G_0^\mathcal{H}(\xi)$. Recall that the condition (36) gives the decomposition (38). Take a multiplicative character $\alpha: H^2(\mathcal{X}, \mathbb{Z}) \rightarrow \mathbb{C}^*$ and set

$$V_{\alpha,j} = \bigoplus_{v \in \mathbb{T}_{\alpha,j}} H^{*-2\nu_v}(\mathcal{X}_v), \quad j = 0, 1, \quad V_\alpha = V_{\alpha,0} \oplus V_{\alpha,1}.$$

Then V_α is the simultaneous eigenspace of $G_0^\mathcal{H}(\xi)$ of eigenvalue α . By (39), for a homogeneous element $\phi \in V_{\alpha,j}$, there exists a unique lift $\hat{\phi} \in z\mathcal{H}_- \cap \mathbb{F}_{\text{lim}}$ such that

$$\hat{\phi} = \phi + O(z), \quad \text{deg } \hat{\phi} = \text{deg } \phi, \quad \hat{\phi} \in V_\alpha \otimes \mathcal{O}(\mathbb{C}^*).$$

By the Claim above, the $H^2(\mathcal{X})$ -action also preserves $z\mathcal{H}_- \cap \mathbb{F}_{\text{lim}}$. Therefore we have $\widehat{\omega \cdot \phi} = \omega \cdot \hat{\phi}$ for a Kähler class ω . Because X is Kähler, the cohomology ring $H^*(\mathcal{X}_v)$ of every inertia component has the Hard Lefschetz property. Hence under the condition (36), the following holds with respect to the grading of the Chen–Ruan cohomology $H_{\text{CR}}^*(\mathcal{X})$.

$$(40) \quad \omega^i: V_{\alpha,j}^{n_\alpha+j-i} \rightarrow V_{\alpha,j}^{n_\alpha+j+i} \quad \text{is an isomorphism} \quad j = 0, 1.$$

We also have the Lefschetz decomposition of $V_{\alpha,j}$:

$$V_{\alpha,j} = \bigoplus_{k \geq 0} \bigoplus_{i=0}^k \omega^i PV_{\alpha,j}^{n_\alpha+j-k}$$

where $PV_{\alpha,j}^{n_\alpha+j-k} = \text{Ker}(\omega^{k+1}: V_{\alpha,j}^{n_\alpha+j-k} \rightarrow V_{\alpha,j}^{n_\alpha+j+k+2})$ is the primitive part. By the property $\widehat{\omega \cdot \phi} = \omega \cdot \widehat{\phi}$, we only need to know $\widehat{\phi}$ for $\phi \in PV_{\alpha,j}^{n_\alpha+j-k}$. For $\phi \in PV_{\alpha,j}^{n_\alpha+j-k}$, we can put

$$\widehat{\phi} = \phi + z\phi_1 + z^2\phi_2 + \dots$$

where $\phi_i \in V_{\alpha}^{n_\alpha+j-k-2i}$. Then $0 = \widehat{\omega^{k+1}\phi} = \sum_{i \geq 1} z^i \omega^{k+1}\phi_i$. This implies $\omega^{k+1}\phi_i = 0$. Note that $\phi_i \in V_{\alpha,0}^{n_\alpha-(k+2i-j)} \oplus V_{\alpha,1}^{n_\alpha+1-(k+2i+1-j)}$. Then the Hard Lefschetz (40) for $V_{\alpha,*}$ implies $\phi_i = 0$ and so $\widehat{\phi} = \phi$. By the Lefschetz decomposition, we have $\widehat{\phi} = \phi$ for every $\phi \in V_{\alpha,j}$. Therefore $z\mathcal{H}_- \cap \mathbb{F}_{\text{lim}} = H_{\text{CR}}^*(\mathcal{X})$ and $z\mathcal{H}_- = H_{\text{CR}}^*(\mathcal{X}) \otimes \mathcal{O}(\mathbb{P}^1 \setminus \{0\})$.

It is easy to show the characterization of v_0 . When v_0 is replaced with λv_0 for some $\lambda \in \mathbb{C}$, the flat metric g is multiplied by λ^2 . Q.E.D.

Remark 3.14. The limiting Hodge structure \mathbb{F}_{lim} depends on the choice of co-ordinates q^1, \dots, q^r on $\overline{U/H^2(X, \mathbb{Z})}$. Another co-ordinate system $\hat{q}^a := c^a q^a \exp(F_a(q))$ with $F_a(0) = 0$ changes \mathbb{F}_{lim} by the multiplication by $\exp(\sum_a \log c^a \text{Log}(M_a)/(2\pi i))$. Under the monodromy invariance (32) for $\mathcal{H}_-, \mathcal{H}_-$ being opposite to \mathbb{F}_{lim} (35) is independent of the choice of a co-ordinate system since $\text{Log}(M_a)$ preserves \mathcal{H}_- .

Remark 3.15. We can normalize the dilaton shift $v_0 \in z\mathcal{H}_-/\mathcal{H}_-$ using the integral structure $F_{\mathbb{Z}}$. The dilaton shift v_0 defines a primitive section s_0 of the quantum D -module via the trivialization (26). Under the condition (37), there exists a one-dimensional subspace $\mathbb{C}A_0$ of the space $\mathcal{S}(\mathcal{X})$ of flat sections which is invariant under every Galois action and contained in the image of $(\text{id} - G^S(\xi))^n$ for some unipotent operator $G^S(\xi)$ with the maximum unipotency $n = \dim_{\mathbb{C}} \mathcal{X}$. (This can be seen from the cohomology framing. See (11).) An integral generator A_0 of this subspace is determined up to sign: In fact, this is given by the structure sheaf of a non-stacky point $A_0 = \pm \mathcal{Z}_K(\mathcal{O}_{\text{pt}})$. The choice $v_0 = \pm \mathbf{1}$ corresponds to the normalization $(s_0, A_0)_F \sim (2\pi i)^n / (2\pi z)^{\frac{n}{2}}$ in the large radius limit.

3.6. Symplectic transformation between Givental spaces

Here we see that Assumption 3.1 gives rise to a symplectic transformation \mathbb{U} between the Givental spaces $\mathcal{H}^{\mathcal{X}_1}$ and $\mathcal{H}^{\mathcal{X}_2}$. The transformation \mathbb{U} was introduced in [27] to describe relationships between the

genus zero Gromov–Witten theories of \mathcal{X}_1 and \mathcal{X}_2 . As we have seen, the genus zero theory defines a semi-infinite variation of Hodge structures $\mathbb{F}_\tau^{\mathcal{X}_i} \subset \mathcal{H}^{\mathcal{X}_i}$ in the Givental spaces. We shall see in (48) that they match under \mathbb{U} : $\mathbb{U}\mathbb{F}_\tau^{\mathcal{X}_1} = \mathbb{F}_\tau^{\mathcal{X}_2}$. This implies that Givental's Lagrangian cones $\mathcal{L}_i \subset \mathcal{H}^{\mathcal{X}_i}$ [26] swept by the semi-infinite subspaces $z\mathbb{F}_\tau^{\mathcal{X}_i}$ are mapped to each other under \mathbb{U} :

$$\mathbb{U}\mathcal{L}_1 = \mathcal{L}_2, \quad \text{where } \mathcal{L}_i := \bigcup_{\tau} z\mathbb{F}_\tau^{\mathcal{X}_i} \subset \mathcal{H}^{\mathcal{X}_i}.$$

The Lagrangian cone $\mathcal{L}_i \subset \mathcal{H}^{\mathcal{X}_i}$ can be also described as the graph of the genus zero descendant potential of \mathcal{X}_i [26] and encodes all the information on genus zero Gromov–Witten theory. In the literature [27, 28, 23], the crepant resolution conjecture was formulated in this way and verified in several examples. See these references for more details and examples of \mathbb{U} .

Take a path $\gamma: [0, 1] \rightarrow \mathcal{M}$ connecting two cusp neighborhoods V_1, V_2 . Then we have the analytic continuation map (20) $P_\gamma: \mathcal{S}(\mathcal{X}_1) \rightarrow \mathcal{S}(\mathcal{X}_2)$ along the path $\hat{\gamma} = (\gamma, 1): [0, 1] \rightarrow \mathcal{M} \times \mathbb{C}^*$. Through the cohomology framing \mathcal{Z}_{coh} (10), P_γ induces the following isomorphism:

$$(41) \quad \mathbb{U}_{\text{coh}}: H_{\text{CR}}^*(\mathcal{X}_1) \rightarrow H_{\text{CR}}^*(\mathcal{X}_2), \quad \mathbb{U}_{\text{coh}} = \mathcal{Z}_{\text{coh}}^{-1} P_\gamma \mathcal{Z}_{\text{coh}}.$$

Recall that the Givental space $\mathcal{H}^{\mathcal{X}_i}$ is identified with the space of (multi-valued) sections of F over $V_i \times \mathbb{C}^*$ which are flat in the V_i direction. Therefore, the analytic continuation along $\hat{\gamma}$ also induces a map between the Givental spaces:

$$(42) \quad \mathbb{U}: \mathcal{H}^{\mathcal{X}_1} \rightarrow \mathcal{H}^{\mathcal{X}_2}.$$

The map \mathbb{U} is an $\mathcal{O}(\mathbb{C}^*)$ -linear isomorphism preserving the pairing $(\cdot, \cdot)_{\mathcal{H}}$ on the Givental spaces. In particular, \mathbb{U} is a symplectic transformation with respect to the symplectic form (29). Recall that the cohomology framing identifies $\phi \in H_{\text{CR}}^*(\mathcal{X}_i)$ with a flat section $L(\tau, z)z^{-\mu_i}z^{\rho_i}\phi$ of $QDM(\mathcal{X}_i)$. Also recall that $\phi(z)$ in the Givental space $\mathcal{H}^{\mathcal{X}}$ corresponds to the flat section $L(\tau, z)\phi(z)$. Therefore, one has the commutative diagram involving “multi-valued” Givental spaces:

$$(43) \quad \begin{array}{ccc} H_{\text{CR}}^*(\mathcal{X}_1) & \xrightarrow{\mathbb{U}_{\text{coh}}} & H_{\text{CR}}^*(\mathcal{X}_2) \\ z^{-\mu_1}z^{\rho_1} \downarrow & & z^{-\mu_2}z^{\rho_2} \downarrow \\ \mathcal{H}^{\mathcal{X}_1} \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{O}(\widehat{\mathbb{C}^*}) & \xrightarrow{\mathbb{U}} & \mathcal{H}^{\mathcal{X}_2} \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{O}(\widehat{\mathbb{C}^*}) \end{array}$$

where $\rho_i = c_1(\mathcal{X}_i)$ and μ_i is the Hodge grading operator of \mathcal{X}_i .

For a rational number $f \in [0, 1)$, we set

$$(44) \quad H_{\text{CR}}^*(\mathcal{X})_f := \bigoplus_{\langle \iota_v \rangle = f} H^{*-2\iota_v}(\mathcal{X}_v) = {}^{12} \bigoplus_{\langle p/2 \rangle = f} H_{\text{CR}}^p(\mathcal{X}).$$

Here $\langle \iota_v \rangle$ is the fractional part of ι_v . Correspondingly, we set

$$\mathcal{H}_f^{\mathcal{X}} := H_{\text{CR}}^*(\mathcal{X})_f \otimes \mathcal{O}(\mathbb{C}^*) \subset \mathcal{H}^{\mathcal{X}}.$$

We list basic properties of \mathbb{U}_{coh} and \mathbb{U} , some of which already appeared in [27, 28]. We will use these later.

Lemma 3.16. *Under Assumption 3.1, the analytic continuation maps \mathbb{U}_{coh} and \mathbb{U} given in (41), (42) satisfy the following:*

$$(45) \quad \mathbb{U}_{\text{coh}}\rho_1 = \rho_2\mathbb{U}_{\text{coh}}, \quad \mathbb{U}\rho_1 = \rho_2\mathbb{U},$$

$$(46) \quad \mathbb{U}_{\text{coh}}H_{\text{CR}}^*(\mathcal{X}_1)_f = H_{\text{CR}}^*(\mathcal{X}_2)_f, \quad \mathbb{U}\mathcal{H}_f^{\mathcal{X}_1} = \mathcal{H}_f^{\mathcal{X}_2},$$

$$(47) \quad \mathbb{U} = z^{-\mu_2}\mathbb{U}_{\text{coh}}z^{\mu_1},$$

$$(48) \quad \mathbb{U}\mathbb{F}_{\tau}^{\mathcal{X}_1} = \mathbb{F}_{\tau}^{\mathcal{X}_2}, \quad \tau \in \mathcal{M}.$$

Here $\mathbb{F}_{\tau}^{\mathcal{X}_i} \subset \mathcal{H}^{\mathcal{X}_i} \cong \mathcal{H}$ is the semi-infinite Hodge structure (30) at $\tau \in \mathcal{M}$ considered as a subspace of the Givental space. The equation (47) shows that \mathbb{U} is degree-preserving, where the grading on $\mathcal{H}^{\mathcal{X}}$ is given by the usual grading on $H_{\text{CR}}^*(\mathcal{X})$ and $\deg z = 2$.

Assume that \mathcal{X}_1 and \mathcal{X}_2 are K -equivalent and related by the diagrams (18), (19) such that $\pi_1 \circ p_1 = \pi_2 \circ p_2$. Let γ be the path in (ii) of Assumption 3.1. Then for a class $\alpha \in H^2(Z, \mathbb{C})$,

$$(49) \quad \mathbb{U}_{\text{coh}}(\pi_1^*\alpha) = (\pi_2^*\alpha)\mathbb{U}_{\text{coh}}, \quad \mathbb{U}(\pi_1^*\alpha) = (\pi_2^*\alpha)\mathbb{U}.$$

Proof. The analytic continuation along $\hat{\gamma} = (\gamma, 1)$ must be equivariant under the monodromy in $z \in \mathbb{C}^*$. A simple calculation shows that the monodromy in z acts on $\mathcal{S}(\mathcal{X}_i) \cong H_{\text{CR}}^*(\mathcal{X}_i)$ by

$$(50) \quad M_i = (-1)^n e^{-2\pi i \rho_i} \bigoplus_{v \in \mathbb{T}_i} e^{2\pi i \iota_v}, \quad n = \dim \mathcal{X}_i,$$

where \mathbb{T}_i is the index set of the inertia component of \mathcal{X}_i . Then $M_2\mathbb{U}_{\text{coh}} = \mathbb{U}_{\text{coh}}M_1$. Taking sufficiently high powers of M_i , we have $e^{-k_0 2\pi i \rho_2}\mathbb{U}_{\text{coh}} = \mathbb{U}_{\text{coh}}e^{-k_0 2\pi i \rho_1}$. This shows the first equation of (45). Therefore we also have $\mathbb{U}_{\text{coh}} \bigoplus_{v \in \mathbb{T}_1} e^{2\pi i \iota_v} = \bigoplus_{v \in \mathbb{T}_2} e^{2\pi i \iota_v}\mathbb{U}_{\text{coh}}$. This shows the first

¹²This equality holds since we ignore cohomology classes of odd parity.

equation of (46). Since \mathbb{U}_{coh} commutes with ρ_i, z^{ρ_i} 's in the commutative diagram (43) cancel each other. This shows (47) and in turn shows the second equations of (45), (46). The equation (48) is a tautological relation since $\mathbb{F}_\tau^{\mathcal{X}_1}$ and $\mathbb{F}_\tau^{\mathcal{X}_2}$ arise from the same subspace \mathbb{F}_τ of \mathcal{H} .

When \mathcal{X}_1 and \mathcal{X}_2 are related by the birational correspondences (18), (19), the analytic continuation P_γ is equivariant under the monodromy (Galois) action coming from a line bundle L on Z . By the formula (11) of the Galois action in terms of \mathcal{Z}_{coh} , we have $\mathbb{U}_{\text{coh}} e^{-2\pi i \pi_1^* c_1(L)} = e^{-2\pi i \pi_2^* c_1(L)} \mathbb{U}_{\text{coh}}$ and (49) follows. Q.E.D.

3.7. Hard Lefschetz condition

We have seen under Assumption 3.1 that quantum cohomologies of \mathcal{X}_1 and \mathcal{X}_2 underlie the same F -manifold \mathcal{M} (Proposition 3.5) and that the F -manifold structure can be (canonically) lifted over V_i to a Frobenius manifold structure by the opposite subspace $\mathcal{H}_-^{\mathcal{X}_i}$ (Propositions 3.9 and Theorem 3.13). Since a Frobenius structure is well-defined over the complement of an analytic subvariety of $\widetilde{\mathcal{M}}$, we can compare the two Frobenius structures arising from different cusps V_1, V_2 . However, we have many examples (e.g. a crepant resolution of $\mathbb{P}(1, 1, 1, 3)$) where they do not coincide [3, 27]. The *Hard Lefschetz condition* introduced by [27] (and adopted by [14]) is a criterion for the two Frobenius structures to match. The point is that the monodromy action coming from line bundles on Z uniquely fixes opposite subspaces under this condition.

In this section, we consider the case where \mathcal{X}_1 and \mathcal{X}_2 are K -equivalent (18) and related by the birational correspondence:

$$\mathcal{X}_1 \xrightarrow{\pi_1} Z \xleftarrow{\pi_2} \mathcal{X}_2$$

such that $\pi_1 \circ p_1 = \pi_2 \circ p_2$.

Definition 3.17. Assume that $H_{\text{CR}}^*(\mathcal{X}_i)$ is graded by integers. We say that $\pi_i: \mathcal{X}_i \rightarrow Z$ satisfies the *Hard Lefschetz condition* if the map

$$(\pi_i^* \omega_Z)^k: H_{\text{CR}}^{n-k}(\mathcal{X}_i) \rightarrow H_{\text{CR}}^{n+k}(\mathcal{X}_i)$$

is an isomorphism for a class ω_Z of an ample line bundle on Z .

Remark 3.18. In the context of crepant resolution conjecture, one can take $\mathcal{X}_1 = \mathcal{X}, Z$ to be the coarse moduli space X of \mathcal{X} and \mathcal{X}_2 to be a crepant resolution Y of X . The Hard Lefschetz condition was originally discussed in [27, 14] for the natural map $\mathcal{X} \rightarrow X$. As was observed in [32], the Hard Lefschetz condition for $\mathcal{X} \rightarrow X$ is equivalent to

$$(51) \quad \iota_v = \iota_{\text{inv}(v)}, \quad \forall v \in \mathbb{T}.$$

For a non-compact \mathcal{X} , the Hard Lefschetz condition for $\mathcal{X} \rightarrow X$ can be defined by this condition (51). It is important to consider non-compact cases, but unfortunately, the discussion in this section does not apply to a non-compact \mathcal{X} .

Remark 3.19. Cataldo–Migliorini [17] showed that when $\mathcal{X}_i = Y$ is a smooth projective variety, $\pi: Y \rightarrow Z$ satisfies the Hard Lefschetz condition if and only if π is semismall. Here a proper morphism $\pi: Y \rightarrow Z$ is said to be *semismall* if $\dim Z^k + 2k \leq \dim Y$, where $Z^k = \{z \in Z; \dim \pi^{-1}(z) = k\}$.

We will consider a generalization of the Hard Lefschetz condition, where we do not assume the integer grading and also include the “bi-centric” case.

Definition 3.20. (i) We say that a pair (V, ω) of a \mathbb{Q} -graded complex vector space V and a nilpotent endomorphism $\omega \in \text{End}(V)$ of degree 2 is *bicentric HL* if there exists a rational number $n \in \mathbb{Q}$ and a graded decomposition $V = V_0 \oplus V_1$ such that $V^p = 0$ unless $p \in n + \mathbb{Z}$ and

$$\omega^k: V_j^{n+j-k} \rightarrow V_j^{n+j+k} \text{ is an isomorphism for } j = 0, 1 \text{ and all } k \geq 0.$$

We call the set $\{n, n + 1\}$ the *bicenter*. Note that this definition contains the “mono-centric” case where V_0 or V_1 vanishes.

(ii) We say that a proper morphism $\pi: \mathcal{X} \rightarrow Z$ satisfies the *generalized Hard Lefschetz condition* if for every rational number $f \in [0, 1)$, the pair $(H_{\text{CR}}^*(\mathcal{X})_f, \pi^*\omega_Z)$ is bicentric HL, where $H_{\text{CR}}^*(\mathcal{X})_f$ is the graded subspace of $H_{\text{CR}}^*(\mathcal{X})$ defined in (44) and ω_Z is a class of an ample line bundle on Z .

Remark 3.21. When π is the natural map $\mathcal{X} \rightarrow X$ to the coarse moduli, the generalized Hard Lefschetz condition for π reads as follows: For every rational number $f \in [0, 1)$, there exists $n_f \in \mathbb{Q}$ such that

$$\langle \iota_v \rangle = f \implies \dim_{\mathbb{C}} \mathcal{X}_v + 2\iota_v = n_f \text{ or } n_f + 1.$$

Here $\{n_f, n_f + 1\}$ is the bicenter of $(H_{\text{CR}}^*(\mathcal{X})_f, \omega_X)$.

Theorem 3.22. *Let $\mathcal{X}_1, \mathcal{X}_2$ be K -equivalent smooth Deligne–Mumford stacks related by the diagrams (18), (19) such that $p_1^*K_{\mathcal{X}_1} = p_2^*K_{\mathcal{X}_2}$ and $\pi_1 \circ p_1 = \pi_2 \circ p_2$. Assume that $\pi_1: \mathcal{X}_1 \rightarrow Z$ satisfies the (generalized) Hard Lefschetz condition. Under Assumption 3.1, the standard opposite subspaces $\mathcal{H}_{\mathcal{X}_1}^{\mathcal{X}_1}, \mathcal{H}_{\mathcal{X}_2}^{\mathcal{X}_2}$ coincide under the analytic continuation along the path γ in (ii) of Assumption 3.1, i.e. $\mathbb{U}(\mathcal{H}_{\mathcal{X}_1}^{\mathcal{X}_1}) = \mathcal{H}_{\mathcal{X}_2}^{\mathcal{X}_2}$. Moreover,*

(i) *If \mathcal{X}_1 or \mathcal{X}_2 does not have generic stabilizers, the Frobenius manifold structures on \mathcal{M} coming from the quantum cohomology of \mathcal{X}_1 and*

\mathcal{X}_2 coincide up to a scalar multiple of the flat metric though the analytic continuation along γ .

(ii) There is a graded isomorphism $(H_{\text{CR}}^*(\mathcal{X}_1), \pi_1^* \omega_Z) \cong (H_{\text{CR}}^*(\mathcal{X}_2), \pi_2^* \omega_Z)$ preserving the actions of ω_Z . In particular, $\pi_2: \mathcal{X}_2 \rightarrow Z$ also satisfies the (generalized) Hard Lefschetz condition.

This theorem is a generalization of a result in [27]. We use the following lemma in the proof.

Lemma 3.23. *Let $V_i, i = 1, 2$ be \mathbb{Q} -graded vector spaces and $\omega_i \in \text{End}(V_i)$ be nilpotent endomorphisms of degree two. Assume that V_1 and V_2 are isomorphic as graded vector spaces and that there exists a (not necessarily graded) linear isomorphism $\mathbb{U}: V_1 \rightarrow V_2$ such that $\mathbb{U}\omega_1 = \omega_2\mathbb{U}$. If (V_1, ω_1) is bicentric HL, then there exists a (not canonical) graded isomorphism $\varphi: V_1 \rightarrow V_2$ such that $\varphi\omega_1 = \omega_2\varphi$. In particular, (V_2, ω_2) is also bicentric HL.*

Proof. Let V be a \mathbb{Q} -graded vector space and ω be a nilpotent operator on V of degree 2. Let $a_1 \geq a_2 \geq \dots \geq a_l$ be lengths of the Jordan cells appearing in the Jordan normal form of ω . Then we can take a basis of V of the form

$$(52) \quad \{\omega^k \phi_j ; 1 \leq j \leq l, 0 \leq k \leq a_j\}, \quad a_1 \geq a_2 \geq \dots \geq a_l$$

such that $\omega^{a_j+1} \phi_j = 0$. Here we can assume that ϕ_j is homogeneous. Set $\deg \phi_j = -a_j + \lambda_j$ for some $\lambda_j \in \mathbb{Q}$. By rearranging the basis, we can assume that $\lambda_j \geq \lambda_{j+1}$ if $a_j = a_{j+1}$. The sequence $\{(a_j, \lambda_j)\}_{j \geq 1}$ is uniquely determined by (V, ω) and we call it the *type* of (V, ω) . It suffices to show that $(V_i, \omega_i), i = 1, 2$ have the same type. Let $\{(a_j^{(i)}, \lambda_j^{(i)})\}_{j \geq 1}$ be the type of (V_i, ω_i) . Since ω_1 and ω_2 are conjugate, we have $a_j := a_j^{(1)} = a_j^{(2)}$. Because (V_1, ω_1) is bicentric HL, there exists $n \in \mathbb{Q}$ such that $\lambda_j^{(1)} = n$ or $n + 1$ for all j . Then the degree spectrum of V_1 is contained in $[-a_1 + n, a_1 + n + 1]$. Since V_1 and V_2 are isomorphic as graded vector spaces, we know that $[-a_j + \lambda_j^{(2)}, a_j + \lambda_j^{(2)}] \subset [-a_1 + n, a_1 + n + 1]$. Therefore, $\lambda_j^{(2)} = n$ or $n + 1$ if $a_j = a_1$. Take $k > 0$ such that $a_1 = \dots = a_k > a_{k+1}$. We calculate

$$\begin{aligned} \dim V_1^{a_1+n+1} + \dim V_1^{-a_1+n} &= k \\ \dim V_2^{a_1+n+1} + \dim V_2^{-a_1+n} &= k + \#\{j > k ; -a_j + \lambda_j^{(2)} = -a_1 + n\} \\ &\quad + \#\{j > k ; a_j + \lambda_j^{(2)} = a_1 + n + 1\}. \end{aligned}$$

Since these are equal, we have $[-a_j + \lambda_j^{(2)}, a_j + \lambda_j^{(2)}] \subset (-a_1 + n, a_1 + n + 1)$ if $j > k$. Therefore,

$$\begin{aligned} \#\{j \leq k; \lambda_j^{(1)} = n + 1\} &= \dim V_1^{a_1+n+1} = \dim V_2^{a_1+n+1} \\ &= \#\{j \leq k; \lambda_j^{(2)} = n + 1\}. \end{aligned}$$

Hence $\lambda_j^{(1)} = \lambda_j^{(2)}$ for $j \leq k$. This shows that (V_1, ω_1) and (V_2, ω_2) contains an isomorphic graded subspace (V', ω') of the type $\{(a_j, \lambda_j^{(1)})\}_{1 \leq j \leq k}$. By taking the quotient by this subspace, one can proceed by induction on dimensions. Q.E.D.

Proof of Theorem 3.22. Take a path $\gamma: [0, 1] \rightarrow \mathcal{M}$ satisfying the condition (ii) of Assumption 3.1. The analytic continuation map P_γ (20) along the path $\hat{\gamma} = (\gamma, 1)$ induces maps \mathbb{U}_{coh} (41) and \mathbb{U} (42). Recall that \mathbb{U}_{coh} splits into isomorphisms $\mathbb{U}_{\text{coh},f}: H_{\text{CR}}^*(\mathcal{X}_1)_f \rightarrow H_{\text{CR}}^*(\mathcal{X}_2)_f$ for each $f \in [0, 1)$ by (46). By (49), we have

$$(53) \quad \mathbb{U}_{\text{coh},f}(\pi_1^* \omega_Z) = (\pi_2^* \omega_Z) \mathbb{U}_{\text{coh},f}$$

for an ample class ω_Z on Z . On the other hand, by the theorem of Lupercio–Poddar [56] and Yasuda [72, 73], $H_{\text{CR}}^*(\mathcal{X}_1)$ and $H_{\text{CR}}^*(\mathcal{X}_2)$ are isomorphic as graded vector spaces when \mathcal{X}_1 and \mathcal{X}_2 are K -equivalent. Thus $H_{\text{CR}}^*(\mathcal{X}_1)_f$ and $H_{\text{CR}}^*(\mathcal{X}_2)_f$ are also isomorphic as graded vector spaces. By Lemma 3.23 and (53), we know that there is a graded isomorphism

$$\varphi: (H_{\text{CR}}^*(\mathcal{X}_1)_f, \pi_1^* \omega_Z) \rightarrow (H_{\text{CR}}^*(\mathcal{X}_2)_f, \pi_2^* \omega_Z)$$

and $(H_{\text{CR}}^*(\mathcal{X}_2)_f, \pi_2^* \omega_Z)$ is also bicentric HL.

In general, a nilpotent operator ω on a vector space V defines a unique (increasing) *weight filtration* $W_i(V)$ of V such that $\omega W_i(V) \subset W_{i-2}(V)$ and that $\omega^i: \text{Gr}_i^W(V) \rightarrow \text{Gr}_{-i}^W(V)$ is an isomorphism. Here $\text{Gr}_i^W(V) = W_i(V)/W_{i-1}(V)$. When V is a graded vector space, ω is of degree two and (V, ω) is bicentric HL with a graded decomposition $V = V_0 \oplus V_1$ and a bicenter $\{n, n + 1\}$ (as in Definition 3.20), the weight filtration of V is given by

$$W_k(V) = V_0^{\geq n-k} \oplus V_1^{\geq n+1-k}.$$

Consider the case $(V, \omega) = (H_{\text{CR}}^*(\mathcal{X}_i)_f, \pi_i^* \omega_Z)$. Since the isomorphism $\mathbb{U}_{\text{coh},f}$ preserves the weight filtration (by (53)) and $(H_{\text{CR}}^*(\mathcal{X}_i)_f, \pi_i^* \omega_Z)$ is bicentric HL, we have

$$(54) \quad \mathbb{U}_{\text{coh},f}(H_{\text{CR}}^p(\mathcal{X}_1)_f) \subset H_{\text{CR}}^{\geq p-1}(\mathcal{X}_2)_f.$$

When $\phi \in H_{\text{CR}}^p(\mathcal{X}_1)$, this together with the formula (47) implies that $U\phi$ cannot contain positive powers in z . Therefore a matrix representation $U(z)$ of \mathbb{U} with respect to a basis of $H_{\text{CR}}^*(\mathcal{X}_i)$ does not contain positive powers in z . Since \mathbb{U} preserves the pairing $(\cdot, \cdot)_{\mathcal{H}}$, the same is true for the inverse $U(z)^{-1}$ which is the adjoint of $U(-z)$ with respect to the Poincaré pairing. Thus we have $\mathbb{U}\mathcal{H}_-^{\mathcal{X}_1} \subset \mathcal{H}_-^{\mathcal{X}_2}$ and $\mathbb{U}^{-1}\mathcal{H}_-^{\mathcal{X}_2} \subset \mathcal{H}_-^{\mathcal{X}_1}$. Hence $\mathbb{U}\mathcal{H}_-^{\mathcal{X}_1} = \mathcal{H}_-^{\mathcal{X}_2}$.

Now we assume \mathcal{X}_i does not have generic stabilizers. Let $\mathcal{H}_- \subset \mathcal{H}$ be the common opposite subspace. Then the dilaton shift $v_0 \in z\mathcal{H}_-/\mathcal{H}_-$ is characterized up to a constant by the condition that v_0 is an eigenvector of $\nabla_{z\partial_z}$ on $z\mathcal{H}_-/\mathcal{H}_-$ of the smallest eigenvalue. This shows (i). The rest of the statements follows from what we already showed. Q.E.D.

Remark 3.24. We used the theorem of Lupercio–Poddar and Yasuda [56, 72, 73] in the proof. However, as [27] did, we can deduce the graded isomorphism $H_{\text{CR}}^*(\mathcal{X}_1) \cong H_{\text{CR}}^*(\mathcal{X}_2)$ from Assumption 3.1 and certain additional assumptions. For example, we can show $H_{\text{CR}}^*(\mathcal{X}_1) \cong H_{\text{CR}}^*(\mathcal{X})$ under the assumption that $\mathcal{H}_-^{\mathcal{X}_2}$ is opposite to the limiting Hodge structure $\mathbb{F}_{\text{lim}}^{\mathcal{X}_1}$ at the cusp of V_1 , i.e.

$$(55) \quad \mathbb{U}(\mathbb{F}_{\text{lim}}^{\mathcal{X}_1}) \oplus \mathcal{H}_-^{\mathcal{X}_2} = \mathcal{H}^{\mathcal{X}_2}.$$

The equality (55) was conjectured to hold for a general crepant resolution $\mathcal{X}_2 = Y \rightarrow X \leftarrow \mathcal{X}_1$ in [28]. Interestingly, under the generalized Hard Lefschetz condition, the equality (55) is a consequence of the weaker Assumption 3.1 and Lupercio–Poddar–Yasuda’s theorem.

By Theorem 3.22 and Cataldo–Migliorini’s theorem [17] (see Remark 3.19), Assumption 3.1 has the following interesting consequences:

- Let \mathcal{X} be a Gorenstein orbifold and $Y \rightarrow X$ be a crepant resolution. Then \mathcal{X} satisfies the Hard Lefschetz condition if and only if $Y \rightarrow X$ is semismall.
- Let X_1 and X_2 be K -equivalent smooth projective varieties related by the diagrams (18), (19) with $\pi_1 \circ p_1 = \pi_2 \circ p_2$. Then $X_1 \rightarrow Z$ is semismall if and only if $X_2 \rightarrow Z$ is semismall.

The author learned from Tom Coates that the first statement has been conjectured by Jim Bryan [10].

3.8. Integral periods (Central charges)

Up to now, we have not used the integral structure $F_{\mathbb{Z}}$ of the global quantum D -module. In this section, we will see that the integral structure defines an integral co-ordinate—integral period—on the global Kähler moduli space. This is called a *central charge* (see (14)) in physics.

For example, using this, we can give a “reason” why the specialization value of quantum parameters should be a root of unity in the crepant resolution conjecture [45, 46]. In this section, we restrict our attention to the case of crepant resolution $\mathcal{X}_1 = \mathcal{X} \rightarrow X \leftarrow Y = \mathcal{X}_2$. Also we assume that Y and \mathcal{X} are Calabi–Yau. The case where $c_1(\mathcal{X})$ is semi-positive can be discussed in a similar way by using the *conformal limit*¹³ introduced in [45, 46]. See [45, 46] for semi-positive case.

Let \mathcal{X} be a Calabi–Yau Gorenstein orbifold of dimension n and $\pi: Y \rightarrow X$ be a crepant resolution of the coarse moduli space X . Note that the Gorenstein assumption implies that $H_{\text{CR}}^*(\mathcal{X})$ is graded by even integers. In the Calabi–Yau case, the base space of the quantum D -module has a distinguished locus where the Euler vector field E vanishes. By the formula (7), this is exactly the small (orbifold) quantum cohomology locus $H_{\text{CR}}^2(\mathcal{X})$ or $H^2(Y)$. Recall that the Euler vector field is globally defined on \mathcal{M} by Section 3.2.

Assumption 3.25. *The locus $\mathcal{M}_0 \subset \mathcal{M}$ where the Euler vector field vanishes is connected. Also the path $\gamma: [0, 1] \rightarrow \mathcal{M}$ in (ii) of Assumption 3.1 can be chosen so that it is contained in \mathcal{M}_0 .*

In Calabi–Yau case ($\rho = 0$), the situation is greatly simplified. The monodromy in $z \in \mathbb{C}^*$ is almost trivial and given by $(-1)^n$ by (50). Over the locus \mathcal{M}_0 , the global quantum D -module gives rise to a *finite dimensional variation of Hodge structures (VHS)*. The finite dimensional VHS arises from the filtration of flat sections by the pole/zero orders at $z = 0$. The space \mathcal{S} of multi-valued ∇ -flat sections of F is single-valued in $w = z^{1/2}$ since the monodromy in z is ± 1 . Moreover, over the locus \mathcal{M}_0 , the flat connection ∇ has a logarithmic pole at $z = 0$ since $\mathcal{U} = \mathcal{A}_E(\tau, 0)$ in (23) is zero. Therefore, a ∇ -flat section $s(\tau, z) \in \mathcal{S}$ is at worst meromorphic at $w = z^{1/2} = 0$. This introduces the decreasing filtration $\mathcal{S} = F_\tau^0(\mathcal{S}) \supset F_\tau^1(\mathcal{S}) \supset \cdots \supset F_\tau^n(\mathcal{S}) \supset 0$ for $\tau \in \mathcal{M}_0$:

$$F_\tau^p(\mathcal{S}) = \{s \in \mathcal{S} ; z^{\frac{n}{2}-p}s(\tau, z) \text{ is regular at } z = 0\}.$$

Note that the factor $z^{\frac{n}{2}}$ kills the monodromy of $s(\tau, z)$ in z . On the neighborhoods V_1, V_2 of cusps, \mathcal{S} is identified with $\mathcal{S}(\mathcal{X}), \mathcal{S}(Y)$ and $F^p(\mathcal{S})$ can be described as follows. Because $E = 0$ on \mathcal{M}_0 , $\nabla_{z\partial_z} = z\partial_z + \mu$ for

¹³This is very close to Y. Ruan’s quantum corrected cohomology ring of Y which has the quantum correction only from the exceptional locus [63]; In the abstract Hodge theory, this is also known as a graded quotient by the Sabbah filtration [64, 38].

quantum D -modules and we have

$$(56) \quad \begin{aligned} F_\tau^p(\mathcal{S}) &\cong \{s \in \mathcal{S}(\mathcal{X}) ; s(\tau, z) = z^{-\mu}\phi, \exists \phi \in H_{\text{CR}}^{\leq 2n-2p}(\mathcal{X})\} \\ &\cong \{s \in \mathcal{S}(Y) ; s(\tau, z) = z^{-\mu}\phi, \exists \phi \in H^{\leq 2n-2p}(Y)\} \end{aligned}$$

on $V_1 \cap \mathcal{M}_0$ and $V_2 \cap \mathcal{M}_0$ respectively. The usual Griffiths transversality and Hodge–Riemann bilinear relation hold for $F_\tau^p(\mathcal{S})$:

$$dF_\tau^p(\mathcal{S}) \subset F_\tau^{p-1}(\mathcal{S}) \otimes \Omega_{\mathcal{M}_0}^1, \quad (F_\tau^p(\mathcal{S}), F_\tau^{n-p+1}(\mathcal{S}))_{\mathcal{S}} = 0.$$

Here the pairing $(\cdot, \cdot)_{\mathcal{S}}$ is defined in the same way as in the case of quantum D -modules (see Definition 2.10). The ∞ VHS \mathbb{F}_τ at $\tau \in \mathcal{M}_0$ can be recovered from $F_\tau^p(\mathcal{S})$ as follows:

$$\mathbb{F}_\tau = (z^{-\frac{n}{2}} F_\tau^n(\mathcal{S}) + z^{-\frac{n}{2}+1} F_\tau^{n-1}(\mathcal{S}) + \dots + z^{\frac{n}{2}} F_\tau^0(\mathcal{S})) \otimes \mathcal{O}(\mathbb{C}).$$

We introduce an integral period on \mathcal{M}_0 corresponding to an element of $\mathcal{S}_{\mathbb{Z}}$, *i.e.* a section of the integral local system $F_{\mathbb{Z}}$. This coincides with the central charge introduced in (14) for quantum D -modules. Recall that the analytic continuation map $\mathcal{S}(\mathcal{X}) \cong \mathcal{S} \cong \mathcal{S}(Y)$ along the path $\hat{\gamma}$ in Assumption 3.1 is equivariant under the Galois action of line bundles of the coarse moduli space X . Take an ample line bundle L on X and consider the corresponding Galois action $M = G^{\mathcal{S}}([L])$ on \mathcal{S} .

Lemma 3.26. (i) $F_\tau^n(\mathcal{S}) \subset \mathcal{S}$ is a one dimensional subspace for a generic $\tau \in \mathcal{M}_0$.

(ii) There exists a unique (up to sign) integral vector $A_0 \in \mathcal{S}_{\mathbb{Z}}$ contained in the image of $(\text{Log}(M) - 1)^n$. Under the K -group framing (13) $\mathcal{Z}_K: K(\mathcal{X}) \rightarrow \mathcal{S}(\mathcal{X})$ (or $K(Y) \rightarrow \mathcal{S}(Y)$), A_0 is identified with the structure sheaf of a non-stacky point $A_0 = \pm \mathcal{Z}_K(\mathcal{O}_{\text{pt}})$.

Proof. Since $\dim F_\tau^n$ is upper semi-continuous, (i) follows from the description (56) of $F_\tau^n(\mathcal{S})$ near the cusps. The operator M corresponds to the unipotent operator $e^{-2\pi i c_1(L)}$ on $H_{\text{CR}}^*(\mathcal{X})$ through the cohomology framing (10), thus $\text{Im}(\text{Log}(M) - 1)^n \cong \text{Im } c_1(L)^n = H^{2n}(\mathcal{X})$ is one-dimensional. This contains an integral vector $\mathcal{Z}_K(\mathcal{O}_{\text{pt}})$. Q.E.D.

By Lemma 3.26, the following definition makes sense.

Definition 3.27. Let $\mathbb{C}_w^* \rightarrow \mathbb{C}^* = \mathbb{C}_z^*$ be the double cover of the z -plane with a co-ordinate $w = z^{1/2}$. Take a flat section $A_0 \in \mathcal{S}_{\mathbb{Z}}$ in Lemma 3.26. A *normalized primitive section* is a section $\tilde{s}_0 \in \Gamma(\mathcal{M}_0 \times \mathbb{C}_w^*, F)$ satisfying

- For every $\tau \in \mathcal{M}_0$, $\tilde{s}_0(\tau, z)$ is the restriction of an element of $F_\tau^n(\mathcal{S})$ to $\{\tau\} \times \mathbb{C}_w^*$.

- $(\tilde{s}_0(\tau, e^{\pi^1 z}), A_0(\tau, z))_F = 1.$

This \tilde{s}_0 is unique up to sign since so is A_0 . The *integral period* Π_A associated to $A \in \mathcal{S}_{\mathbb{Z}}$ is the function on \mathcal{M}_0 defined by

$$(57) \quad \Pi_A(\tau) := (\tilde{s}_0(\tau, e^{\pi^1 z}), A(\tau, z))_F, \quad \tau \in \mathcal{M}_0.$$

We compute the normalized primitive section and integral periods for the quantum D -modules of \mathcal{X} and Y . Using the fundamental solution $L(\tau, z)$ in Proposition 2.8, we define the *J-function* by

$$J(\tau, -z) := L(\tau, z)^\dagger \mathbf{1},$$

where $L(\tau, z)^\dagger$ is the adjoint with respect to the Poincaré pairing. The J -function has the following expression:

$$J(\tau, -z) = e^{-\tau_0,2/z} \left(1 - \frac{\tau'}{z} + \sum_{\substack{d \in \text{Eff}_{\mathcal{X}}, 1 \leq k \leq N \\ d=0 \Rightarrow m \geq 2}} \left\langle \tau', \dots, \tau', \frac{\phi_k}{z(z + \psi_{m+1})} \right\rangle_{0, m+1, d} \frac{e^{\langle \tau_0, 2, d \rangle}}{m!} \phi^k \right).$$

Here $\tau = \tau_{0,2} + \tau'$ is the decomposition in (4). (This can be derived from (9) and the String equation [1, Theorem 8.3.1].) When \mathcal{X} is Calabi–Yau and $\tau \in H_{\text{CR}}^2(\mathcal{X})$, the J -function is homogeneous of degree zero and is of the form

$$(58) \quad J(\tau, -z) = 1 - \frac{\tau}{z} + \sum_{k \geq 2} \frac{\alpha_k(\tau)}{z^k}, \quad \alpha_k(\tau) \in H_{\text{CR}}^{2k}(\mathcal{X}).$$

Proposition 3.28. *(This proposition applies to the resolution Y as well.) The normalized primitive section of the quantum D -module of \mathcal{X} is given by*

$$\tilde{s}_0(\tau, z) = \frac{(2\pi z)^{\frac{n}{2}}}{(-2\pi)^n} \mathbf{1}.$$

Therefore, the integral period Π_A (57) associated to an integral flat section $A = Z_K(V)$, $V \in K(\mathcal{X})$ equals the central charge $Z(V)$ (14) of V . This is a component of the J -function:

$$\Pi_A = Z(V) = (2\pi)^{-\frac{n}{2}} \mathbf{i}^{-n} (J(\tau, -1), \Psi(V))_{\text{orb}}, \quad \tau \in H_{\text{CR}}^2(\mathcal{X}),$$

where $\Psi(V)$ was defined in (13) and $J(\tau, -z)$ is the J -function.

Proof. By (56), \tilde{s}_0 satisfies the first condition in Definition 3.27. From $A_0 = \mathcal{Z}_K(\mathcal{O}_{\text{pt}}) = L(\tau, z)((2\pi i)^n / (2\pi z)^{\frac{n}{2}})[\text{pt}]$ and the formula (58) for the J -function, the second condition follows. The rest of the statements just follows from the definition (57) of Π_A with the formulas (13), (14), (58) and $\mu^\dagger = -\mu$. Q.E.D.

Remark 3.29. The above calculation shows that the “normalized” primitive section is (up to a function in z) nothing but the primitive section $s_0 = \mathbf{1}$ associated to the standard opposite subspace and dilation shift (see Section 3.4). The existence of a canonical (normalized) primitive section along the locus \mathcal{M}_0 does not mean that the Frobenius manifold structures of \mathcal{X} and Y are the same. In fact, the primitive sections s_0 of \mathcal{X} and Y may differ outside the locus $\mathcal{M}_0 \subset \mathcal{M}$.

Corollary 3.30. *Under the Assumption 3.1 and Conjecture 3.2, the central charges of the corresponding K -group elements define the same function (up to sign) on \mathcal{M}_0 :*

$$Z^Y(V) = \pm Z^{\mathcal{X}}(\mathbb{U}_K^{-1}(V)), \quad V \in K(Y),$$

where $Z^{\mathcal{X}}$ and Z^Y are the central charges (14) of \mathcal{X} and Y respectively and $\mathbb{U}_K = \mathbb{U}_{K,\gamma}: K(\mathcal{X}) \cong K(Y)$ is the isomorphism in Conjecture 3.2. The sign \pm depends on the sign of $\mathbb{U}_K(\mathcal{O}_{\text{pt}}) = \pm \mathcal{O}_{\text{pt}}$ ¹⁴.

It is interesting to study what integral periods are *affine linear* functions on $H^2_{\text{CR}}(\mathcal{X})$ or $H^2(Y)$. For example, there exists an affine co-ordinate system on $H^2(\mathcal{X}) \oplus \bigoplus_{\text{codim } \mathcal{X}_v=2} H^0(\mathcal{X}_v) \subset H^2_{\text{CR}}(\mathcal{X})$ or on $H^2(Y)$ consisting of integral periods [45, Proposition 6.3], [46, Proposition 5.5]. If we have a stratum \mathcal{X}_v of codimension ≥ 3 with $\iota_v = 1$, the corresponding linear projection $H^2_{\text{CR}}(\mathcal{X}) \rightarrow H^0(\mathcal{X}_v) = \mathbb{C}$ may not be written as an affine linear combination of integral periods. Also, an affine linear integral period on $H^2(Y)$ may not correspond to an affine linear integral period on $H^2_{\text{CR}}(\mathcal{X})$. In the next section, we will examine some local examples.

3.9. Local examples

We consider the crepant resolution conjecture for $\mathcal{X} = [\mathbb{C}^n/G]$ where $G \subset SL(n, \mathbb{C})$ is a finite subgroup and $n = 2$ or 3 . A standard crepant resolution of $X = \mathbb{C}^n/G$ is given by the G -Hilbert scheme [9]:

$$\pi: Y := G\text{-Hilb}(\mathbb{C}^n) \rightarrow X = \mathbb{C}^n/G.$$

¹⁴The author guesses that the sign should be plus.

Moreover, an equivalence of derived categories $D(Y) \cong D(\mathcal{X}) := D^G(\mathbb{C}^n)$ is given by the Fourier–Mukai transformation $\Phi: D(Y) \rightarrow D(\mathcal{X})$ [9]:

$$\Phi = \mathbf{R}q_* \circ p^*, \quad Y \xleftarrow{p} \mathcal{Z} \xrightarrow{q} \mathbb{C}^n$$

where $\mathcal{Z} \subset Y \times \mathbb{C}^n$ is the universal subscheme and p and q are natural projections. It would be natural to conjecture that our K -group isomorphism \mathbb{U}_K comes from this derived equivalence:

$$\mathbb{U}_K^{-1}: K_E(Y) \cong K_0^G(\mathbb{C}^n), \quad [V] \mapsto [\mathbf{R}q_*(p^*V)],$$

where $E = \pi^{-1}(0) \subset Y$ is the exceptional set. Recall that we need to use compactly supported K -groups in order to get well-defined central charges. For a rational curve $\mathbb{P}^1 \cong C \subset E$ in the exceptional set, the central charge of the class $[\mathcal{O}_C(-1)] \in K_E(Y)$ is given by (cf. Example 2.14)

$$Z^Y(\mathcal{O}_C(-1)) = -\frac{1}{2\pi i} \tau \cap [C]$$

for $\tau \in H^2(Y)$. Let $\tau_C := \tau \cap [C]$, $\tau \in H^2(Y)$ be the co-ordinate on $H^2(Y)$ and ϱ_C be the virtual representation of G given by the Fourier–Mukai transform $[\varrho_C \otimes \mathcal{O}_0] = [\mathbf{R}q_*(p^*\mathcal{O}_C(-1))]$. Corollary 3.30 suggests the following conjecture:

Conjecture 3.31. *The small quantum cohomology (or D -modules) of \mathcal{X} and Y are isomorphic under the co-ordinate change*

$$(59) \quad \tau_C = -2\pi i Z^{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho_C)$$

where the right-hand side is the central charge function on $H_{\text{CR}}^2(\mathcal{X})$. See (15) and (16) for formulas of $Z^{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho_C)$. In particular, the quantum variable $q_C = \exp(\tau_C)$ specializes to $\exp(-2\pi i(\dim \varrho_C)/|G|)$ at the large radius limit point of \mathcal{X} .

Remark 3.32. (i) Because \mathcal{X} is not compact, the characterization of the vector A_0 in Lemma 3.26 does not hold. However, we can expect that the conclusion of Corollary 3.30 still holds because the K -group class¹⁵ $[\mathcal{O}_{\text{pt}}]$ of a non-stacky point should correspond to each other (i.e. $\mathbb{U}_K([\mathcal{O}_{\text{pt}}]) = [\mathcal{O}_{\text{pt}}]$) under a birational transformation.

(ii) Since the H^2 -variables do not have the degree, we can expect that the co-ordinate change above is also correct for the \mathbb{C}^* -equivariant quantum cohomology. Here \mathbb{C}^* acts on \mathbb{C}^n diagonally. In dimension two, the non-equivariant quantum product is constant in τ , so it is interesting to study the equivariant version.

¹⁵This corresponds to $[\mathcal{O}_0 \otimes \varrho_{\text{reg}}]$ in $K_0^G(\mathbb{C}^n)$.

(iii) The specialization of q_C to a root of unity comes from the fact that the central charges (15), (16) of $[\mathcal{O}_0 \otimes \varrho_C] = \mathbb{U}_K^{-1}[\mathcal{O}_C(-1)]$ take rational values at the orbifold large radius limit point $\tau = 0$. In [45, 46], the rationality of the central charge of $\mathbb{U}_K^{-1}[\mathcal{O}_C(-1)]$ at the large radius limit was also discussed without assuming the precise form of the K -group framing. When the coarse moduli space X is projective, under the assumption that $H^*(\mathcal{X})$ is generated by $H^2(\mathcal{X})$ and the condition (37), the rationality here is forced only by the monodromy consideration [45, 46].

We have two cases.

(Case 1) *When the Hard Lefschetz condition holds for $\mathcal{X} \rightarrow X$.* Then we have [13, Lemma 3.4.1]

- $n = 2$ or
- $n = 3$ and G is conjugate to a subgroup of $SL(2, \mathbb{C})$ or
- $n = 3$ and G is conjugate to a subgroup of $SO(3, \mathbb{R})$.

In these cases, every inertia component has age $\iota_v = 1$ and the small quantum cohomology is already “big” (ignoring the unit direction), so the above conjecture determines the full relationships of quantum cohomology algebras. Because all the central charges $Z^{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho)$ are affine linear on $H_{\text{CR}}^2(\mathcal{X})$ (the third term in (16) does not exist), the co-ordinate change (59) preserves the flat structure on the base and the Frobenius structures match. Each irreducible component C of the exceptional set E is a rational curve and corresponds to a non-trivial irreducible representation ϱ_C under the Fourier–Mukai transformation¹⁶ (see [48, 34, 6]). The formula (59) agrees with the conjecture of Bryan–Gholampour [11, 13, 14]. The conjecture has been proved for A_n surface singularities $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_n]$ [24] and for $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2]$ and $[\mathbb{C}^3/A_4]$ [12] (where $G = A_4$ is the alternating group; this is the only case where the non-abelian crepant resolution conjecture has been proved).

(Case 2) *When the Hard Lefschetz condition fails for $\mathcal{X} \rightarrow X$.* This happens only when $n = 3$. In this case, since we have the component with age ≥ 2 , the above conjecture does not give a full co-ordinate change between Frobenius manifolds (see Remark 3.33 below). As we can see from (16), integral periods can be non-linear functions on $H_{\text{CR}}^2(\mathcal{X})$, so the co-ordinate change (59) can be also non-linear. Consider the case $\mathcal{X} = \mathbb{C}^3/\mathbb{Z}_3$, where \mathbb{Z}_3 acts on \mathbb{C}^3 by the weight $\frac{1}{3}(1, 1, 1)$. Then Y is the total space of the canonical bundle of \mathbb{P}^2 with the exceptional set

¹⁶The author thanks Samuel Boissiere for explaining this for $G \subset SO(3, \mathbb{R})$.

$E = \mathbb{P}^2$. The Fourier–Mukai transformation is given by the diagram

$$Y = \mathcal{O}_{\mathbb{P}^2}(-3) \xleftarrow{p} \mathcal{Z} = \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{q} \mathbb{C}^3.$$

Let ϱ_1, ϱ_2 be the representations of \mathbb{Z}_3 such that $\varrho_k(1 \bmod 3) = e^{2\pi i k/3}$. For a degree one rational curve $\mathbb{P}^1 \cong C \subset E$, the Fourier–Mukai transform of $\mathcal{O}_C(-1)$ gives the representation $\varrho_C = 2\varrho_1 \oplus \varrho_2$. Thus the predicted co-ordinate change is

$$(60) \quad \tau_C = -2\pi i - \frac{2\pi\sqrt{3}}{3\Gamma(\frac{2}{3})^3} \alpha^2 t + \frac{2\pi\sqrt{3}}{\Gamma(\frac{1}{3})^3} \alpha \frac{\partial F_0^\mathcal{X}}{\partial t},$$

where t is a co-ordinate on the twisted sector $H_{\text{CR}}^2(\mathcal{X})$ dual to $\mathbf{1}_{\frac{1}{3}}$, $\alpha = e^{2\pi i/3}$ and $F_0^\mathcal{X}$ is the genus zero potential of \mathcal{X} (see (17)). Since we have [27, 24]:

$$F_0^\mathcal{X}(t) = \frac{1}{3 \cdot 3!} t^3 - \frac{1}{3^3 \cdot 6!} t^6 + \frac{1}{3^2 \cdot 9!} t^9 - \frac{1093}{3^5 \cdot 12!} t^{12} + \dots,$$

the co-ordinate change (60) is quite non-linear. This (60) agrees with the computation in [27, 23] up to the Galois actions $\tau_C \mapsto \tau_C + 2\pi i, t \mapsto \alpha^2 t$.

Coates [23] studied other non-Hard Lefschetz examples $[\mathbb{C}^2/\mathbb{Z}_4]$ with weight $\frac{1}{4}(1, 1, 2)$ and $[\mathbb{C}^3/\mathbb{Z}_5]$ with weight $\frac{1}{5}(1, 1, 3)$. It would be an interesting exercise to compare Conjecture 3.31 with Coates' calculations.

Remark 3.33. In the second case, we can predict the full relationships between the small quantum cohomology by considering the central charges of $[\mathcal{O}_S] \in K_E(Y)$ associated to surfaces $S \subset E$ in Corollary 3.30. Note that $Z^Y(\mathcal{O}_S)$ contains the information of the derivative of the potential F_0^Y (see Example 2.14, (ii)). The co-ordinate change of big quantum cohomology can be also determined by \mathbb{U}_K in principle, but the formula could be very complicated.

References

- [1] D. Abramovich, T. Graber and A. Vistoli, Gromov–Witten theory of Deligne–Mumford stacks, *Amer. J. Math.*, **130** (2008), 1337–1398; arXiv:math/0603151[math.AG].
- [2] A. Adem and Y. Ruan, Twisted orbifold K -theory, *Comm. Math. Phys.*, **237** (2003), 533–556.
- [3] M. Aganagic, V. Bouchard and A. Klemm, Topological strings and (almost) modular forms, *Comm. Math. Phys.*, **277** (2008), 771–819; arXiv:hep-th/0607100.

- [4] S. Barannikov, Quantum periods. I. Semi-infinite variations of Hodge structures, *Internat. Math. Res. Notices*, **2001** (2001), 1243–1264.
- [5] S. Boissière, E. Mann and F. Perroni, The cohomological crepant resolution conjecture for $P(1, 3, 4, 4)$, *Internat. J. Math.*, **20** (2009), 791–801; arXiv:0712.3248.
- [6] S. Boissière and A. Sarti, Contraction of excess fibres between the McKay correspondences in dimensions two and three, *Ann. Inst. Fourier (Grenoble)*, **57** (2007), 1839–1861.
- [7] L. A. Borisov and R. P. Horja, Mellin–Barnes integrals as Fourier–Mukai transforms, *Adv. Math.*, **207** (2006), 876–927.
- [8] T. Bridgeland, Stability conditions on triangulated categories, *Ann. of Math. (2)*, **166** (2007), 317–345.
- [9] T. Bridgeland, A. King and M. Reid, The McKay correspondence as an equivalence of derived categories, *J. Amer. Math. Soc.*, **14** (2001), 535–554.
- [10] J. Bryan, An e-mail to T. Coates, December, 2006.
- [11] J. Bryan and A. Gholampour, Root systems and the quantum cohomology of ADE resolutions, *Algebra Number Theory*, **2** (2008), 369–390; arXiv:0707.1337.
- [12] J. Bryan and A. Gholampour, Hurwitz–Hodge integrals, the E_6 , D_4 root systems, and the crepant resolution conjecture, *Adv. Math.*, **221** (2009), 1047–1068; arXiv:0708.4244.
- [13] J. Bryan and A. Gholampour, The quantum McKay correspondence for polyhedral singularities, *Invent. Math.*, **178** (2009), 655–681; arXiv:0803.3766.
- [14] J. Bryan and T. Graber, The crepant resolution conjecture, In: *Algebraic Geometry—Seattle 2005. Part 1*, *Proc. Sympos. Pure Math.*, **80**, Part 1, Amer. Math. Soc., Providence, RI, 2009, pp. 23–42; arXiv:math/0610129[math.AG].
- [15] J. Bryan, T. Graber and R. Pandharipande, The orbifold quantum cohomology of C_2/\mathbb{Z}_3 and Hurwitz–Hodge integrals, *J. Algebraic Geom.*, **17** (2008), 1–28.
- [16] P. Candelas, X. C. de la Ossa, P. S. Green and L. Parkes, A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory, *Nuclear Phys. B*, **359** (1991), 21–74.
- [17] M. de Cataldo and L. Migliorini, The Hard Lefschetz Theorem and the topology of semismall maps, *Ann. Sci. École Norm. Sup. (4)*, **35** (2002), 759–772; arXiv:math/0006187.
- [18] S. Cecotti and C. Vafa, Topological—anti-topological fusion, *Nuclear Phys. B*, **367** (1991), 359–461.
- [19] B. Chen, A.-M. Li and G. Zhao, Ruan’s conjecture on singular symplectic flops, preprint, arXiv:0804.3143.
- [20] B. Chen, A.-M. Li and Q. Zhang and G. Zhao, Singular symplectic flops and Ruan cohomology, *Topology*, **48** (2009), 1–22; arXiv:0804.3144.

- [21] W. Chen and Y. Ruan, Orbifold Gromov–Witten theory, In: Orbifolds in Mathematics and Physics, Madison, WI, 2001, *Contemp. Math.*, **310**, Amer. Math. Soc., Providence, RI, 2002, pp. 25–85.
- [22] S. Chowla and A. Selberg, On Epstein’s zeta-function, *J. Reine Angew. Math.*, **227** (1967), 86–110.
- [23] T. Coates, On the crepant resolution conjecture in the local case, *Comm. Math. Phys.*, **287** (2009), 1071–1108; a longer preprint version: Wall-crossing in toric Gromov–Witten theory II: local examples, arXiv:0804.2592.
- [24] T. Coates, A. Corti, H. Iritani and H.-H. Tseng, Computing genus-zero twisted Gromov–Witten invariants, *Duke Math. J.*, **147** (2009), 377–438; arXiv:math/0702234.
- [25] T. Coates, A. Corti, H. Iritani and H.-H. Tseng, Gromov–Witten theory of toric stacks, in preparation.
- [26] T. Coates and A. Givental, Quantum Riemann–Roch, Lefschetz and Serre, *Ann. of Math. (2)*, **165** (2007), 15–53.
- [27] T. Coates, H. Iritani and H.-H. Tseng, Wall-crossing in toric Gromov–Witten theory I: crepant examples, *Geom. Topol.*, **13** (2009), 2675–2744; arXiv:math/0611550[math.AG].
- [28] T. Coates and Y. Ruan, Quantum cohomology and crepant resolutions: A conjecture, preprint, arXiv:0710.5901.
- [29] P. Deligne, Équations Différentielles à Points Singuliers Réguliers, *Lecture Notes in Math.*, **163**, Springer-Verlag, Berlin-New York, 1970.
- [30] M. R. Douglas, Dirichlet branes, homological mirror symmetry, and stability, In: *Proceedings of the International Congress of Mathematicians, Vol. III, Beijing, 2002*, Higher Ed. Press, Beijing, 2002, pp. 395–408.
- [31] B. Dubrovin, Geometry of 2D topological field theories, In: *Integrable Systems and Quantum Groups, Montecatini Terme, 1993*, *Lecture Notes in Math.*, **1620**, Springer-Verlag, Berlin, 1996, pp. 120–348.
- [32] J. Fernandez, Hodge structures for orbifold cohomology, *Proc. Amer. Math. Soc.*, **134** (2006), 2511–2520.
- [33] A. Givental, Gromov–Witten invariants and quantization of quadratic Hamiltonians, *Mosc. Math. J.*, **1** (2001), 551–568, 645.
- [34] Y. Gomi, I. Nakamura and K.-I. Shinoda, Coinvariant algebras of finite subgroups of $SL(3, \mathbb{C})$, *Canad. J. Math.*, **56** (2004), 495–528.
- [35] T. Graber and R. Pandharipande, Localization of virtual classes, *Invent. Math.*, **135** (1999), 487–518.
- [36] C. Hertling and Yu. I. Manin, Weak Frobenius manifolds, *Internat. Math. Res. Notices*, **1999** (1999), 277–286.
- [37] C. Hertling, tt^* -geometry, Frobenius manifolds, their connections, and the construction for singularities, *J. Reine Angew. Math.*, **555** (2003), 77–161.
- [38] C. Hertling and C. Sevenheck, Nilpotent orbits of a generalization of Hodge structures, *J. Reine Angew. Math.*, **609** (2007), 23–80.
- [39] K. Hori and C. Vafa, Mirror symmetry, preprint, arXiv:hep-th/0002222.

- [40] R. P. Horja, Hypergeometric functions and mirror symmetry in toric varieties, preprint, arXiv:math/9912109[math.AG].
- [41] R. P. Horja, Derived category automorphisms from mirror symmetry, *Duke Math. J.*, **127** (2005), 1–34.
- [42] S. Hosono, Local mirror symmetry and type IIA monodromy of Calabi–Yau manifolds, *Adv. Theor. Math. Phys.*, **4** (2000), 335–376.
- [43] S. Hosono, Central charges, symplectic forms, and hypergeometric series in local mirror symmetry, In: *Mirror Symmetry. V*, AMS/IP Stud. Adv. Math., **38**, Amer. Math. Soc., Providence, RI, 2006, pp. 405–439.
- [44] S. Hosono and Y. Konishi, Higher genus Gromov–Witten invariants of the Grassmannian, and the Pfaffian Calabi–Yau threefolds, *Adv. Theor. Math. Phys.*, **13** (2009), 463–495; arXiv:0704.2928.
- [45] H. Iritani, Real and integral structures in quantum cohomology I: toric orbifolds, preprint, arXiv:0712.2204.
- [46] H. Iritani, An integral structure in quantum cohomology and mirror symmetry for toric orbifolds, *Adv. Math.*, **222** (2009), 1016–1079; arXiv:0903.1463.
- [47] H. Iritani, tt^* -geometry in quantum cohomology, preprint, arXiv:0906.1307.
- [48] Y. Ito and I. Nakamura, Hilbert schemes and simple singularities, In: *New Trends in Algebraic Geometry*, Warwick, 1996, London Math. Soc. Lecture Note Ser., **264**, Cambridge Univ. Press, Cambridge, 1999, pp. 151–233.
- [49] L. Katzarkov, M. Kontsevich and T. Pantev, Hodge theoretic aspects of mirror symmetry, In: *From Hodge Theory to Integrability and TQFT tt^* -Geometry*, Proc. Sympos. Pure Math., **78**, Amer. Math. Soc., Providence, RI, 2008, pp. 87–174; arXiv:0806.0107.
- [50] Y. Kawamata, D -equivalence and K -equivalence, *J. Differential Geom.*, **61** (2002), 147–171.
- [51] Y. Kawamata, Log crepant birational maps and derived categories, *J. Math. Sci. Univ. Tokyo*, **12** (2005), 211–231.
- [52] T. Kawasaki, The Riemann–Roch theorem for complex V -manifolds, *Osaka J. Math.*, **16** (1979), 151–159.
- [53] T. Kawasaki, The index of elliptic operators over V -manifolds, *Nagoya Math. J.*, **84** (1981), 135–157.
- [54] Y.-P. Lee, H.-W. Lin and C.-L. Wang, Flops, motives and invariance of quantum rings, preprint, arXiv:math/0608370[math.AG].
- [55] A.-M. Li and Y. Ruan, Symplectic surgery and Gromov–Witten invariants of Calabi–Yau 3-folds, *Invent. Math.*, **145** (2001), 151–218.
- [56] E. Lupercio and M. Poddar, The global McKay–Ruan correspondence via motivic integration, *Bull. London Math. Soc.*, **36** (2004), 509–515.
- [57] Yu. I. Manin, Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces, *Amer. Math. Soc. Colloq. Publ.*, **47**, Amer. Math. Soc., Providence, RI, 1999.

- [58] R. Pandharipande, Rational curves on hypersurfaces (after Givental), Séminaire Bourbaki, **1007/98**, Astérisque, **252** (1998), Exp. No. 848, 5, 307–340.
- [59] F. Perroni, Orbifold Cohomology of ADE-singularities, Ph.D. thesis at SISSA, Trieste, arXiv:math/0510528; short published version: Chen–Ruan cohomology of ADE singularities, Internat. J. Math., **18** (2007), 1009–1059.
- [60] A. Pressley and G. Segal, Loop Groups, Oxford Math. Monogr., The Clarendon Press, Oxford Univ. Press, 1986.
- [61] Einer Andreas Rødland, The Pfaffian Calabi–Yau, its Mirror and their link to the Grassmannian $G(2, 7)$, Compositio Math., **122** (2000), 135–149.
- [62] Y. Ruan, Stringy geometry and topology of orbifolds, In: Symposium in Honor of C. H. Clemens, Salt Lake City, UT, 2000, Contemp. Math., **312**, Amer. Math. Soc., Providence, RI, 2002, pp. 187–233.
- [63] Y. Ruan, Cohomology ring of crepant resolution of orbifolds, In: Gromov–Witten Theory of Spin Curves and Orbifolds, Contemp. Math., **403**, Amer. Math. Soc., Providence, RI, 2006, pp. 117–126.
- [64] C. Sabbah, Hypergeometric period for a tame polynomial, preprint, arXiv:math/9805077[math.AG]; a short version published in: C. R. Acad. Sci. Paris Sér. I Math., **328** (1999), 603–608.
- [65] C. Sabbah, Polarizable Twistor D -Modules, Astérisque, **300**, Soc. Math. France, 2005.
- [66] K. Saito, Period mapping associated to a primitive form, Publ. Res. Inst. Math. Sci., **19** (1983), 1231–1264.
- [67] M. Saito, On the structure of Brieskorn lattice, Ann. Inst. Fourier (Grenoble), **39** (1989), 27–72.
- [68] W. Schmid, Variation of Hodge structure: the singularities of the period mapping, Invent. Math., **22** (1973), 211–319.
- [69] G. Segal, Equivariant K -theory, Inst. Hautes Études Sci. Publ. Math., **34** (1968), 129–151.
- [70] C. T. Simpson, Mixed twistor structures, preprint, arXiv:math/9705006 [math.AG].
- [71] B. Totaro, The resolution property of schemes and stacks, J. Reine Angew. Math., **577** (2004), 1–22.
- [72] T. Yasuda, Twisted jets, motivic measure and orbifold cohomology, Compos. Math., **140** (2004), 396–422.
- [73] T. Yasuda, Motivic integration over Deligne–Mumford stacks, Adv. Math., **207** (2006), 707–761.

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