

On non-abelian Lubin–Tate theory via vanishing cycles

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Abstract.

We give a purely local proof, in the depth 0 case, of the result by Harris–Taylor which asserts that the local Langlands correspondence for GL_n is realized in the vanishing cycle cohomology of the deformation spaces of one-dimensional formal modules of height n . Our proof is given by establishing the direct geometric link with the Deligne–Lusztig theory for $GL_n(\mathbb{F}_q)$.

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§1. Introduction

Let p be a prime, and K be a finite extension of the p -adic field \mathbb{Q}_p , with the ring of integers \mathcal{O} and the residue field k of cardinality q . The proof of the local Langlands correspondence for $GL_n(K)$, by Harris–Taylor [HT], was achieved by showing that the desired correspondence is realized in the ℓ -adic vanishing cycle cohomology groups of the deformation spaces of formal \mathcal{O} -modules of height n with Drinfeld

Received April 20, 2009.

Revised October 05, 2009.

2000 *Mathematics Subject Classification*. Primary: 11S37, Secondary: 11G18, 20C33, 22E50.

level structures (known as *non-abelian Lubin–Tate theory* or the conjecture of Deligne–Carayol [Ca2]). As these deformation spaces occur as complete local rings of certain unitary Shimura varieties at the “supersingular” points, they made an essential use of the fact that global Langlands correspondences are realized in the ℓ -adic étale cohomology groups of these Shimura varieties over CM fields. In this article, we give a purely local approach to this non-abelian Lubin–Tate theory, in the special case of depth 0 or level \mathfrak{p} , by computing the local equation of the deformation space and constructing its suitable resolution to calculate the vanishing cycle cohomology directly. We show that, in this case, the non-abelian Lubin–Tate theory for supercuspidal representations of $GL_n(K)$ is essentially equivalent to the Deligne–Lusztig theory for GL_n of the residue field k , which realizes the cuspidal representations of $GL_n(k)$ in the ℓ -adic cohomology groups of certain varieties over an algebraic closure \bar{k} of k .

To state our theorems precisely, let K, \mathcal{O}, k as above and fix $n \geq 1$. Let K^{ur} be the maximal unramified extension of K , and let W be the completion of the ring of integers \mathcal{O}^{ur} of K^{ur} (sometimes denoted $W_{\mathcal{O}}(\bar{k})$ in the literature). Let $\eta, \bar{\eta}$ be the spectra of $\text{Frac } W$ and its fixed algebraic closure, respectively.

Firstly, let X be the spectrum of the deformation ring of formal \mathcal{O} -module of height n with level \mathfrak{p} structure ([Dr]), which is a scheme of relative dimension $n - 1$ over W . We are interested in the ℓ -adic étale cohomology groups $H^i(X_{\bar{\eta}}, \bar{\mathbb{Q}}_{\ell})$ ($\ell \neq \text{char } k$) of the geometric generic fiber $X_{\bar{\eta}} := X \times_{\text{Spec } W} \bar{\eta}$, which are finite dimensional $GL_n(k) \times I_K$ -modules, where I_K is the inertia group of K . Secondly, let DL be the Deligne–Lusztig variety for $GL_n(k)$, associated to the element of the Weyl group of GL_n that corresponds to the cyclic permutation $(1, \dots, n)$ in the symmetric group of n letters, or equivalently to a non-split torus T with $T(k) \cong k_n^{\times}$ where k_n is the extension of k of degree n ([DL]). This DL is a smooth affine variety over \bar{k} with actions of $GL_n(k)$ and $T(k) \cong k_n^{\times}$, hence we can regard $H_c^i(DL, \bar{\mathbb{Q}}_{\ell})$ as a $GL_n(k) \times I_K$ -module by the canonical surjection $I_K \rightarrow k_n^{\times}$.

We denote the alternating sums of these cohomology groups as follows:

$$H^*(X_{\bar{\eta}}) := \sum_i (-1)^i [H^i(X_{\bar{\eta}}, \bar{\mathbb{Q}}_{\ell})],$$

$$H_c^*(DL) := \sum_i (-1)^i [H_c^i(DL, \bar{\mathbb{Q}}_{\ell})],$$

which are regarded as elements of the Grothendieck group of $GL_n(k) \times I_K$ -modules. Then our main theorem on the vanishing cycle cohomology groups of X can be stated as follows (Theorem 6.16):

- Theorem 1.1.** (i) We have the equality $H^*(X_{\overline{\eta}}) = H_c^*(DL)$.
- (ii) Among the $H^i(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$, cuspidal representations π of $GL_n(k)$ and generic inertia characters χ of I_K (here generic means χ does not factor through any k_m^\times with $m \mid n$, $m < n$ via the norm map $k_n^\times \rightarrow k_m^\times$) occur only in $H^{n-1}(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$, where they are coupled as $\bigoplus \pi_\chi \otimes \chi$ by the Deligne-Lusztig correspondence $\chi \leftrightarrow \pi_\chi$ characterized by

$$\pi_\chi \otimes \text{St} = \text{Ind}_{T(k)}^{GL_n(k)} \chi,$$

where St is the Steinberg representation of $GL_n(k)$.

The correspondence in the part (ii) can essentially be deduced from one of the main theorems of Harris-Taylor [HT] (Theorem VII.1.5), which was proven via highly nontrivial global arguments, but in this article we first prove the part (i) of the above theorem by a local geometric argument, and apply the results of Deligne-Lusztig theory. The fact that the supercuspidal representations appear only in the degree $n - 1$ (in the limit of cohomology groups for all levels \mathfrak{p}^m) was remarked by Faltings [Fal], as a refinement of the results of Harris-Taylor, and recently proved by Mieda [Mi] via purely local argument.

To see that the results of Harris-Taylor imply the above theorem, one only needs to spell out the depth 0 case of the local Langlands correspondence. Here, the supercuspidal representations of $GL_n(K)$ are obtained as compact inductions of the pull back of cuspidal representations of $GL_n(k)$ to $GL_n(\mathcal{O})$. The irreducible n -dimensional representations of the Weil group W_K are obtained by extending the generic tame inertia characters χ as in the theorem from I_K to the Weil group W_L of the unramified extension L of degree n over K , and then inducing them from W_L to W_K . The local Langlands correspondence (up to twists by unramified characters) boils down to the correspondence mentioned in (ii) of the above theorem. This case gives all the *tamely ramified* irreducible representations of W_K .

To prove the above theorem by purely local arguments, we construct a suitable model of the deformation space X and compute the cohomology of the geometric generic fiber $X_{\overline{\eta}}$ in terms of vanishing cycle sheaves on the special fiber. In its course we obtain important information concerning the geometry of X as the following:

Theorem 1.2. Let ϖ be a uniformizer of \mathcal{O} .

- (i) (Proposition 3.5) The W -scheme X is isomorphic to

$$\text{Spec } W[[X_1, \dots, X_n]] / (P(X_1, \dots, X_n) - \varpi),$$

where $P \in W[[X_1, \dots, X_n]]$ is of the form:

$$\text{(unit)} \cdot \prod_{(a_i \bmod \mathfrak{p})_{i \in k^n \setminus \{0\}}} ([a_1](X_1) +_{\bar{\Sigma}} \dots +_{\bar{\Sigma}} [a_n](X_n))$$

where $[a_i]$ and $+_{\bar{\Sigma}}$ denote the formal \mathcal{O} -multiplication and addition of a formal \mathcal{O} -module over $W[[X_1, \dots, X_n]]$ obtained by lifting the universal formal \mathcal{O} -module over X .

- (ii) (Theorem 4.2) There exists a generalized semistable model Z_{st} of X over W , i.e. a proper W -morphism $Z_{st} \rightarrow X$ which is an isomorphism on the generic fibers and Z_{st} being generalized semistable. Here generalized semistable means that its complete local rings at all the closed points are isomorphic over W to

$$W[[T_1, \dots, T_n]] / (T_1^{e_1} \dots T_d^{e_d} - \varpi) \quad (d \leq n),$$

where the integers e_i ($1 \leq i \leq d$) are all prime to $\text{char } k$.

- (iii) (Proposition 6.15) Over the tamely ramified extension $W_n := W(\varpi^{1/(q^n-1)})$ of W , there is a model of X whose special fiber contains a smooth affine variety over \bar{k} which is isomorphic to DL as schemes with right $GL_n(k) \times I_K$ -action.

The part (i) of this theorem gives the integral local equations of the relevant unitary Shimura varieties at supersingular points (similar equations can be given for the deformation spaces of formal \mathcal{O} -modules with level \mathfrak{p}^m structures for any $m \geq 1$), and in the special case $K = \mathbb{Q}_p$ and $n = 2$, it gives the “integral” version of Katz–Mazur’s description of the bad reduction of modular curves $X(p^m)$ ([KM] Theorem 13.8.4). The resolution constructed in the proof of the part (ii) of this theorem can be used to give a generalized semistable model of the unitary Shimura variety with level \mathfrak{p} structure, which suggests a more “local” approach for computing the cohomology of these Shimura varieties. The part (iii) of the theorem is the basis of the proof of Theorem 1.1 above, and is obtained by normalizing the base change of a relevant part of generalized semistable model to W_n .

The vanishing cycle cohomology of the coverings of Lubin–Tate spaces is known to incorporate the *local Jacquet–Langlands correspondence* as well as local Langlands correspondence ([HT]). The realization of Jacquet–Langlands correspondence was proved via local arguments by M. Strauch ([St1], [St2], [St3]) using the period map and the trace formula on rigid analytic spaces. We hope to clarify how our work is related to other works in the field ([Bo2], [Da], [Fa1], [FGL], [Mi]) in the near future.

Acknowledgements. This work was the PhD thesis of the author, submitted to the University of Tokyo in January 2004. The May 2004 version of this article was accepted for publication in the *Annales de l’Institut Fourier*, but the author was unable to make the necessary revision for a long time. One of the reasons was that the author wishes to present a more canonical and moduli-theoretic approach to the results in this article in the near future. However, as the explicit calculations presented in this article may be of some interest, here we left this article close to its original form. The author heartfully thanks the editor of this volume for the acceptance of this article.

The author would like to thank his adviser Takeshi Saito for his encouragements and many improvements of the proofs. The author is also grateful to his previous adviser Kazuya Kato for his warm encouragements during the graduate program. The problem treated here grew from the numerous discussions with Tetsushi Ito. Also this work is a generalization of some computations in the modular curve case ($n = 2$, $K = \mathbb{Q}_p$) done by B. Edixhoven in his unpublished manuscript [Ed], whom author would like to thank sincerely for showing him the manuscript. The author would like to thank G. Lusztig and Y. Mieda for their assistance in Deligne–Lusztig theory and étale cohomology, K. Ban, S. DeBacker, M. Strauch for helpful comments and discussions, Y. Yasufuku for the manuscript [Ya] from which he learned many things about formal groups. The author is indebted to the referee of this article for his very careful reading and some essential corrections. Finally, a large part of this work was done at Harvard University, and the author would like to thank Richard Taylor for many invaluable discussions and constant encouragement during the development of this article.

Notation. For $j \geq 1$, we denote the group of j -th roots of unity by μ_j . Cohomology groups are all ℓ -adic étale cohomology groups, where we fix a prime ℓ different from the residue characteristic p . For a representation V of a group, we denote the corresponding element in a suitable Grothendieck group by $[V]$. For a ring A , we denote its group of units by A^\times . For a field F , we denote a fixed separable closure of F by \bar{F} . For a finite field $k \cong \mathbb{F}_q$ with q elements and $n \geq 1$, we denote by $k_n \cong \mathbb{F}_{q^n}$ the unique extension of k of degree n . For a scheme X and its point $x \in X$ (resp. a geometric point x of X), we denote the Zariski local ring (resp. strict local ring) at x by $\mathcal{O}_{X,x}$ or \mathcal{O}_x . Sometimes we refer to the elements of the coordinate rings as “coordinates”, but when there is a risk of confusion we distinguish them by capital/lower cases. For a ring A and indeterminates X_1, \dots, X_n we denote $A[X_1, \dots, X_n]$, $A[[X_1, \dots, X_n]]$ respectively by $A[X_{\leq n}]$, $A[[X_{\leq n}]]$.

§2. The deformation spaces of formal \mathcal{O} -modules

Let p be a fixed prime, and K be a finite extension of the p -adic field \mathbb{Q}_p . Equivalently, it is a complete discrete valuation field of characteristic zero with a finite residue field. Let \mathcal{O} be the ring of integers of K , and \mathfrak{p} be its maximal ideal. Let q be the cardinality of the residue field, which we denote by $k := \mathcal{O}/\mathfrak{p} \cong \mathbb{F}_q$. These are fixed throughout the paper, with the exception of Section 6.1.

We denote the maximal unramified extension of K by K^{ur} , and its completion by $\widehat{K} := \widehat{K}^{\text{ur}}$. We denote the ring of integers of \widehat{K} by W , which is a complete discrete valuation ring with the residue field $\overline{k} = \overline{\mathbb{F}}_q$. The uniformizers of \mathcal{O} are also uniformizers of W . We often fix a uniformizer ϖ of \mathcal{O} .

Here we recall from [Dr] the basic facts on the deformation theory of one-dimensional formal \mathcal{O} -modules. The author is greatly indebted to the expository article of Yasufuku [Ya], which gives a detailed account of Drinfeld's theory.

2.1. Formal \mathcal{O} -modules

For a (commutative) \mathcal{O} -algebra A , by a *formal \mathcal{O} -module* over A , we mean a pair $\Sigma = (F, [\cdot])$, where $F(X, Y) \in A[[X, Y]]$ is a one-dimensional commutative formal group law and $[\cdot] : \mathcal{O} \ni a \mapsto [a] \in \text{End}(F)$ is an injective ring homomorphism such that $[a](X) \equiv aX \pmod{X^2}$. Here $\text{End}(F) := \text{Hom}(F, F)$, where

$$\text{Hom}(F, G) := \{f(X) \in X \cdot A[[X]] \mid f(F(X, Y)) = G(f(X), f(Y))\}$$

for formal group laws F, G over A . The homomorphisms between formal \mathcal{O} -modules $\Sigma = (F, [\cdot])$, $\Sigma' = (F', [\cdot]')$ are the elements of $\text{Hom}(F, F')$ which commute with \mathcal{O} -multiplications. For any A -algebra B and a formal \mathcal{O} -module Σ over A , a formal \mathcal{O} -module $\Sigma \otimes_A B$ over B is defined by the images of F and $[a]$ under the induced homomorphisms $A[[X, Y]] \rightarrow B[[X, Y]]$ and $A[[X]] \rightarrow B[[X]]$. For $\Sigma = (F, [\cdot])$, we often use the notation:

$$X +_{\Sigma} Y := F(X, Y), \quad [\cdot]_{\Sigma} := [\cdot].$$

Example 2.1. The *additive group* $\mathbb{G}_a = (F, [\cdot])$, over arbitrary A , is defined by $F(X, Y) = X + Y$ and $[a](X) = aX$ for every $a \in \mathcal{O}$.

Proposition 2.2. (Drinfeld [Dr]) *For a formal \mathcal{O} -module Σ over $\overline{\mathbb{F}}_q$, not isomorphic to the additive group \mathbb{G}_a , there is a unique integer $n \geq 1$ (called the height of Σ) such that if ϖ is a uniformizer of \mathcal{O} then $[\varpi]_{\Sigma}(X) = u \cdot X^{q^n}$ for some $u \in \overline{\mathbb{F}}_q[[X]]^{\times}$. We define the height of \mathbb{G}_a to*

be ∞ . The formal \mathcal{O} -modules over $\overline{\mathbb{F}}_q$ are classified up to isomorphism by their heights.

Let $n \geq 1$, and fix a formal \mathcal{O} -module Σ_n of height n over $\overline{\mathbb{F}}_q$, which is unique up to isomorphism. If ϖ is a uniformizer of \mathcal{O} , then we can (and will) choose $\Sigma_n = (F, [\cdot])$ to be ϖ -normal, i.e. it satisfies the following:

- (i) $[\varpi](X) = X^{q^n}$.
- (ii) $F(X, Y) \in \mathbb{F}_{q^n}[[X, Y]]$, $F(X, Y) \equiv X + Y \pmod{\deg q^n}$.
- (iii) $[a](X) \in \mathbb{F}_{q^n}[[X]]$, $[a](X) \equiv aX \pmod{X^{q^n}}$ for every $a \in \mathcal{O}$.

2.2. Deformations of formal \mathcal{O} -modules

Let \mathcal{C} be the category of complete noetherian local W -algebras (A, \mathfrak{m}_A) such that the structure morphisms $W \rightarrow A$ induce isomorphisms between the residue fields $\overline{k} = W/\mathfrak{p}W \xrightarrow{\cong} A/\mathfrak{m}_A$. We identify \overline{k} and A/\mathfrak{m}_A by this isomorphism. The morphisms in \mathcal{C} are local W -homomorphisms. For a formal \mathcal{O} -module Σ over $A \in \mathcal{C}$, its reduction $\text{mod } \mathfrak{m}_A$ is defined as $\overline{\Sigma} := \Sigma \otimes_A \overline{k}$, which is a formal \mathcal{O} -module over \overline{k} .

Let us fix an integer $n \geq 1$ in the rest of the paper. For $A \in \mathcal{C}$, a deformation of Σ_n to A is a pair (Σ, i) , consisting of a formal \mathcal{O} -module Σ over A and an isomorphism $i : \Sigma_n \xrightarrow{\cong} \overline{\Sigma}$ as formal \mathcal{O} -modules over \overline{k} . Two deformations (Σ, i) and (Σ', i') are equivalent if there is an isomorphism $f : \Sigma \cong \Sigma'$ such that $i' = (f \text{ mod } \mathfrak{m}_A) \circ i$. The deformation functor \mathcal{F}_0 from \mathcal{C} to the category of sets is defined by sending $(A, \mathfrak{m}_A) \in \mathcal{C}$ to the set of equivalence classes of deformations of Σ_n over A .

Proposition 2.3. (Drinfeld [Dr]) *The deformation functor \mathcal{F}_0 is representable by a ring $A_0 \in \mathcal{C}$, which is isomorphic to the formal power series ring of $n - 1$ variables $W[[T_1, \dots, T_{n-1}]]$ over W . We denote the universal formal \mathcal{O} -module over A_0 by Σ_n^{univ} .*

2.3. Deformations with Drinfeld level structure

For a formal \mathcal{O} -module Σ over $A \in \mathcal{C}$, the maximal ideal \mathfrak{m}_A of A is endowed with an \mathcal{O} -module structure by defining the addition and the \mathcal{O} -multiplication by

$$x +_{\Sigma} y = F(x, y), \quad [a](x) = [a]_{\Sigma}(x) \quad (x, y \in \mathfrak{m}_A, a \in \mathcal{O}).$$

We denote this \mathcal{O} -module by \mathfrak{m}_{Σ} . A homomorphism $f : \Sigma \rightarrow \Sigma'$ of formal \mathcal{O} -modules over A induces an \mathcal{O} -homomorphism $f_{\mathfrak{m}} : \mathfrak{m}_{\Sigma} \rightarrow \mathfrak{m}_{\Sigma'}$ of the corresponding \mathcal{O} -modules.

For a formal \mathcal{O} -module $\Sigma = (F, [\cdot])$ over $A \in \mathcal{C}$ of height n and an integer $m \geq 1$, a Drinfeld level \mathfrak{p}^m structure on Σ is defined to be

an \mathcal{O} -module homomorphism $\varphi : (\mathfrak{p}^{-m}/\mathcal{O})^n \rightarrow \mathfrak{m}_\Sigma$, which satisfies the divisibility:

$$\prod_{x \in (\mathfrak{p}^{-m}/\mathcal{O})^n} (X - \varphi(x)) \mid [\varpi^m](X)$$

in $A[[X]]$, for some (equivalently, every) uniformizer ϖ of \mathcal{O} .

Let us fix a uniformizer ϖ of \mathcal{O} and choose Σ_n to be ϖ -normal. If we denote the left hand side of the above by $P_\varphi(X) := \prod(X - \varphi(x))$, and let $[\varpi^m](X) = U(X)P_\varphi(X)$ with $U(X) \in A[[X]]$, then the constant term u_Σ of $U(X)$ lies in $1 + \mathfrak{m}_A$, because we have $[\varpi^m](X) \equiv P_\varphi(X) \equiv X^{q^{mn}} \pmod{\mathfrak{m}_A}$.

Also, once we fix a uniformizer ϖ of \mathcal{O} , we have a standard basis $\{e_1, \dots, e_n\}$ of $(\mathfrak{p}^{-m}/\mathcal{O})^n$ as a free $\mathcal{O}/\mathfrak{p}^m$ -module, where

$$e_i := (0, \dots, 0, \varpi^{-m}, 0, \dots, 0).$$

We call $\varphi(e_1), \dots, \varphi(e_n) \in \mathfrak{m}_A$ the *formal parameters* of φ .

For $A \in \mathcal{C}$, a *deformation of Σ_n with level \mathfrak{p}^m structure* over A is a triple (Σ, i, φ) consisting of a usual deformation (Σ, i) over A and a Drinfeld level \mathfrak{p}^m structure φ of Σ . Two deformations $(\Sigma, i, \varphi), (\Sigma', i', \varphi')$ are *equivalent* if there is an equivalence f of (Σ, i) to (Σ', i') such that $\varphi' = f_m \circ \varphi$. The deformation functor \mathcal{F}_m from \mathcal{C} to the category of sets is defined by sending $A \in \mathcal{C}$ to the set of equivalence classes of deformations of Σ_n with level \mathfrak{p}^m structure over A .

Proposition 2.4. (Drinfeld [Dr]) *For every integer $m \geq 1$, the deformation functor \mathcal{F}_m is represented by an n -dimensional regular local ring A_m . The local W -algebra homomorphism $A_0 \rightarrow A_m$, representing the obvious forgetting morphism of functors $\mathcal{F}_m \rightarrow \mathcal{F}_0$, is finite and flat, and the universal object over A_m is a level \mathfrak{p}^m structure φ_m on $\Sigma_n^{\text{univ}} \otimes_{A_0} A_m$. The formal parameters $X_1, \dots, X_n \in \mathfrak{m}_{A_m}$ of the universal level \mathfrak{p}^m structure φ_m gives a set of regular parameters of A_m .*

This functor \mathcal{F}_m naturally factors through the category of sets with right $GL_n(\mathcal{O}/\mathfrak{p}^m)$ -action as follows. For $A \in \mathcal{C}$, the set $\mathcal{F}_m(A)$ has a natural right $GL_n(\mathcal{O}/\mathfrak{p}^m)$ -action induced by the action of $g \in GL_n(\mathcal{O}/\mathfrak{p}^m)$ on the deformations defined as

$$(\Sigma, i, \varphi) \longmapsto (\Sigma, i, \varphi \circ g).$$

This results in a right $GL_n(\mathcal{O}/\mathfrak{p}^m)$ -action on $\text{Spec } A_m$. In terms of formal parameters X_1, \dots, X_n of the universal level \mathfrak{p}^m structure on A_m , this coincides with the left action given by the “linear” action of $GL_n(\mathcal{O}/\mathfrak{p}^m)$ on the row vector (X_1, \dots, X_n) from the right, as elements

of $\mathfrak{m}_{\Sigma_n^{\text{univ}} \otimes A_m}$, i.e. the addition and the $\mathcal{O}/\mathfrak{p}^m$ -multiplications on X_i 's are the operations of $\Sigma_n^{\text{univ}} \otimes A_m$. The finite flat covering $\text{Spec } A_m \rightarrow \text{Spec } A_0$ is a Galois étale covering on the generic fibers, with the Galois group $GL_n(\mathcal{O}/\mathfrak{p}^m)$.

2.4. Realization as a complete local ring of a Shimura variety

Here we briefly recall from [HT] the realization of this deformation ring as a complete local ring of certain Shimura variety, especially the proof of Lemma II.2.7, given in p.114 of the book. The detailed definition of the relevant Shimura variety is described carefully in [HT], which we omit here as it has little importance for us. In [HT] Section III.4, the proper flat integral model $X_{U^p, m}$ of some Shimura variety over $\text{Spec } \mathcal{O}_{F, w}$ is defined, where U^p is a compact open subgroup of $G(\mathbb{A}^{\infty, p})$ for certain reductive group G over \mathbb{Q} , and $m = (m_1, \dots, m_r) \in (\mathbb{Z}_{\geq 0})^r$ is a multi-index.

Now we can take the local field F_w to be our K , and the index m_1 to be our m . Writing $\text{Sh} := X_{U^p, m}$, Lemma III.4.1(1) of [HT] tells us that the completion $\widehat{\mathcal{O}}_{\text{Sh}, \bar{s}}$ of the strict local ring of Sh at any geometric point \bar{s} centered at a closed point s with $h(s) = 0$ (a “supersingular” point, i.e. where the étale height $h(s)$ of the corresponding Barsotti–Tate group is 0), is isomorphic to our universal deformation ring A_m . Note that the existence of such a closed point is ensured by Lemma III 4.3 of [HT]. This will be used in Section 4.2.

§3. The level \mathfrak{p} deformation space and the first blow-up

We fix $n \geq 1$, and we are interested in the deformation space $X := \text{Spec } A_1$ of formal \mathcal{O} -modules over \bar{k} of height n with level \mathfrak{p} structures. This X is a regular flat scheme over $S := \text{Spec } W$ of relative dimension $n - 1$ with a (formally) smooth generic fiber. We denote the universal formal \mathcal{O} -module over A_0 by $\Sigma^{\text{univ}} := \Sigma_n^{\text{univ}}$. From this section on, we denote the ring A_1 simply by A , and the maximal ideal of A by $\mathfrak{m} := (X_1, \dots, X_n)$, where X_i are the formal parameters of the universal level \mathfrak{p} structure on $\Sigma^{\text{univ}} \otimes A$, associated to our choice of uniformizer ϖ of \mathcal{O} .

3.1. The equation of the space

We start by computing the defining equation of this space $X = \text{Spec } A$. By Proposition 2.4, the formal parameters X_1, \dots, X_n give a

set of regular parameters, therefore we have a surjective local homomorphism of local W -algebras

$$\tilde{A} := W[[\tilde{X}_1, \dots, \tilde{X}_n]] \longrightarrow A, \quad \tilde{X}_i \longmapsto X_i \quad (1 \leq i \leq n),$$

where $(\tilde{A}, \tilde{\mathfrak{m}}) \in \mathcal{C}$ is a formal power series ring in $\tilde{X}_1, \dots, \tilde{X}_n$ over W with the maximal ideal $\tilde{\mathfrak{m}} := (\varpi, \tilde{X}_1, \dots, \tilde{X}_n)$.

Let I be the kernel of this surjection, so that $A \cong \tilde{A}/I$. Note that \tilde{A} is an $(n + 1)$ -dimensional regular local ring, and A is regular by Proposition 2.4. Hence the ideal I has height one, therefore principal, generated by an element t which is part of a system of regular parameters of \tilde{A} , i.e. an element $t \in I \setminus \tilde{\mathfrak{m}}^2$ ([Mat], Theorem 14.2). But for any element $t' \in I \setminus \tilde{\mathfrak{m}}^2$, we have $t'\tilde{\mathfrak{m}} = (t') \cap \tilde{\mathfrak{m}}^2$, thus the map $(t')/t'\tilde{\mathfrak{m}} \rightarrow I/\tilde{\mathfrak{m}}I$ is an injection between 1-dimensional \bar{k} -vector spaces, therefore an isomorphism, which in turn gives $(t') = I$ by Nakayama's lemma. Therefore, to determine I we only need to find any element $t' \in I \setminus \tilde{\mathfrak{m}}^2$, and for this we observe:

Proposition 3.1. *We have the following equality in A :*

$$\varpi = u \cdot \prod_{\underline{a} \in k^n \setminus \{0\}} ([a_1](X_1) +_{\Sigma^{\text{univ}}} \dots +_{\Sigma^{\text{univ}}} [a_n](X_n))$$

with $u \in 1 + \mathfrak{m} \subset A^\times$, where $+_{\Sigma^{\text{univ}}}, [\cdot]$ denote the operations in $\mathfrak{m}_{\Sigma^{\text{univ}} \otimes A}$, and we used the notation $\underline{a} := (a_1, \dots, a_n)$ and $\underline{0} := (0, \dots, 0)$. (Note that $[a_i](X_i)$ are well-defined because $[\varpi](X_i) = 0$.) In particular, we have $\varpi \in \mathfrak{m}^{q^n - 1}$.

Proof. For the universal Drinfeld structure $(\Sigma^{\text{univ}} \otimes A, i, \varphi)$ over A , by definition we have:

$$[\varpi](T) = U(T)P_\varphi(T), \quad U(T) \in A[[T]], \quad P_\varphi(T) = \prod_{x \in (\mathfrak{p}^{-1}/\mathcal{O})^n} (T - \varphi(x)).$$

As remarked before in Section 3.3, the constant term u of $U(T)$ is in $1 + \mathfrak{m}$. By comparing the leading terms (i.e. the coefficients of T), we have the equality:

$$\varpi = u \cdot \prod_{x \in (\mathfrak{p}^{-1}/\mathcal{O})^n \setminus \{0\}} \varphi(x)$$

in A . The definition of the formal parameters gives:

$$\begin{aligned} \prod_{x \in (\mathfrak{p}^{-1}/\mathcal{O})^n \setminus \{0\}} \varphi(x) &= \prod_{\underline{a} \in k^n \setminus \{0\}} \varphi(a_1 e_1 + \cdots + a_n e_n) \\ &= \prod_{\underline{a} \in k^n \setminus \{0\}} ([a_1](X_1) +_{\Sigma^{\text{univ}}} \cdots +_{\Sigma^{\text{univ}}} [a_n](X_n)). \end{aligned}$$

Q.E.D.

In order to define an element $t \in \tilde{A}$ which vanishes in A , first we lift the universal formal \mathcal{O} -module Σ^{univ} to \tilde{A} . By definition of the deformation space, defining such a lift $\tilde{\Sigma}^{\text{univ}}$ over \tilde{A} is equivalent to giving a local W -algebra homomorphism f making the following diagram commute:

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\quad} & A \\ & \searrow \scriptstyle f & \uparrow \\ & & A_0 = W[[T_1, \dots, T_{n-1}]] \end{array}$$

where the right vertical arrow is the canonical map defined in Proposition 2.4. The existence of such an f is ensured by the formal smoothness of A_0 over W (finding such f only amounts to defining the images of T_i by finding a power series of X_1, \dots, X_n with coefficients in W which represent the images of T_i in A).

Definition 3.2. We choose and fix one such f , and define $\tilde{\Sigma}^{\text{univ}} := \Sigma^{\text{univ}} \otimes_{A_0, f} \tilde{A}$.

Note that this does not mean that $\tilde{X}_1, \dots, \tilde{X}_n$ are formal parameters of a level \mathfrak{p} structure on $\tilde{\Sigma}^{\text{univ}}$ (that would amount to giving a section $A \rightarrow \tilde{A}$), thus $[\varpi]_{\tilde{\Sigma}^{\text{univ}}}(\tilde{X}_i)$ does not vanish and $[a](\tilde{X}_i)$ for $a \in k$ is not well-defined.

Definition 3.3. For each $a \in k^\times$, let $\tilde{a} \in \mu_{q-1} \subset \mathcal{O}^\times$ be its multiplicative lift, and set $\tilde{0} = 0$. For each $\underline{a} = (a_1, \dots, a_n) \in k^n \setminus \{0\}$, define a formal power series:

$$P_{\underline{a}}(\tilde{X}_1, \dots, \tilde{X}_n) := [\tilde{a}_1](\tilde{X}_1) +_{\tilde{\Sigma}^{\text{univ}}} \cdots +_{\tilde{\Sigma}^{\text{univ}}} [\tilde{a}_n](\tilde{X}_n) \in \tilde{\mathfrak{m}} \subset \tilde{A},$$

where $+_{\tilde{\Sigma}^{\text{univ}}}, [\cdot]$ denotes the operations in $\tilde{\mathfrak{m}}_{\tilde{\Sigma}^{\text{univ}}}$.

We record some easy properties of these power series $P_{\underline{a}}$.

Proposition 3.4. *The following are true :*

- (i) $P_{\underline{a}} \equiv \tilde{a}_1 \tilde{X}_1 + \cdots + \tilde{a}_n \tilde{X}_n \pmod{(\tilde{X}_1, \dots, \tilde{X}_n)^2}$.
- (ii) If $\underline{a} = c \cdot \underline{a}'$ for $c \in k^\times$, then $P_{\underline{a}} = u_c \cdot P_{\underline{a}'}$ for a unit $u_c \in \tilde{A}^\times$.
- (iii) If $a_{j+1} = \cdots = a_n = 0$ for some $j < n$, then $P_{\underline{a}} \in (\tilde{X}_1, \dots, \tilde{X}_j)$.

Proof. Parts (i) and (iii) follow from the definition of formal \mathcal{O} -modules and $\tilde{0} = 0$. For (ii), note that $P_{\underline{a}} = [\tilde{c}](P_{\underline{a}'})$, but as $\tilde{c} \in \mathcal{O}^\times$, we have $[\tilde{c}](x) = u_c \cdot x$ for any $x \in \tilde{\mathfrak{m}}$, with a unit $u_c \in \tilde{c} + \tilde{\mathfrak{m}}$ depending on x . Q.E.D.

Now we can define an element of \tilde{A} reducing to 0 in A using Proposition 3.1:

Proposition 3.5. *We have an W -algebra isomorphism*

$$\tilde{A}/(P - \varpi) \cong A,$$

where $\tilde{A} := W[[\tilde{X}_1, \dots, \tilde{X}_n]]$, and $P \in \tilde{A}$ is a formal power series of the form:

$$P(\tilde{X}_1, \dots, \tilde{X}_n) := \tilde{u} \cdot \prod_{\underline{a} \in k^n \setminus \{0\}} P_{\underline{a}}(\tilde{X}_1, \dots, \tilde{X}_n)$$

with $\tilde{u} \in 1 + \tilde{\mathfrak{m}} \subset \tilde{A}^\times$.

Proof. We define \tilde{u} as any lift of $u \in A$ in Proposition 3.1 to \tilde{A} , which is a unit satisfying the asserted property because $\tilde{A} \rightarrow A$ is a local homomorphism. The element $P - \varpi \in \tilde{A}$ reduces to 0 in A by Proposition 3.1, and by the discussion in the beginning of this section, we only need to make sure that $P - \varpi \notin \tilde{\mathfrak{m}}^2 = (\varpi, \tilde{X}_1, \dots, \tilde{X}_n)^2$. This is clear because $P \in (\tilde{X}_1, \dots, \tilde{X}_n)^{q^n - 1}$ and $P - \varpi \equiv -\varpi \pmod{\tilde{\mathfrak{m}}^2}$ unless when $q = 2$ and $n = 1$, in which case $P - \varpi \equiv \tilde{u} \tilde{X}_1 - \varpi$. Q.E.D.

Remark 3.6. This argument can also be applied to the deformation spaces for higher level \mathfrak{p}^m structures, giving similar equations with P of the form

$$P = (\text{unit}) \cdot \prod_{\underline{a} \in (\mathcal{O}/\mathfrak{p}^m)^n \setminus (\mathfrak{p}/\mathfrak{p}^m)^n} P_{\underline{a}}.$$

Here we recall from Section 2.3 that the left action of $(a_{ij}) \in GL_n(k)$ on A on the formal parameters X_1, \dots, X_n is defined by

$$(3.1) \quad X_j \longmapsto [a_{1j}](X_1) + \sum_{\text{univ}} \cdots + \sum_{\text{univ}} [a_{nj}](X_n),$$

therefore by definition we can regard it as:

$$(3.2) \quad \tilde{X}_j \text{ mod } I \longmapsto P_{(a_{ij})_i}(\tilde{X}_1, \dots, \tilde{X}_n) \text{ mod } I$$

where $(a_{ij})_i := (a_{1j}, \dots, a_{nj})$ and $I := (P - \varpi)$. Note that this does not lift to the action of $GL_n(k)$ on \tilde{A} , as \tilde{a} are not additive lifts.

3.2. The special fiber

Now we investigate the special fiber of $X = \text{Spec } A$. As X is a scheme over $S := \text{Spec } W$, we use the notation $s := \text{Spec } \bar{k}$ and $X_s := X \times_S s$. By Proposition 3.5, we have:

$$(3.3) \quad X_s = \text{Spec } \bar{k}[[\tilde{X}_1, \dots, \tilde{X}_n]] / \left(\prod_{\underline{a} \in k^n \setminus \{0\}} (P_{\underline{a}} \bmod \mathfrak{p}) \right).$$

Definition 3.7. For each $\underline{a} \in k^n \setminus \{0\}$, we denote by $Y_{\underline{a}}$ the closed subscheme of X_s defined by $(P_{\underline{a}} \bmod \mathfrak{p}) = 0$, or equivalently, the closed subscheme of X defined by $(P_{\underline{a}} \bmod I) = 0$ (note that $(P_{\underline{a}} \bmod I)$ divides ϖ in A).

Note that, in the regular local ring $\bar{k}[[\tilde{X}_1, \dots, \tilde{X}_n]]$ with the maximal ideal $\bar{\mathfrak{m}} := (\tilde{X}_1, \dots, \tilde{X}_n)$, we have

$$(P_{\underline{a}} \bmod \mathfrak{p}) \equiv a_1 \tilde{X}_1 + \dots + a_n \tilde{X}_n \pmod{\bar{\mathfrak{m}}^2}$$

by Proposition 3.4(i), therefore that $(P_{\underline{a}} \bmod \mathfrak{p}) \in \bar{\mathfrak{m}} \setminus \bar{\mathfrak{m}}^2$. This shows that the quotient $\bar{k}[[\tilde{X}_1, \dots, \tilde{X}_n]] / (P_{\underline{a}} \bmod \mathfrak{p})$ is an $(n - 1)$ -dimensional regular local ring, thus its spectrum $Y_{\underline{a}}$ is an irreducible and reduced \bar{k} -scheme of dimension $n - 1$.

By Proposition 3.4(ii), we see that $Y_{\underline{a}} = Y_{\underline{a}'}$ when $\underline{a} = c \cdot \underline{a}'$ for $c \in k^\times$, and Proposition 3.4(i) ensures that $Y_{\underline{a}} \neq Y_{\underline{a}'}$ if otherwise. Therefore we introduce the following notation to label these closed subschemes of X_s , by k -rational hyperplanes of a projective $(n - 1)$ -space.

Definition 3.8. Let $\mathbb{P} := \mathbb{P}^{n-1}$ be an $(n - 1)$ -dimensional projective space over \bar{k} , equipped with a set of projective coordinates $(X_1^* : \dots : X_n^*)$. For each $\underline{a} \in k^n \setminus \{0\}$, we define a k -rational hyperplane:

$$M_{\underline{a}} : a_1 X_1^* + \dots + a_n X_n^* = 0$$

of \mathbb{P} . For any such hyperplane $M = M_{\underline{a}}$, we denote $Y_M := Y_{\underline{a}}$, which is well-defined by the above remark.

We will later identify this \mathbb{P} with the exceptional divisor of our first blow-up. The following proposition is clearly seen from the equation (3.3) and the remarks before Definition 3.8.

Proposition 3.9. *The correspondence $M \mapsto Y_M$ gives a bijection from the set of k -rational hyperplanes of \mathbb{P} to the set of irreducible components of X_s . In particular, there are $(q^n - 1)/(q - 1)$ irreducible components of X_s . Each irreducible component Y_M has multiplicity $|k^\times| = q - 1$ in X_s .*

We will also need the closed subschemes of X_s of higher codimension, i.e. the intersections of Y_M 's, in Section 4.

Definition 3.10. (i) For any k -rational linear subspace $N \subset \mathbb{P}$, we define a reduced closed subscheme Y_N of X_s as

$$Y_N := \bigcap_{N \subset M} Y_M.$$

(ii) For $1 \leq h \leq n - 1$, let $Y^{[h]} := \bigcup_N Y_N$, where N runs through all the k -rational linear subspaces of \mathbb{P} of dimension $h - 1$. Define $Y^{[0]}$ as the unique closed point x of X . Note that $Y^{[n-1]} = \bigcup_M Y_M = X_s^{\text{red}}$.

The way closed subschemes Y_M intersect with each other correspond precisely to how the hyperplanes M intersect inside \mathbb{P} . More precisely, we have the following.

Lemma 3.11. *Let $1 \leq h \leq n - 1$, and $N \subset \mathbb{P}$ be a k -rational linear subspace of dimension $h - 1$. If we choose a set of linearly independent k -rational hyperplanes M_1, \dots, M_{n-h} such that $N = \bigcap_{i=1}^{n-h} M_i$ in \mathbb{P} , then we have $Y_N = \bigcap_{i=1}^{n-h} Y_{M_i}$.*

Therefore the defining ideal of Y_N is $(P_{\underline{a}_1}, \dots, P_{\underline{a}_{n-h}})$ if $M_i = M_{\underline{a}_i}$. In particular, the closed immersion $Y_N \subset X$ is a regular immersion of codimension $n - h$, hence Y_N is a smooth \bar{k} -scheme of dimension h .

Proof. As the other inclusion is trivial, it is enough to show $Y_N \supset \bigcap_{i=1}^{n-h} Y_{M_i}$, or $Y_M \supset \bigcap_{i=1}^{n-h} Y_{M_i}$ for any k -rational hyperplane M containing N . For this, by the $GL_n(k)$ -action we can assume that $N \subset \mathbb{P}$ is defined by $X_1^* = \dots = X_{n-h}^* = 0$ and M_i is defined by $X_i^* = 0$ for $1 \leq i \leq n - h$, without loss of generality. In this case Y_{M_i} is defined by $P_{\underline{1}_i} = \tilde{X}_i = 0$ where $\underline{1}_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i -th entry, therefore $\bigcap_{i=1}^{n-h} Y_{M_i}$ is defined by the ideal (X_1, \dots, X_{n-h}) of A . Now for any M containing N is defined by an equation $\sum_{i=1}^n a_i X_i^* = 0$ with $a_{n-h+1} = \dots = a_n = 0$, which shows $P_{\underline{a}} \in (\tilde{X}_1, \dots, \tilde{X}_{n-h})$ by Proposition 3.4(iii), hence $Y_M \supset \bigcap_{i=1}^{n-h} Y_{M_i}$. Q.E.D.

The action (3.2) of $GL_n(k)$ on the formal parameters X_1, \dots, X_n and the Definition 3.3 of $P_{\underline{a}}$ show that the right $GL_n(k)$ -action on X

permutes the components Y_M in an obvious way, by acting on the set of indices through $PGL_n(k)$. More precisely, for $g \in GL_n(k)$ and $\underline{a} \in k^n \setminus \{0\}$, let $\underline{a} \mapsto \underline{a}g$ be the linear action on the row vector $\{\underline{a}\}$. This gives the action $M \mapsto Mg$ on the hyperplanes of \mathbb{P} , induced by the right linear $PGL_n(k)$ -action on \mathbb{P} through the projective coordinates. Then $GL_n(k)$ acts on the set of irreducible components of X_s by $Y_M \mapsto Y_{Mg}$. Similarly we see that $Y^{[h]}$ is stable under this action which maps $Y_N \mapsto Y_{Ng}$.

3.3. Some preliminaries on blow-ups of schemes

Before we start blowing up our space, we collect some facts on blow-up of schemes which will be used in the sequel. Let X be a noetherian scheme and \mathcal{I} be a coherent sheaf of ideals in \mathcal{O}_X , and $Y := \text{Spec}(\mathcal{O}_X/\mathcal{I})$ be the closed subscheme defined by \mathcal{I} . The *blow-up* of X at Y is defined as the scheme $X' := \text{Proj}(\bigoplus_{i \geq 0} \mathcal{I}^i)$, which is projective over X . The structure morphism $p : X' \rightarrow X$ is an isomorphism outside the inverse image $Y' := p^{-1}(Y)$ of Y (the *exceptional divisor*). As a projective scheme over Y , we have $Y' = \text{Proj}(\bigoplus_{i \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1})$. Therefore, if $Y \subset X$ is a *regular immersion* of codimension r , in other words when \mathcal{I} is locally generated by a regular sequence of sections of \mathcal{O}_X of length r , then the graded $\mathcal{O}_X/\mathcal{I}$ -algebra $\bigoplus_{i \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ is naturally isomorphic to the symmetric algebra generated by the conormal sheaf $\mathcal{N}_{Y/X} = \mathcal{I}/\mathcal{I}^2$, which is a locally free $\mathcal{O}_X/\mathcal{I}$ -module of rank r , thus $Y' \cong \mathbb{P}(\mathcal{N}_{Y/X})$ is a projective space bundle (\mathbb{P}^{r-1} -bundle) over Y ([EGA] IV, 16.9 and 19.4), and if \mathcal{I} is locally generated by a regular sequence X_1, \dots, X_r , then $(X_1^* : \dots : X_r^*)$ gives a set of projective coordinates of this projective space bundle where $X_i^* := X_i \text{ mod } \mathcal{I}^2$.

We will need some commutativity between blow-ups and completions.

Lemma 3.12. *Let X, \mathcal{I}, X' as above, and $f : Z \rightarrow X$ be a flat morphism. Let $\mathcal{I}' := \text{Im}(f^*\mathcal{I} \rightarrow \mathcal{O}_Z)$, where $f^*\mathcal{I} = f^{-1}\mathcal{I} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Z$, and Z' be the blow-up of Z at the subscheme defined by \mathcal{I}' . Then the canonical morphism $Z' \rightarrow Z \times_X X'$ defined by the universality of blow-ups is an isomorphism.*

Proof. As f is flat, the sheaf \mathcal{O}_Z is a flat $f^{-1}\mathcal{O}_X$ -algebra, which shows that $f^*\mathcal{I} \rightarrow \mathcal{O}_Z$ is an injection and $f^*\mathcal{I} \cong \mathcal{I}'$. Now $Z \times_X X' = Z \times_X \text{Proj}(\bigoplus_{i \geq 0} \mathcal{I}^i) = \text{Proj}(\mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_X} \bigoplus_{i \geq 0} (f^{-1}\mathcal{I})^i) = \text{Proj}(\bigoplus_{i \geq 0} (f^*\mathcal{I})^i)$, which is isomorphic to $Z' = \text{Proj}(\bigoplus_{i \geq 0} (\mathcal{I}')^i)$. Q.E.D.

Corollary 3.13. *Let X, \mathcal{I} as above. For $x \in X$, let $\widehat{X}_x := \text{Spec } \widehat{\mathcal{O}}_{X,x}$ be the complete local ring (or the completion of the strict local ring) of*

X at x , and X'_x the blow-up of \widehat{X}_x at the subscheme defined by $\mathcal{I} \cdot \mathcal{O}_{\widehat{X}_x}$. Then $X'_x \cong \widehat{X}_x \times_X X'$.

3.4. The first blow-up Z_1

We first blow-up the unique closed point $x = Y^{[0]}$ of X . By Proposition 3.5, we have the description

$$X = \text{Spec } W[[\widetilde{X}_1, \dots, \widetilde{X}_n]]/(P - \varpi)$$

$$P(\widetilde{X}_1, \dots, \widetilde{X}_n) = \widetilde{u} \cdot \prod_{\underline{a} \in k^n \setminus \{0\}} P_{\underline{a}}(\widetilde{X}_1, \dots, \widetilde{X}_n)$$

with $\widetilde{u} \in 1 + \widetilde{\mathfrak{m}} \subset W[[\widetilde{X}_1, \dots, \widetilde{X}_n]]^\times$. The universal formal parameters on X are given by $X_i = \widetilde{X}_i \bmod (P - \varpi)$ for $1 \leq i \leq n$.

Definition 3.14. Let $Z_1 \rightarrow X$ be the blow-up of X at the unique closed point $x := (X_1, \dots, X_n)$ of X , and $Y_{\mathbb{P}}$ be its exceptional divisor, i.e. the inverse image of x in Z_1 .

As X is the spectrum of an n -dimensional regular local ring, the exceptional divisor $Y_{\mathbb{P}}$ is an $(n - 1)$ -dimensional projective space \mathbb{P}^{n-1} over \bar{k} , equipped with a set of projective coordinates $(X_1^* : \dots : X_n^*)$, where $X_i^* := X_i \bmod \mathfrak{m}^2$. We identify $Y_{\mathbb{P}}$ with the \mathbb{P} that we introduced in Definition 3.8 for the indexing purpose.

As the center x of the blow-up is $GL_n(k)$ -invariant, the blow-up Z_1 inherits the right $GL_n(k)$ -action on X . The action on the exceptional divisor $Y_{\mathbb{P}} \cong \mathbb{P}$ is the obvious one:

Proposition 3.15. *The right action of $GL_n(k)$ on $Y_{\mathbb{P}} \cong \mathbb{P}$, in terms of the projective coordinates $(X_1^* : \dots : X_n^*)$, is the right linear action through $PGL_n(k)$.*

Proof. As the action of $GL_n(k)$ on $Y_{\mathbb{P}} \cong \mathbb{P}(\mathfrak{m}/\mathfrak{m}^2)$ is induced from the action on the n -dimensional k -vector space $\mathfrak{m}/\mathfrak{m}^2$ generated by $X_i^* = X_i \bmod \mathfrak{m}^2$ for $1 \leq i \leq n$, it suffices to see that the left action of $(a_{ij}) \in GL_n(k)$ on X_i^* is defined by

$$X_j^* \longmapsto a_{1j}X_1^* + \dots + a_{nj}X_n^*,$$

which is readily confirmed by reducing the left action (3.1) on A modulo \mathfrak{m}^2 . Q.E.D.

Now we analyze how the proper transforms of the other components Y_M intersect the exceptional divisor $Y_{\mathbb{P}}$ in Z_1 .

Definition 3.16. For a k -linear subspace $N \subset \mathbb{P}$ of dimension $h - 1$, with $1 \leq h \leq n - 1$, we denote the proper transform of Y_N in Z_1 by $Y_{N,1}$.

For $1 \leq h \leq n - 1$, let $Y_1^{[h]}$ be the proper transform of $Y^{[h]}$ in Z_1 . Then $Y_1^{[h]} = \bigcup_N Y_{N,1}$ for $1 \leq h \leq n - 1$, where N runs through all k -linear subspaces of \mathbb{P} of dimension $h - 1$.

Proposition 3.17. *Let Z_1 be the model of X defined in Definition 3.14.*

- (i) *In the special fiber of Z_1 , the exceptional divisor $Y_{\mathbb{P}}$ has multiplicity $q^n - 1$.*
- (ii) *For a k -linear subspace $N \subset \mathbb{P} \cong Y_{\mathbb{P}}$, we have $Y_{\mathbb{P}} \cap Y_{N,1} = N$.*

Proof. These properties can be checked by looking at the completions along the exceptional divisor of the affine open sets of Z_1 . We can reduce to the case where N is a hyperplane $M = M_{\underline{a}}$ in (ii), and assume moreover that $a_n \neq 0$, without loss of generality. We look at the affine open set that is the spectrum of

$$W[V_{\leq n-1}][[\tilde{X}_1, \dots, \tilde{X}_n]] / (P(\tilde{X}_1, \dots, \tilde{X}_n) - \varpi, V_i \tilde{X}_n - \tilde{X}_i),$$

where $W[V_{\leq n-1}] = W[V_1, \dots, V_{n-1}]$. We take the completion of this ring along the exceptional divisor $\tilde{X}_n = 0$ (which is an affine space \mathbb{A}^{n-1} with the coordinate ring $k[V_1, \dots, V_{n-1}]$) to get the spectrum of

$$B_1 := W[V_{\leq n-1}][[\tilde{X}_n]] / (P(V_1 \tilde{X}_n, \dots, V_{n-1} \tilde{X}_n, \tilde{X}_n) - \varpi)$$

with $P = \tilde{u} \cdot \prod_{\underline{a} \in k^n \setminus \{0\}} P_{\underline{a}}$ where

$$\tilde{u} \in W[V_{\leq n-1}][[\tilde{X}_n]]^\times, \quad \tilde{u} \equiv 1 \pmod{\tilde{X}_n}.$$

Now by the congruence Proposition 3.4(i), we have

$$P_{\underline{a}}(V_1 \tilde{X}_n, \dots, V_{n-1} \tilde{X}_n, \tilde{X}_n) = \tilde{X}_n \cdot P'_{\underline{a}}$$

with some $P'_{\underline{a}} \in W[V_{\leq n-1}][[\tilde{X}_n]]$, which satisfies

$$(3.4) \quad P'_{\underline{a}}(V_1, \dots, V_{n-1}, \tilde{X}_n) \equiv \sum_{i=1}^{n-1} \tilde{a}_i V_i + \tilde{a}_n \pmod{\tilde{X}_n},$$

hence we can write

$$B_1 = W[V_{\leq n-1}][[\tilde{X}_n]] / (\tilde{u} \cdot \tilde{X}_n^{q^n - 1} \cdot \prod_{\underline{a} \in k^n \setminus \{0\}} P'_{\underline{a}} - \varpi).$$

Therefore the special fiber of this affine scheme is the spectrum of:

$$B_1 \otimes_W \bar{k} = \bar{k}[V_{\leq n-1}][[\tilde{X}_n]]/(\tilde{X}_n^{q^n-1} \cdot \prod_{\underline{a} \in k^n \setminus \{0\}} (P'_{\underline{a}} \bmod \mathfrak{p})).$$

In this expression, the exceptional divisor $\tilde{X}_n = 0$ clearly has multiplicity $q^n - 1$ in the special fiber $\text{Spec}(B_1 \otimes_W \bar{k})$. Also the inverse image in $\text{Spec} B_1$ of the proper transform $Y_{M,1}$ of Y_M for $M = M_{\underline{a}}$ is defined by $P'_{\underline{a}}(V_1, \dots, V_{n-1}, \tilde{X}_n) = 0$, therefore by the above congruence it intersects $\tilde{X}_n = 0$ at the hyperplane $\sum_{i=0}^{n-1} a_i V_i + a_n = 0$ in \mathbb{A}^{n-1} . As the situations in the completions of the other affine open sets of Z_1 are checked in exactly the same way, we conclude that $Y_{M,1}$ intersects $Y_{\mathbb{P}}$ at the hyperplane $M_{\underline{a}} \subset \mathbb{P} \cong Y_{\mathbb{P}}$. Q.E.D.

The following simple stratification $Y_{\mathbb{P}}^{(h)}$ on $Y_{\mathbb{P}} \cong \mathbb{P}$ by locally closed subschemes of dimension $h - 1$ will be used in the analysis of this model in the subsequent sections.

Definition 3.18. For $1 \leq h \leq n$, let $Y_{\mathbb{P}}^{[h]}$ denote the reduced closed subscheme of $Y_{\mathbb{P}} \cong \mathbb{P}$ which is the union of all k -rational linear subspaces of dimension $h - 1$. Then by Proposition 3.17(ii), we have $Y_{\mathbb{P}} \cap Y_1^{[h]} = Y_{\mathbb{P}}^{[h]}$ for $1 \leq h \leq n - 1$.

Also, set $Y_{\mathbb{P}}^{[0]} := \emptyset$, and $Y_{\mathbb{P}}^{(h)} := Y_{\mathbb{P}}^{[h]} - Y_{\mathbb{P}}^{[h-1]}$ for $1 \leq h \leq n$.

Then $Y_{\mathbb{P}}^{(h)}$ is a smooth locally closed subscheme of $Y_{\mathbb{P}} \cong \mathbb{P}$ of dimension $h - 1$, namely the disjoint union

$$Y_{\mathbb{P}}^{(h)} = \coprod_N N^0,$$

where $N \subset \mathbb{P}$ runs through the set of k -rational linear subspaces of dimension $h - 1$, and N^0 is the complement in N of all k -rational proper linear subspaces of N . This stratification is clearly stable under the $GL_n(k)$ -action.

§4. A generalized semistable model Z_{st}

Now we construct a generalized semistable model of X over S by blowing up the model Z_1 further. Recall that, in the special fiber of Z_1 , there is one proper component $Y_{\mathbb{P}} \cong \mathbb{P}$ with multiplicity $q^n - 1$ and a set of projective coordinates $(X_1^* : \dots : X_n^*)$ on it, and $(q^n - 1)/(q - 1)$ other components, each with multiplicity $q - 1$, intersecting $Y_{\mathbb{P}}$ at each of the k -rational hyperplanes. While the intersections of reduced components

are pairwise transversal, more than n components meet at a point, so we need to blow-up several more times to get a generalized semistable model. The successive blow-up performed here is analogous to the one performed to obtain the irreducible components of the special fiber of the p -adic upper half space (see for example [It], Section 4).

4.1. A generalized semistable model

Recall that Z_1 was the blow-up of X at the closed point $x = Y^{[0]}$.

Definition 4.1. We define inductively $Z_{h+1} \rightarrow Z_h$ for $1 \leq h \leq n-2$ as the blow-up of Z_h at the proper transform of $Y^{[h]}$ in Z_h . We write $Z_{st} := Z_{n-1}$.

The scheme Z_2 is the blow-up of Z_1 at $Y_1^{[1]} = \coprod_N Y_{N,1}$, where N runs through all the k -rational points of \mathbb{P} . Similarly, the blow-up $Z_{h+1} \rightarrow Z_h$ is centered at $\coprod_N Y_{N,h}$, where N runs through all the k -rational linear subspaces of \mathbb{P} of dimension $h-1$ and $Y_{N,h}$ denotes the proper transform of Y_N in Z_h .

For a k -rational point N , as Y_N has codimension $n-1$ in Z_1 , the inverse image $Y_{N,2}$ of $Y_{N,1}$ in Z_2 is a \mathbb{P}^{n-2} -bundle over $Y_{N,1}$, which is an irreducible component of the special fiber of Z_2 . As $(q^{n-1}-1)/(q-1) = |\mathbb{P}^{n-2}(k)|$ components meet at Y_N , the component $Y_{N,2}$ has multiplicity $q^{n-1}-1$ in the special fiber of Z_2 . Similarly, for a k -rational linear subspace N of dimension $h-1$, with $1 \leq h \leq n-1$, the proper transform $Y_{N,h}$ in Z_h is a regular immersion of codimension $n-h$ in Z_h , hence its inverse image $Y_{N,h+1}$ in Z_{h+1} is a \mathbb{P}^{n-h-1} -bundle over $Y_{N,h}$.

Eventually, for a k -rational linear subspace N of dimension $h-1$, with $1 \leq h \leq n-1$, the proper transform $Y_{N,st}$ of Y_N in Z_{st} is an irreducible component of the special fiber of Z_{st} , whose multiplicity is $q^{n-h}-1$. The inverse image $Y_{\mathbb{P},st}$ of $Y_{\mathbb{P}}$ in Z_{st} is the unique proper component of the special fiber of Z_{st} , whose multiplicity is q^n-1 . This $Y_{\mathbb{P},st}$ is isomorphic to the B^{n-1} in the notation of [It], Section 4, and $Y_{N,st} \cap Y_{\mathbb{P},st}$ is isomorphic to $B^h \times B^{n-h-2}$.

Note that all the blow-ups performed here are $GL_n(k)$ -equivariant, so that the models $Z_1, Z_2, \dots, Z_{n-1} = Z_{st}$ all inherit the right $GL_n(k)$ -action on X , and $g \in GL_n(k)$ sends $Y_{N,st}$ to $Y_{Ng,st}$.

Now our main result here is:

Theorem 4.2. *The W -scheme $Z_{st} = Z_{n-1}$ is generalized semistable. This means that, at every closed point of Z_{st} , its complete local ring is isomorphic to*

$$W[[T_1, \dots, T_n]] / (T_1^{e_1} \dots T_d^{e_d} - \varpi) \quad (d \leq n),$$

where integers e_i are all prime to $\text{char } k$.

Proof. We start by describing the complete local rings at all the closed points of Z_1 , i.e. all the closed points of $Y_{\mathbb{P}}$. By the obvious $GL_n(k)$ -symmetry, it is enough to look at each points on the completion along the exceptional divisor of the affine piece that was defined in the proof of Proposition 3.17:

$$\text{Spec } B_1 = \text{Spec } W[V_{\leq n-1}][[\tilde{X}_n]] / (\tilde{u} \cdot \tilde{X}_n^{q^n-1} \cdot \prod_{\underline{a} \in k^n \setminus \{0\}} P'_{\underline{a}} - \varpi),$$

where $\tilde{u} \in W[V_1, \dots, V_{n-1}][[\tilde{X}_n]]^\times$. Hereafter in this proof, we denote the coordinates of a point in the corresponding lower case alphabets to distinguish them from the elements of the coordinate rings. Let $x \in \text{Spec } B_1$ be a point which is closed in Z_1 and $(v_1, \dots, v_{n-1}) \in \bar{k}^{n-1}$ the affine coordinates of x . The complete local ring $\hat{\mathcal{O}}_x$ depends on whether the value of $P'_{\underline{a}}$ at x is a unit or not for each \underline{a} , which in turn depends on how many of the v_i 's have k -rational linear relations among them, by the equations (3.4).

If the image of x in Z_1 lies in $Y_{\mathbb{P}}^{(n)}$, i.e. if there is no k -rational relation of the form

$$a_1 v_1 + \dots + a_{n-1} v_{n-1} + a_n = 0 \quad (a_1, \dots, a_n \in k)$$

among v_1, \dots, v_{n-1} , then we have $\hat{\mathcal{O}}_x \cong W[[\tilde{V}_1, \dots, \tilde{V}_{n-1}, \tilde{X}_n]]/(\tilde{u} \cdot \tilde{X}_n^{q^n-1} - \varpi)$ with $\tilde{u} \in W[[\tilde{V}_1, \dots, \tilde{V}_{n-1}, \tilde{X}_n]]^\times$, where \tilde{V}_i is a translation of V_i which vanishes at x . If $x \in Y_{\mathbb{P}}^{(m)}$ with $m \leq n-1$, i.e. there are exactly m of the k -rational linear dependences between v_1, \dots, v_{n-1} , then by $GL_n(k)$ -symmetry, it suffices to treat the case where $v_1 = \dots = v_m = 0$ and no k -rational linear relation between v_{m+1}, \dots, v_{n-1} , by using the $GL_n(k)$ -action. In this case the value of $P'_{\underline{a}}$ at x is a non-unit if and only if $a_{m+1} = \dots = a_n = 0$. Thus we have

$$\hat{\mathcal{O}}_x \cong W[[\tilde{V}_1, \dots, \tilde{V}_{n-1}, \tilde{X}_n]] / (\tilde{u} \cdot \tilde{X}_n^{q^n-1} \cdot \prod_{\underline{a} \in k^m \setminus \{0\}} P'_{(\underline{a}, 0)} - \varpi)$$

with $\tilde{u} \in W[[\tilde{V}_1, \dots, \tilde{V}_{n-1}, \tilde{X}_n]]^\times$, where \tilde{V}_i is a translation of V_i which vanishes at x and $(\underline{a}, 0) := (a_1, \dots, a_m, 0, \dots, 0)$.

We have $P'_{(\underline{a}, 0)}(\tilde{V}_1, \dots, \tilde{V}_{n-1}, \tilde{X}_n) \in (\tilde{V}_1, \dots, \tilde{V}_m)$ by Proposition 3.4(i), (iii), and

$$(4.1) \quad P'_{(\underline{a}, 0)}(\tilde{V}_1, \dots, \tilde{V}_{n-1}, \tilde{X}_n) \equiv \sum_{i=1}^m \tilde{a}_i \tilde{V}_i \pmod{(\tilde{V}_1, \dots, \tilde{V}_m)^2}.$$

As a special case where $m = 1$, we see that

$$\widehat{\mathcal{O}}_x \cong W[[\widetilde{V}_1, \dots, \widetilde{V}_{n-1}, \widetilde{X}_n]] / (\widetilde{u} \cdot \widetilde{X}_n^{q^n-1} \cdot \prod_{a \in k^\times} P'_{(a, \mathfrak{Q})} - \varpi)$$

is already generalized semistable, because by Proposition 3.4 we have

$$\prod_{a \in k^\times} P'_{(a, \mathfrak{Q})} = (\text{unit}) \cdot \widetilde{V}_1^{q-1}.$$

Now by Corollary 3.13, we can analyze the situation after the following blow-ups by blowing up these complete local rings further. As we go through the procedure of the successive blow-ups, the first time we touch the point $x \in Y_{\mathbb{P}^{(n-m)}}$ satisfying $v_1 = \dots = v_m = 0$ is when we blow-up $Y_{N, n-m}$, where N is the codimension m linear subspace $X_1^* = \dots = X_m^* = 0$ on which x lies on. Then by the proof of Lemma 3.11 we see that $Y_{N, n-m}$ is defined by the ideal (V_1, \dots, V_m) , which is a regular sequence in $\widehat{\mathcal{O}}_x$. After the blowing up and completing along the new exceptional divisor, we obtain the affine pieces of the form:

$$\text{Spec } W[U_{\leq m-1}][[\widetilde{V}_m, \dots, \widetilde{V}_{n-1}, \widetilde{X}_n]] / I, \\ I = (\widetilde{u} \cdot \widetilde{X}_n^{q^n-1} Q - \varpi),$$

$$Q = \prod_{\underline{a} \in k^m \setminus \{0\}} P'_{(\underline{a}, \mathfrak{Q})}(U_1 \widetilde{V}_m, \dots, U_{m-1} \widetilde{V}_m, \widetilde{V}_m, \dots, \widetilde{V}_{n-1}, \widetilde{X}_n)$$

with a unit \widetilde{u} , and by the congruence (4.1) we can pull out the V_m as we did in (3.4):

$$P'_{(\underline{a}, \mathfrak{Q})}(U_1 \widetilde{V}_m, \dots, U_{m-1} \widetilde{V}_m, \widetilde{V}_m, \dots, \widetilde{V}_{n-1}, \widetilde{X}_n) = \widetilde{V}_m \cdot P''_{\underline{a}}$$

for some $P''_{\underline{a}} \in W[U_{\leq m-1}][[\widetilde{V}_m, \dots, \widetilde{V}_{n-1}, \widetilde{X}_n]]$ which satisfies

$$P''_{\underline{a}}(U_1, \dots, U_{m-1}, \widetilde{V}_m, \dots, \widetilde{V}_{n-1}, \widetilde{X}_n) \equiv \sum_{i=1}^{m-1} \widetilde{a}_i U_i + \widetilde{a}_m \pmod{\widetilde{V}_m},$$

hence we can write the above ideal I in the form

$$I = (\widetilde{u} \cdot \widetilde{X}_n^{q^n-1} \cdot \widetilde{V}_m^{q^m-1} \cdot \prod_{\underline{a} \in k^m \setminus \{0\}} P''_{\underline{a}} - \varpi).$$

Then if we look at the closed point y on the exceptional divisor of this blow-up, which is a \mathbb{P}^{m-1} -bundle over $Y_{N, n-m}$ with a set of projective coordinates $(\widetilde{V}_1^* : \dots : \widetilde{V}_m^*)$ or affine coordinates (u_1, \dots, u_{m-1}) with

$U_i = \tilde{V}_i^*/\tilde{V}_m^*$ on the fibers, the complete local ring $\hat{\mathcal{O}}_y$ will depend on how many k -rational linear relations there are among the fiber coordinates u_1, \dots, u_{m-1} of y . Assume that y lies on a k -rational linear subspace of codimension $l \leq m - 1$, say $U_1 = \dots = U_l = 0$ and no k -rational linear relation among u_{l+1}, \dots, u_{m-1} , without loss of generality because of the action of the parabolic subgroup of $GL_n(k)$ preserving N . Then the complete local ring will have the form:

$$\hat{\mathcal{O}}_y \cong W[[\tilde{U}_1, \dots, \tilde{U}_{m-1}, \tilde{V}_m, \dots, \tilde{V}_{n-1}, \tilde{X}_n]] / J,$$

$$J = (\tilde{u} \cdot \tilde{X}_n^{q^n-1} \cdot \tilde{V}_m^{q^m-1} \prod_{\underline{a} \in k^l \setminus \{0\}} P''_{(\underline{a}, 0)} - \varpi)$$

with $\tilde{u} \in W[[\tilde{U}_1, \dots, \tilde{U}_{m-1}, \tilde{V}_m, \dots, \tilde{V}_{n-1}, \tilde{X}_n]]^\times$, and it is generalized semistable if $l = 0$ or 1 . As we repeat the blow-up process, we get a sequence

$$n = n_0, m = n_1, l = n_2, n_3, \dots$$

until there is no more blow-up to be done when $n_t = 1$ for some integer t . This proves that Z_{n-1} is generalized semistable at all closed points.

Q.E.D.

4.2. Relationship with a generalized semistable model of Shimura varieties

In order to apply the computations of nearby cycle sheaves for the varieties with generalized semistable reduction due to T. Saito [Sa] (we will recall his results in Section 6.2), we need to compare what we have constructed with some scheme of finite type over $S = \text{Spec } W$. For this, first we can approximate the scheme X by a scheme of finite type over S , so that its completion at the origin would be isomorphic to X . Then by performing the successive blow-ups of this scheme by the corresponding closed subschemes and observing that it has generalized semistable reduction (i.e. etale locally etale over $\text{Spec } W[T_1, \dots, T_n]/(T_1^{e_1} \dots T_d^{e_d} - \varpi)$ ($d \leq n$)), we can show that the results concerning the nearby cycles of schemes of finite type are applicable to our situation. We need the following:

Proposition 4.3. *There is a scheme U of finite type over S and a closed point $x \in U$ such that X is isomorphic to the spectrum of the complete local ring $\hat{\mathcal{O}}_{U,x}$ of U at x . Moreover, there is a closed subscheme $V^{[h]}$ of U for each $0 \leq h \leq n-1$, and $Y^{[h]} \cong X \times_U V^{[h]}$ via the morphism $X \cong \text{Spec } \hat{\mathcal{O}}_{U,x} \rightarrow U$.*

This proposition can be proven by a purely local argument. For the existence of U and x , we can refer to [St3], Theorem 2.3.1, where the

proof attributed to L. Fargues, using Artin's approximation theorem [Ar] and Faltings' theory of strict \mathcal{O} -modules [Fa2], is given. As this U comes with an approximation of strict \mathcal{O} -module $\Sigma^{\text{univ}}[\mathfrak{p}]$, its reduced closed subschemes $V^{[h]} = \bigcup_M V_M$, where M runs through all k -subspaces of $(\mathfrak{p}^{-1}/\mathcal{O})^n$ of dimension $n - h$, can be given in the exactly the same way as in the proof of Lemma 4.4 below (based on Lemma 9 of [Man]). (Or, once we have an approximation of the finite covering $X \rightarrow \text{Spec } A_0 = \text{Spec } W[[T_1, \dots, T_{n-1}]]$ by a finite covering of the strict henselization of $W[[T_1, \dots, T_{n-1}]]$ at the origin, we can use the fact that we could choose T_i so that $Y^{[h]}$ is the reduced closed subscheme of X associated to the pull-back of the closed subscheme of $\text{Spec } A_0$ defined by $(\mathfrak{p}, T_1, \dots, T_{n-1-h})$, i.e. the closure of the set of closed points where the reduction of Σ^{univ} has height $\geq n - h$).

Here we explain a proof using the integral model of unitary Shimura varieties (we have already recalled in Section 2.4 that this gives the existence of U and x). This has an advantage that it spells out the corresponding blow-ups of Shimura varieties, and calculations in the last section imply that we have a generalized semistable model of Shimura varieties in this case.

We use the notation from [HT], especially Chapter III.4. The integral model $X_{U^p, m}$ of Shimura varieties defined in p.109 of [HT] is a proper flat scheme of relative dimension $n - 1$ over $\mathcal{O}_{F, w} = \mathcal{O}$, and its special fiber $\overline{X}_{U^p, m} = X_{U^p, m} \times_{\text{Spec } \mathcal{O}} \text{Spec } k$ has a stratification by reduced closed subschemes $\overline{X}_{U^p, m}^{[h]}$ of dimension h for $0 \leq h \leq n - 1$. This $\overline{X}_{U^p, m}^{[h]}$ is the closure of the set of closed points where the associated 1-dimensional Barsotti-Tate \mathcal{O} -module \mathcal{G} has etale height $\leq h$ (defined in p.111 of [HT]).

We fix a multi-index m with $m_1 = 1$, and denote $X_{U^p, m}$ simply by Sh , and similarly $\overline{\text{Sh}} := \overline{X}_{U^p, m}$, $\overline{\text{Sh}}^{[h]} := \overline{X}_{U^p, m}^{[h]}$ etc. We recalled in Section 2.4 that the completion $\widehat{\mathcal{O}}_{\text{Sh}, \overline{s}}$ of the strict local ring of Sh at any geometric closed point \overline{s} centered in $s \in \overline{\text{Sh}}^{[0]}$ is isomorphic to our deformation ring A , i.e. $X = \text{Spec } A \cong \text{Spec } \widehat{\mathcal{O}}_{\text{Sh}, \overline{s}}$. We fix the resulting morphism

$$X \cong \text{Spec } \widehat{\mathcal{O}}_{\text{Sh}, \overline{s}} \longrightarrow \text{Sh}.$$

Lemma 4.4. *We have $Y^{[h]} = X \times_{\text{Sh}} \overline{\text{Sh}}^{[h]}$.*

Proof. We can show directly that $Y^{[h]}$ is the reduced subscheme of X associated to the locus where the deformation of Σ_n has height $\geq n - h$. Here we give a proof closer in spirit to how we defined $Y^{[h]}$. In [Man] 3.2 (see Remark 10(2)), the following decomposition of $\overline{\text{Sh}}^{[h]}$

is constructed (their intersections with the open strata are defined in [HT], p.115, denoted $\overline{X}_{Up,m,M}$). If M is a free $k = \mathcal{O}/\mathfrak{p}$ -submodule of $\mathfrak{p}^{-1}\Lambda_{11}/\Lambda_{11}$ of rank $n - h$ (here $\Lambda_{11} \cong (\mathcal{O}^n)^\vee$ is a free \mathcal{O} module of rank n used in the definition of the Shimura variety), then $\overline{\text{Sh}}_M^{[h]}$ is the unique closed subscheme of Sh such that, for any scheme T over Sh , a T -valued point $T \rightarrow \text{Sh}$ factors through $\overline{\text{Sh}}_M^{[h]}$ if and only if the base change of the Drinfeld level \mathfrak{p} structure α on the universal Barsotti–Tate \mathcal{O} -module \mathcal{G} on Sh of height n :

$$\alpha_T : (\mathfrak{p}^{-1}\Lambda_{11}/\Lambda_{11})_T \longrightarrow \mathcal{G}[\mathfrak{p}]_T$$

vanishes on M_T . Then $\overline{\text{Sh}}^{[h]} = \bigcup_M \overline{\text{Sh}}_M^{[h]}$, where M runs through all k -subspaces of $\mathfrak{p}^{-1}\Lambda_{11}/\Lambda_{11}$ of rank $n - h$.

Now, the universal formal \mathcal{O} -module $\Sigma^{\text{univ}} \otimes A$ and the universal level \mathfrak{p} structure φ are the pull-backs under our morphism $X \rightarrow \text{Sh}$ of the Barsotti–Tate \mathcal{O} -module \mathcal{G}/Sh and the level \mathfrak{p} structure α . Therefore, for any scheme $T \rightarrow \text{Sh}$ pulled back to $T_X \rightarrow X$, the condition for T_X to factor through $X \times_{\text{Sh}} \overline{\text{Sh}}_M^{[h]}$ translates as follows. Expressing the elements of $(\mathfrak{p}^{-1}\Lambda_{11}/\Lambda_{11})_{T_X} \cong (k^n)^\vee$ by the standard basis, vanishing of α_{T_X} on M_{T_X} is written as:

$$(a_1, \dots, a_n) \in M \implies [a_1](X_1) +_{\Sigma^{\text{univ}}} \dots +_{\Sigma^{\text{univ}}} [a_n](X_n) = 0.$$

This RHS is exactly the defining equation of $P_{\underline{a}}$. If we take a basis $\underline{a}_1, \dots, \underline{a}_{n-h}$ of M and define the hyperplanes $M_i := M_{\underline{a}_i}$ of \mathbb{P} , then $N := \bigcap_{i=1}^{n-h} M_i \subset \mathbb{P}$ is a k -rational linear subspace of \mathbb{P} of dimension $h - 1$. We see that $X \times_{\text{Sh}} \overline{\text{Sh}}_M^{[h]}$ is the closed subscheme of X defined by the ideal $(P_{\underline{a}_1}, \dots, P_{\underline{a}_{n-h}})$, namely Y_N by Lemma 3.11. As M runs through the rank $n - h$ free submodules of $(k^n)^\vee$, clearly N runs through all k -rational linear subspaces of \mathbb{P} of dimension $h - 1$. Q.E.D.

Definition 4.5. We denote the blow-up of Sh at $\overline{\text{Sh}}^{[0]}$ by Sh_1 , and inductively define $\text{Sh}_{h+1} \rightarrow \text{Sh}_h$ as the blow-up of Sh_h at the proper transform of $\overline{\text{Sh}}^{[h]}$ inside Sh_h , for $1 \leq h \leq n - 2$.

Lemma 4.6. *We have canonically $Z_{\text{st}} = Z_{n-1} \cong X \times_{\text{Sh}} \text{Sh}_{n-1}$.*

Proof. We show that $Z_h \rightarrow \text{Sh}_h$ is flat and $Z_h \cong X \times_{\text{Sh}} \text{Sh}_h$ by induction. Corollary 3.13 shows that it is true for $h = 1$, the strict henselisation and the completion being flat. The above lemma shows that the center of blow-up for $Z_{h+1} \rightarrow Z_h$ is the pull back of the center of blow-up for $\text{Sh}_{h+1} \rightarrow \text{Sh}_h$. Therefore we have $Z_{h+1} \cong Z_h \times_{\text{Sh}_h}$

Sh_{h+1} by Lemma 3.12, therefore $Z_{h+1} \cong X \times_{\text{Sh}} \text{Sh}_{h+1}$ by our inductive hypothesis. Q.E.D.

Now if we denote the base change of $\text{Sh}, \text{Sh}_{n-1}$ by $S = \text{Spec } W \rightarrow \text{Spec } \mathcal{O}$ by $\text{Sh}_S, \text{Sh}_{\text{st}}$, we have a diagram of S -schemes

$$\begin{array}{ccccccc}
 Y_{\mathbb{P},\text{st}} & \longrightarrow & Z_{\text{st}} & \longrightarrow & X & & \\
 \cong \downarrow & & f \downarrow & & \downarrow & \searrow & \\
 Y_{\bar{s},\text{st}} & \longrightarrow & \text{Sh}_{\text{st}} & \longrightarrow & \text{Sh}_S & \longrightarrow & S
 \end{array}$$

whose second square is cartesian by the above lemma. We can regard the geometric closed point $\bar{s} \rightarrow s \in \text{Sh}$ that we chose before as the closed point of Sh_S , and the inverse image of the closed point $x \in X$ under $Z_{\text{st}} \rightarrow X$, namely the unique proper component $Y_{\mathbb{P},\text{st}}$ of $Z_{\text{st}} \times_{\text{Spec } W} \text{Spec } \bar{k}$, is isomorphic under $f : Z_{\text{st}} \rightarrow \text{Sh}_{\text{st}}$ to the inverse image $Y_{\bar{s},\text{st}}$ of \bar{s} under $\text{Sh}_{\text{st}} \rightarrow \text{Sh}_S$.

Lemma 4.7. *For all closed points $z \in Y_{\mathbb{P},\text{st}} \subset Z_{\text{st}}$, the local W -homomorphism between the complete local rings $\widehat{\mathcal{O}}_{\text{Sh}_{\text{st}},f(z)} \rightarrow \widehat{\mathcal{O}}_{Z_{\text{st}},z}$ induced by f is an isomorphism.*

Proof. The morphism $\text{Sh}_{\text{st}} \rightarrow \text{Sh}_S$ maps $f(z)$ to \bar{s} , hence it induces a local homomorphism between complete local rings $\widehat{\mathcal{O}}_{\text{Sh}_S,\bar{s}} \rightarrow \widehat{\mathcal{O}}_{\text{Sh}_{\text{st}},f(z)}$, or $\text{Spec } \widehat{\mathcal{O}}_{\text{Sh}_{\text{st}},f(z)} \rightarrow X$. This gives $\text{Spec } \widehat{\mathcal{O}}_{\text{Sh}_{\text{st}},f(z)} \rightarrow Z_{\text{st}} \cong X \times_{\text{Sh}_S} \text{Sh}_{\text{st}}$ whose image of the closed point must be z as $f^{-1}(f(z)) = \{z\}$, therefore the desired inverse $\widehat{\mathcal{O}}_{Z_{\text{st}},z} \rightarrow \widehat{\mathcal{O}}_{\text{Sh}_{\text{st}},f(z)}$ of the local homomorphism in the proposition. Q.E.D.

Therefore we have:

Proposition 4.8. (i) *The proper flat S -scheme Sh_{st} has generalized semistable reduction at the points in $Y_{\bar{s},\text{st}}$ of the special fiber, in the sense that for any $x \in Y_{\bar{s},\text{st}}$ there exists an etale morphism $U \rightarrow \text{Sh}_{\text{st}}$ with $x \in \text{Im } U$ and an etale S -morphism:*

$$\varphi_U : U \longrightarrow \text{Spec } W[T_1, \dots, T_n] / (T_1^{e_1} \cdots T_d^{e_d} - \varpi) \quad (d \leq n),$$

where integers e_i are all prime to $\text{char } k$.

(ii) *The canonical base change morphisms of nearby cycle sheaves $f^*(R^i \psi \Lambda|_{Y_{\bar{s},\text{st}}}) \rightarrow R^i \psi \Lambda|_{Y_{\mathbb{P},\text{st}}}$ (for $\Lambda = \overline{\mathbb{Q}}_\ell$) are isomorphisms for all i .*

Proof. (i) This follows from Lemma 4.7, using the fact that we can characterize the generalized semistable reduction by looking at the completion of the strict local rings. From the lack of immediate reference, we give a sketch of its proof. Assume that the complete local ring at a closed point x of the scheme X of finite type over S is isomorphic to $W[[T_1, \dots, T_n]]/(T_1^{e_1} \cdots T_d^{e_d} - \varpi)$ ($d \leq n$), where the integers e_i are all prime to $\text{char } k$. Then it is a regular scheme with its reduced special fiber being a normal crossing divisor with all the multiplicities prime to $\text{char } k$ (i.e. the generators t_1, \dots, t_d of the components passing through x are the part of a regular system of parameters). Therefore, some étale neighborhood of x in X is a regular scheme with its reduced special fiber being a normal crossing divisor with all the multiplicities prime to $\text{char } k$. Then by sending T_i to the local generators t_i of the divisors crossing at x we obtain an unramified map from an étale neighborhood $\text{Spec } A$ of x in X to $\text{Spec } W[[T_1, \dots, T_n]]$. By [EGA] IV, (18.4.7), we can decompose this morphism into an étale morphism $\text{Spec } B \rightarrow \text{Spec } W[[T_1, \dots, T_n]]$ and a closed immersion $\text{Spec } A \rightarrow \text{Spec } B$. Hence $A = B/I$ for an ideal I of B , and the inverse image of I in $W[[T_1, \dots, T_n]]$ must be of the form $(T_1^{e_1} \cdots T_d^{e_d} - u\varpi)$ with a unit u . This induces an étale map $\text{Spec } A \rightarrow \text{Spec } W[[T_1, \dots, T_n]]/(T_1^{e_1} \cdots T_d^{e_d} - u\varpi)$, and as all the e_i 's are prime to p , we can take $u = 1$ étale locally, hence we obtain the desired étale map.

(ii) This follows from the regular base change theorem for nearby cycle sheaves ([Fu], Cor. 7.1.6), as f is regular (because Sh_S is excellent and $X \rightarrow \text{Sh}_S$ is regular). (Or, using Lemma 4.7, the morphisms induced on the stalks are isomorphisms by the formal invariance theorem ([Fu], Cor. 7.1.7), i.e. the isomorphism $(R^i \psi \Lambda)_z \cong H^i(\text{Spec}(\widehat{\mathcal{O}}_z)_{\bar{\eta}}, \Lambda)$, where $\bar{\eta}$ denotes the geometric generic fiber.) Q.E.D.

Remark 4.9. We can show that the whole Sh_{st} has generalized semistable reduction in a similar way.

§5. A model Z_n containing the Deligne–Lusztig variety

5.1. Base change and normalization

Now we proceed to construct a model of X over a tamely ramified extension of W , namely $W_n := W(\varpi_n)$ where $\varpi_n := \varpi^{1/(q^n-1)}$, which includes a Deligne–Lusztig variety inside the special fiber. This is done simply by taking the base change of Z_1 over $S_n := \text{Spec } W_n$ and normalizing.

- Definition 5.1.** (i) Let Z_n be the normalization of $Z_1 \times_S S_n$.
 (ii) Let U_n and Y_n respectively be the inverse images of $Y_{\mathbb{P}}^{(n)}$ and $Y_{\mathbb{P}}$ under the morphism $Z_n \rightarrow Z_1$. The \bar{k} -scheme Y_n is a proper subvariety of the special fiber of Z_n , and U_n is an open subvariety of Y_n .

First we define an open affine subscheme $\text{Spec } C_1$ of $\text{Spec } B_1$ (the completion of an affine open of Z_1 along the exceptional divisor), which has $Y_{\mathbb{P}}^{(n)}$ as the special fiber. As $Y_{\mathbb{P}}^{(n)}$ is the complement in $Y_{\mathbb{P}}$ of the intersection with all the other irreducible components, namely $Y_{M,1}$ for all M , we only need to invert the equations which reduces mod \mathfrak{p} to the defining equations of $Y_{M,1}$. We start from the affine subscheme defined in the proof of Proposition 3.17, namely:

$$\text{Spec } B_1 = W[V_{\leq n-1}][[\tilde{X}_n]] / (\tilde{u} \cdot \tilde{X}_n^{q^n-1} \cdot \prod_{\underline{a} \in k^n \setminus \{0\}} P'_{\underline{a}} - \varpi),$$

with $\tilde{u} \in 1 + (\tilde{X}_n) \subset W[V_{\leq n-1}][[\tilde{X}_n]]^\times$. The exceptional divisor inside $\text{Spec}(B_1 \otimes_W \bar{k})$ is the affine space \mathbb{A}^{n-1} , the complement of the hyperplane $X_n^* = 0$ in \mathbb{P} . Now in order to remove all the k -rational hyperplanes $\sum_{i=0}^{n-1} \tilde{a}_i V_i + a_n = 0$ of \mathbb{A}^{n-1} from the special fiber, we invert the element $\prod_{\underline{a} \in k^n \setminus \{0\}} P'_{\underline{a}}$ and define:

$$C_1 := B_1 \left[\frac{1}{\prod P'_{\underline{a}}} \right] = W[V_{\leq n-1}][[\tilde{X}_n]] \left[\frac{1}{\prod P'_{\underline{a}}} \right] / (\tilde{u} \cdot \tilde{X}_n^{q^n-1} \cdot \prod P'_{\underline{a}} - \varpi),$$

because $Y_{M,1}$ for $M = M_{\underline{a}}$ was defined by $P'_{\underline{a}} = 0$.

Now we describe the equation of the variety U_n , which is achieved by normalizing the ring $C_1 \otimes_W W_n$, which is:

$$W_n[V_{\leq n-1}][[\tilde{X}_n]] \left[\frac{1}{\prod P'_{\underline{a}}} \right] / (\tilde{u} \cdot \tilde{X}_n^{q^n-1} \cdot \prod P'_{\underline{a}} - \varpi_n^{q^n-1}).$$

For this we adjoin the element $V_n := \varpi_n / \tilde{X}_n$ of its total quotient ring to this ring, as the equation:

$$\varpi_n^{q^n-1} = \tilde{u} \cdot \tilde{X}_n^{q^n-1} \cdot \prod P'_{\underline{a}}, \text{ i.e. } (\varpi_n / \tilde{X}_n)^{q^n-1} = \tilde{u} \cdot \prod P'_{\underline{a}}$$

shows that $V_n = \varpi_n / \tilde{X}_n$ is integral over $C_1 \otimes_W W_n$, and moreover it is a unit, as $V_n^{q^n-1} = \tilde{u} \cdot \prod P'_{\underline{a}}$ is inverted. By using the fact that inverting $\prod P'_{\underline{a}}$ is equivalent to inverting V_n , we compute the ring $C_1 \otimes_W W_n[V_n]$

as follows:

$$\begin{aligned} & W_n[V_{\leq n}] \left[\frac{1}{V_n} \right] [[\tilde{X}_n]] / (\varpi_n - V_n \tilde{X}_n, \tilde{u} \cdot \tilde{X}_n^{q^n-1} \prod P'_a - \varpi_n^{q^n-1}) \\ &= W_n[V_{\leq n}] \left[\frac{1}{V_n} \right] [[\tilde{X}_n]] / (\varpi_n - V_n \tilde{X}_n, \tilde{X}_n^{q^n-1}(\tilde{u} \cdot \prod P'_a - V_n^{q^n-1})). \end{aligned}$$

Now we claim the following:

Proposition 5.2. *Consider the spectrum $\text{Spec } C_n$ of the ring:*

$$C_n = W_n[V_{\leq n}] \left[\frac{1}{V_n} \right] [[\tilde{X}_n]] / (\varpi_n - V_n \tilde{X}_n, \tilde{u} \cdot \prod P'_a - V_n^{q^n-1}),$$

which is a closed subscheme of $\text{Spec}(C_1 \otimes_W W_n[V_n])$.

- (i) $\text{Spec } C_n$ is regular.
- (ii) $\text{Spec } C_n$ is the normalization of $C_1 \otimes_W W_n$.

Proof. (i) It can be seen by applying Jacobian computation directly, but here we prove it by showing that $\text{Spec } C_n$ is formally smooth over S_n . The generic fiber is apparently formally smooth as it is unchanged from that of $\text{Spec } B_1 \times_S S_n$ and $\text{Spec } B_1$ was the completion of an affine open of Z_1 , which in turn had the same generic fiber as X . The special fiber of $\text{Spec } C_n$ (equal to the locus of $\tilde{X}_n = 0$, as $V_n = \varpi_n/\tilde{X}_n$ is a unit), i.e. the spectrum of the ring:

$$C_n \otimes_{W_n} \bar{k} = \bar{k}[V_1, \dots, V_n] \left[\frac{1}{V_n} \right] / \left(\prod (P'_a \bmod \tilde{X}_n) - V_n^{q^n-1} \right)$$

(here we used $\tilde{u} \in 1 + (\tilde{X}_n)$), is a smooth affine variety, because we have

$$\prod (P'_a \bmod \tilde{X}_n) = \prod_{a \in k^n \setminus \{0\}} (a_1 V_1 + \dots + a_{n-1} V_{n-1} + a_n)$$

by (3.4). Now it remains to show that $\text{Spec } C_n$ is flat over S_n . Starting from the flat W_n -algebra $W_n[V_1, \dots, V_n][1/V_n][[\tilde{X}_n]]$, we will apply the following lemma twice:

Lemma 5.3. *Let (A, \mathfrak{m}) be a noetherian local ring, and let B be a noetherian flat A -algebra such that every maximal ideal of B lies over \mathfrak{m} . If $b \in B$ is $B/\mathfrak{m}B$ -regular (i.e. multiplication by b on $B/\mathfrak{m}B$ is injective), then $B/(b)$ is flat over A .*

Proof. It is a special case of [Mat], Th. 22.6.

Q.E.D.

Letting $A = W_n$, first for $B = W_n[V_1, \dots, V_n][1/V_n][[\tilde{X}_n]]$, we see that $B/\varpi_n B = \bar{k}[V_1, \dots, V_n][1/V_n][[\tilde{X}_n]]$ is an integral domain, where $\varpi_n - V_n \tilde{X}_n \in B$ does not reduce to zero, hence we have the flatness of $B/(\varpi_n - V_n \tilde{X}_n)$.

Next, letting $B = W_n[V_1, \dots, V_n][1/V_n][[\tilde{X}_n]] / (\varpi_n - V_n \tilde{X}_n)$, and seeing that

$$B/\varpi_n B = \bar{k}[V_1, \dots, V_n] \left[\frac{1}{V_n} \right] [[\tilde{X}_n]] / (V_n \tilde{X}_n) = \bar{k}[V_1, \dots, V_n] \left[\frac{1}{V_n} \right]$$

is again an integral domain where $\tilde{u} \cdot \prod P'_a - V_n^{q^n - 1} \in B$ does not reduce to zero, flatness of C_n over W_n follows as desired.

(ii) Clearly the generic point of C_n coincides with that of $C_1 \otimes_W W_n[V_n]$, i.e. that of $C_1 \otimes_W W_n$, and C_n is finite over $C_1 \otimes_W W_n$. Hence the assertion follows from (i). Q.E.D.

Now $\text{Spec}(C_1 \otimes_W W_n)$ is an affine open of $\text{Spec} B_1 \times_S S_n$, which is the completion along the exceptional divisor of an affine open of $Z_1 \times_S S_n$. Therefore $\text{Spec} C_n$ is an affine open of the completion along the exceptional divisor of the normalization Z_n . Moreover, as the special fiber of $\text{Spec} C_1$ is naturally isomorphic to $Y_{\mathbb{P}}^{(n)}$, which is contained in the exceptional divisor, the special fiber $\text{Spec}(C_n \otimes_{W_n} \bar{k})$ must be isomorphic to the inverse image of $Y_{\mathbb{P}}^{(n)}$ inside Z_n , namely U_n .

By changing the coordinates on U_n as follows:

$$X'_i := V_i/V_n \quad (1 \leq i \leq n-1), \quad X'_n := 1/V_n$$

we have the following obvious equality of rings:

$$R := \bar{k}[V_1, \dots, V_n] \left[\frac{1}{V_n} \right] = \bar{k}[X'_1, \dots, X'_n] \left[\frac{1}{X'_n} \right].$$

Hence we have the following description of $U_n \cong \text{Spec}(C_n \otimes_{W_n} \bar{k})$:

$$U_n = \text{Spec} \bar{k}[X'_1, \dots, X'_n] \left[\frac{1}{X'_n} \right] / I,$$

where the following equality of ideals is true in the ring R

$$\begin{aligned} I &:= \left(\prod_{\mathfrak{a} \in k^n \setminus \{0\}} (a_1 V_1 + \cdots + a_{n-1} V_{n-1} + a_n) - V_n^{q^n - 1} \right) \\ &= \left(\prod_{\mathfrak{a} \in k^n \setminus \{0\}} \left(a_1 \frac{X'_1}{X'_n} + \cdots + a_{n-1} \frac{X'_{n-1}}{X'_n} + a_n \right) - \left(\frac{1}{X'_n} \right)^{q^n - 1} \right) \\ &= \left(\prod_{\mathfrak{a} \in k^n \setminus \{0\}} (a_1 X'_1 + \cdots + a_n X'_n) - 1 \right). \end{aligned}$$

Note that, by unwinding the definitions, we see that the coordinates X'_1, \dots, X'_n are related to the original $\tilde{X}_1, \dots, \tilde{X}_n$ through $X'_i = \tilde{X}_i / \varpi_n$ for each $1 \leq i \leq n$.

Proposition 5.4. *The equation of U_n is given by:*

$$U_n = \text{Spec } \bar{k}[X'_1, \dots, X'_n] / \left(\prod_{\mathfrak{a} \in k^n \setminus \{0\}} (a_1 X'_1 + \cdots + a_n X'_n) - 1 \right)$$

which is a smooth affine variety over \bar{k} .

5.2. Action of $GL_n(k)$ and the inertia group

Summing up, we have following diagram of schemes:

$$\begin{array}{ccccccc} U_n & \longrightarrow & Y_n & \longrightarrow & Z_n & \longrightarrow & S_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_{\mathbb{P}}^{(n)} & \longrightarrow & Y_{\mathbb{P}} & \longrightarrow & Z_1 & \longrightarrow & S. \end{array}$$

Here all the vertical maps are surjective. The W -schemes Z_1 and Z_n are normal models of X and $X \times_S S_n$ respectively. The \bar{k} -schemes $Y_{\mathbb{P}}$ and Y_n are the unique proper components (with multiplicity $q^n - 1, 1$ respectively) of their special fibers, and $Y_{\mathbb{P}}^{(n)}$ and U_n are open affine subvarieties of them. Also:

- (i) The right action of $GL_n(k)$ on X extend to all the schemes in the above diagrams, and all maps are $GL_n(k)$ -equivariant.
- (ii) The special fiber of Z_n , and therefore also Y_n and U_n , has the geometric inertia action of I_K , which factors through the finite quotient:

$$I_K \rightarrow \text{Gal}(\text{Frac } W_n / \text{Frac } W) \cong k_n^\times.$$

Now we describe these actions on U_n explicitly by the coordinates introduced above. We denote the image of X'_i in the coordinate ring of U_n by $\overline{X'_i}$.

Proposition 5.5. (i) *The group $GL_n(k)$ acts on U_n via right linear action on the row vector $(\overline{X'_1}, \dots, \overline{X'_n})$.*

(ii) *The covering $f : U_n \rightarrow Y_{\mathbb{P}}^{(n)}$ induced from the finite map $Z_n \rightarrow Z_1$ is a finite Galois étale covering with the automorphism group $\text{Gal}(\text{Frac } W_n / \text{Frac } W)$, which is canonically isomorphic to $\mu_{q^n-1} \cong k_n^\times$. Its action on the coordinate ring is described by $\overline{X'_i} \mapsto \zeta^{-1} \overline{X'_i}$ for all $1 \leq i \leq n$, where $\zeta \in \mu_{q^n-1}$ denotes the image under the above canonical isomorphism.*

Proof. (i) This is easily seen by substituting $\tilde{X}_j = \varpi_n X'_j$ to the original action (3.2) and using Proposition 3.4(i) to see that the action of $(a_{ij}) \in GL_n(k) \bmod \varpi_n^2$ is

$$\varpi_n \overline{X'_j} \mapsto a_{1j}(\varpi_n \overline{X'_1}) + \dots + a_{nj}(\varpi_n \overline{X'_n})$$

and dividing both sides by ϖ_n .

(ii) This is also clear by the relation $X'_i = \tilde{X}_i / \varpi_n$ for each $1 \leq i \leq n$, and the inertia action on S_n being defined by $\varpi_n \mapsto \zeta \varpi_n$ for $\zeta \in \mu_{q^n-1} \cong \text{Gal}(\text{Frac } W_n / \text{Frac } W)$. Q.E.D.

We will observe in the next chapter that the finite étale covering $U_n \rightarrow Y_{\mathbb{P}}^{(n)}$ of smooth affine varieties over \bar{k} , with the right action of $GL_n(k) \times I_K$, is isomorphic to the Deligne–Lusztig variety for $GL_n(k)$ and its maximally non-split torus T with $T(k) \cong k_n^\times$, defined in [DL], Sections 2.1 and 2.2.

§6. Vanishing cycles

6.1. Review on vanishing cycle cohomology

Here we recall the method of calculating vanishing cycle cohomology of strict local rings over strict henselian discrete valuation ring via the special fiber of a model. A similar argument can be found in [Br].

In this subsection, let K be a strict henselian discrete valuation field, let \mathcal{O} be its ring of integers and k be its residue field with $\text{char } k = p > 0$. We denote the generic point and the closed point of $S := \text{Spec } \mathcal{O}$ by $\eta := \text{Spec } K$ and $s := \text{Spec } k$, and the geometric generic point by $\bar{\eta} = \text{Spec } \bar{K}$, where \bar{K} is a separable closure of K . We let $\bar{S} := \text{Spec } \mathcal{O}_{\bar{K}}$, where $\mathcal{O}_{\bar{K}}$ is the integral closure of \mathcal{O} in \bar{K} , which is a non-discrete valuation ring. The generic point and the closed point of \bar{S} are respectively $\bar{\eta}$ and s .

We consider an n -dimensional strict henselian local \mathcal{O} -algebra A with local structure homomorphism $\mathcal{O} \rightarrow A$, and let $X := \text{Spec } A$. We denote the base change of X/S over $s, \eta, \bar{\eta}$, or \bar{S} respectively by $X_s, X_\eta, X_{\bar{\eta}}$, or \bar{X} . The above schemes fit into the following diagrams:

$$\begin{array}{ccc}
 s & \longrightarrow & \bar{S} \longleftarrow \bar{\eta} \\
 \parallel & & \downarrow \qquad \downarrow \\
 s & \longrightarrow & S \longleftarrow \eta
 \end{array}
 \qquad
 \begin{array}{ccccc}
 X_s & \xrightarrow{\bar{i}} & \bar{X} & \xleftarrow{\bar{j}} & X_{\bar{\eta}} \\
 \parallel & & \downarrow & & \downarrow \\
 X_s & \xrightarrow{i} & X & \xleftarrow{j} & X_\eta
 \end{array}$$

We are interested in the ℓ -adic étale cohomology groups $H^*(X_{\bar{\eta}}, \overline{\mathbb{Q}}_\ell)$ of the $(n - 1)$ -dimensional affine scheme $X_{\bar{\eta}}$ over $\bar{\eta} = \text{Spec } \bar{K}$, where ℓ is a prime not equal to p . Hereafter we denote the constant ℓ -adic sheaf $\overline{\mathbb{Q}}_\ell$ simply by Λ . By the Leray spectral sequence for \bar{j} , we have canonical isomorphisms:

$$H^i(X_{\bar{\eta}}, \Lambda) \cong \mathbb{H}^i(\bar{X}, R\bar{j}_* \Lambda) \cong (R^i \bar{j}_* \Lambda)_x,$$

where \mathbb{H} denotes the hypercohomology and $R\bar{j}_* \Lambda$ is the object in the derived category of ℓ -adic sheaves on \bar{X} , and x denotes the unique closed point of X and \bar{X} . We denote by $(R^i \bar{j}_* \Lambda)_x$ the stalk at x of the ℓ -adic sheaf $R^i \bar{j}_* \Lambda$ over \bar{X} . Constructibility of the ℓ -adic sheaf $R\bar{j}_* \Lambda$ (from which follows that $H^i(X_{\bar{\eta}}, \Lambda)$ has finite rank) follows from [SGA4.1/2], [Finitude], under the hypothesis that X is the strict local ring of a scheme locally of finite type over S .

We can try to compute these cohomology groups by constructing good models of X , obtained by repeated blow-ups and normalizations over ramified extensions of K .

Let K' be a finite extension of K and $S' := \text{Spec } \mathcal{O}'$ be the normalization of S in K' . Let $X' := X \times_S S'$, $\eta' := \text{Spec } K'$, $X'_{\eta'} := X' \times_{S'} \eta'$, $j' : X'_{\eta'} \rightarrow X'$ and let x denote also the closed point of X' . Suppose we have a proper morphism $p : Z \rightarrow X'$ over S' which is an isomorphism on the generic fibers (i.e. $p|_{Z_{\eta'}} = \text{id}$ where $Z_{\eta'} := Z \times_{S'} \eta'$), and denote the inverse image of x by $Y := p^{-1}(x)$. Let i_Z, j_Z be the inclusions $Y \rightarrow Z$ and $X'_{\eta'} \rightarrow Z$ respectively, and we denote the objects obtained from p, Z, i_Z, j_Z by the base change under $\bar{S} \rightarrow S'$ respectively by $\bar{p}, \bar{Z}, \bar{i}_Z, \bar{j}_Z$. Note that $Y = \bar{p}^{-1}(x)$ is a proper subscheme of the special fiber $Z_s := Z \times_S s$ of Z . We can describe the situation by the

following diagrams:

$$\begin{array}{ccc}
 Y & \xrightarrow{i_Z} & Z & \xleftarrow{j_Z} & X'_{\eta'} \\
 \downarrow & & \downarrow p & & \parallel \\
 x & \longrightarrow & X' & \xleftarrow{j'} & X'_{\eta'}
 \end{array}
 \qquad
 \begin{array}{ccc}
 Y & \xrightarrow{\bar{i}_Z} & \bar{Z} & \xleftarrow{\bar{j}_Z} & X_{\bar{\eta}} \\
 \downarrow & & \downarrow \bar{p} & & \parallel \\
 x & \longrightarrow & \bar{X} & \xleftarrow{\bar{j}} & X_{\bar{\eta}}.
 \end{array}$$

Then, if we denote by $R\psi\Lambda$ the nearby cycle sheaves $\bar{i}_Z^* R\bar{j}_{Z*}\Lambda$ in the derived category of ℓ -adic sheaves on Y , the proper base change theorem ([SGA4] Exposé XIII) gives the following canonical isomorphisms:

$$(R\bar{j}_*\Lambda)_x \cong (R\bar{p}_* R\bar{j}_{Z*}\Lambda)_x \cong R\Gamma(Y, R\psi\Lambda).$$

Therefore we have:

Proposition 6.1. *We have a canonical isomorphism $H^i(X_{\bar{\eta}}, \Lambda) \cong \mathbb{H}^i(Y, R\psi\Lambda)$. (As it is canonical, if a group G acts on X and $Z \rightarrow X$ is G -equivariant, then this isomorphism is $G \times I_K$ -equivariant. This holds for similar isomorphisms in what follows.)*

We will also need the following consequence of the proper base change theorem:

Proposition 6.2. *Let $K'_1 \subset K'_2$ be successive finite extensions of K , and suppose we have the above situation over each field:*

$$\begin{array}{ccc}
 Y_1 & \xrightarrow{i_{Z_1}} & Z_1 & \xleftarrow{j_{Z_1}} & X'_{\eta'_1} \\
 \downarrow & & \downarrow p_1 & & \parallel \\
 x & \longrightarrow & X'_1 & \xleftarrow{j'_1} & X'_{\eta'_1}
 \end{array}
 \qquad
 \begin{array}{ccc}
 Y_2 & \xrightarrow{i_{Z_2}} & Z_2 & \xleftarrow{j_{Z_2}} & X'_{\eta'_2} \\
 \downarrow & & \downarrow p_2 & & \parallel \\
 x & \longrightarrow & X'_2 & \xleftarrow{j'_2} & X'_{\eta'_2}
 \end{array}$$

with a proper morphism $f : Z_2 \rightarrow Z_1$ which induces the isomorphism $X'_{\eta'_1} \times \eta'_2 \cong X'_{\eta'_2}$. If we denote the nearby cycle sheaves on Y_1, Y_2 respectively by $R\psi_1\Lambda, R\psi_2\Lambda$, we have a canonical isomorphism $R\psi_1\Lambda \cong Rf_* R\psi_2\Lambda$. In particular, if f is finite, we have $R^i\psi_1\Lambda \cong f_* R^i\psi_2\Lambda$ for each i .

Lastly, in order to apply the results from Section 4 to compute the vanishing cycle cohomology, we need the results of T. Saito on the nearby cycle sheaves on generalized semistable schemes, following [Sa] (see also [RZ]).

Let X be a scheme locally of finite type over the strict henselian trait $S = \text{Spec } \mathcal{O}$. We denote the inclusions $X_s \rightarrow X$ and $X_{\eta} \rightarrow X$

respectively by i and j , and denote by $\overline{X}, X_{\overline{\eta}}, \overline{i}, \overline{j}$ the objects obtained by the base change under $\overline{S} \rightarrow S$ of the corresponding objects. Consider the nearby cycle sheaves $R^q\psi\Lambda := \overline{i}^* R^q\overline{j}_*\Lambda$, which are constructible ℓ -adic sheaves on X_s with the action of the inertia group $I_K = \text{Gal}(\overline{K}/K)$ of K .

Assume that X has generalized semistable reduction in the sense of Proposition 4.8(i). Let $\{Y_i\}_{i \in I}$ be the irreducible components of the special fiber X_s , and for any finite subset $J \subset I$, we denote $Y_J := \bigcap_{i \in J} Y_i$ and $Y_J^0 := Y_J \setminus \bigcup_{i \notin J} Y_i$. Let e_i denote the multiplicity of Y_i in X_s for $i \in I$. By Proposition 6 of [Sa], we have:

Proposition 6.3. *Let $d = d_J$ be the greatest common divisor of $\{e_i\}_{i \in J}$, and K_d be the unique tamely ramified extension over K of degree d . Then we have the following canonical isomorphisms as constructible ℓ -adic sheaves with I_K -action:*

$$R^0\psi\Lambda|_{Y_J^0} \cong \text{Ind}_{I_{K_d}}^{I_K} \Lambda \quad (\text{etale locally}),$$

$$R^q\psi\Lambda|_{Y_J^0} \cong R^0\psi\Lambda|_{Y_J^0} \otimes_{\mathbb{Z}_\ell} \bigwedge^q \text{Hom}\left(\text{Ker}\left(\bigoplus_{i \in J} \mathbb{Z}_\ell \rightarrow \mathbb{Z}_\ell\right)(1), \Lambda\right),$$

where the map $\bigoplus_{i \in J} \mathbb{Z}_\ell \rightarrow \mathbb{Z}_\ell$ is defined by $1_i \mapsto e_i$, and (1) denotes the Tate twist.

Corollary 6.4. (i) *Each $R^q\psi\Lambda$ is locally constant on Y_J^0 of rank equal to $d \cdot \binom{|J|-1}{q}$, where $\binom{|J|-1}{q} = 0$ for $q > |J| - 1$ by convention. Moreover, the inertia group I_K acts on any $R^q\psi\Lambda$ through the finite cyclic Galois group $I_K/I_{K_d} \cong \mu_d$.*

(ii) (Cor. 1 to Prop. 6 of [Sa]) *On Y_J^0 with $|J| \neq 1$, in the Grothendieck group of smooth ℓ -adic sheaves with I_K -action, the following alternating sum vanishes:*

$$[R\psi\Lambda|_{Y_J^0}] = \sum_q (-1)^q [R^q\psi\Lambda|_{Y_J^0}] = 0.$$

6.2. Review of the Deligne–Lusztig theory

Here we recall some results of the Deligne–Lusztig theory [DL] that are relevant to our case (see also [Se]).

Let G be a connected reductive group defined over a finite field k , and $F : G \rightarrow G$ be the Frobenius morphism. For an F -stable maximal torus T and a Borel subgroup containing T , Deligne and Lusztig construct a G^F -equivariant finite etale Galois covering of smooth varieties over \overline{k} with right G^F -actions:

$$f : \widetilde{X}_{TCB} \rightarrow X_{TCB},$$

with Galois group T^F . By decomposing the ℓ -adic sheaf as $f_*\overline{\mathbb{Q}}_\ell \cong \bigoplus_\theta \mathcal{F}_\theta$, where θ runs through the characters of T^F , we define a virtual G_F -representation

$$R_T^\theta := \sum_i (-1)^i H_c^i(X_{TCB}, \mathcal{F}_\theta),$$

which turns out to be independent of the choice of B , and moreover depends only on the G^F -conjugacy class of T and on the orbit of θ under $(N(T)/T)^F$ where $N(T)$ is the normalizer of T . Note that as \tilde{X}_{TCB} has right $G^F \times T^F$ -action, $H_c^i(\tilde{X}_{TCB}, \Lambda)$ is a left $G^F \times T^F$ -module, and $H_c^i(X_{TCB}, \mathcal{F}_\theta) = H_c^i(\tilde{X}_{TCB}, \Lambda)(\theta)$, where we denote by $V(\theta)$ the maximal direct summand of V on which T^F acts by θ .

- Definition 6.5.** (i) Let T, T' be two F -stable maximal tori of G , and θ, θ' be characters of T^F, T'^F , respectively. The pairs $(T, \theta), (T', \theta')$ are said to be *geometrically conjugate* if the pairs $(T, \theta \circ N), (T', \theta' \circ N)$, where N is the norm from T^{F^n} to T^F (resp. T'^{F^n} to T'^F), are G^{F^n} -conjugate for some integer n . Here the *norm* N for T is the map $\prod_{i=0}^{n-1} F^i : T^{F^n} \rightarrow T^F$.
- (ii) The character θ of T^F is said to be in *general position* if it is not fixed by any non-trivial element of $(N(T)/T)^F$.

Here we summarize some of the main theorems in this theory:

- Theorem 6.6.** (i) *Every irreducible representation π of G^F occurs in some R_T^θ , i.e. $\langle \pi, R_T^\theta \rangle \neq 0$ where \langle, \rangle is the natural inner product on the Grothendieck group of representations of G^F ([DL], Cor. 7.7).*
- (ii) *If (T, θ) and (T', θ') are not geometrically conjugate, no irreducible representation of G^F occurs in both R_T^θ and $R_{T'}^{\theta'}$, ([DL], Cor. 6.3).*
- (iii) *If we put, for two F -stable maximal tori T, T' ,*

$$N(T, T') := \{g \in G \mid Tg = gT'\}, \quad W(T, T')^F := T^F \setminus N(T, T')^F$$

then we have ([DL], Th. 6.8):

$$\langle R_T^\theta, R_{T'}^{\theta'} \rangle = |\{w \in W(T, T')^F \mid \theta w = w\theta'\}|.$$

Therefore, if (T, θ) is in general position, one of $\pm R_T^\theta$ (namely $(-1)^{\sigma(G) - \sigma(T)} R_T^\theta$ by [DL], Prop. 7.3) is an irreducible representation of G^F . If moreover T is not contained in any F -stable proper parabolic subgroup of G , then $(-1)^{\sigma(G) - \sigma(T)} R_T^\theta$ is a cuspidal representation.

- (iv) The \bar{k} -variety X_{TCB} is affine as long as q is larger than the Coxeter number of G . In this case, if θ is in general position then we have $H_c^i(X_{TCB}, \mathcal{F}_\theta) = 0$ for $i \neq l(w)$, where $l(w)$ is the length of the element $w \in W$ of the Weyl group W such that $(B, F(B))$ is in the relative position w (i.e. it is in the G -orbit of $(B, \tilde{w}B\tilde{w}^{-1})$ for a representative $\tilde{w} \in N(T)$ of w).

Now we specialize to the case $G = GL_n$, and a torus T associated to the element w of the Weyl group corresponding to the cyclic permutation $(1, \dots, n)$ in the symmetric group of n letters, i.e. a torus T such that $T^F \cong k_n^\times$.

Definition 6.7. We denote by DL the Deligne–Lusztig variety \tilde{X}_{TCB} for $G = GL_n, T^F = k_n^\times$ and $w = (1, \dots, n)$. If V is a finite dimensional representation of $GL_n(k) \times k_n^\times$ and θ is a character of k_n^\times , we denote by V_{cusp} (resp. $V(\theta)$) the maximal direct summand of V on which $GL_n(k)$ acts by cuspidal representations (resp. on which T^F acts by θ , often considered simply as a $GL_n(k)$ -representation). A character θ of k_n^\times is in *general position* if and only if it does not factor through the norm map $k_n^\times \rightarrow k_m^\times$ for any $m \mid n$ with $m \neq n$. We denote by C the set of all characters of k_n^\times in general position.

Proposition 6.8. (i) ([DL], Prop. 7.3, Th. 8.3, and Cor. 9.9)
 If $\theta \in C$, then:

$$H_c^i(DL, \Lambda)(\theta) \cong \begin{cases} \pi_\theta \otimes \theta & (i = n - 1), \\ 0 & (i \neq n - 1). \end{cases}$$

Here π_θ is an irreducible cuspidal representation of $GL_n(k)$ which is characterized by $\pi_\theta \otimes \text{St} = \text{Ind}_{T(k)}^{GL_n(k)} \theta$, where St is the Steinberg representation of $GL_n(k)$. All cuspidal representations of $GL_n(k)$ arise in this way.

- (ii) If $\theta \notin C$, then no cuspidal representation of $GL_n(k)$ occur in the cohomology groups $H_c^i(DL, \Lambda)(\theta)$ for any i .

Here, (ii) follows from the slightly stronger version of the disjointness theorem (above Theorem 6.6(ii)), which is stated only in terms of the alternating sums in [DL]. It can be shown that the disjointness holds between each cohomology groups, by replacing the alternating sums in the proof of Th. 6.2 of [DL] by each of the cohomology groups.

Corollary 6.9. As $GL_n(k) \times k_n^\times$ -representations, we have:

$$H_c^i(DL, \Lambda)_{\text{cusp}} \cong \begin{cases} \bigoplus_{\theta \in C} \pi_\theta \otimes \theta & (i = n - 1), \\ 0 & (i \neq n - 1). \end{cases}$$

6.3. Computing the vanishing cycles

Now we return to our original space $X = \text{Spec } A$ and look at the cohomology of $X_{\bar{\eta}}$, where $\eta = \text{Spec } \widehat{K}^{\text{ur}}$ is the generic point of $\text{Spec } W$. We analyze the cohomology groups $H^i(X_{\bar{\eta}}, \Lambda)$, which are finite dimensional representations of $GL_n(k) \times I_K$.

6.3.1. *Using the first blow-up*

We start with the computation of cohomology groups using the model Z_1 of X introduced in Section 3. By Proposition 6.1, we have

$$H^i(X_{\bar{\eta}}, \Lambda) \cong \mathbb{H}^i(Y_{\mathbb{P}}, R\psi\Lambda)$$

for all i , as $GL_n(k) \times I_K$ -representations. Here we show that the cuspidal representations of $GL_n(k)$ only live in the cohomology of nearby cycle sheaves on the open subscheme $Y_{\mathbb{P}}^{(n)}$ of $Y_{\mathbb{P}}$. (Recall from the end of Section 3.4 the stratification $Y_{\mathbb{P}}^{(h)} \subset Y_{\mathbb{P}} \cong \mathbb{P}$ for $1 \leq h \leq n$.)

For a k -rational linear subspace $N \subset \mathbb{P}$ of dimension $h - 1$ and $g \in GL_n(k)$, the right $GL_n(k)$ -action on $Y_{\mathbb{P}}$ induces $N^0 \cong (Ng)^0$ and $g^*R\psi\Lambda|_{(Ng)^0} \cong R\psi\Lambda|_{N^0}$. Therefore by a standard argument (see [Bo1] Lemme 13.2, [DL] Prop. 8.2 or [HT], pp.115–117), we have the following:

Proposition 6.10. *For a k -rational linear subspace $N \subset \mathbb{P}$ of dimension $h - 1$, with $1 \leq h \leq n$, let P_N be the stabilizer of N , a parabolic subgroup of $GL_n(k)$. Then we have for every i, j :*

$$H_c^i(Y_{\mathbb{P}}^{(h)}, R^j\psi\Lambda) \cong \text{Ind}_{P_N}^{GL_n(k)} H_c^i(N^0, R^j\psi\Lambda).$$

Corollary 6.11. *We have $H^i(X_{\bar{\eta}}, \Lambda)_{\text{cusp}} \cong \mathbb{H}_c^i(Y_{\mathbb{P}}^{(n)}, R\psi\Lambda)_{\text{cusp}}$ for all i .*

Proof. The unipotent radical U_N of P_N acts trivially on N^0 , hence also on $H_c^i(N^0, R^j\psi\Lambda)$ by Proposition 6.3. Therefore Proposition 6.10 shows that $H_c^i(Y_{\mathbb{P}}^{(h)}, R^j\psi\Lambda)$ is parabolically induced from a representation of a Levi subgroup of P_N , i.e. $H_c^i(Y_{\mathbb{P}}^{(h)}, R^j\psi\Lambda)_{\text{cusp}} = 0$ for all i, j and $h \neq n$. By using the long exact sequences:

$$\begin{aligned} \dots \longrightarrow H_c^i(Y_{\mathbb{P}}^{(h)}, R^j\psi\Lambda) \longrightarrow H^i(Y_{\mathbb{P}}^{[h]}, R^j\psi\Lambda) \longrightarrow \\ H^i(Y_{\mathbb{P}}^{[h-1]}, R^j\psi\Lambda|_{Y_{\mathbb{P}}^{h+1}}) \longrightarrow \dots \end{aligned}$$

recursively for $1 \leq h \leq n - 1$, we have $H^i(Y_{\mathbb{P}}^{[n-1]}, R^j\psi\Lambda)_{\text{cusp}} = 0$ for any j , hence

$$H^i(X_{\bar{\eta}}, \Lambda)_{\text{cusp}} \cong \mathbb{H}^i(Y_{\mathbb{P}}, R\psi\Lambda)_{\text{cusp}} \cong \mathbb{H}_c^i(Y_{\mathbb{P}}^{(n)}, R\psi\Lambda)_{\text{cusp}}.$$

Q.E.D.

6.3.2. *Using the generalized semistable model*

Now we make use of the generalized semistable model Z_{st} constructed in Section 4. By Proposition 6.1, we have

$$H^i(X_{\bar{\eta}}, \Lambda) \cong \mathbb{H}^i(Y_{\mathbb{P}, \text{st}}, R\psi\Lambda)$$

for all i . We will appeal to Saito’s results in Section 6.1 through Proposition 4.8.

Definition 6.12. Let \mathcal{G} be the Grothendieck group of finite dimensional vector spaces over $\Lambda = \overline{\mathbb{Q}}_\ell$ with left action of $GL_n(k) \times I_K$. We regard the characters of k_n^\times as characters of I_K , by composing the canonical surjection $I_K \rightarrow k_n^\times$.

Definition 6.13. We denote the alternating sum of the cohomology groups, regarded as elements of \mathcal{G} , as follows:

$$H^*(X_{\bar{\eta}}) := \sum_i (-1)^i [H^i(X_{\bar{\eta}}, \Lambda)],$$

$$H^*(Y_{\mathbb{P}}^{(n)}, R^j\psi\Lambda) := \sum_i (-1)^i [H_c^i(Y_{\mathbb{P}}^{(n)}, R^j\psi\Lambda)].$$

Proposition 6.14. (i) $H^*(X_{\bar{\eta}}) = H^*(Y_{\mathbb{P}}^{(n)}, R^0\psi\Lambda)$.
 (ii) For each degree i and $\chi \in C$, we have:

$$H^i(X_{\bar{\eta}}, \Lambda)_{\text{cusp}} \cong H_c^i(Y_{\mathbb{P}}^{(n)}, R^0\psi\Lambda)_{\text{cusp}},$$

$$H^i(X_{\bar{\eta}}, \Lambda)(\chi) \cong H_c^i(Y_{\mathbb{P}}^{(n)}, R^0\psi\Lambda)(\chi).$$

Proof. Note that $R^j\psi\Lambda|_{Y_{\mathbb{P}}^{(n)}} = 0$ for $j > 0$ by Corollary 6.4(i), hence for all i :

$$(6.1) \quad \mathbb{H}_c^i(Y_{\mathbb{P}}^{(n)}, R\psi\Lambda) \cong H_c^i(Y_{\mathbb{P}}^{(n)}, R^0\psi\Lambda).$$

(i) By Proposition 4.8, we can calculate $\mathbb{H}^i(Y_{\mathbb{P}, \text{st}}, R\psi\Lambda)$ on $Y_{\bar{s}, \text{st}} \subset \text{Sh}_{\text{st}}$, where we apply Corollary 6.4(ii) to see that

$$\sum_i (-1)^j [R^j\psi\Lambda|_{Y_{\bar{s}, \text{st}} \setminus Y_{\mathbb{P}}^{(n)}}] = 0,$$

thus:

$$H^*(X_{\bar{\eta}}) = \sum_i (-1)^i [\mathbb{H}^i(Y_{\mathbb{P}, \text{st}}, R\psi\Lambda)]$$

$$= \sum_i (-1)^i [\mathbb{H}_c^i(Y_{\mathbb{P}}^{(n)}, R\psi\Lambda)] = H^*(Y_{\mathbb{P}}^{(n)}, R^0\psi\Lambda).$$

(ii) The first equality follows from Corollary 6.11 and (6.1). The second is proven in an exactly similar way as in Corollary 6.11, except that we use Corollary 6.4(i), instead of Proposition 6.10, to see that $H_c^i(Y_{\bar{s},st} \setminus Y_{\mathbb{P}}^{(n)}, R^j\psi\Lambda)(\chi) = 0$ for all i, j (here note that multiplicities of the components intersecting $Y_{\mathbb{P}}$ are of the form $q^m - 1$ with $m < n$, hence outside $Y_{\mathbb{P}}^{(n)}$ the d in Corollary 6.4(i) is strictly less than $q^n - 1$ and inertia cannot act by $\chi \in C$). Use (6.1) to conclude the proof. Q.E.D.

6.3.3. Using the model Z_n

As we have seen in Proposition 6.14, the part of $H^i(X_{\bar{\eta}}, \Lambda)$ we are interested in comes from the cohomology $H_c^i(Y_{\mathbb{P}}^{(n)}, R\psi\Lambda)$ of nearby cycle sheaves on $Y_{\mathbb{P}}^{(n)}$ (it does not matter whether we regard $Y_{\mathbb{P}}^{(n)}$ as a subvariety of Z_1 or of Z_{st} , in view of Proposition 6.2). Because $R^j\psi\Lambda|_{Y_{\mathbb{P}}^{(n)}} = 0$ for $j > 0$ by Corollary 6.4(i) (or by Proposition 6.14), we can concentrate on $H_c^i(Y_{\mathbb{P}}^{(n)}, R^0\psi\Lambda)$.

We compute this using the normalization U_n and the finite étale covering $f : U_n \rightarrow Y_{\mathbb{P}}^{(n)}$. By Proposition 6.2 we see that $R^i\psi\Lambda|_{Y_{\mathbb{P}}^{(n)}} \cong f_*R^i\psi\Lambda|_{U_n}$ for each i , and as U_n is the special fiber of the formally smooth S_n -scheme $\text{Spec } C_n$, we have

$$R^i\psi\Lambda|_{U_n} \cong \begin{cases} \Lambda & (i = 0) \\ 0 & (i > 0) \end{cases}, \quad R^i\psi\Lambda|_{Y_{\mathbb{P}}^{(n)}} \cong \begin{cases} f_*\Lambda & (i = 0) \\ 0 & (i > 0) \end{cases}.$$

Therefore we have a canonical $GL_n(k) \times I_K$ -equivariant isomorphism

$$(6.2) \quad H_c^i(Y_{\mathbb{P}}^{(n)}, R^0\psi\Lambda) \cong H_c^i(U_n, \Lambda).$$

Now we connect our result in Chapter 5 to the Deligne–Lusztig theory by comparing their explicit equations, which proves the third part of Theorem 1.2:

Proposition 6.15. *There is a $GL_n(k) \times k_n^\times$ -equivariant isomorphism $U_n \cong DL$ of varieties over \bar{k} , where DL is the Deligne–Lusztig variety defined in Section 6.2.*

Proof. This is readily seen by comparing our equation in Proposition 5.4 and the $GL_n(k) \times k_n^\times$ -action in Proposition 5.5 with the equation and group actions of DL given in Section 2.2 of [DL]. Q.E.D.

Therefore we can invoke the Deligne–Lusztig theory to derive our main theorem. Let us denote the alternating sum of cohomology groups of DL as:

$$H_c^*(DL) := \sum_i (-1)^i [H_c^i(DL, \Lambda)] = \sum_{\theta} R_T^\theta \otimes [\theta] \in \mathcal{G}.$$

For a character $\chi \in C$ of $T^F \cong k_n^\times$ in general position, considered as a character of I_K , recall the corresponding cuspidal representation π_χ , defined in Proposition 6.8.

Theorem 6.16. (i) $H^*(X_{\bar{\eta}}) = H_c^*(DL)$.

(ii) For each degree i and $\chi \in C$, we have:

$$H^i(X_{\bar{\eta}}, \Lambda)_{\text{cusp}} \cong \begin{cases} \bigoplus_{\chi \in C} \pi_\chi \otimes \chi & (i = n - 1), \\ 0 & (i \neq n - 1), \end{cases}$$

$$H^i(X_{\bar{\eta}}, \Lambda)(\chi) \cong \begin{cases} \pi_\chi \otimes \chi & (i = n - 1), \\ 0 & (i \neq n - 1). \end{cases}$$

Proof. We combine the isomorphism (6.2) and Proposition 6.15 with the Proposition 6.14, and use the Deligne–Lusztig theory (Proposition 6.8 and Corollary 6.9) for (ii). Q.E.D.

Note added in proof: In the proof of Proposition 6.10, we need a little more argument to prove that U_N acts trivially on $R^j\psi\Lambda$. We use Z_{st} to apply Proposition 6.3. Although the first isomorphism of Proposition 6.3 holds only étale locally, it shows that the canonical morphism $\Lambda \rightarrow R^0\psi\Lambda|_{Y_J^0}$ is an isomorphism if $d = 1$. For general d , by Proposition 6.2, the sheaf $R^0\psi\Lambda$ is a push forward from a $d = 1$ situation, namely the normalization of the base change to tamely ramified extension of W of degree d (similar to what is done in §5.1), which restricts to a finite étale covering of degree d on Y_J^0 . Thus U_N , being a p -group, acts trivially on $R^0\psi\Lambda|_{Y_J^0}$. In the second isomorphism of Proposition 6.3, the group U_N can only act on the index set J , but J is a partial flag of linear subspaces of \mathbb{P} containing N , and U_N fixes each element of J .

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