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A note on Fano surfaces of nodal cubic threefolds

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Abstract.

We study the Picard variety of the Fano surface of nodal and mildly cuspidal cubic threefolds in arbitrary characteristic by relating divisors on the Fano surface to divisors on the symmetric product of a curve of genus 4.

§1. Introduction

Cubic threefolds have been studied extensively, first by the classical geometers starting with Fano [13] and later by Clemens and Griffiths [6] and many others. Nodal cubic threefolds were already considered by Clemens and Griffiths. In their paper the intermediate Jacobian plays a central role. In this note we come back to these nodal threefolds, but we let the Picard variety of the Fano surface of lines on the cubic threefolds replace the intermediate Jacobian. This has the advantage that it works in all characteristics, including characteristic 2. We relate divisors on the Fano surface directly to curves on the symmetric product of a non-hyperelliptic curve of genus 4.

We begin by showing that for a canonically embedded non-hyperelliptic curve $C \subset \mathbb{P}^3$ of genus 4 the linear system of cubics passing through C maps $\mathbb{P}^3$ to a cubic threefold with a node or a mild cusp. The Fano surface of lines on such a cubic threefold is a non-normal surface whose normalization equals $\mathrm{Sym}^2 C$. We analyze for linear systems on a curve C the associated trace divisors on $\mathrm{Sym}^2 C$ and their intersection theory and apply this to divisors on the Fano surface. We analyze the Picard variety of the Fano surface. As an application we get a variation of the proof by Collino that the general cubic threefold is not rational and this variation works also in characteristic 2. We also analyze the standard compactification of the Picard variety of the Fano

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surface, compare the Clemens–Griffiths map $S \to \mathrm{Pic}^0(S)$ to the Abel–Jacobi map $\mathrm{Sym}^2 C \to \mathrm{Pic}^0(C)$ and derive a formula for the algebraic equivalence class of the Abel–Jacobi image of the Fano surface.

We work over an algebraically closed field k of arbitrary characteristic.

§2. The nodal and mildly cuspidal cubic threefold

We start with a non-hyperelliptic curve C of genus 4. The canonical map $\phi_K : C \to \mathbb{P}^3$ has as image a curve of degree 6. There is a unique quadric Q in $\mathbb{P}^3$ passing through $\phi_K(C)$. If Q is smooth then the two rulings of Q determine two 1-dimensional linear systems $|D_1|$ and $|D_2|$ on C of degree 3 (g_3^1's) with $D_1+D_2 \sim K$. This determines two embeddings $\gamma_i : C \to \mathrm{Sym}^2 C$ of C into the symmetric product of C, by sending $P \in C$ to the effective divisor of degree 2 in $|D_i - P|$. The images $\gamma_1(C)$ and $\gamma_2(C)$ are disjoint. If Q is singular then it is of rank 3 and the two g_3^1's on C coincide and we find only one embedding $\gamma : C \to \mathrm{Sym}^2 C$ sending P to the pair $P_1 + P_2$ such that $P + P_1 + P_2$ is a divisor of the unique g_3^1.

There is a 4-dimensional linear system Π of cubic surfaces passing through $\phi_K(C)$. This defines a rational map

$$\rho : \mathbb{P}^3 \to \mathbb{P}^4 = \Pi^\vee,$$

sending a point $p \in \mathbb{P}^3 - \phi_K(C)$ to the hyperplane $\{H \in \Pi : p \in H\}$. Note that Π is the projective space of the kernel U of the surjective linear map $\mathrm{Sym}^3(H^0(C, \omega_C)) \to H^0(C, \omega_C^{\otimes 3})$, where ω_C is the relative dualizing sheaf of C. The image is a cubic hypersurface X in $\mathbb{P}^4$. To see this, one may restrict ρ to a general plane T in $\mathbb{P}^3$ and obtain the map given by the cubics passing through six general points, the intersection points of $\phi_K(C)$ with T. The image of T is a Del Pezzo surface of degree 3 and this is a linear section of X.

We need a precise description of the map ρ. Observe that a cubic hypersurface containing $\phi_K(C)$ that contains also a point of $P \in Q - \phi_K(C)$ automatically contains Q for reasons of degree. Therefore the image of such a point is independent of the choice of $P \in Q - \phi_K(C)$ and the open part $Q - \phi_K(C)$ of the quadric Q is contracted. But $\phi_K(C)$ is blown up to a $\mathbb{P}^1$-bundle, the projectivized normal bundle of $\phi_K(C)$. Thus the image X can be obtained by first blowing up $\mathbb{P}^3$ along $\phi_K(C)$ and then blowing down the proper transform Q' of Q. So X has one singular point x_0, the image of Q'. If Q is smooth then the point x_0 is a node singularity (type A_1). If Q is singular then the point x_0 is a cusp singularity (type A_2 if $\mathrm{char}(k) \neq 2$). We therefore have the diagram

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\pi"] \arrow[dr, "b"'] & & X \subset \mathbb{P}^4 \\ & \mathbb{P}^3 \arrow[ur, "\rho"'] & \end{tikzcd}$$

where $\tilde{X}$ is the proper transform of $\mathbb{P}^3$ under the blow up map b along $\phi_K(C)$ and π is the blow down map of the proper transform Q' of the quadric surface Q.

A point $P_1 + P_2$ of $\mathrm{Sym}^2 C$ determines a line $\ell \subset \mathbb{P}^3$, namely the line connecting $\phi_K(P_1)$ and $\phi_K(P_2)$ and we interpret this as the tangent line at $\phi_K(P_i)$ to $\phi_K(C)$ if $P_1 = P_2$. This line is contained in Q if and only if $(P_1, P_2) \in \gamma_1(C) \cup \gamma_2(C)$ (resp. $\gamma(C)$).

Next we need a precise description of the lines in $\mathbb{P}^4 = \Pi^\vee$, in particular those that lie on X. A line in $\mathbb{P}^4$ can be given by a three-dimensional subspace of the 5-dimensional space U.

A chord L of $\phi_K(C)$ that is not contained in Q determines a line in $\mathbb{P}^4$, that is, the subspace W of U consisting of elements vanishing in two points of the chord different from the intersection points of L with $\phi_K(C)$. Since the elements of U vanish already in $L \cap \phi_K(C)$ this means that the elements of W vanish on L. The linear system Π restricted to L has projective dimension 1, hence maps the chord to a line in $\mathbb{P}^4$. This line is contained in X, and does not pass through x_0.

The other lines can be obtained as follows. Take a point $P \in \phi_K(C)$. The line of a ruling of Q through P determines a point $(P_1, P_2) \in \mathrm{Sym}^2 C$ with $P_1 + P_2 + P$ a divisor of a g^1_3 and a line, namely the image of the exceptional line over P in the blow-up of $\mathbb{P}^3$. Or, differently, the corresponding 3-dimensional space of U is the space of cubic surfaces in $\mathbb{P}^3$ containing Q and a hyperplane through P. These are lines through the singular point x_0. So in case Q is smooth the Fano surface S of lines is $\mathrm{Sym}^2 C$ with $\gamma_1(C)$ and $\gamma_2(C)$ identified. In case Q is singular the Fano surface has a cusp singularity along a curve isomorphic to C. Its normalization is $\mathrm{Sym}^2 C$ and let $\nu : \tilde{S} = \mathrm{Sym}^2 C \to S$ be the normalization map.

We shall identify C with its image $\phi_K(C) \subset \mathbb{P}^3$. For $p \in C$ we denote by E_p the exceptional line over p. For $s \in S$ we denote by ℓ_s the corresponding line in X. To summarize the above description of lines on X, we have: if $p + q$ is a point of $\mathrm{Sym}^2 C$ not on $\gamma_1(C) \cup \gamma_2(C)$ then $\ell_{\nu(p+q)} = \pi_*(\tilde{\overline{pq}})$, the push forward by the map π of the proper transform of the secant line $\overline{pq}$. If $p + q \in \gamma_i(C)$ then $\ell_{\nu(p+q)} = \pi_*(E_{\gamma_i^{-1}(p+q)})$.

If X is a cubic threefold with one singularity x which is resolved by a quadric of rank 3 such that the tangent cone at x intersects X along a smooth curve not passing through x then we call X mildly cuspidal. In [6] (see also [9] and [5]) it is proved that every nodal or mildly cuspidal cubic in $\mathbb{P}^4$ is obtained by blowing up $\mathbb{P}^3$ along a canonically embedded non-hyperelliptic curve C of genus 4 and by blowing down the proper transform of the unique quadric containing C. This extends without problems to characteristic 2. We therefore have, cf. also [5] Corollary 3.3:

Proposition 2.1. *For a non-hyperelliptic curve C of genus 4 with canonical image contained in a smooth quadric (resp. singular quadric) the linear system of cubics passing through the canonical image of C maps $\mathbb{P}^3$ to a nodal (resp. mildly cuspidal) cubic in $\mathbb{P}^4$. This defines an isomorphism between the moduli space $\mathcal{M}_4 - \mathcal{H}_4$ of non-hyperelliptic curves of genus 4 and the moduli space of nodal or mildly cuspidal cubic threefolds in $\mathbb{P}^4$.*

§3. The symmetric square of a curve

For a smooth curve F of genus g and $p \in F$ a point of F we define the divisor $X_p = \{p + q : q \in F\}$ on $\mathrm{Sym}^2 F$ (image of a fiber from the ordinary product) and we denote by x its class for algebraic equivalence. If we denote by $j_p : F \to \mathrm{Sym}^2 F$ the inclusion defined by $j_p(q) = p + q$ then X_p is the isomorphic image of F under j_p. We shall write algebraic equivalence as $\stackrel{a}{=}$.

We write $\Delta = \{p + p : p \in F\} \subset \mathrm{Sym}^2 F$ for the diagonal divisor and δ for its class. We have the intersections on $\mathrm{Sym}^2 F$:

$$x^2 = 1, \quad x \cdot \delta = 2 \quad \text{and} \quad \delta^2 = 4(1 - g).$$

A divisor (class) $A = \sum_i p_i - \sum_j q_j$ on F defines a divisor (class) $S_A = \sum_i X_{p_i} - \sum_j X_{q_j}$ on $\mathrm{Sym}^2 F$. The map $A \mapsto S_A$ is obviously linear in A.

Lemma 3.1. *For a smooth curve F of genus g the map $O(A) \mapsto O(S_A)$ defines an isomorphism $i : \mathrm{Pic}^0(F) \xrightarrow{\sim} \mathrm{Pic}^0(\mathrm{Sym}^2 F)$ with inverse map the $j_p^* : \mathrm{Pic}^0(\mathrm{Sym}^2 F) \to \mathrm{Pic}^0(F)$ for a fixed $p \in F$.*

Proof. We have $j_p^* \circ i = 1$ showing that i is an injection. To show that j_p^* is the inverse map to i it suffices to prove that i is onto. The result follows from the fact that $\mathrm{Pic}^0(\mathrm{Sym}^2 F)$ and $\mathrm{Pic}^0(F)$ have the same dimension. This follows from the fact that $\dim H^1(\mathrm{Sym}^2 F, O_{\mathrm{Sym}^2 F}) = g$ as a standard calculation shows.

Q.E.D.

Definition 3.2. For a linear system Γ of degree d and rank 1 (a g^1_d), we define a *trace divisor* on $\mathrm{Sym}^2 F$ of pairs contained in Γ by

$$T_\Gamma = \{p+q :\ \Gamma \geq p+q\}.$$

We shall denote linear equivalence by $\sim$. Then we have:

Lemma 3.3. *If A is a degree d divisor on F defining a linear system $\Gamma = |A|$ which is a g^1_d then in* $\mathrm{Pic}(\mathrm{Sym}^2 F)$ *we have*

$$\Delta \sim 2S_A - 2T_\Gamma.$$

In particular, the divisor class Δ is divisible by 2 in $\mathrm{Pic}(\mathrm{Sym}^2 F)$*, i.e. there is a divisor class $\Delta/2$ such that*

$$T_\Gamma \sim S_A - \Delta/2.$$

Proof. It suffices to prove the result in the case where Γ is base point free g^1_d. Indeed, otherwise the elements of the g^1_d have the form $D + D_0$, where D is an element of a base point free g^1_{d-n} and $D_0 = \sum_{i=1}^n p_i$ is the base divisor. Then $A \sim A' + D_0$, where A' is the divisor of the g^1_{d-n}. Therefore the points of T_Γ have the form i) points $a+b$ with $D \geq a+b$ for some $D \in \Gamma'$, the g^1_{d-n}, and ii) points $a+p_i$, some $a \in F$ and some i with $1 \leq i \leq n$. The first are points of $T_{\Gamma'}$ while the latter are points of X_{p_i}, $i = 1, \ldots, n$. We conclude that $T_\Gamma = T_{\Gamma'} + \sum_{i=1}^n X_{p_i}$ and since $S_A \sim S_{A'} + \sum_{i=1}^n X_{p_i}$, the result for Γ' implies it for Γ.

Assume therefore that Γ is base point free g^1_d and let $\phi : F \to \mathbb{P}^1$ be the map defined by the g^1_d. Take $\Phi = \phi \times \phi : F \times F \to \mathbb{P}^1 \times \mathbb{P}^1$. Let $\beta : F \times F \to \mathrm{Sym}^2 F$ be the canonical map. We denote by $\Delta_{\mathbb{P}^1 \times \mathbb{P}^1}$ (resp. $\Delta_{F \times F}$) the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ (resp. $F \times F$). Then $\Phi^* \Delta_{\mathbb{P}^1 \times \mathbb{P}^1} = \Delta_{F \times F} + \beta^* T_\Gamma$. Now, $\Delta_{\mathbb{P}^1 \times \mathbb{P}^1}$ is linearly equivalent to $f^1 + f^2$, where f^i, $i = 1, 2$, are the fibers of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $A = a_1 + \cdots + a_d$ be a fiber divisor of the map $\phi : F \to \mathbb{P}_1$. Then $\Phi^* \Delta_{\mathbb{P}^1 \times \mathbb{P}^1} \sim \Phi^*(f^1 + f^2) = \sum_{i=1}^d \beta^* X_{a_i} = \beta^* S_A$. We also have $\beta^* \Delta = 2\Delta_{F \times F}$. Therefore

$$\beta^* 2T_\Gamma = 2\Phi^* \Delta_{\mathbb{P}^1 \times \mathbb{P}^1} - 2\Delta_{F \times F} \sim 2\beta^* S_A - \beta^* \Delta.$$

Since β^* is an injection the result follows. Q.E.D.

Remark 3.4. Among all divisor classes A on $\mathrm{Sym}^2 F$ with $A \overset{a}{=} \delta/2$, the above defined divisor class $\Delta/2$ has the characteristic property that $j_p^*(\Delta/2) = p$ for every $p \in F$. Indeed, one has $j_p^*(\Delta/2) = j_p^*(S_A - T_\Gamma) = A - (A - p) = p$.

§4. Divisors on $\mathrm{Sym}^2 C$

We shall assume for the rest of the paper that Q is smooth, or equivalently, that the curve C has two different g^1_3's, say $|D_1|$ and $|D_2|$ with $D_1 + D_2 \sim K$, the canonical divisor of C. The corresponding cubic threefold X is then a nodal cubic threefold, see Section 2. Let R_1 and R_2 be the two rulings of Q. If $p \in C$ is a point and ℓ_i a line in the ruling R_i through p then ℓ_i cuts out on C a divisor $p + p_i + q_i$ in one of the two g^1_3's. The map γ_i, see Section 2, sends p to $p_i + q_i \in \mathrm{Sym}^2 C$. We shall write C_i for the image curve $\gamma_i(C)$ on $\mathrm{Sym}^2 C$. The map $\gamma_2\gamma_1^{-1}$: $C_1 \to C_2$ sends $p_1+q_1 \in \mathrm{Sym}^2 C$ to the complementary point p_2+q_2. For complementary points $p_1+q_1 \in C_1$ and $p_2+q_2 \in C_2$ we have $\ell_{\nu(p_1+q_1)} = \ell_{\nu(p_2+q_2)}$. Therefore the normalization map $\nu : \tilde{S} = \mathrm{Sym}^2 C \to S$ glues the complementary points of the curves C_1 and C_2.

We observe now that $C_i = T_{|D_i|}$, i.e. C_i is a trace divisor on $\mathrm{Sym}^2 C$, and hence by Lemma 3.3 we have

$$C_i \sim S_{D_i} - \Delta/2. \tag{1}$$

This relation yields the following corollary, cf. also [7].

Corollary 4.1. *The divisors C_1 and C_2 are algebraically equivalent on $\mathrm{Sym}^2 C$, but not linearly equivalent.*

Proof. We have $S_{D_i} \stackrel{a}{=} 3x$, and hence $C_i \stackrel{a}{=} 3x - \delta/2$ which proves that $C_1 \stackrel{a}{=} C_2$. If we assume that $C_i \sim C_2$ then $S_{D_1} - \Delta/2 \sim S_{D_2} - \Delta/2$, and so $S_{D_1} \sim S_{D_2}$. But then Lemma 3.1 implies that $D_1 \sim D_2$, a contradiction since we assumed that D_1 and D_2 define two different g^1_3's. Q.E.D.

Since the curve C is not hyperelliptic we have $h^0(C, O(K - p - q)) = 2$, for every $p, q \in C$. We introduce now the following notation:

Notation 4.2. For $p + q \in \mathrm{Sym}^2 C$ we set $D_{p+q} = T_\Gamma$ with Γ the g^1_4 defined by $K - p - q$.

So D_{p+q} is the trace divisor on $\mathrm{Sym}^2 C$ corresponding to the projection of the curve C with center the secant line $\overline{pq}$. By Lemma 3.3, we have

$$D_{p+q} \sim S_{K-p-q} - \Delta/2\,. \tag{2}$$

If $p + q \notin C_1 \cup C_2$ then $|K - p - q|$ is base point free and defines a $4:1$ map from C to $\mathbb{P}^1$. If $p + q \in C_i$ then the linear system $|K - p - q|$ has the base point $\gamma_i^{-1}(p + q)$.

Lemma 4.3. *If $p+q \in C_1$ then $D_{p+q} = C_2 + X_{\gamma_1^{-1}(p+q)}$ and, similarly, if $p+q \in C_2$ then $D_{p+q} = C_1 + X_{\gamma_2^{-1}(p+q)}$.*

Proof. If $p+q \in C_1$ then the linear system $|K-p-q|$ has the base point $\gamma_1^{-1}(p+q)$. The elements in $|K-p-q|$ have the form $D+\gamma_1^{-1}(p+q)$, where $D \in |K-p-q-\gamma_1^{-1}(p+q)| = |K-D_1| = |D_2|$. Then, as in the first paragraph of the proof of Lemma 3.3, we have that $D_{p+q} = T_{|D_2|} + X_{\gamma_1^{-1}(p+q)} = C_2 + X_{\gamma_1^{-1}(p+q)}$. Q.E.D.

We now compute several intersection numbers on $\mathrm{Sym}^2 C$.

Inters 4.4. $[C_i]\cdot[C_j] = (3x-\delta/2)^2 = 9-6-3 = 0$.

Inters 4.5. $[X_p]\cdot[C_i] = x\cdot(3x-\delta/2) = 2$.
The two points of intersection are $p+a$, $p+b$, where a and b are defined by $\gamma_i(p) = a+b$.

Inters 4.6. $[D_{p+q}]\cdot[C_i] = (4x-\delta/2)\cdot(3x-\delta/2) = 12-7-3 = 2$.
If $p+q \notin C_1 \cup C_2$, the two points of intersection are the $\gamma_i(p)$ and $\gamma_i(q)$. Note therefore that, in this case, the divisor D_{p+q} intersects the curves C_1 and C_2 in complementary points: $\gamma_1(p) = \gamma_1\gamma_2^{-1}(\gamma_2(p))$ and $\gamma_1(q) = \gamma_1\gamma_2^{-1}(\gamma_2(q))$. This indicates that the divisor D_{a+b}, $a+b \notin C_1 \cup C_2$, is the pull back of a Cartier divisor from S - we will see this later in a more rigorous way. If $p+q \in C_1$ then, by Lemma 4.3, we have that $D_{p+q} = C_2 + X_{\gamma_1^{-1}(p+q)}$ and the points of intersection are the two points of intersection of $X_{\gamma_1^{-1}(p+q)}$ with C_i, that is, the points $p+\gamma_1^{-1}(p+q)$ and $q+\gamma_1^{-1}(p+q)$, see Inters 4.5.

Inters 4.7. $D_{p+q}\cdot X_a = (4x-\delta/2)\cdot x = 4-1 = 3$.
If $p+q \notin C_1 \cup C_2$ it corresponds to the three points $a+b_i$, where b_i are the three additional points of intersection with C of the plane defined by the points p, q and a. If $p+q \in C_1$ then $D_{p+q} = C_2 + X_{\gamma_1^{-1}(p+q)}$, see Lemma 4.3, and the intersection corresponds to the two points of intersection of C_2 with X_a, see Inters 4.5, plus the point of intersection of $X_{\gamma_1^{-1}(p+q)}$ with X_a, i.e. the point $\gamma_1^{-1}(p+q)+a$.

Inters 4.8. $[D_{p+q}]\cdot[D_{p'+q'}] = (4x-\delta/2)^2 = 16-8-3 = 5$.
If $p+q$, $p'+q' \notin C_1 \cup C_2$ it corresponds to the five common secant lines to the lines $l_{\nu(p+q)}$ and $l_{\nu(p'+q')}$ in X, cf. [6]. If $p+q \in C_1$ then $D_{p+q} = C_2 + X_{\gamma_1^{-1}(p+q)}$, see Lemma 4.3, and the intersection corresponds to the sum of the Inters 4.6 and Inters 4.7.

§5. Divisors on the Fano surface

Let X be again a nodal threefold with S the Fano surface. For each $s \in S$ we have the divisor

$$D_s = \{s' \in S,\ l_{s'} \cap l_s \neq \emptyset\}$$

on S as defined in [6]. Let $s \in S$ so that $s = \nu(p+q)$ for some $p+q \in \mathrm{Sym}^2 C$, where $\nu : \mathrm{Sym}^2 C \to S$ is the normalization map. The following proposition relates the divisor D_s on S with the trace divisor D_{p+q} on $\mathrm{Sym}^2 C$.

Proposition 5.1. *Let* $\mathrm{sing}(S)$ *be the singular locus of* S *viewed as a Weil divisor on* S. *We have*

(1) *If* $p+q \notin C_1 \cup C_2$ *then* $D_{\nu(p+q)} = \nu_* D_{p+q}$.

(2) *If* $p+q \in C_1$ *then* $D_{\nu(p+q)} = \nu_* D_{p+q} = \mathrm{sing}(S) + \nu_* X_{\gamma_1^{-1}(p+q)}$. *Similarly, if* $p+q \in C_2$ *then* $D_{\nu(p+q)} = \nu_* D_{p+q} = \mathrm{sing}(S) + \nu_* X_{\gamma_2^{-1}(p+q)}$.

Proof. We start by proving the first claim. The points $a, b \in C$ belong to the same fiber of the projection to $\mathbb{P}^1$ defined by the $g_4^1 = |K - p - q|$ if and only if there is a hyperplane section H on C with $H \geq p+q+a+b$. This is equivalent to saying that the line $\overline{ab}$ intersects the line $\overline{pq}$. If $p+q \notin C_1 \cup C_2$ then the secant $\overline{pq}$ corresponds, via the rational map ρ, to the line $l_{\nu(p+q)} = \pi_* \tilde{\overline{pq}}$ of X. The point p (resp. q) of $\overline{pq}$ corresponds to the intersection x_p (resp. x_q) of $l_{\nu(p+q)}$ with $\pi_* E_p$ (resp. $\pi_* E_q$).

Apart from the line $\pi_* E_p$ $(= l_{\nu(\gamma_1(p))} = l_{\nu(\gamma_2(p))})$, the lines in X which intersect the line $l_{\nu(p+q)}$ at x_p are the lines $l_{\nu(p+a)}$, where $p+a$ is one of three points of intersection of D_{p+q} with X_p, see Inters 4.7. This is because the proper transform of two lines through p passes from the same point of E_p if and only if the plane they span contains the tangent line $T_p C$. Let $S_p = [X_p \cap D_{p+q}] \cup \{\gamma_1(p), \gamma_2(p)\} \subset D_{p+q}$. Then $a+b \in S_p$ if and only if $l_{\nu(a+b)}$ is a line in X that intersects $l_{\nu(p+q)}$ at x_p. Note that $p+q \in D_{p+q}$ if and only if the tangents to C at p and q are coplanar and in this case $l_{\nu(p+q)}$ intersects itself. This gives a characterization of the lines of second type, see [6] Lemma 10.7. Similarly, the set of $s \in S$ such that the line l_s intersects the line $l_{\nu(p+q)}$ at x_q is the image of the set $S_q = [X_q \cap D_{p+q}] \cup \{\gamma_1(q), \gamma_2(q)\} \subset D_{p+q}$.

We set $U = D_{p+q} \backslash [S_p \cup S_q]$ and let U' be the set of points s in the divisor $D_{\nu(p+q)}$ such that l_s intersects the line $l_{\nu(p+q)}$ at a point different from x_p and x_q. We shall show that $a+b \in U$ if and only if $\nu(a+b) \in U'$, which yields the first claim. We claim that $a+b \in U$ if and only if the line $\overline{ab}$ intersects $\overline{pq}$ at a point t different than p and q. Indeed, the line $\overline{ab}$ intersects $\overline{pq}$ since $a+b \in D_{p+q}$. As $a+b \notin X_p + X_q$ we have $\{a,b\} \cap \{p,q\} = \emptyset$ and so if we assume that the point of intersection is p or q then the line $\overline{ab}$ intersects the curve C at 3 points and hence it is a line in a ruling. But then, Inters 4.6 yields that $a+b \in \{\gamma_i(p), \gamma_i(q),\ i=1,2\}$, a contradiction since $a+b \in U$. Hence the lines $l_{\nu(a+b)}$ and $l_{\nu(p+q)}$ are intersecting lines with point of intersection $\alpha(t) \neq x_p, x_q$. Therefore $\nu(a+b) \in U'$ and vice versa.

The second claim follows easily from Lemma 4.3. The curve $\mathrm{sing}(S)$ corresponds to lines intersecting $l_{\nu(p+q)}$ at the singular point of the threefold and $\nu_* X_{\gamma_i^{-1}(p+q)}$ corresponds to lines intersecting $l_{\nu(p+q)}$ at the other points. Q.E.D.

Since for every $p+q \in \mathrm{Sym}^2 C$ the divisors D_{p+q} have algebraic equivalence class $4x - \delta/2$, see relation (2), we have the following corollary.

Corollary 5.2. *For every $s \in S$ the divisors D_s are algebraically equivalent.*

Remark 5.3. Note that for $p+q \notin C_1 \cup C_2$ the divisor D_{p+q} has an involution which sends the point $a+b$ to the residual point in the linear system $|K-p-q|$. The induced involution on $D_{\nu(p+q)}$ is the one defined in [6].

For a 2-plane V in $\mathbb{P}^4$ the set of lines in $\mathbb{P}^4$ meeting V defines a Cartier divisor C_V on S. The corresponding divisor class is the pull back to S via the natural embedding $S \longrightarrow \mathrm{Gr}(2,5)$ of the natural ample line bundle on the Grassmannian. Let $p_1+q_1 \in \mathrm{Sym}^2 C$, but $\notin C_1 \cup C_2$ and choose a generic plane H in $\mathbb{P}^3$ containing the secant $\overline{p_1q_1}$ and intersecting the curve $C \subset \mathbb{P}^3$ in four additional distinct points p_2, q_2, p_3, q_3 different from p_1, q_1. We may assume that p_2+q_2, $p_3+q_3 \notin C_1 \cup C_2$ and that the lines $\overline{p_iq_i}$, $i=1,2,3$, meet at three distinct points not on C. Therefore, their image under the rational map ρ are three intersecting lines in $\mathbb{P}^4$ which define a 2-plane V_0. Note that the rational map ρ embeds the plane H in a hyperplane of $\mathbb{P}^4$ but does not send it to a 2-plane in $\mathbb{P}^4$. Then the plane V_0 intersects X in the sum of the three lines $\sum_{i=1,2,3} l_{\nu(p_i+q_i)}$ and hence $C_{V_0} = \sum_{i=1,2,3} D_{\nu(p_i+q_i)}$ is a Cartier divisor on S. Now, by Proposition 5.1 and Inters 4.6 the divisor $D_{\nu(p_1+q_1)}$ intersects the singular locus of S at the divisor $A = \nu(\gamma_1(p_1)+\gamma_1(q_1))$, while the divisor $D_{\nu(p_2+q_2)} + D_{\nu(p_3+q_3)}$ intersects the singular locus of S at the divisor $B = \nu(\gamma_1(p_2)+\gamma_1(p_3)+\gamma_1(q_2)+\gamma_1(q_3))$. Since $\mathrm{supp}A \cap \mathrm{supp}B = \emptyset$, the divisor $D_{\nu(p_1+q_1)}$ is a Cartier divisor. Hence, if

$p+q \notin C_1 \cup C_2$ then the divisor $D_{\nu(p+q)}$ defines a line bundle $\mathcal{O}(D_{\nu(p+q)})$ on the singular surface S. Combining this with Proposition 5.1 we have:

Corollary 5.4. *If $s \in S - \operatorname{sing}(S)$ so that $s = \nu(p+q)$ with $p+q \notin C_1 \cup C_2$, then D_s is a Cartier divisor on S and $\mathcal{O}(D_{p+q}) = \nu^*\mathcal{O}(D_s)$.*

Remark 5.5. If $s \in \operatorname{sing}(S)$ with $s = \nu(p+q)$ but $p+q \in C_1 \cup C_2$, then $D_s = \operatorname{sing}S + \nu_* X_{\gamma_i^{-1}(p+q)}$, $i = 1$ or 2 (see Proposition 5.1), is not a Cartier divisor on S. For example, $\nu_* X_{\gamma_i^{-1}(p+q)}$ is not a Cartier divisor since for $s \in C$ the divisor X_s does not intersect the curves C_i, $i = 1, 2$, at complementary points.

§6. The Picard variety of the Fano surface

We now analyze the Picard variety of the Fano surface of our nodal cubic threefold.

Proposition 6.1. *The pull back map $\nu^* : \operatorname{Pic}^0(S) \to \operatorname{Pic}^0(\tilde{S})$ is onto.*

Proof. By Lemma 3.1 the group $\operatorname{Pic}^0(\tilde{S})$ is generated by the classes of divisors of the form S_{a-b} with $a, b \in C$. Choosing a point $c \in C$ with $a+c$, $b+c \notin C_1 \cup C_2$ we get by relation (2) and Corollary 5.4 that S_{a-b} is linearly equivalent to

$$[S_{K-b-c} - \Delta/2] - [S_{K-a-c} - \Delta/2] = D_{b+c} - D_{a+c} = \nu^*(D_{\nu(b+c)} - D_{\nu(a+c)}).$$

Q.E.D.

Remark 6.2. A line bundle L on $\operatorname{Sym}^2 C$ defining an element of $\operatorname{Pic}^0(\operatorname{Sym}^2 C)$ restricts to the same line bundle on C_1 and C_2, that is, $\gamma_1^*(L) \cong \gamma_2^*(L)$. Indeed, for $p \in C$ the intersection of X_p with C_i is $s_i + p$, $t_i + p$, where $\gamma_i(p) = s_i + t_i$, see Inters 4.5. Therefore $\gamma_i^*(X_p) = s_i + t_i$ and $\gamma_i^*(S_p) \sim D_i - p$ because $s_i + t_i + p \sim D_i$. So $\gamma_i^*(S_{p-q}) \sim q - p$ and since these divisors generate $\operatorname{Pic}^0(\operatorname{Sym}^2 C)$ the result follows. Given now L we can glue $L|C_1$ with $L|C_2$ to obtain a line bundle on S which under ν pulls back to L. This proves the surjectivity of Proposition 6.1 in a different way.

Corollary 6.3. *The semi-abelian variety $\operatorname{Pic}^0(S)$ is isomorphic to the $\mathbb{G}_m$-extension of $\operatorname{Pic}^0(C)$ given by $D_1 - D_2$, the difference of the two g_3^1's.*

Proof. The kernel of the surjective map $\nu^* : \operatorname{Pic}^0(S) \to \operatorname{Pic}^0(\tilde{S}) \overset{i}{\cong} \operatorname{Pic}^0(C)$ is the algebraic torus $\mathbb{G}_m$. More precisely, the fibre over $[L] \in$

$\mathrm{Pic}^0(\tilde{S})$ consists of the isomorphisms $\gamma_1^*(L) \cong \gamma_2^*(L)$. Note that $\gamma_1^*(L)$ and $\gamma_2^*(L)$ are isomorphic line bundles on C as the map

$$\mathrm{Pic}^0(C) \xrightarrow{i} \mathrm{Pic}^0(\tilde{S}) \xrightarrow{\gamma_j^*} \mathrm{Pic}^0(C)$$

is given on divisors as $\sum n_i p_i \mapsto \sum n_i(D_j - p_i)$ (cf. Remark 6.2), hence by multiplication by -1. Let $\mathcal{L}$ be a universal line bundle on $\mathrm{Pic}^0(C) \times \tilde{S}$ constructed via the Abel–Jacobi map $u : \tilde{S} \to \mathrm{Pic}^0(C)$ with $u(p+q) = O(p_0 + q_0 - p - q)$ for fixed $p_0, q_0 \in C$. It has the properties $\mathcal{L}|[L] \times \tilde{S} = O(S_L)$ and $\mathcal{L}|\mathrm{Pic}^0 C \times \{p+q\} = O(p_0 + q_0 - p - q)$. The line bundle $(1 \times \gamma_2)^*(\mathcal{L}) \otimes (1 \times \gamma_1)^*(\mathcal{L})^{-1}$ on $\mathrm{Pic}^0(C) \times C$ is trivial on each fibre C and hence the pull back of a line bundle on $\mathrm{Pic}^0(C)$. To determine which one, we can restrict to a fibre $\mathrm{Pic}^0(C) \times \{p\}$ and then it is seen to equal $O(D_1 - D_2)$, since $(1 \times \gamma_j)^*(\mathcal{L})|\mathrm{Pic}^0(C) \times \{p\} = O(p_0 + q_0 - \gamma_j(p)) = O(p_0 + q_0 - D_j + p))$. Hence the $\mathbb{G}_m$-extension is obtained by deleting the zero-section from the line bundle $O(D_1 - D_2)$. Q.E.D.

Remark 6.4. Note that we do not require isomorphisms of $\mathbb{G}_m$-extensions to be the identity on $\mathbb{G}_m$, hence $O(D_1 - D_2)$ and $O(D_2 - D_1)$ define isomorphic extensions.

As a corollary we now can deduce that the general cubic threefold is not rational, cf. [8], but with no assumptions on the characteristic.

Corollary 6.5. *The general cubic threefold is not rational.*

Proof. Let $\mathcal{X} \to B$ be a cubic threefold over the spectrum of a discrete valuation ring such that the generic fibre X_η is a smooth cubic threefold and the special fibre X_s is a nodal cubic threefold. Then the Picard variety of the Fano surface S of $\mathcal{X}$ is a semi-stable abelian variety $\mathcal{A}$ of dimension 5 with generic fibre A_η a principally polarized abelian variety and as special fibre A_s a $\mathbb{G}_m$-extension of $\mathrm{Jac}(C)$ given by $\pm(D_1 - D_2)$. If $\mathcal{X}$ were rational then A_η would be the Jacobian of a curve of compact type. But if $D_1 \neq D_2$ then $D_1 - D_2$ is not of the form $p - q$ for points $p, q \in C$. Hence A_s is not a limit of a Jacobian, and thus A_η cannot be a Jacobian. Q.E.D.

Remark 6.6. If X is mildly cuspidal then $\mathrm{Pic}^0(S)$ is an extension of $\mathrm{Pic}^0(C)$ by an additive group, hence not the Jacobian of a curve of compact type.

§7. The compactified Jacobian of a one-nodal curve

Since the Picard group $\mathrm{Pic}^0(S)$ is very similar to the Jacobian of a one-nodal curve we first review compactifications of the latter. References are [16, 15, 2, 3, 11, 4] for example.

Let F be a curve with one node x_0 and let $\mu : \tilde{F} \to F$ be the normalization, where $\tilde{F}$ is a smooth curve of genus g with $\mu^{-1}(x_0) = \{x_1, x_2\}$. Then there is a short exact sequence of algebraic groups

$$1 \to \mathbb{G}_m \to \mathrm{Pic}^0(F) \to \mathrm{Pic}^0(\tilde{F}) \to 0$$

and the extension class of this semi-abelian variety is given by the class of $\pm(x_1 - x_2) \in \mathrm{Pic}^0(\tilde{F})/\pm 1$, where we identify $\mathrm{Pic}^0(\tilde{F})$ with its dual abelian variety. We shall write

$$J := \mathrm{Pic}^0(\tilde{F}), \quad G := \mathrm{Pic}^0(F).$$

There are two ways to compactify G, one by explicitly constructing a geometric compactification (the rank-1-compactification), the other one by the moduli interpretation. In the case at hand they lead to the same result. We begin by defining the compactified Jacobian G^c of F as the moduli space of rank 1 torsion-free sheaves on F of degree 0; here the degree is defined by $\deg(M) = \chi(M) - \chi(O_F)$. It contains $G = \mathrm{Pic}^0(F)$ as an open part.

The direct construction of the compactified Jacobian as a variety is obtained as follows. Take the $\mathbb{P}^1$-bundle $P = \mathbb{P}(L \oplus O)$ with the projection $q : P \to J$ over J where $L = O(x_1 - x_2)$.

Recall that in order to lift a morphism $\alpha : X \to J$ for a variety X to a morphism $\tilde{\alpha} : X \to P$ one must give an invertible sheaf M on X and a surjective map of sheaves $\tau : \alpha^*(L \oplus O) \to M$, see [14], Ch. II, Prop. 7.12. The $\mathbb{P}^1$-bundle P contains two effective divisors T_1 and T_2 given by $\mathbb{P}(L \oplus 0)$ and $\mathbb{P}(0 \oplus O)$. There exist two sections $t_i : \mathrm{Pic}^0(C) \to P$ $(i = 1, 2)$ of q with image T_i with t_1 corresponding to the projection $L \oplus O \to L$ and t_2 to $L \oplus O \to O$. By deleting T_1 and T_2 from P one gets G back.

Since $O(T_i) \otimes O(1)^{-1}$ is trivial on the fibers of q it is the pull back of a line bundle Λ_i on J. One determines Λ_1 by taking the pull back of the relation $O(T_1) \otimes O(1)^{-1} = q^*\Lambda_1$ under t_2 and one finds $\Lambda_1 = O$ and similarly one gets $\Lambda_2 = L^{-1}$. In particular we get

$$O(T_1) \cong O(1), \quad O(T_2) \cong O(1) \otimes q^*(L^{-1}). \tag{3}$$

The compactification G^c is the non-normal variety obtained by glueing T_1 to T_2 over a translation by $x_1 - x_2$. The smooth part can be identified with G and the singular locus Σ with $\mathrm{Pic}^{-1}(\tilde{F})$ by associating to a line bundle N on $\tilde{F}$ of degree -1 the torsion-free sheaf $\mu_*(N)$ of rank 1 on F:

$$\mu_* : \mathrm{Pic}^{-1}(\tilde{F}) \xrightarrow{\sim} \Sigma, \qquad N \mapsto \mu_*(N).$$

Note that $\chi(\tilde{F}) = \chi(F) + 1$.

There is a natural action of J by translation on itself and on $\mathrm{Pic}^{-1}(\tilde{F})$ and this results in an action of G on G^c extending the action on itself as one easily checks.

Moreover, for a line bundle Λ on $\tilde{F}$ the fibre under μ^* can be interpreted as the pairs (Λ, λ) where $\lambda : \Lambda_{x_1} \xrightarrow{\sim} \Lambda_{x_2}$. If we choose generators for the fibres Λ_{x_i} the map λ can be identified with a non-zero scalar. Letting this scalar go to 0 or infinity gives the two extra points on the fibre of G^c over Λ; these have as their images in Σ the points corresponding to the torsion-free sheaves $\mu_*(\Lambda \otimes O(-x_1))$ and $\mu_*(\Lambda \otimes O(-x_2)$.

After choosing a smooth point p_0 on F with inverse image $\tilde{p}_0$ on $\tilde{F}$ we can define an Abel–Jacobi map

$$u : \tilde{F} \to J, \quad \tilde{p} \to O(\tilde{p}_0 - \tilde{p})$$

and it can be lifted to a map $\tilde{u} : \tilde{F} \to P$ which is given by an invertible sheaf M on $\tilde{F}$ and a surjection $u^*(L \oplus O) \to M$ with $M = \tilde{u}^*(O(1))$. We take $M = O(x_1) \cong O(x_2) \otimes u^*(L)$. Then $M \otimes u^*(L \oplus O)^\vee \cong O(x_1) \oplus O(x_2)$ and this has a canonical section $1 \oplus 1$, giving the desired surjection $u^*(L \oplus O) \to M$.

Note that by equation (3) for $i = 1, 2$ we have $\tilde{u}^*O(T_i) = O(x_i)$ as $\tilde{u}^*O(1) = O(x_1)$ and $u^*L^{-1} = O(x_2 - x_1)$. It follows that $\tilde{u}(\tilde{F}) = \tilde{u}(x_i)$ and $\tilde{F} - \{x_1, x_2\}$ is mapped into G under $\tilde{u}$. Note also that $q(\tilde{u}(x_1)) - q(\tilde{u}(x_2)) = u(x_1) - u(x_2)$, the class of $x_1 - x_2$. Thus the morphism $\tilde{u}$ descends to an Abel-Jacobi map $\bar{u} : F \to G^c$. It has a moduli interpretation via the direct construction as follows. The ideal sheaf I_Δ of the diagonal on $F \times F$ is a torsion-free sheaf of degree -1 for the curve $F \times F \xrightarrow{\mathrm{pr}} F$, with pr the first projection. Then the sheaf $I_\Delta \otimes \mathrm{pr}^*O(p_0)$ defines the morphism $\bar{u}$; we refer to [11].

We now calculate the algebraic equivalence class of the curve $\tilde{u}(\tilde{F})$ in P.

Proposition 7.1. *The algebraic equivalence class γ of the curve $\tilde{u}(\tilde{F})$ in P is given by*

$$\tilde{u}(\tilde{F}) \stackrel{a}{=} q^*(p) + q^*(F) \cdot \eta,$$

with $q : P \to J$ the projection and η the class of $O(1)$ on P and p a point of J.

Proof. From equation (3) we have $T_1 \stackrel{a}{=} \eta$ and $T_2 \stackrel{a}{=} \eta - q^*(L) \stackrel{a}{=} \eta$. Since $\eta^2 = 0$ we have $\gamma \stackrel{a}{=} q^*(a_0) + q^*(a_1)\eta$ with a_i a class of dimension i on J satisfying $a_0 = q_*(\gamma\eta)$ and $a_1 = q_*(\gamma)$. We have

$$q_*(\gamma) = q_* \tilde{u}_* 1_{\tilde{F}} = u_* 1_F = [\tilde{F}]$$

and $q_*(\gamma\eta)$ equals the class of a point p and the formula follows. Q.E.D.

§8. The compactified Picard of the Fano surface

As we saw in Section 6 the semi-abelian variety $\mathrm{Pic}^0(S)$ is isomorphic to the $\mathbb{G}_m$-extension of $\mathrm{Pic}^0(C)$ with extension class $D_1 - D_2$. This $\mathbb{G}_m$-extension G can be realized by considering the line bundle L on $\mathrm{Pic}^0(C)$ associated to the divisor class of $D_1 - D_2$ and deleting the zero section. It is an algebraic group since it can be identified with the theta group of L, cf. [15, 12].

Just as in the preceding section there are two ways for compactifying G: one by considering the moduli of rank 1 torsion-free sheaves on S and secondly by glueing two sections of the $\mathbb{P}^1$-bundle defined by G (the rank-1-compactification). The result is the same.

We consider the corresponding $\mathbb{P}^1$-bundle $q : P = \mathbb{P}(L \oplus O) \to \mathrm{Pic}^0(C)$. The $\mathbb{P}^1$-bundle P contains two effective divisors T_1 and T_2 given by $\mathbb{P}(L \oplus 0)$ and $\mathbb{P}(0 \oplus O)$. There exist two sections $t_i : \mathrm{Pic}^0(C) \to P$ $(i = 1, 2)$ of q with image T_i. Then t_1 corresponds to the projection $L \oplus O \to L$ and t_2 to $L \oplus O \to O$. Since $O(T_i) \otimes O(1)^{-1}$ is trivial on the fibers of q it is the pull back of a line bundle Λ_i on $\mathrm{Pic}^0(C)$ and one determines $\Lambda_1 \cong O$ by pulling $O(T_1) \otimes O(1)^{-1}$ under t_2, and similarly $\Lambda_2 \cong L^{-1}$. We thus get as in equation (3)

$$O(T_1) = O(1), \qquad O(T_2) \cong O(1) \otimes q^*(L)^{-1}. \tag{4}$$

We construct a non-normal variety G^c by glueing T_1 with T_2 by a translation over $D_1 - D_2$ in $\mathrm{Pic}^0(C)$. It contains G as a open subvariety and the singular locus Σ is isomorphic to $\mathrm{Pic}^0(C)$.

We may interpret G alternatively as a $\mathbb{G}_m$-extension of $\mathrm{Pic}^0(\mathrm{Sym}^2 C) = \mathrm{Pic}^0(S)$. Then we have the following interpretation for the compactification G^c obtained here, cf. [2], Section 3. We consider the moduli space of rank 1 torsion-free sheaves of O_S-modules on S of first Chern class 0. If N is such a sheaf then outside the singular locus $\mathrm{sing}(S)$ the sheaf N is locally free. Along $\mathrm{sing}(S)$ we have that $N \cong O_{S,\mathrm{sing}(S)}$ or $N \cong \nu_*(O_{\tilde{S}})|\mathrm{sing}(S)$.

Let $c = 3x - \delta/2$ be the class (for algebraic equivalence) of C_1 and C_2 on $\tilde{S} = \mathrm{Sym}^2 C$. Recall that ν is the normalization map $\nu : \tilde{S} \to S$.

For a line bundle N in $\mathrm{Pic}^{-c}(\mathrm{Sym}^2C)$ the direct image $\nu_*(N)$ is a torsion free sheaf of rank 1 of first Chern class 0 which is not locally free. So our situation is very similar to the one of one-nodal curves and we have a morphism

$$\nu_* : \mathrm{Pic}^{-c}(\mathrm{Sym}^2C) \to \Sigma$$

that is an isomorphism. We have a natural action of G on G^c extending the action on itself. On Σ this action is compatible with the action of $\mathrm{Pic}^0(\tilde{S}) \cong \mathrm{Pic}^0(C)$ on $\mathrm{Pic}^{-c}(\mathrm{Sym}^2C)$

We also have Abel–Jacobi maps here. If we pick a base point p_0+q_0 on Sym^2C we have the map

$$u : \mathrm{Sym}^2C \to \mathrm{Pic}^0(C), \qquad p+q \mapsto O(p_0+q_0-p-q).$$

As in the case of the compactified Jacobian there is a lift of u to a map $\tilde{u} : \mathrm{Sym}^2C \to P$ given by a surjection $u^*(L\oplus O) \to M$ with M an invertible sheaf on Sym^2C. In fact, take $M = O(C_1) = O(C_2)\otimes L$. Then we have

$$M \otimes u^*(L\oplus O)^\vee = O(C_1)\oplus O(C_2)$$

and this has a canonical section $1\oplus 1$ defining $u^*(L\oplus O) \to M$. This choice of M is dictated by the fact that we want $\tilde{u}^{-1}(T_i) = C_i$, and $u^*(L) = O(C_1-C_2)$ and moreover that $\tilde{u}^*(O(1))$ should be equal to M. Note that the restriction of $\tilde{u}$ on $\mathrm{Sym}^2C - C_1 - C_2$ is the map defined by the canonical rational section 1 of the divisor $C_1 - C_2$. Moreover, if $p_1 + q_1$, p_2+q_2 are complementary points on the curves C_1, C_2 respectively, see Section 4, then $q\tilde{u}(p_1+q_1) - q\tilde{u}(p_2+q_2) = u(p_1+q_1-p_2-q_2) = O(D_1-D_2)$. This implies that the map $\tilde{u}$ descends to an Abel–Jacobi map $\bar{u} : S \to G^c$.

Finally we calculate the class of $\tilde{u}(\mathrm{Sym}^2C)$ in P modulo algebraic equivalence.

Proposition 8.1. *Let $\gamma = \tilde{u}(\mathrm{Sym}^2C)$. Then the algebraic equivalence class of γ in P is given by*

$$\gamma = q^*[C] + \frac{1}{2}q^*[C*C]\cdot\eta,$$

where $C*C$ is the Pontryagin product and η the class of $O(1)$.

Proof. As above we have $[T_1] = \eta$ and since $L = O(D_1 - D_2)$ it is algebraically equivalent to 0, hence by relation (4) we have $[T_2] = \eta$ too. We can write our class γ as $q^*(a_1) + q^*(a_2)\cdot\eta$ with a_i a dimension i cycle on $\mathrm{Pic}^0(C)$. We have $a_2 = q_*(\gamma) = q_*\tilde{u}_*1_{\mathrm{Sym}^2C} = u_*1_{\mathrm{Sym}^2C} = \frac{1}{2}[C*C]$ and $q_*(\gamma\cdot\eta) = [C]$. The result follows. Q.E.D.

§9. The Clemens–Griffiths map

For smooth Fano surfaces it is natural to consider the map

$$S \to \mathrm{Pic}^0(S), \qquad s \mapsto O(D_s - D_{s_0}),$$

where s_0 is a fixed base point in S, see [6]. This map embeds S into $\mathrm{Pic}^0(S)$. There is an analogue of this map for the singular Fano surface S. We choose $p_0 + q_0 \in \mathrm{Sym}^2 C - C_1 - C_2$ and let $s_0 = \nu(p_0 + q_0) \in S - \mathrm{sing}(S)$. We consider the incidence variety $I = \{(s,t) \text{ with } t \in D_s\} \subset S \times S$. If $\pi_i : S \times S \to S$ is the i-projection then $I - \pi_1^* D_{s_0} \subset S \times S \overset{\pi_2}{\to} S$ is a family of Cartier divisors over $S - \mathrm{sing}(S)$, see Proposition 5.1, and therefore this defines a map $u_0 : S - \mathrm{sing}(S) \to \mathrm{Pic}^0(S)$.

Lemma 9.1. *Let $q : \mathrm{Pic}^0(S) = G \to \mathrm{Pic}^0(C)$ be the natural projection and $\nu : \mathrm{Sym}^2 C \to S$ the normalization map. Then we have the equality of maps*

$$q \circ u_0 \circ \nu = u : \mathrm{Sym}^2 C - C_1 - C_2 \to \mathrm{Pic}^0(C).$$

Proof. Let $i : \mathrm{Pic}^0(\mathrm{Sym}^2 C) \to \mathrm{Pic}^0(C)$ be the natural isomorphism given in 3.1. Then by Corollary 5.4 we have

$$q \circ u_0 \circ \nu(p+q) = i\nu^*(D_{\nu(p+q)} - D_{s_0}) = i(D_{p+q} - D_{p_0+q_0}).$$

By relation (2) we have

$$D_{p+q} - D_{p_0+q_0} \sim [S_{K-p-q} - \Delta/2] - [S_{K-p_0-q_0} - \Delta/2] \sim S_{p_0+q_0-p-q}.$$

and the result follows by the definition of i. Q.E.D.

The above lemma basically says that u_0 is a lift to $G = \mathrm{Pic}^0(S)$ of the usual Abel–Jacobi map $u : \mathrm{Sym}^2 C \to \mathrm{Pic}^0(C)$. The next proposition shows that it coincides with the generalized Abel–Jacobi map $\bar{u}$.

Proposition 9.2. *Let $u_0 : S - \mathrm{sing}(S) \to \mathrm{Pic}^0(S) = G$ with $u_0(s) = O(D_s - D_{s_0})$ be the Clemens–Griffiths map for the singular Fano surface. Then u_0 coincides with $\bar{u}$ on $S - \mathrm{sing}(S)$, with $\bar{u}$ the Abel–Jacobi map defined in section 8.*

Proof. Recall that the restriction to $\mathrm{Sym}^2 C - C_1 - C_2$ of the lifting $\tilde{u}$ of the Abel–Jacobi map u is given by the canonical rational section of $O(C_1 - C_2)$ on $\mathrm{Sym}^2 C$, as we saw in the preceding section. For $p + q \in \mathrm{Sym}^2 C$ not on C_1 nor on C_2 and $s = \nu(p+q)$ we have seen that the divisor $D_{p+q} - D_{p_0+q_0}$ on $\mathrm{Sym}^2 C$ descends to the Cartier divisor $D_s - D_{s_0}$ on S. The image $u_0(s)$ for $s \in S - \mathrm{sing}(S)$ is given by the class of the pull back $L = \nu^* O(D_s - D_{s_0})$ and an isomorphism $\gamma_1^* L \cong \gamma_2^* L$ on C. Let 1 be the canonical rational section of $O(D_{p+q} - D_{p_0+q_0})$. The pull back L is isomorphic to $O(D_{p+q} - D_{p_0+q_0})$ and the glueing is given by the ratio $\gamma_2^*(1)/\gamma_1^*(1)$ of the two sections. To determine now the embedding $u_0 : S - \mathrm{sing}(S) \to G = \mathrm{Pic}^0(S)$ we have to carry out the above construction for the family of Cartier divisors $I - \pi_1^* D_{s_0} \subset S \times S \overset{\pi_2}{\to} S$ over $S - \mathrm{sing}(S)$ which defines the map u_0.

Let $\mathcal{D} = \{(p+q, r+s)$ with $p+q \in D_{r+s}\} \subset \mathrm{Sym}^2 C \times \mathrm{Sym}^2 C$. Note that $(\nu \times \nu)_* \mathcal{D} = I$, see Proposition 5.1. Take $1 \times \gamma_i : \mathrm{Sym}^2 C \times C \to \mathrm{Sym}^2 C \times \mathrm{Sym}^2 C$ and we have

$$(1 \times \gamma_i)^* (\mathcal{D} - \pi_1^* D_{p_0+q_0}) | \{p+q\} \times C = p + q - p_0 - q_0,$$

(see Inters 4.6) and

$$(1 \times \gamma_i)^* (\mathcal{D} - \pi_1^* D_{p_0+q_0}) | \mathrm{Sym}^2 C \times \{p\} = D_{p_i'+p_i''} - D_{p_0+q_0}.$$

with $p_i' + p_i'' = \gamma_i(p)$. Let 1 be the canonical rational section of $O(\mathcal{D})$. Then $\gamma_2^*(1)/\gamma_1^*(1)$ is up to a non-zero multiplicative scalar the canonical rational section 1 of $O(A)$ with A the divisor

$$A = (1 \times \gamma_2)^* (\mathcal{D} - \pi_1^* D_{p_0+q_0}) - (1 \times \gamma_1)^* (\mathcal{D} - \pi_1^* D_{p_0+q_0})$$

on $\mathrm{Sym}^2 C \times C$. But $A | \{p+q\} \times C$ is the zero divisor for every $p+q \in \mathrm{Sym}^2 C$. Hence A is the pull back from $\mathrm{Sym}^2 C$ of the divisor $A | \mathrm{Sym}^2 C \times \{p\} = D_{p_2'+p_2''} - D_{p_1'+p_1''} = C_1 - C_2$, see Lemma 4.3. Therefore the section $\gamma_2^*(1)/\gamma_1^*(1)$ which gives the glueing over $S - \mathrm{sing}(S)$ is (up to a non-zero scalar) the pull back of the canonical rational section of $O(C_1 - C_2)$ and hence the result. Q.E.D.

§10. The limit of the Clemens–Griffiths map

Assume that we have a family $\mathcal{X} \to \Delta$, with Δ the spectrum of a discrete valuation ring (or an open unit disc in the complex case) with generic fibre X_η, a smooth cubic threefold and special fibre X_0, a nodal cubic threefold. Let $\mathcal{S} \to \Delta$ be the corresponding family of Fano surfaces with $\mathcal{S}_0$ the non-normal Fano surface of X_0. We may assume that the family $\mathcal{S} \to \Delta$ has a section $\sigma : \Delta \to \mathcal{S}$ with $\sigma(0) \in S_0 - \mathrm{sing}(S_0)$.

The map $S_\eta \times S_\eta \to \mathrm{Pic}^0 S_\eta$ given by $(s, s') \to [D_s - D_{s'}]$ has generic degree 6 and has as image a divisor Θ that defines a principal polarization.

We consider the correspondence I_η on $S_\eta \times_{\Delta_\eta} S_\eta$ given by pairs (s_1, s_2) with $s_2 \in D_{s_1}$. This gives us a relatively effective divisor $\mathcal{D}$ over

S_η via the projection on the second factor S_η. In turn this defines an embedding $\phi_\eta : S_\eta \to \mathrm{Pic}^0(S_\eta/\Delta_\eta)$ that sends s to $D_s - D_\sigma$. The family ϕ_η is a flat family as it is irreducible and the base Δ_η is 1-dimensional. We consider the rank-1-compactification of the relative Picard variety with special fibre G^c and let F be the flat limit of $\phi_\eta(S_\eta)$ in the special fibre.

Proposition 10.1. *The flat limit F of the Fano surface coincides with the Abel–Jacobi image $\bar{u}(S_0)$.*

Proof. In characteristic 0 it is well-known that the cohomology class of the fibre $\phi_\eta(S_\eta)$ is $\theta^3/3!$, with θ the polarization class on the Picard variety. In positive characteristic we can lift the hypersurface X to characteristic 0 and deduce the result from this. Hence the cohomology class of the limit $F \subset G^c$ is $\theta_0^3/3!$ where θ_0 is the limit of the polarization. If $\tau : P \to G^c$ is the normalization map then the class of $\tau^*\theta_0$ is equal to $q^*\xi + \eta$, where ξ is the polarization on $\mathrm{Pic}^0(C)$ and η as in Proposition 8.1, cf. [15]. Note that $(\tau^*\theta_0)^3/3! = q^*\xi^3/3! + q^*\xi^2/2! \cdot \eta$ because $\eta^2 = 0$ in cohomology, which is exactly the cohomology class of $\bar{u}(S_0)$, see Proposition 8.1.

Let $\mathcal{S}^*$ be the subscheme of $\mathcal{S}$ obtained by removing the singular points of S_0. The correspondence I_η extends to I^* on $\mathcal{S}^*$ in a natural way by adding the points $(s_1, s_2) \in S_0 \times S_0$ with $s_1 \in D_{s_2}$ and the map ϕ_η extends also naturally to $\phi^* : \mathcal{S}^* \to \mathrm{Pic}^0(\mathcal{S}/\Delta)$ using the section σ. Then $\phi_\eta(S_\eta) \subset \phi^*(\mathcal{S}^*)$ and hence $\phi^*(S_0^*)$ is contained in the limit F. Note that $S_0^* = S_0 - \mathrm{sing}(S_0)$ and $\phi^*(S_0^*) = u_0(S_0 - \mathrm{sing}(S_0) = \bar{u}(S_0 - \mathrm{sing}(S_0))$, see Proposition 9.2. Hence $\bar{u}(S_0)$ is contained in F and hence is a component of F. But since F and $\bar{u}(S_0)$ are effective cycles and have the same homology class they should be equal, as the intersection number of θ_0^3 with $F - \bar{u}(S_0)$ otherwise would be positive since θ_0 is ample. See also [10] Sections 7 and 8.

Q.E.D.

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