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A gap theorem for ancient solutions to the Ricci flow

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Abstract.

We outline the proof of the gap theorem stating that any ancient solution to the Ricci flow with Perelman's reduced volume whose asymptotic limit is sufficiently close to that of the Gaussian soliton must be isometric to the Euclidean space for all time. This is the main result of the author's paper [Yo].

§1. Introduction

Let $g(t), t \in [0, T]$ be a smooth one-parameter family of Riemannian metrics on a manifold M. We call $(M, g(t))$ a *Ricci flow* if it satisfies Hamilton's evolution equation

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}(g(t)), \tag{1.1}$$

where Ric is the Ricci tensor of $g(t)$. We will also use $R := \mathrm{tr}\mathrm{Ric}$ to denote the scalar curvature.

In his seminal paper [Pe], Perelman introduced a comparison geometric approach to the Ricci flow, which we now briefly discuss. Given a Ricci flow $(M, g(t)), t \in [0, T]$, let us consider the backward Ricci flow $(M, g(\tau))$, where $\tau := T - t$ is the reverse time parameter. He endowed the space-time $\tilde{M} := M \times S^N \times (0, T)$ with the metric $\tilde{g}$ defined by

$$\tilde{g} := g(\tau) + \tau g_{S^N} + \Big(R + \frac{N}{2\tau}\Big) d\tau.$$

Here (S^N, g_{S^N}) is the $N(>> 1)$-dimensional round sphere with constant curvature $\frac{1}{2N}$. A remarkable fact is that $(\tilde{M}, \tilde{g})$ can be thought of as an '∞-dimensional Ricci-flat space' (e.g. [We]).

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Now let us recall the Bishop–Gromov comparison theorem: *for a complete Riemannian manifold* (M^{n+1}, g) *with non-negative Ricci curvature, the ratio* $\mathrm{Area}\,\partial B(p,r)/\alpha_n r^n$ *is non-increasing in* $r > 0$. Here α_n stands for the area of the unit sphere in the Euclidean space $(\mathbb{R}^{n+1}, g_E)$.

Perelman applied this to $(\tilde{M}, \tilde{g})$ formally and observed that

$$\begin{aligned}
&\frac{\mathrm{Area}\,\partial B_{\tilde{M}}(\tilde{p}, r)}{\alpha_{n+N} r^{n+N}} \\
\approx &\frac{\alpha_N}{(2N\tau)^{n/2}\alpha_{n+N}} \int_M \left(1 - \frac{1}{N}\ell(\cdot,\tau) + O(N^{-2})\right)^N d\mu_{g(\tau)} \\
\to &\int_M (4\pi\tau)^{-n/2} e^{-\ell(\cdot,\tau)} d\mu_{g(\tau)} \quad \text{as } N \to \infty
\end{aligned}$$

for $\tilde{p} = (p, s, 0) \in M \times S^N \times \{0\}$ and $r := \sqrt{2N\tau}$. Here $d\mu_{g(\tau)}$ denotes the volume element induced by $g(\tau)$. See Definition 2.4 below for the definition of $\ell(q, \tau)$.

In view of the above derivation, it is natural to expect that the *reduced volume* defined by

$$\tilde{V}_{(p,0)}(\tau) := \int_M (4\pi\tau)^{-n/2} e^{-\ell(\cdot,\tau)} d\mu_{g(\tau)} \tag{1.2}$$

is non-increasing in $\tau > 0$. As is stated in Theorem 2.5 below, this is the case.

In order to state our main theorem, we need some terminologies. We say that a backward Ricci flow $(M, g(\tau))$ is an *ancient solution* when $g(\tau)$ exists for all $\tau \in [0, \infty)$. For a given ancient solution $(M, g(\tau))$, the limit $\widetilde{\mathcal{V}}(g) := \lim_{\tau\to\infty} \tilde{V}_{(p,0)}(\tau)$ will be called the *asymptotic reduced volume* of $(M, g(\tau))$. This is a Ricci flow analogue of the *asymptotic volume ratio* $\nu(g) := \lim_{r\to\infty} \mathrm{Vol}\, B(p,r)/\omega_n r^n$ of a Riemannian manifold (M^n, g) with non-negative Ricci curvature, where $\omega_n := \alpha_{n-1}/n$. We will see in Lemma 3.1 below that $\widetilde{\mathcal{V}}(g)$ is independent of the base point $p \in M$. Finally, the *Gaussian soliton* is the trivial Ricci flow $(\mathbb{R}^n, g_E)$ on the Euclidean space. The reduced volume $\tilde{V}_{(p,0)}(\tau)$ is identically 1 for the Gaussian soliton.

Now we state the main theorem of the present article.

Theorem 1.1 ([Yo]). *There exists* $\varepsilon_n > 0$ *depending only on* $n \geq 2$ *such that: let* $(M^n, g(\tau)), \tau \in [0, \infty)$ *be a complete ancient solution to the Ricci flow on an* n*-manifold* M *with Ricci curvature bounded below. Suppose that the asymptotic reduced volume* $\widetilde{\mathcal{V}}(g)$ *of* $(M^n, g(\tau))$ *is greater than* $1 - \varepsilon_n$. *Then* $(M^n, g(\tau)), \tau \in [0, \infty)$ *is the Gaussian soliton.*

By regarding a Ricci-flat metric as an ancient solution to the Ricci flow as in Theorem 1.1, we recover the following result.

Corollary 1.2 (Anderson [An1, Gap Lemma 3.1]). *There exists $\varepsilon_n > 0$ such that: let (M^n, g) be an n-dimensional complete Ricci-flat Riemannian manifold. Suppose that the asymptotic volume ratio $\nu(g)$ of (M^n, g) is greater than $1 - \varepsilon_n$. Then (M^n, g) is isometric to $(\mathbb{R}^n, g_E)$.*

Next, we would like to apply Theorem 1.1 to shrinkers. We call a triple (M, g, f) a *gradient shrinking Ricci soliton* when

$$\mathrm{Ric} + \mathrm{Hess}\, f - \frac{1}{2\lambda} g = 0 \tag{1.3}$$

holds for some positive constant $\lambda > 0$. Shrinking Ricci solitons are typical examples of ancient solutions to the Ricci flow. In what follows, we implicitly normalize the potential function $f \in C^\infty(M)$ by adding a constant so that $\lambda(|\nabla f|^2 + R) = f$, or equivalently,

$$\lambda(2\Delta f - |\nabla f|^2 + R) + f - n = 0. \tag{1.4}$$

Since the left-hand side of (1.4), the integrand of Perelman's $\mathcal{W}$-entropy (see Definition 2.3 below), is known to be constant for gradient Ricci solitons ([Vol2-I, Proposition 1.15]), this is always possible.

Corollary 1.3. *Let (M^n, g, f) be a complete gradient shrinking Ricci soliton with Ricci curvature bounded below. Then*

(1) *the normalized f-volume (or the Gaussian density)*

$$\mathrm{Vol}_f(M) := \int_M (4\pi\lambda)^{-n/2} e^{-f} d\mu_g$$

of (M^n, g, f) does not exceed 1.

(2) *Suppose that $\mathrm{Vol}_f(M) > 1 - \varepsilon_n$, then (M^n, g, f) is the Gaussian soliton $(\mathbb{R}^n, g_E, \frac{|\cdot|^2}{4})$ up to the scaling $\lambda > 0$. Here ε_n is the constant obtained in Theorem 1.1.*

The statements in Corollary 1.3 are intimately related to the results of Carrillo–Ni [CaNi]. In particular, part (2) of Corollary 1.3 proves the following conjecture when the Ricci curvature is bounded below. (The assumption on the Ricci curvature might be superfluous.)

Conjecture 1.4 (Carrillo–Ni [CaNi]). *The normalized f-volume of a gradient shrinking Ricci soliton is equal to 1 if and only it is the Gaussian soliton.*

The organization of this article is as follows. In the next section, we review definitions and Perelman's results in [Pe]. We will do this for the super Ricci flow. In Section 3, we prove some lemmas. In Section 4 and 5, we present the proofs of Theorem 1.1 and Corollary 1.3, respectively.

In this short article, we cannot provide the whole proof. Please consult the author's paper [Yo] for a detailed proof.

§2. Comparison geometry of super Ricci flows

In this section, Perelman's approach, sometimes called *reduced geometry* (e.g. [Ni]), is recalled in more general situation than the original. The main references are [Pe, §§6, 7] and [Ye, MoTi, Vol2-I], etc. Among them, Ye ([Ye]) deals with much wider class of the Ricci flows than others where the sectional curvature is assumed to be bounded. The curvature assumption made in [Ye] is the same as that of our Theorem 1.1.

A smooth one-parameter family of Riemannian metrics $(M, g(\tau))$, $\tau \in [0, T)$ is called a *super Ricci flow* when it satisfies

$$\frac{\partial}{\partial \tau} g \leq 2\mathrm{Ric}(g(\tau)). \tag{2.1}$$

Super Ricci flow was introduced by McCann–Topping [McTo] who studied the time-dependent analogue of the condition that characterizes Ricci non-negativity of Riemannian manifolds (e.g. [ReSt]). See also [Lo] for this topic.

Apart from the backward Ricci flow, a basic and important example of the super Ricci flow is provided by

Example 2.1. $g(\tau) := (1 + 2C\tau)g_0, \tau \in [0, \frac{1}{|C|-C})$ for some fixed Riemannian metric g_0 with Ricci curvature bounded from below by a constant $C \in \mathbb{R}$.

As was pointed out by the author in [Yo], we can straightforwardly generalize Perelman's reduced geometry to the super Ricci flow if we impose the following assumptions on $2h := \frac{\partial}{\partial \tau} g$.

Assumption 2.2. Letting $H := \mathrm{tr}_{g(\tau)} h$, h satisfies

(1) contracted second Bianchi identity $2\mathrm{div} h = dH$ and
(2) heat-like equation $-\mathrm{tr}_{g(\tau)} \frac{\partial}{\partial \tau} h = \Delta_{g(\tau)} H$, or equivalently,

$$-\frac{\partial}{\partial \tau} H = \Delta_{g(\tau)} H + 2|h|^2. \tag{2.2}$$

We know that the backward Ricci flow ($h = \mathrm{Ric}$) and the one in Example 2.1 ($h = (\frac{1}{2C} + \tau)^{-1} g(\tau)$) above satisfy Assumption 2.2. In fact, (2.2) is nothing but the evolution equation for the scalar curvature R along the Ricci flow.

Let $(M, g(\tau)), \tau \in [0, T)$ be a super Ricci flow on a *closed* manifold M^n. We now define *Perelman's* $\mathcal{W}$*-entropy* for the super Ricci flow.

Definition 2.3. For $\tau > 0$ and a smooth positive function $u := (4\pi\tau)^{-n/2}e^{-f} > 0$ on M, define

$$\mathcal{W}(g(\tau), f, \tau) := \int_M \Big[\tau(2\Delta f - |\nabla f|^2 + H) + f - n\Big] u \, d\mu_{g(\tau)}. \tag{2.3}$$

We evolve u by the conjugate heat equation $\frac{\partial}{\partial \tau} u = \Delta_{g(\tau)} u - Hu$. Then by using the integration by parts, we can derive the monotonicity formula for $\mathcal{W}$-entropy that slightly generalizes the original one obtained by Perelman for the Ricci flow:

$$\begin{aligned}
&\frac{d}{d\tau}\mathcal{W}(g(\tau), f, \tau) \\
=& -2\tau \int_M \Big[\Big|h + \operatorname{Hess} f - \frac{1}{2\tau} g\Big|^2 + (dH - 2\operatorname{div} h)(\nabla f) \\
&\qquad + (\operatorname{Ric} - h)(\nabla f, \nabla f) - \frac{1}{2}\Big(\frac{\partial H}{\partial \tau} + \Delta H + 2|h|^2\Big)\Big] u \, d\mu_{g(\tau)} \\
\le& 0.
\end{aligned}$$

In view of the above formula, the conditions in Assumption 2.2 seem to be reasonable ones to impose on our super Ricci flows. Although they are somewhat restrictive, the class of super Ricci flows satisfying Assumption 2.2 includes that of Ricci flows and of manifolds with Ricci curvature bounded below.

Throughout this article, we denote by $(M, g(\tau)), \tau \in [0, T)$ a complete super, or backward Ricci flow on an n-manifold M satisfying Assumption 2.2. It is also assumed that the time-derivative $\frac{\partial}{\partial \tau} g$ is bounded from below, that is, for any compact interval $[\tau_1, \tau_2] \subset [0, T)$, we can find $K = K(\tau_1, \tau_2) \ge 0$ such that $\frac{\partial}{\partial \tau} g \ge -Kg(\tau)$ for all $\tau \in [\tau_1, \tau_2]$.

Now we define the reduced volume of the super Ricci flow. Fix $p \in M$ and $[\tau_1, \tau_2] \subset [0, T)$.

Definition 2.4. We define the $\mathcal{L}$*-length* of a curve $\gamma : [\tau_1, \tau_2] \to M$ by

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau}\Big(\Big|\frac{d\gamma}{d\tau}\Big|^2_{g(\tau)} + H(\gamma(\tau), \tau)\Big) d\tau$$

and the $\mathcal{L}$*-distance* between (p, τ_1) and $(q, \tau_2) \in M \times [0, T)$ is, by definition,

$$L_{(p,\tau_1)}(q, \tau_2) := \inf_\gamma \mathcal{L}(\gamma),$$

where we take the infimum over all curves $\gamma : [\tau_1, \tau_2] \to M$ from p to q. The assumption on the lower bound of $\frac{\partial}{\partial \tau} g$ is employed here so as to guarantee that the $\mathcal{L}$-distance between any two points is always achieved by a minimal $\mathcal{L}$-geodesic (e.g. [MoTi]). A curve $\gamma : [\tau_1, \tau_2] \to M$ is called an *$\mathcal{L}$-geodesic* when it satisfies

$$(2.4) \qquad 2\nabla_X X + \frac{X}{\tau} - \nabla H + 4h(X, \cdot) = 0, \text{ where } X := \frac{d\gamma}{d\tau}.$$

Finally, with $\ell(q,\tau) := \frac{1}{2\sqrt{\tau}} L_{(p,0)}(q,\tau)$ being called the *reduced distance*, the *reduced volume* $\tilde{V}_{(p,0)}(\tau)$ based at $(p, 0)$ is defined by (1.2).

We now state the main theorem of this section.

Theorem 2.5 (cf. [Pe, Ye, Yo]). *Let $(M^n, g(\tau)), \tau \in [0, T)$ be a complete super Ricci flow satisfying Assumption 2.2 with time-derivative $\frac{\partial}{\partial \tau} g$ bounded below. Then for any $p \in M$, $\tilde{V}_{(p,0)}(\tau)$ is non-increasing in $\tau > 0$, $\lim_{\tau \to 0+} \tilde{V}_{(p,0)}(\tau) = 1$, and hence $\tilde{V}_{(p,0)}(\tau) \leq 1$ for all $\tau > 0$.*

Moreover, $\tilde{V}_{(p,0)}(\bar{\tau}) = 1$ for some $\bar{\tau} > 0$ if and only if $(M^n, g(\tau)), \tau \in [0, \bar{\tau}]$ is the Gaussian soliton.

We close this section by considering a static super Ricci flow. This is an important example.

Lemma 2.6 ([Vol2-I, Lemma 8.10]). *Let (M^n, g) be a complete Riemannian manifold of non-negative Ricci curvature regarded as a static super Ricci flow, i.e., $\frac{\partial}{\partial \tau} g = 0 \leq 2\mathrm{Ric}$. Then for any $p \in M$ and $\tau > 0$, we have*

$$\tilde{V}_{(p,0)}(\tau) = \int_M (4\pi\tau)^{-n/2} \exp\Big(-\frac{d(p,q)^2}{4\tau}\Big) d\mu(q),$$

and

$$\lim_{\tau \to \infty} \tilde{V}_{(p,0)}(\tau) = \nu(g).$$

Here $\nu(g)$ denotes the asymptotic volume ratio of (M^n, g) as before.

Lemma 2.6 tells us that Corollary 1.2 indeed follows from Theorem 1.1.

§3. Preliminary lemmas

In this section, we present two lemmas that will be needed in the proof of the main theorem.

Given an ancient super Ricci flow $(M, g(\tau)), \tau \in [0, \infty)$, it is natural to expect that the asymptotic reduced volume $\tilde{\mathcal{V}}(g) := \lim_{\tau \to \infty} \tilde{V}^g_{(p,0)}(\tau)$

is independent of the choice of the base point $(p,0)\in M\times[0,\infty)$, as the asymptotic volume ratio $\nu(g)$ is. We first prove the following

Lemma 3.1. *Let $(M,g(\tau)),\tau\in[0,\infty)$ be a complete ancient super Ricci flow satisfying Assumption 2.2 with time-derivative $\frac{\partial}{\partial\tau}g$ bounded from below. Take $(p_k,\tau_k)\in M\times[0,\infty)$ for $k=1,2$ with $\tau_2\geq\tau_1$ and put $g_k(\tau):=g(\tau+\tau_k),\tau\in[0,\infty)$. Then we have*

$$\lim_{\tau\to\infty}\tilde{V}^{g_2}_{(p_2,0)}(\tau)\geq\lim_{\tau\to\infty}\tilde{V}^{g_1}_{(p_1,0)}(\tau).$$

In particular, $\widetilde{\mathcal{V}}(g)$ does not depend on the base point $p\in M$.

Proof. Put $\tau_\Delta:=\tau_2-\tau_1\geq 0$ to notice that $g_1(\tau+\tau_\Delta)=g_2(\tau)$.

The main ingredient in the proof of the lemma is the following inequality: for any $(p,\tau_*),(q,\bar\tau)\in M\times[0,\infty)$ with $\bar\tau>\tau_*$,

$$\frac{1}{2\sqrt{\bar\tau+\tau_\Delta}}L^{g_1}_{(p,\tau_*+\tau_\Delta)}(q,\bar\tau+\tau_\Delta)\geq\frac{1}{2\sqrt{\bar\tau}}L^{g_2}_{(p,\tau_*)}(q,\bar\tau). \tag{3.1}$$

This is verified as follows:

$$\begin{aligned}\text{LHS}=&\frac{1}{2\sqrt{\bar\tau+\tau_\Delta}}\inf_\gamma\left\{\int_{\tau_*}^{\bar\tau}\sqrt{\tau+\tau_\Delta}\Big(|\gamma'|^2_{g_2(\tau)}+H_{g_2(\tau)}(\gamma(\tau))\Big)d\tau\right\}\\ \geq&\frac{1}{2\sqrt{\bar\tau}}\inf_\gamma\left\{\int_{\tau_*}^{\bar\tau}\sqrt{\tau}\Big(|\gamma'|^2_{g_2(\tau)}+H_{g_2(\tau)}(\gamma(\tau))\Big)d\tau\right\}\\ =&\text{RHS}.\end{aligned}$$

Here the infimum runs over all curves $\gamma:[\tau_*,\bar\tau]\to M$ with $\gamma(\tau_*)=p$ and $\gamma(\bar\tau)=q$. We also utilized the fact that *for any ancient super Ricci flow satisfying (2.2), H is non-negative on $M\times[0,\infty)$* (cf. [Ch], [Yo]).

Having established the key inequality (3.1), the rest of the proof is routine. Q.E.D.

Lemma 3.2. *Let $(M,g(\tau)),\tau\in[0,\infty)$ be a complete ancient super Ricci flow satisfying Assumption 2.2 with time-derivative $\frac{\partial}{\partial\tau}g$ bounded below. Lift $g(\tau)$ to the universal covering $\bar M$ of M to obtain the lifted ancient super Ricci flow $(\bar M,\bar g(\tau))$. If the asymptotic reduced volume $\widetilde{\mathcal{V}}(g)$ of $(M,g(\tau))$ is strictly positive, then*

$$|\pi_1(M)|=\widetilde{\mathcal{V}}(\bar g)\widetilde{\mathcal{V}}(g)^{-1}<+\infty.$$

The proof of Lemma 3.2 is a modification of that of [An2, Theorem 1.1], a simple fundamental domain argument. Now let us state the following immediate corollary that follows from Lemma 3.2 combined with Lemma 2.6.

Corollary 3.3 (Anderson [An2], Li [Li]). *Let (M,g) be a complete Riemmanian manifold with non-negative Ricci curvature and $(\bar{M},\bar{g})$ be the universal covering of (M,g). If (M,g) has Euclidean volume growth, i.e., $\nu(g)>0$, then*

$$|\pi_1(M)| = \nu(\bar{g})\nu(g)^{-1} < +\infty.$$

§4. Proof of the main theorem

In this section, we describe the proof of Theorem 1.1. It is by contradiction and follows the same line as those of Perelman's pseudolocality theorem [Pe, Theorem 10.1] and Ni's ε-regularity theorem [Ni, Theorem 4.4].

Proof of Theorem 1.1. Fix some $\bar{\tau}>0$. Assume that we have a sequence $\{(M_k^n, g_k(\tau))\}_{k\in\mathbb{Z}^+}, \tau\in[0,\infty)$ of complete non-Gaussian ancient solutions to the Ricci flow with Ricci curvature bounded below such that

$$\widetilde{\mathcal{V}}(g_k) > 1-k^{-1} \text{ for all } k\in\mathbb{Z}^+.$$

Due to Lemma 3.2, we know that $(M_k, g_k(\tau))$ are non-flat Ricci flows.

Applying Perelman's point picking argument in [Pe, §10], we can find a point $(p_k,\tau_k)\in M_k\times[0,\infty)$ such that the norm of the curvature tensor Rm of $g_k(\tau)$ satisfies

$$|\mathrm{Rm}|(x,\tau)\le 2Q_k$$

for any $(x,\tau)\in B_{g_k(\tau_k)}(p_k, kQ_k^{-1/2})\times[\tau_k, \tau_k+Q_k^{-1}(\bar{\tau}+1)]$, where $Q_k := |\mathrm{Rm}|(p_k,\tau_k)>0$.

Consider the rescaled Ricci flows

$$\tilde{g}_k(\tau) := Q_k g_k\Big(\frac{\tau}{Q_k}+\tau_k\Big),\ \tau\in[0,\bar{\tau}+1].$$

By construction, every $\tilde{g}_k(\tau)$ has $|\mathrm{Rm}|(p_k,0)=1$ and $|\mathrm{Rm}|\le 2$ on $B_0(p_k,k)\times[0,\bar{\tau}+1]$, while Lemma 3.1 yields that $\tilde{V}^{\tilde{g}_k}_{(p_k,0)}(\bar{\tau})>1-k^{-1}$.

Case 1. We suppose that each $(M_k^n,\tilde{g}_k(\tau))$ has a uniform lower bound for the injectivity radius at $(p_k,0)$. Then, according to Hamilton's compactness theorem for the Ricci flow [Ha], we can find a subsequence of $\{(M_k^n,\tilde{g}_k(\tau),p_k)\}_{k\in\mathbb{Z}^+}$ converging to a limit Ricci flow denoted by $(M_\infty^n, g_\infty(\tau), p_\infty), \tau\in[0,\bar{\tau}]$. We know that $|\mathrm{Rm}|(p_\infty,0)=1$. However, it is shown that $\tilde{V}^{g_\infty}_{(p_\infty,0)}(\bar{\tau})=1$, which implies that the limit is isometric to the Euclidean space by Theorem 2.5. This is the desired contradiction.

Case 2. If the injectivity radius of $(M_k^n, g_k(0))$ at $p_k \in M_k$, denoted by λ_k, goes to 0 as $k \to \infty$, we consider the further rescaled flow

$$\widehat{g}_k(\tau) := \lambda_k^{-2}\tilde{g}_k(\lambda_k^2\tau), \ \tau \in [0, \lambda_k^{-2}(\bar{\tau}+1)],$$

whose injectivity radius at the base point is 1. Again, due to the compactness theorem of Hamilton [Ha], we can find a limit Ricci flow $(M_\infty^n, g_\infty(\tau), p_\infty)$ which turns out to be the Gaussian soliton. But the injectivity radius at $(p_\infty, 0)$ must be equal to 1. We arrived at the contradiction.

Now the proof of Theorem 1.1 is complete. Q.E.D.

§5. A gap theorem for gradient shrinking Ricci solitons

In this section, we give a sketch of the proof of Corollary 1.3.

Proof of Corollary 1.3. We first construct an ancient solution to the Ricci flow from the given data (M, g, f) (e.g. [CLN, Theorem 4.1]). Define a one-parameter family of diffeomorphisms $\varphi_\tau : M \to M, \tau \in (0, \infty)$ by

$$\frac{d}{d\tau}\varphi_\tau = \frac{\lambda}{\tau}\nabla f \circ \varphi_\tau \text{ and } \varphi_\lambda = \mathrm{id}_M.$$

It is easily seen that the gradient vector field ∇f is complete due to the assumption on the lower bound for Ric. Then we pull back g by φ_τ^{-1} and let $g_0(\tau) := \frac{\tau}{\lambda}(\varphi_\tau^{-1})^* g, \tau \in (0, \infty)$. The soliton equation (1.3) implies that $(M, g_0(\tau))$ is a backward Ricci flow with $g_0(\lambda) = g$. Finally, put $g_S(\tau) := g(\tau + s), \tau \in [0, \infty)$ for some positive $s > 0$.

Then the corollary follows immediately from Theorem 1.1 and the following proposition. Q.E.D.

Proposition 5.1 ([Yo]). *Let $(M, g_S(\tau)), \tau \in [0, \infty)$ be an ancient solution to the Ricci flow determined by a complete gradient shrinking Ricci soliton (M, g, f) with Ricci curvature bounded below. Then*

$$\widetilde{\mathcal{V}}(g_S) = \mathrm{Vol}_f(M). \tag{5.1}$$

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