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Standard bases and algebraic local cohomology for zero dimensional ideals

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Abstract.

Zero-dimensional ideals in the formal power series and the associated vector space consisting of algebraic local cohomology classes are considered in the context of Grothendieck local duality. An algorithmic strategy for computing relative Čech cohomology representations of the algebraic local cohomology classes are described. A new algorithmic method for computing standard bases of a given zero-dimensional ideal is derived by using algebraic local cohomology and the Grothendieck local duality.

§ Introduction

We consider standard bases of zero-dimensional ideals from the point of view of the Grothendieck local residue theory. In [29], we study the Jacobi ideals of isolated hypersurface singularities by using duality and describe, in particular, an effective method for solving membership problems for Jacobi ideals in local rings. In this paper, we give, using the same framework as in [29], a new method for treating standard bases of zero-dimensional ideals.

The theory of standard bases for ideals in power series rings was introduced in 1964 by H. Hironaka in the celebrated work [15] on the resolution of singularities. Since then, standard bases have been extensively utilized in various fields ([8, 10, 20]). There are now two classical and widely used methods that compute standard bases of ideals, in local rings, generated by polynomials. One method is based on Mora's tangent

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cone algorithm and the other is based on Lazard's homogenization technique ([6, 7, 17, 19, 20]). For the zero-dimensional case, there is another approach, called duality method to deal with ideals in local rings, which has also been extensively studied by several authors([3, 18, 21]) in the context of computer algebra. The basis of the duality approach goes back essentially to Macaulay's inverse system theory and its interpretation given by Gröbner([11, 12]).

In this paper, we adopt another classical duality, the Grothendieck duality on local residues, for treating standard bases of zero-dimensional ideals. Key ingredient in this approach is the concept of algebraic local cohomology which was introduced by A. Grothendieck in the context of Algebraic Geometry ([14]) and, independently in 1960 by M. Sato in the theory of hyperfunctions ([25]). We show that the use of algebraic local cohomology provides an efficient method for computing standard bases.

In Section 1, we briefly recall a local duality between the local cohomology supported at a point and power series rings, and introduce a representation of algebraic local cohomology classes using relative Čech cohomology classes. We then introduce a finite dimensional vector space, that gives rise to the dual space of the quotient by a zero-dimensional ideal of formal power series rings, as a subspace of the algebraic local cohomology (cf. [9, 22]). In section 2, we introduce a term ordering and a filtration on algebraic local cohomology classes. We describe a basic strategy for computing algebraic local cohomology classes. In section 3, we propose, by using algebraic local cohomology, a method for computing standard bases of zero dimensional ideals. In appendix, we present the algorithm that computes algebraic local cohomology classes. We also give an example for illustration and timing data of the computations.

§1. The setup

Let X be a neighborhood of the origin O of the n-dimensional complex space $\mathbb{C}^n$ and let Ω_X^n be the sheaf on X of holomorphic differential n-forms. Let $\mathcal{H}^n_{\{O\}}(\Omega_X^n)$ (resp. $\mathcal{H}^n_{[O]}(\Omega_X^n)$) be the local cohomology (resp. the algebraic local cohomology) supported at the origin O. Then, the space $\mathcal{H}^n_{\{O\}}(\Omega_X^n)$ naturally has a structure of Fréchet–Schwartz topological vector space ([4]) and the space $\mathcal{H}^n_{[O]}(\Omega_X^n)$ has a structure of dual Fréchet–Schwartz topological vector space. Recall that the topological vector space $\mathcal{H}^n_{\{O\}}(\Omega_X^n)$ and the space $\mathcal{O}_{X,O}$ of convergent power series at the origin are mutually strong dual via the Grothendieck local residue pairing. The same assertion holds for the space $\mathcal{H}^n_{[O]}(\Omega_X^n)$ and the space

$\hat{\mathcal{O}}_{X,O}$ of formal power series at the origin ([16]). This implies in particular the following fact which plays an important role in our approach ([26]).

Fact. Any algebraic local cohomology class in $\mathcal{H}^n_{[O]}(\Omega^n_X)$ can be regarded as an analytic linear functional that acts on the space of formal power series at the origin.

Now let $U_0 = X$ and $U_j = \{x = (x_1, \ldots, x_n) \in X \mid x_j \neq 0\}$ for $j = 1, \ldots, n$. Consider the pair $(X, X - \{O\})$ and its relative covering $(\mathcal{U}, \mathcal{U}')$ where $\mathcal{U} = \{U_0, U_1, \ldots, U_n\}$ and $\mathcal{U}' = \{U_1, \ldots, U_n\}$. Then, any section of $\mathcal{H}^n_{\{O\}}(\Omega^n_X)$ can be represented as, by taking X sufficiently small if necessary, an element of relative Čech cohomology class in $\check{\mathrm{H}}^n_{\{O\}}(X, \Omega^n_X)$ defined by Leray covering $(\mathcal{U}, \mathcal{U}')$. Since $\mathcal{H}^n_{[O]}(\Omega^n_X)$ is a subspace of $\mathcal{H}^n_{\{O\}}(\Omega^n_X)$, any algebraic local cohomology class in $\mathcal{H}^n_{[O]}(\Omega^n_X)$ can be represented as a finite sum of the form

$$\sum_\lambda c_\lambda \left[\frac{1}{x^\lambda} dx\right] = \sum c_{(l_1 \ldots l_n)} \left[\frac{1}{x_1^{l_1} \ldots x_n^{l_n}} dx\right],$$

where $\lambda = (l_1, \ldots, l_n) \in \mathbb{N}^n_+$, $dx = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ and $[\frac{1}{x^\lambda} dx]$ is a relative Čech cohomology class in $\check{\mathrm{H}}^n_{\{O\}}(X, \Omega^n_X)$. We also use the notation $(\sum[\frac{1}{x^\lambda}])dx$ or $[\sum \frac{1}{x^\lambda}]dx$ for representing algebraic local cohomology classes in $\mathcal{H}^n_{[O]}(\Omega^n_X)$. Note that

$$x^\kappa[\frac{1}{x^\lambda} dx] = \begin{cases} [\frac{1}{x^{\lambda-\kappa}} dx], & l_i > k_i, i = 1, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

where $\kappa = (k_1, \ldots, k_n) \in \mathbb{N}^n$, $\lambda = (l_1, \ldots, l_n) \in \mathbb{N}^n_+$, and $\lambda - \kappa = (l_1 - k_1, \ldots, l_n - k_n)$ (see [23, 27]).

Let us assume that a set F of m polynomials $f_1(x), f_2(x), ..., f_m(x)$ with coefficients in K satisfying $\{x \in X \mid f_1(x) = \cdots = f_m(x) = 0\} = \{O\}$ are given, where $K = \mathbb{Q}$ or $\mathbb{C}$. We introduce a vector space V to be the set of algebraic local cohomology classes $\sum_\lambda c_\lambda \left[\frac{1}{x^\lambda} dx\right]$ with coefficients c_λ in K,

$$V = \{\psi \in \check{\mathrm{H}}^n_{\{O\}}(X, \Omega^n_X) \mid \psi = \sum_\lambda c_\lambda \left[\frac{1}{x^\lambda} dx\right], \ c_\lambda \in K\}.$$

We define a vector space W to be the set of algebraic local cohomology classes in V that are annihilated by these m polynomials $f_1(x)$, $f_2(x)$,

$\dots, f_m(x)$,

$$W = \{\psi \in V \mid f_i(x)\psi = 0,\, i = 1, 2, ..., m\}.$$

Then, the vector space W is the dual vector space of $K[[x]]/\hat{I}$, where $K[[x]]$ stand for the space of formal power series with coefficients in K and $\hat{I}$ is the ideal, in $K[[x]]$, generated by the set

$$F = \{f_1(x), f_2(x), \dots, f_m(x)\}.$$

Since the duality is induced by local residues, we have the non-degenerate residue pairing,

$$\mathrm{res}_O(\ ,\ ) : K[[x]]/\hat{I} \times W \longrightarrow K,$$

which can be regarded as a special case of the Grothendieck local duality ([13]).

The non-degeneracy of the pairing implies the fact that, a formal power series $h \in K[[x]]$ is in the ideal $\hat{I}$ if and only if $\mathrm{res}_O(h, \psi) = 0,\ \forall\psi \in W$. Thus, the ideal $\hat{I}$ is completely determined by the space W of algebraic local cohomology classes via local residues ([29]).

§2. Term ordering on algebraic local cohomology

For generators $f_i(x) = \sum_\alpha a_{i,\alpha}x^\alpha$ $(a_{i,\alpha} \in K)$ of the ideal $\hat{I}$, denote by E_F the set of exponents in f_i,

$$E_F = \{\alpha \in \mathbb{N}^n \mid \exists i, \text{ s.t.}, a_{i,\alpha} \neq 0\}.$$

Set

$$K_S = \mathbb{N}^n \setminus \{\alpha + \alpha' \in \mathbb{N}^n \mid \alpha \in E_F,\, \alpha' \in \mathbb{N}^n\}.$$

Let ℓ_D be a map from $\mathbb{N}^n$ to $\mathbb{N}^n_+$ defined by $\ell_D(\kappa) = \kappa + \mathbf{1}$ for $\kappa \in \mathbb{N}^n$, with $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^n$. We define Λ_S to be the image by ℓ_D of the set K_S. It is easy to see that an algebraic local cohomology $[\frac{1}{x^\lambda}dx]$ with exponent λ is in W if and only if $\lambda \in \Lambda_S$ (cf. Proposition 1 in [29]). Let

$$\Psi_S = \{[\frac{1}{x^\lambda}dx] \mid \lambda \in \Lambda_S\},\ W_S = \mathrm{Span}_K\{[\frac{1}{x^\lambda}dx] \mid \lambda \in \Lambda_S\}.$$

We introduce a term ordering and associated filtration on the space V of algebraic local cohomologies. The aim is to derive an algorithm that compute a space W_P of cohomology classes satisfying

$$W = W_S \oplus W_P$$

by using the notion of induced filtrations on W.

Definition 2.1 (a term order for algebraic local cohomology classes). *For two multi-indices $\lambda = (l_1, \ldots, l_n)$ and $\lambda' = (l'_1, \ldots, l'_n)$ in $\mathbb{N}^n_+$, we denote*

$$\left[\frac{1}{x^{\lambda'}}dx\right] \prec \left[\frac{1}{x^{\lambda}}dx\right] \text{ or } \lambda' \prec \lambda$$

if $|\lambda'| < |\lambda|$, or if $|\lambda'| = |\lambda|$ and there exists $j \in \mathbb{N}_+$ so that $l'_i = l_i$ for $i < j$ and $l'_j < l_j$, where $|\lambda| = l_1 + \cdots + l_n$ for a multi-index $\lambda = (l_1, \ldots, l_n) \in \mathbb{N}^n$.

For a given algebraic local coholomogy class of the form

$$c_\lambda \left[\frac{1}{x^\lambda}dx\right] + \sum_{\lambda' \prec \lambda} c_{\lambda'} \left[\frac{1}{x^{\lambda'}}dx\right], \ c_\lambda \neq 0,$$

we call $\left[\frac{1}{x^\lambda}dx\right]$ the head term and $\left[\frac{1}{x^{\lambda'}}dx\right]$, $\lambda' \prec \lambda$ the lower terms. We denote the head term of a cohomology class ψ by $\mathrm{ht}(\psi)$.

Let Λ_H denote the set of exponents of head terms in W :

$$\Lambda_H = \{\lambda \in \mathbb{N}^n_+ \mid \exists \psi \in W, \text{ s.t. } \mathrm{ht}(\psi) = [\frac{1}{x^\lambda}dx]\}.$$

Let

$$V^{(\lambda)} = \{\psi \in V \mid \mathrm{ht}(\psi) \prec [\frac{1}{x^\lambda}dx]\} \text{ and } W^{(\lambda)} = V^{(\lambda)} \cap W.$$

Then, the subspaces $W^{(\lambda)}$, $\lambda \in \mathbb{N}^n_+$ define a filtration on the finite dimensional vector space W. Let

$$\Lambda_H^{(\lambda)} = \{\lambda' \in \Lambda_H \mid \lambda' \prec \lambda\},$$

It is clear from the definition that there exists, in the space W, an algebraic local cohomology class of the form

$$\psi = [\frac{1}{x^\lambda}dx] + \sum_{\lambda' \prec \lambda, \lambda' \notin \Lambda_H^{(\lambda)}} c_{\lambda,\lambda'}[\frac{1}{x^{\lambda'}}dx] \tag{1}$$

if and only if $\lambda \in \Lambda_H$.

A natural strategy for computing a basis of the vector spave W is thus to find algebraic local cohomology classes from bottom to top by executing the following steps.

- Find a candidate of head term λ.
- Construct an associated set L_λ of candidates of lower terms that satisfies $L_\lambda \cap \Lambda_H^{(\lambda)} = \emptyset$.

- Set

$$\psi = [\frac{1}{x^\lambda}dx] + \sum_{\lambda' \in L_\lambda} c_{\lambda,\lambda'}[\frac{1}{x^{\lambda'}}dx].$$

- Solve the system of equations

$$f_1\psi = f_2\psi = \cdots = f_m\psi = 0.$$

Now let us recall the following result, which will be exploited several times in an algorithm to compute a basis of the space W.

Lemma 1. ([28, 29]) If $\psi \in W$, so are $x_j\psi, j = 1, 2, ..., n$.

The lemma 1 yields the followings ([28, 29]).

Lemma 2. Let $\lambda = (l_1, \ldots, l_n) \in \mathbb{N}_+^n$. If $\lambda \in \Lambda_H$, then, for each $j = 1, 2, ..., n$, $\lambda - \varepsilon_j = (l_1, l_2, .., l_{j-1}, l_j - 1, l_{j+1}, ..., l_n)$ is in $\Lambda_H^{(\lambda)}$, provided $l_j > 1$.

Lemma 3. Let $\lambda = (l_1, \ldots, l_n) \in \mathbb{N}_+^n$. If $\lambda \notin \Lambda_H$, then, $(\lambda + \mathbb{N}^n) \cap \Lambda_H = \emptyset$.

Let Λ_T denote the set of exponents of all terms that appear in W and let $\Lambda_T^{(\lambda)}$ denote the set of exponents of all terms that appear in $W^{(\lambda)}$.

Let $\lambda = (l_1, \ldots, l_n) \in \mathbb{N}_+^n$. Assume that we have already constructed a basis of the vector space $W_S + W^{(\lambda)}$ and λ is a candidate of the head term of an algebraic local cohomology class in W. We may also assume here that $\lambda \notin \Lambda_S$. Let us consider an algebraic local cohomology class ψ of the form

$$\psi = [\frac{1}{x^\lambda}dx] + \sum_{\lambda' \prec \lambda, \lambda' \notin \Lambda_H^{(\lambda)}} c_{\lambda,\lambda'}[\frac{1}{x^{\lambda'}}dx]. \tag{2}$$

Suppose that ψ is in W, then the exponent $\lambda' = (l'_1, l'_2, ...l'_n)$ of each lower term also satisfies the following conditions.

$$\text{If } l'_j > 1, \text{ then } \lambda' - \varepsilon_j \text{ is in } \Lambda_T^{(\lambda)} \cup \Lambda_S, j = 1, 2, ..., n.$$

Note that there is a possibility, for lower terms λ' of ψ, that $\lambda' - \varepsilon_j$ is in $\Lambda_H^{(\lambda)}$.

Assume in addition now that, all algebraic local cohomology classes in the basis of the vector space $W_S + W^{(\lambda)}$ we have already constructed are of the form

$$\psi_\alpha = \left[\frac{1}{x^\alpha}dx\right] + \sum_{\beta \prec \alpha, \beta \notin \Lambda_H^{(\alpha)}} c_{\alpha,\beta}\left[\frac{1}{x^\beta}dx\right], \tag{3}$$

where $\alpha \in \Lambda_H^{(\lambda)}$. Assume also that a set L_λ of exponents of candidates of lower terms of ψ that enjoyes the property $L_\lambda \cap \Lambda_H^{(\lambda)} = \emptyset$ is given. For the algebraic local cohomology

$$\psi = [\frac{1}{x^\lambda}dx] + \sum_{\lambda' \in L_\lambda} c_{\lambda,\lambda'}[\frac{1}{x^{\lambda'}}dx],$$

we set, for each $j = 1, 2, ..., n$, $H_{x_j} = \{\alpha \in \Lambda_H^{(\lambda)} \mid \alpha + \varepsilon_j \in L_\lambda\}$ and set

$$\rho_{x_j}(\psi) = x_j\psi - \sum_{\alpha \in H_{x_j}} c_{\lambda,\alpha+\varepsilon_j}\psi_\alpha, \; j = 1, 2, ..., n.$$

It follows from the lemma 1 that if ψ is in W so are $\rho_{x_j}(\psi)$. Furthermore, since we have assumed that the basis we have constructed are of the form (3), we have $\rho_{x_j}(\psi) = 0, \; j = 1, 2, ..., n$. The neccesary condition is also utilised in the algorithm.

By introducing three lists FH, HList and LList, we describe first few steps of the algorithm for illustration.

A set $F = \{f_1, f_2, ... f_m\}$ of m polynomials with coefficients in K is given. Let E_F be the set of all exponents in f_i, $i = 1, \ldots, m$. Let $\{x^{\gamma_0}, x^{\gamma_1}, \ldots, x^{\gamma_r}\}$ be the minimal basis of the monomial ideal generated by x^κ, $\kappa \in E_F$, where $\gamma_0 \prec \gamma_1 \prec \cdots \prec \gamma_r$ with respect to the total degree lexicographical ordering. Let $\Gamma_0 = \{\gamma_0, \gamma_1, \ldots \gamma_r\}$. Then, the first candidate of an algebraic local cohomology class ψ in $W \setminus W_S$ whose head term $\mathrm{ht}(\psi)$ is smallest in $W_H \setminus W_S$ is of the form

$$\psi = [\frac{1}{x^{\lambda_1}}dx] + c_{\lambda_0}[\frac{1}{x^{\lambda_0}}dx],$$

where $\lambda_1 = \ell_D(\gamma_1)$ and $\lambda_0 = \ell_D(\gamma_0)$. Set FH $= [\gamma_0]$, HList $=$ LList $= [\,]$. Solve a system of simultaneous linear equations

$$f_i\psi = 0, \, i = 1, \ldots, m.$$

- If the system has no solution, $\gamma_1 \notin \Lambda_H$ and $(\gamma_1 + \mathbb{N}^n) \cap \Lambda_H = \emptyset$ (cf. Lemma 3). Then the next candidate of the exponent of the head term of an algebraic local cohomology class ψ in W

is $\lambda_2 = \ell_D(\gamma_2)$ and the exponents of the possible lower terms are given by λ_1 and λ_0.

$$\psi = [\frac{1}{x^{\lambda_2}}dx] + c_{\lambda_1}[\frac{1}{x^{\lambda_1}}dx] + c_{\lambda_0}[\frac{1}{x^{\lambda_0}}dx].$$

Add γ_1 to the top of FH.

- If the system has a solution, then ψ is in W, $\gamma_1 \in \Lambda_H$, and the next candidate of the exponent of the head term is the term next to λ_1 which is the smallest one, or λ_2. The corresponding possible candidates of lower terms differ by cases. Add γ_1 to the top of HList and γ_0 to the top of LList.

By continuing the same kind of arguments, one can construct a basis of the vector space in an algorithmic way. Besides the algebraic local cohomology classes, the computation provides the following three lists.

$$\text{HList} = \{\kappa \in \mathbb{N}^n \mid \ell_D(\kappa) \text{ is an exponent of a head term in } \Psi_P.\},$$

$$\text{LList} = \{\kappa \in \mathbb{N}^n \mid \ell_D(\kappa) \text{ is exponents of lower terms in } \Psi_P.\},$$

$$\text{FH} = \{\kappa \in \mathbb{N}^n \mid \kappa = \gamma_0 \text{ or } \ell_D(\kappa) \text{ was nominated for a candidate for a head terms in } \Psi_P \text{ whereas } \kappa \notin \text{HList}.\}.$$

In the next section, these three lists will play a key role to construct standard bases.

§3. Standard bases

In this section, we propose a new method for computing standard bases. The key idea is the use of the Grothendieck local duality ([5, 26]).

Recall that, there is a natural residue pairing, denoted by $\mathrm{res}_{\{O\}}(\ ,\ )$, between the quotient space $K[[x]]/\hat{I}$ and the vector space W,

$$\mathrm{res}_{\{O\}}(\ ,\ ) : K[[x]]/\hat{I} \times W \longrightarrow K.$$

The pairing is nondegenerate according to the Grothendieck local duality theorem ([13]). As we have already mentioned, the result above implies in particular that a given formal power series $p \in K[[x]]$ is in the ideal $\hat{I}$ generated by m polynomials $F = \{f_1, f_2, ..., f_m\}$ if and only if p satisfies

$$\mathrm{res}_{\{O\}}(p, \psi) = 0, \quad \forall \psi \in \Psi_S \cup \Psi_P, \tag{4}$$

where Ψ_S, Ψ_P are bases of the vector spaces W_S, W_P respectively ([29]).

Now we consider the total-degree reverse lexicographical ordering on $K[[x]]$ and let $\widehat{\mathrm{ht}}(p)$ denote, with respect to this ordering, the head term of formal power series p in $K[[x]]$.

Theorem 1. *The polynomials* x^τ *for* $\tau \in$ FH *with* $\tau \notin$ LList *and*

$$p_\tau(x) := x^\tau - \sum_{\kappa \in \mathrm{HList}} c_{\ell_D(\kappa),\ell_D(\tau)} x^\kappa$$

for $\tau \in$ FH $\cap$ LList *give rise to the standard basis of the ideal* $\hat{I}$ *with respect to total-degree reverse lexicographical ordering.*

Proof. It is clear from the definition that the monomial ideal in $K[[x]]$ generated by $\widehat{\mathrm{ht}}(f), f \in \hat{I}$ coincides with that generated by $x^\tau, \tau \in$ FH. That is, the list FH is the set of exponents of head terms of the standard basis. By the condition (4), it is obvious that if $\tau \in$ FH is not in LList, then the monomial $p_\tau(x) := x^\tau$ itself is in the ideal $\hat{I}$, and if $\tau \in$ FH is in LList, then

$$p_\tau(x) := x^\tau - \sum_{\kappa \in \mathrm{HList}} c_{\ell_D(\kappa),\ell_D(\tau)} x^\kappa$$

is also in $\hat{I}$.

The theorem says that once one has a basis of the space W of algebraic local cohomology classes, one can directly compute a standard basis of the ideal $\hat{I}$ from the basis. One can also derive an algorithm that compute, from algebraic local cohomology classes, Gröbner basis of the primary component supported at the origin of the zero dimensional ideal, when the ideal in question is generated by polynomials. Note that the algorightm is free from primary decomposition algorithm. These algorithms have been implemeted by the authors in Risa/Asir ([1, 28, 29]).

Example. Let J be the Jacobi ideal in the ring $K[[x, y]]$ of Z_{12} singularity $f = x^3y + xy^4 + x^2y^3$,

$$J = (3x^2y + y^4 + 2xy^3, x^3 + 4xy^3 + 3x^2y^2).$$

One obtains 4 algebraic local cohomology classes $\psi_{(1,5)}, \psi_{(2,4)}, \psi_{(1,6)}$, and $\psi_{(1,7)}$ as elements of the basis Ψ_P by Algorithm 1 in appendix, where

$$\psi_{(1,5)} = [\frac{1}{xy^5} - \frac{1}{3}\frac{1}{x^3y^2}]dx \wedge dy, \quad \psi_{(2,4)} = [\frac{1}{x^2y^4} - 4\frac{1}{x^4y} - \frac{2}{3}\frac{1}{x^3y^2}]dx \wedge dy,$$

$$\psi_{(1,6)} = [\frac{1}{xy^6} - \frac{1}{3}\frac{1}{x^3y^3} + \frac{1}{x^4y}]dx \wedge dy,$$

$$\psi_{(1,7)} = [\frac{1}{xy^7} - \frac{1}{3}\frac{1}{x^3y^4} + \frac{7}{33}\frac{1}{x^2y^5} + \frac{4}{3}\frac{1}{x^5y} - \frac{14}{99}\frac{1}{x^3y^3}$$

$$+\frac{5}{33}\frac{1}{x^4y^2}+\frac{14}{33}\frac{1}{x^4y}]dx \wedge dy.$$

Let M be the coefficient matrix $(c_{(\ell_D(\kappa),\ell_D(\kappa'))})_{\kappa\in\mathrm{HList},\kappa'\in\mathrm{List}}$ of algebraic local cohomology classes in Ψ_P with respect to classes with exponents in $\Lambda_T \setminus \Lambda_S$ arranged according to the term ordering $\prec$, where List $=$ HList $\cup$ LList and $c_{(\ell_D(\kappa),\ell_D(\kappa))} = 1$ for $\kappa \in$ HList. Then M is

$$M = \begin{pmatrix} 1 & -\frac{1}{3} & \frac{7}{33} & 0 & \frac{4}{3} & \frac{5}{33} & -\frac{14}{99} & 0 & 0 & \frac{14}{33} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -4 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} \end{pmatrix},$$

which is arranged according to

$$\mathrm{List} = [(0,6),(2,3),(1,4),(0,5),(4,0),(2,2),(1,3),(0,4),(3,0),(2,1)].$$

From the way of construction, M is in a reduced row echelon form.

Since we have FH $= [(0,7),(1,4),(3,0),(2,1)]$ and $(0,7) \notin$ LList, we look at the 10-th, third and the final column of M.

From the tenth column ${}^t(\frac{14}{33},1,-4,0)$, we have $x^3+4xy^3-y^5-\frac{14}{33}y^6$.

From the third column ${}^t(\frac{7}{33},0,0,0)$, we have $xy^4-\frac{7}{33}y^6$.

From the final column ${}^t(0,0,-\frac{2}{3},-\frac{1}{3})$, coefficients of the cohomology class $[\frac{1}{x^3y^2}]dx \wedge dy$, we have $x^2y+\frac{2}{3}xy^3+\frac{1}{3}y^4$ as a constituent of the standard basis of J with respect to the degree reverse lexicographical ordering.

Thus,

$$\{x^2y+\frac{2}{3}xy^3+\frac{1}{3}y^4, x^3-\frac{14}{33}y^6-y^5+4xy^3, xy^4-\frac{7}{33}y^6, y^7\}$$

is the standard basis of J with respect to the degree reverse lexicographical ordering.

Example. Let $f = x^3+x^2y^3+(a+by)y^{9+p}$, with $a \neq 0, p > 0$. For $p = 1,2,3,...$, these polynomials define an infinite series $J_{3,k}$ of bimodal singularities. Consider the Jacobi ideal $J = (f_x, f_y)$ in the ring $K[[x,y]]$ with parameters a and b. Since $\mathrm{E}_F = \{(2,0),(1,3),(0,8+p),(0,9+p)\}$, we have

$$\mathrm{K}_S = [(0,0),(0,1),(1,0),(0,2),(1,1),(0,3),(1,2),$$

$$(0,4),(0,5),\cdots,(0,8+p)],$$

which implies $\Gamma = [(2,0),(1,3),(0,9+p)]$. Thus, the basis Ψ_S consists of the following algebraic local cohomology classes

$$[\frac{1}{xy}],[\frac{1}{xy^2}],[\frac{1}{x^2y}],[\frac{1}{xy^3}],[\frac{1}{x^2y^2}],[\frac{1}{xy^4}],[\frac{1}{x^2y^3}],[\frac{1}{xy^5}],[\frac{1}{xy^6}],\dots,[\frac{1}{xy^{8+p}}].$$

One finds, as a basis Ψ_P of W_P, 5 algebraic local cohomology classes

$$\begin{aligned}
\psi_{(2,4)} &= [\frac{1}{x^2y^4}] - \frac{2}{3}[\frac{1}{x^3y}],\ \psi_{(2,5)} = [\frac{1}{x^2y^5}] - \frac{2}{3}[\frac{1}{x^3y^2}],\\
\psi_{(1,9+p)} &= [\frac{1}{xy^{9+p}}] + \frac{1}{2}(9+p)a[\frac{1}{x^2y^6}] - \frac{1}{3}(9+p)a[\frac{1}{x^3y^3}],\\
\psi_{(1,10+p)} &= [\frac{1}{xy^{10+p}}] + \frac{1}{2}(9+p)a[\frac{1}{x^2y^7}] + \frac{1}{2}(10+p)b[\frac{1}{x^2y^6}]\\
&\quad -\frac{1}{3}(9+p)a[\frac{1}{x^3y^4}] - \frac{1}{3}(10+p)b[\frac{1}{x^3y^3}] + \frac{2}{9}(9+p)a[\frac{1}{x^4y}],\\
\psi_{(1,11+p)} &= [\frac{1}{xy^{11+p}}] + \frac{1}{2}(9+p)a[\frac{1}{x^2y^8}] + \frac{1}{2}(10+p)b[\frac{1}{x^2y^7}]\\
&\quad -\frac{1}{3}(9+p)a[\frac{1}{x^3y^5}] - \frac{1}{3}(10+p)b[\frac{1}{x^3y^4}]\\
&\quad +\frac{2}{9}(9+p)a[\frac{1}{x^4y^2}] + \frac{2}{9}(10+p)b[\frac{1}{x^4y}].
\end{aligned}$$

We omit the notation $dx \wedge dy$ of differential 2-forms in the representations of algebraic local cohomology classes.

Since $\mathrm{FH} = [(2,0),(1,5),(0,11+p)]$, the standard basis of J is

$$\{x^2 + \frac{2}{3}xy^3, xy^5 - \frac{9+p}{2}ay^{8+p} - \frac{10+p}{2}by^{9+p}, y^{11+p}\}.$$

Note that the method presented in this paper can be extendable to handle parametric cases. The detail will appear elsewhere.

§4. Appendix

We give an algorithm for computing a basis of the vector space W and an example to illustrate the algorithm. We give timing data of the computations of W, standard basis and Gröbner basis of the primary component.

Algorithm

Let $\Gamma = \Gamma_0 \setminus \{\gamma_0\}$ where $\Gamma_0 = \{\gamma_0, \gamma_1, \ldots, \gamma_r\}$ is the set of exponents of the minimal basis of the monomial basis generated by x^κ, $\kappa \in E_F$. Let $\{d_1, \ldots, d_g\}$ be the set of total degrees of exponents in Γ, that is, $\{d_1, \ldots, d_g\} = \{|\gamma_1|, \ldots, |\gamma_r|\}$, where $d_1 < \cdots < d_g$ and $g = \sharp\{|\gamma_1|, \ldots, |\gamma_r|\}$. Let classify exponents in Γ with respect to the total degree and make a list G of lists as follows:

$$\mathrm{G} = [\mathrm{G}_{d_1}, \ldots, \mathrm{G}_{d_g}],$$

where G_{d_i} is the list $\{\gamma \in \Gamma \mid |\gamma| = d_i\}$, $i = 1, \ldots, g$.

The lists of exponents FH, HH, and NH are used in the algorithm to construct a list of candidates of head term CH that consists of terms of the same total degree. The list of exponents LL, UU, RR, EL NL are used for constructing CL, which is the list of candidates of lower terms for a chosen candidate of head terms. We assume that the elements in each list are always ordered by $\prec$.

Algorithm 1 (A basis of W).

Input: $f_1, f_2, \ldots f_m$
Output: a basis Ψ_S *of* W_S*, a basis* Ψ_P *of* W_P
and lists HList,LList *and* FH

/* *Initialization* */

LL := [ ], UU := [ ], RR := [ ]
HH := [ ], LList := [ ], HList := [ ]

/* *Computations of* Ψ_S *: initialization of* FH, FG *and* G */

set E_F*: the set of all exponents in* f_i, $i = 1, \ldots, m$.
compute the set $\{\gamma_0, \ldots, \gamma_r\}$ *of exponents of the Gröbner basis of the monomial ideal generated by* x^κ, $\kappa \in \mathrm{E}_F$.
$\Gamma := [\gamma_1, \ldots, \gamma_r]$
compute K_S.
construct Ψ_S.
FH := $[\gamma_0]$, FG := FH, EL := FH
G := $[\mathrm{G}_{d_1}, \ldots, \mathrm{G}_{d_g}]$, *where* $\mathrm{FH} \cup \mathrm{G}_{d_1} \cup \cdots \cup \mathrm{G}_{d_g} = \{\gamma_0, \ldots, \gamma_r\}$
Flag := 0

/* *Main block* */
while Flag = 0{
if HH = [ ]{
if G = [ ], Flag := 1 *and terminate the computation.*
else if G $\neq$ [ ], CH := car(G)*: the first element of* G,
G := cdr(G)

```
    }
if HH ≠ [ ]{
    NH := {κ + ε_j | κ ∈ HH, j = 1,...,n} − {α + α' | α ∈
    FH, α' ∈ ℕ^n}
    if G = [ ]{
        if NH ≠ [ ], CH := NH
        else if NH = [ ], Flag := 1 and terminate the com-
        putation.
    }
    if G ≠ [ ]{
        let deg_h and deg_g be the degrees of the first elements
        in HH and G.
        if deg_g − deg_h = 1{
            CH := NH ∪ car(G)
            G := cdr(G): a list obtained by removing the
            first element from G.
        }
        else if deg_g − deg_h > 1{
            CH := NH
        }
    }
    HH := [ ]
}
while CH ≠ [ ]{
    κ := car(CH), first element of CH
    if UU ≠ [ ]{
        EL := append([κ' ∈ UU | κ' ≺ κ], EL)
        UU := [κ' ∈ UU | κ ≺ κ']
    }
    if LL ≠ [ ]{
        NL := {κ' + ε_j | κ' ∈ LL, κ' + ε_j − ε_i ∈ LList ∪ K_S ∪
        HList, j = 1, 2, ..., n, i = 1, 2, .., j − 1, j + 1, ..., n}
        EL := append([κ' ∈ NL | κ' ≺ κ], EL)
        UU := append([κ' ∈ NL | κ ≺ κ'], UU)
    }
    CL := append(EL, append(RR, LList))
    Execute the procedure "Solve" for κ, CL.
    if the systems in the procedure Solve have a solution ψ {
        ψ_λ := ψ
        add ψ_λ in Ψ_P.
        add κ in HH.
        add κ in HList.
```

```
            LL := [κ' ∈ EL | c_{ℓ_D(κ),ℓ_D(κ')} ≠ 0],
            if LL ≠ [ ]{
                LList := append(LL, LList)
                RR := EL − LL
                EL := [ ]
            }
        }
        else{
            add κ in FH.
            add κ in EL.
            LL := [ ]
        }
        CH := cdr(CH)
       }
}
```

The use of the following procedure that executes the actual computation to construct algebraic local cohomology classes saves the cost of computation considerably.

Procedure Solve

Let $\lambda = \ell_D(\kappa), \mathrm{L}_\lambda = \ell_D(\mathrm{CL} - \mathrm{FG})$, where FG is a list defined to be $\mathrm{FG} = \Gamma_0 \cap \mathrm{FH}$.
Set $\psi_0 = [\frac{1}{x^\lambda}dx] + \sum_{\lambda' \in \mathrm{L}_\lambda} c_{\lambda,\lambda'}[\frac{1}{x^{\lambda'}}dx]$.
Compute $\rho_{x_j}(\psi_0) := x_j\psi_0 - \sum_{\alpha \in \mathrm{H}_{x_j}} c_{\lambda,\alpha+\varepsilon_j}\psi_\alpha$,
where $\mathrm{H}_{x_j} := \{\alpha \in \Lambda_H^{(\lambda)} \mid \alpha + \varepsilon_j \in \mathrm{L}_\lambda\}$.
Solve the simultaneous equations derived from the condition that the coefficients of lower terms of $\rho_{x_j}(\psi_0)$, not belonging to Ψ_S, are equal to zero for all j.
If the equations have solutions,
put

$$\psi = \psi_0 + \sum_{\lambda' \in \ell_D(\mathrm{FG})} c_{\lambda,\lambda'}[\frac{1}{x^{\lambda'}}dx].$$

Solve the system of linear equations

$$f_i\psi = 0, \quad i = 1, \dots, m.$$

Then, the algebraic local cohomology class $\psi_\lambda = \psi$ is in W if and only if the system is solvable.

Note. Basic version of the algorithm was implemented by T. Abe, the first and second authors ([1, 28]) and the present version has been implemented by the third author in the computer algebra system Risa/Asir ([24]).

Example

We give an examples to illustrate Algorithm 1.

Let $f = x^4y + y^8 + xy^7$. The function $f_0 = x^4y + y^8$ defines N_{25} quasihomogeneous singularity at the origin of $\mathbb{C}^2$ (cf. [2]). The monomial xy^7 is the smallest upper monomial in the monomial basis of the quotient space $K[[x, y]]/(f_x, f_y)$. Let us compute W for f and the standard basis of the Jacobi ideal $J = (f_x, f_y)$. We omit the notations of differential two forms $dx \wedge dy$ in the representations of cohomology classes for the sake of simplicity.

The partial derivatives of the function f are $f_x = 4x^4y + y^7$ and $f_y = x^4 + 8y^7 + 7xy^6$. We have $\Gamma_0 = \{(3,1),(4,0),(0,7),(1,6)\}$ and $\Gamma = \{(4,0),(0,7),(1,6)\}$.
Initialization: FH = FG = EL = $[(3,1)]$, G = $[[(4,0)],[(0,7),(1,6)]]$, LL = UU = RR = HH = HList = LList = [ ].

Since HH = [ ] and G $\neq$ [ ], CH = $[(4,0)]$ and $G = [[(0,7),(1,6)]]$

CH = $[(4,0)]$

$\kappa = (4,0)$, CH = [ ].

CL = $[(3,1)]$.

Put $\psi = [\frac{1}{x^5y} + c_{(4,2)}\frac{1}{x^4y^2}]$.

$f_x\psi = 0$, $f_y\psi = 0$ have no solution.

FH = $[(3,1),(4,0)]$, EL = $[(3,1),(4,0)]$, LL = [ ].

Since HH = [ ] and G $\neq$ [ ], CH = $[(0,7),(1,6)]$ and G = [ ].

CH = $[(0,7),(1,6)]$.

$\kappa = (0,7)$, CH = $[(1,6)]$.

CL = $[(3,1),(4,0)]$.

Put $\psi = [\frac{1}{xy^8} + c_{(4,2)}\frac{1}{x^4y^2} + c_{(5,1)}\frac{1}{x^5y}]$.

From $f_x\psi = 0$ and $f_y\psi = 0$,

$c_{(4,2)} = -\frac{1}{4}$ and $c_{(5,1)} = -8$.

$\psi = [\frac{1}{xy^8} - 8\frac{1}{x^5y} - \frac{1}{4}\frac{1}{x^4y^2}]$

HH = $[(0,7)]$, HList = $[(0,7)]$.

LL = $[(3,1),(4,0)]$.

LList = $[(3,1),(4,0)]$, RR = [ ], EL = [ ].

$\kappa = (1,6)$, CH $= [\ ]$.
NL $= \{(3,2),(4,1),(5,0)\}$.
EL $= [(3,2),(4,1),(5,0)]$.
UU $= [\ ]$.
CL $= [(3,1),(4,0),(3,2),(4,1),(5,0)]$.
Put $\psi_0 = [\frac{1}{x^2y^7} + c_{(4,3)}\frac{1}{x^4y^3} + c_{(5,2)}\frac{1}{x^5y^2} + c_{(6,1)}\frac{1}{x^6y}]$.
From $\rho_x(\psi_0) = \rho_y(\psi_0) = 0$,
$c_{(5,2)} = c_{(6,1)} = c_{(4,3)} = c_{(5,2)} = 0$.
Put $\psi = [\frac{1}{x^2y^7} + c_{(4,2)}\frac{1}{x^4y^2} + c_{(5,1)}\frac{1}{x^5y}]$.
From $f_x\psi = 0$ and $f_y\psi = 0$,
$c_{(4,2)} = 0$ and $c_{(5,1)} = -7$.
$\psi = [\frac{1}{x^2y^7} - 7\frac{1}{x^5y}]$
HH $= [(0,7),(1,6)]$, HList $= [(1,6),(0,7)]$.
LL $= [\ ]$.

Since HH $\neq [\ ]$ and G $= [\ ]$,
NH $= \{(0,8),(1,7),(2,6)\}$,CH $= [(0,8),(1,7),(2,6)]$ and HH $= [\ ]$.

CH $= [(0,8),(1,7),(2,6)]$.
$\kappa = (0,8)$, CH $= [(1,7),(2,6)]$.
CL $= [(3,2),(4,1),(5,0),(3,1),(4,0)]$.
Put $\psi_0 = [\frac{1}{xy^9} + c_{(4,3)}\frac{1}{x^4y^3} + c_{(5,2)}\frac{1}{x^5y^2} + c_{(6,1)}\frac{1}{x^6y}]$.
$\rho_x(\psi_0) = \rho_y(\psi_0) = 0$ have no solution.
FH $= [(3,1),(4,0),(0,8)]$.
EL $= [(3,2),(4,1),(5,0),(0,8)]$.
LL $= [\ ]$.
$\kappa = (1,7)$, CH $= [(2,6)]$.
CL $= [(3,1),(4,0),(3,2),(4,1),(5,0),(0,8)]$.
Put $\psi_0 = [\frac{1}{x^2y^8} + c_{(4,3)}\frac{1}{x^4y^3} + c_{(5,2)}\frac{1}{x^5y^2} + c_{(6,1)}\frac{1}{x^6y} + c_{(1,9)}\frac{1}{xy^9}]$
From $\rho_x(\psi_0) = \rho_y(\psi_0) = 0$, $c_{(4,3)} = \frac{27}{128}$, $c_{(5,2)} = -\frac{1}{4}$, $c_{(6,1)} = -8$ and $c_{(1,9)} = -\frac{27}{32}$.
Put $\psi = [\frac{1}{x^2y^8} + \frac{27}{128}\frac{1}{x^4y^3} - \frac{1}{4}\frac{1}{x^5y^2} - 8\frac{1}{x^6y} - \frac{27}{32}\frac{1}{xy^9} + c_{(4,2)}\frac{1}{x^4y^2} + c_{(5,1)}\frac{1}{x^5y}]$.
From $f_x\psi = 0$ and $f_y\psi = 0$, $c_{(4,2)} = c_{(5,1)} = 0$.
$\psi = [\frac{1}{x^2y^8} + \frac{27}{128}\frac{1}{x^4y^3} - \frac{1}{4}\frac{1}{x^5y^2} - 8\frac{1}{x^6y} - \frac{27}{32}\frac{1}{xy^9}]$.
HH $= [(1,7)]$, HList $= [(1,7),(1,6),(0,7)]$.
LL $= [(3,2),(4,1),(5,0),(0,8)]$.
LList $= [(3,1),(4,0),(3,2),(4,1),(5,0),(0,8)]$.
RR $= [\ ]$, EL $= [\ ]$
$\kappa = (2,6)$, CH $= [\ ]$.
NL $= [(3,3),(4,2),(5,1),(6,0),(0,9),(1,8)]$.

EL $= [(3,3),(4,2),(5,1),(6,0)]$.
UU $= [(0,9),(1,8)]$.
CL $= [(3,1),(4,0),(3,2),(4,1),(5,0),(3,3),(4,2),$
$(5,1),(6,0),(0,8)]$.
Put $\psi_0 = [\frac{1}{x^3y^7} + c_{(4,3)}\frac{1}{x^4y^3} + c_{(5,2)}\frac{1}{x^5y^2} + c_{(6,1)}\frac{1}{x^6y}$
$+c_{(4,4)}\frac{1}{x^4y^4} + c_{(5,3)}\frac{1}{x^5y^3} + c_{(6,2)}\frac{1}{x^6y^2} + c_{(7,1)}\frac{1}{x^7y} + c_{(1,9)}\frac{1}{xy^9}]$.
From $\rho_x(\psi_0) = \rho_y(\psi_0) = 0$, $c_{(4,3)} = c_{(5,2)} = c_{(4,4)} = c_{(5,3)} = c_{(6,2)} = c_{(7,1)} = c_{(1,9)} = 0$ and $c_{(6,1)} = -7$.
Put $\psi = [\frac{1}{x^3y^7} - 7\frac{1}{x^6y} + c_{(4,2)}\frac{1}{x^4y^2} + c_{(5,1)}\frac{1}{x^5y}]$.
From $f_x\psi = 0$ and $f_y\psi = 0$, $c_{(4,2)} = c_{(5,1)} = 0$.
$\psi = [\frac{1}{x^3y^7} - 7\frac{1}{x^6y}]$.
HH $= [(1,7),(2,6)]$, HList $= [(2,6),(1,7),(1,6),(0,7)]$.
LL $= [\,]$.

Since HH $\neq [\,]$ and G $= [\,]$, NH $= \{(2,7)\}$, CH $= [(2,7)]$, HH $= [\,]$.

CH $= [(2,7)]$.

$\kappa = (2,7)$, CH $= [\,]$.
EL $= [(3,3),(4,2),(5,1),(6,0),(0,9),(1,8)]$.
UU $= [\,]$.
CL $= [(3,1),(4,0),(3,2),(4,1),(5,0),(3,3),(4,2),(5,1),$
$(6,0),(0,8),(0,9),(1,8)]$.
Put $\psi_0 = [\frac{1}{x^3y^8} + c_{(4,3)}\frac{1}{x^4y^3} + c_{(5,2)}\frac{1}{x^5y^2} + c_{(6,1)}\frac{1}{x^6y}$
$+ c_{(4,4)}\frac{1}{x^4y^4} + c_{(5,3)}\frac{1}{x^5y^3} + c_{(6,2)}\frac{1}{x^6y^2} + c_{(7,1)}\frac{1}{x^7y}$
$+ c_{(1,9)}\frac{1}{xy^9} + c_{(1,10)}\frac{1}{xy^{10}} + c_{(2,9)}\frac{1}{x^2y^9}]$.
From $\rho_x(\psi_0) = \rho_y(\psi_0) = 0$,
$c_{(5,2)} = c_{(6,1)} = c_{(1,9)} = c_{(4,3)} = 0$,
$c_{(5,2)} = \frac{27}{128}$, $c_{(4,4)} = -\frac{729}{4096}$, $c_{(6,2)} = -\frac{1}{4}$,
$c_{(7,1)} = -8$, $c_{(1,10)} = \frac{729}{1024}$ and $c_{(2,9)} = -\frac{27}{32}$.
Put $\psi = [\frac{1}{x^3y^8} - \frac{729}{4096}\frac{1}{x^4y^4} + \frac{27}{128}\frac{1}{x^5y^3} - \frac{1}{4}\frac{1}{x^6y^2}$
$-8\frac{1}{x^7y} + \frac{729}{1024}\frac{1}{xy^{10}} - \frac{27}{32}\frac{1}{x^2y^9} + c_{(4,2)}\frac{1}{x^4y^2} + c_{(5,1)}\frac{1}{x^5y}]$.
From $f_x\psi = 0$ and $f_y\psi = 0$, $c_{(4,2)} = c_{(5,1)} = 0$.
$\psi = [\frac{1}{x^3y^8} - \frac{729}{4096}\frac{1}{x^4y^4} + \frac{27}{128}\frac{1}{x^5y^3} - \frac{1}{4}\frac{1}{x^6y^2}$
$-8\frac{1}{x^7y} + \frac{729}{1024}\frac{1}{xy^{10}} - \frac{27}{32}\frac{1}{x^2y^9}]$.
HH $= [(2,7)]$, HList $= [(2,7),(2,6),(1,7),(1,6),(0,7)]$.
LL $= [(3,3),(4,2),(5,1),(6,0),(0,9),(1,8)]$.
LList $= [(3,1),(4,0),(3,2),(4,1),(5,0),(3,3),(4,2),$
$(5,1),(6,0),(0,8),(0,9),(1,8)]$.
RR $= [\,]$, EL $= [\,]$.

Since HH $\neq [\,]$, NH $= [\,]$.

Thus, the computation terminates.

The algorithm outputs five algebraic local cohomology classes in Ψ_P. Since

$$\mathrm{HList} = [(2,7),(2,6),(1,7),(1,6),(0,7)],$$

$$\mathrm{LList} = [(3,1),(4,0),(3,2),(4,1),(5,0),(3,3),(4,2),(5,1),(6,0),\\ (0,8),(0,9),(1,8)]$$

and

$$\mathrm{FH} = [(3,1),(4,0),(0,8)],$$

the standard basis of the Jacobi ideal J is

$$\{x^3y + \frac{1}{4}y^7, x^4 + 8y^7 + 7xy^6, y^8 + \frac{27}{32}xy^7\}.$$

Timing data

We give timing data for the computation of W, standard basis and Gröbner basis of typical hypersurface isolated singularities E_{6k+2}, Q_{6k+5} and $J_{k,0}$ ([2]). The second column in the table is the defining polynomial of the singularity and the third column is Milnor number μ of the singularity. The fourth, fifth and seventh column are the timing data for computing W, standard basis and Gröbner basis respectively. In these three column, the first line is CPU time and the second is GC time. The time is given in second. We use the version 20051019 of Risa/Asir for the test on Dell Dimension4700C, Pentium(R)4 CPU 3.00GHz, 2.99GHz, 504MB RAM.

$E_{6k+2} : x^3 + y^{3k+2} + xy^{2k+2}$.

k	$f(x,y)$	μ	W	standard	Gröbner
5	$x^3 + y^{17} + xy^{12}$	32	0.04688 +0.01563	0	0
10	$x^3 + y^{22}x + y^{32}$	62	0.1719 +0.2344	0	0 +0.01563
20	$x^3 + y^{42}x + y^{62}$	122	2.406 +2.188	0	0.03125 +0.03125
30	$x^3 + y^{62}x + y^{92}$	182	11.14 +9.219	0 +0.01563	0.07813 +0.07813
40	$x^3 + y^{82}x + y^{122}$	242	34.14 +22.33	0.03125 +0.01563	0.2344 +0.125

$J_{k,0} : x^3 + x^2y^k + y^{3k} + xy^{2k+1}$.

k	$f(x,y)$	μ	W	standard	Gröbner
5	$x^3 + y^5x^2 + y^{11}x + y^{15}$	28	0.04688 +0.01563	0	0
10	$x^3 + y^{10}x^2 + y^{21}x + y^{30}$	58	0.2344 +0.2031	0	0.04688
20	$x^3 + y^{20}x^2 + y^{41}x + y^{60}$	118	3.109 +1.797	0.03125 +0.03125	0.1719 +0.1094
30	$x^3 + y^{30}x^2 + y^{61}x + y^{90}$	170	13.69 +4.828	0.1406 +0.01563	0.8125 +0.1719
40	$x^3 + y^{40}x^2 + y^{81}x + y^{120}$	238	39.28 +9.609	0.3594 +0.03125	2.313 +0.4844

$Q_{6k+5} : x^3 + yz^2 + xy^{2k+1} + y^{3k+2}$.

k	$f(x,y,z)$	μ	W	standard	Gröbner
5	$x^3 + y^{11}x + y^{17} + z^2y$	35	0.0625 +0.0625	0	0
10	$x^3 + y^{21}x + y^{32} + z^2y$	65	0.4063 +0.2656	0.01563	0.01563 +0.01563
20	$x^3 + y^{41}x + y^{62} + z^2y$	125	3.25 +3.078	0.01563	0.0625 +0.04688
30	$x^3 + y^{61}x + y^{92} + z^2y$	185	13.75 +8.797	0.03125	0.125 +0.1094
40	$x^3 + y^{81}x + y^{122} + z^2y$	245	38.39 +19.75	0.04688	0.2344 +0.1719

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