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Čech–Dolbeault cohomology and the $\bar{\partial}$-Thom class

Tatsuo Suwa

The Čech–de Rham cohomology theory for C^∞ manifolds was initiated by A. Weil [17] and is fully explained in [5]. Combined with the Chern–Weil theory of characteristic classes, this cohomology theory, especially its relative version, is particularly suited to describe the localization theory of characteristic classes related to Chern polynomials. The techniques are effective even on some singular varieties (e.g., [11], [12], [14], [16]). We may express the Poincaré, Alexander and Thom homomorphisms in terms of this cohomology as well, see [6] for a combinatorial definition of these homomorphisms. Note also that the Thom class of a vector bundle can be naturally defined in this cohomology and, in the case of complex vector bundles, it coincides with the localization of the top Chern class of the pull-back bundle with respect to the diagonal section [14]. This point of view helps us very much in clarifying the local behavior of a section near its zeros.

In this article, we discuss complex analytic analogues of the above, replacing the Čech–de Rham cohomology by the "Čech–Dolbeault cohomology" and the Chern classes by the classes introduced by M. Atiyah in [2]. To be a little more specific, we first develop a theory of Čech–Dolbeault cohomology, its relative version and related topics in Sections 1, 2 and 3. In Section 4, we introduce the notion of "$\bar{\partial}$-integration along the fiber" for later use. We recall, in Sections 5 and 6, the Atiyah classes for holomorphic vector bundles and the localization theory of these classes, which is exploited in detail in [1]. In Section 7, we define the "$\bar{\partial}$-Thom class" of a holomorphic vector bundle as the localization of the top Atiyah class of the pull-back bundle with respect to the diagonal section. We then prove the "$\bar{\partial}$-Thom isomorphism", and in this situation, we have a satisfactory expression of the "$\bar{\partial}$-Alexander homomorphism". Finally, in Section 8, we indicate a way of getting a good description of the $\bar{\partial}$-Alexander homomorphism in a more general setting.

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For this, we use the idea of deformation to the normal bundle, which is a special case of the graph construction of R. MacPherson. Here we follow the description of W. Fulton [7].

The materials in this article may be used to give a simple geometric proof of the holomorphic Lefschetz fixed point formula [10].

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§1. Čech–Dolbeault cohomology

Let M be a complex manifold of dimension n. For an open set U of M, we denote by $A^{p,q}(U)$ the vector space of C^∞ (p,q)-forms on U. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be an open covering of M. We assume that I is an ordered set such that if $U_{\alpha_0\dots\alpha_r} \neq \emptyset$, the induced order on the subset $\{\alpha_0,\dots,\alpha_r\}$ is total. We set

$$I^{(r)} = \{\,(\alpha_0,\dots,\alpha_r) \mid \alpha_0 < \cdots < \alpha_r,\ \alpha_\nu \in I\,\}$$

and denote by $C^r(\mathcal{U}, A^{p,q})$ the direct product

$$C^r(\mathcal{U}, A^{p,q}) = \prod_{(\alpha_0,\dots,\alpha_r)\in I^{(r)}} A^{p,q}(U_{\alpha_0\dots\alpha_r}),$$

where we set $U_{\alpha_0\dots\alpha_r} = U_{\alpha_0}\cap\cdots\cap U_{\alpha_r}$. Thus an element σ in $C^r(\mathcal{U}, A^{p,q})$ assigns to each $(\alpha_0,\dots,\alpha_r)$ in $I^{(r)}$ a form $\sigma_{\alpha_0\dots\alpha_r}$ in $A^{p,q}(U_{\alpha_0\dots\alpha_r})$. The coboundary operator $\delta : C^r(\mathcal{U}, A^{p,q}) \to C^{r+1}(\mathcal{U}, A^{p,q})$ is defined as in the usual Čech theory, i.e.,

$$(\delta\sigma)_{\alpha_0\dots\alpha_{r+1}} = \sum_{\nu=0}^{r+1}(-1)^\nu \sigma_{\alpha_0\dots\widehat{\alpha_\nu}\dots\alpha_{r+1}},$$

where $\widehat{\ }$ means the letter under it is to be omitted and each form $\sigma_{\alpha_0\dots\widehat{\alpha_\nu}\dots\alpha_{r+1}}$ is to be restricted to $U_{\alpha_0\dots\alpha_{r+1}}$. This together with the operator

$$\bar\partial : C^r(\mathcal{U}, A^{p,q}) \longrightarrow C^r(\mathcal{U}, A^{p,q+1})$$

make $C^\bullet(\mathcal{U}, A^{p,\bullet})$ a double complex for each $p = 0,\dots,n$. The simple complex associated to this is denoted by $(A^{p,\bullet}(\mathcal{U}), \bar D)$ or simply by $A^{p,\bullet}(\mathcal{U})$. Thus

$$A^{p,q}(\mathcal{U}) = \bigoplus_{q'+r=q} C^r(\mathcal{U}, A^{p,q'})$$

and the differential $\bar D = \bar D^q : A^{p,q}(\mathcal U) \to A^{p,q+1}(\mathcal U)$ is given by

$$(1.1) \qquad (\bar D\sigma)_{\alpha_0\ldots\alpha_r} = \sum_{\nu=0}^{r}(-1)^\nu \sigma_{\alpha_0\ldots\widehat{\alpha_\nu}\ldots\alpha_r} + (-1)^r\,\bar\partial\sigma_{\alpha_0\ldots\alpha_r}.$$

In particular, for small values of r, we have

$$\begin{aligned}(\bar D\sigma)_{\alpha_0} &= \bar\partial\sigma_{\alpha_0}, \quad (\bar D\sigma)_{\alpha_0\alpha_1} = \sigma_{\alpha_1} - \sigma_{\alpha_0} - \bar\partial\sigma_{\alpha_0\alpha_1} \quad \text{and}\\ (\bar D\sigma)_{\alpha_0\alpha_1\alpha_2} &= \sigma_{\alpha_1\alpha_2} - \sigma_{\alpha_0\alpha_2} + \sigma_{\alpha_0\alpha_1} + \bar\partial\sigma_{\alpha_0\alpha_1\alpha_2}.\end{aligned}$$

We call $(A^{p,\bullet}(\mathcal U), \bar D)$ the p-th Čech–Dolbeault complex and its q-th cohomology, denoted by $H^{p,q}_{\bar D}(\mathcal U)$, the Čech–Dolbeault cohomology of type (p,q) associated to the covering $\mathcal U$. We denote by $H^{p,q}_{\bar\partial}(M)$ the usual Dolbeault cohomology of type (p,q) of M.

Theorem 1.2. *The restriction map $A^{p,q}(M) \to C^0(\mathcal U, A^{p,q}) \subset A^{p,q}(\mathcal U)$ induces an isomorphism*

$$H^{p,q}_{\bar\partial}(M) \xrightarrow{\sim} H^{p,q}_{\bar D}(\mathcal U).$$

Proof. We consider one of the spectral sequences associated to the double complex $C^\bullet(\mathcal U, A^{p,\bullet})$:

$${}'E_2^{q,r} = H^q_{\bar\partial} H^r_\delta(C^\bullet(\mathcal U, A^{p,\bullet})) \Longrightarrow H^{p,q+r}_{\bar D}(\mathcal U).$$

Since the Čech complex of (p,q)-forms on $\mathcal U$ is acyclic, we have $H^r_\delta(C^\bullet(\mathcal U, A^{p,\bullet})) = 0$ for $r > 0$, while $H^0_\delta(C^\bullet(\mathcal U, A^{p,\bullet})) = A^{p,\bullet}(M)$, the Dolbeault complex. Hence we see that $H^{p,q}_{\bar D}(\mathcal U) \simeq {}'E_2^{q,0} = H^{p,q}_{\bar\partial}(M)$.
Q.E.D.

We say that a covering $\mathcal U = \{U_\alpha\}$ of M is *analytically good*, if every non-empty intersection $U_{\alpha_0\ldots\alpha_r}$ is biholomorphic to a domain of holomorphy in $\mathbb C^n$.

Theorem 1.3. *If $\mathcal U$ is analytically good, we have an isomorphism*

$$H^{p,q}_{\bar D}(\mathcal U) \simeq H^q(M, \Omega^p),$$

where Ω^p denotes the sheaf of germs of holomorphic p-forms on M.

Proof. We consider the other spectral sequence associated to the double complex $C^\bullet(\mathcal U, A^{p,\bullet})$:

$${}''E_2^{q,r} = H^q_\delta H^r_{\bar\partial}(C^\bullet(\mathcal U, A^{p,\bullet})) \Longrightarrow H^{p,q+r}_{\bar D}(\mathcal U).$$

Since $\mathcal{U}$ is analytically good, by the $\bar\partial$-Poincaré lemma, $H^r_{\bar\partial}(C^\bullet(\mathcal{U},A^{p,\bullet}))=0$ for $r>0$. While $H^0_{\bar\partial}(C^\bullet(\mathcal{U},A^{p,\bullet}))=C^\bullet(\mathcal{U},\Omega^p)$. Hence we see that $H^{p,q}_{\bar D}(\mathcal{U})\simeq {}''E_2^{q,0}=H^q(\mathcal{U},\Omega^p)$, which is isomorphic to $H^q(M,\Omega^p)$, as $\mathcal{U}$ is analytically good. Q.E.D.

Since every complex manifold admits an analytically good covering, from Theorems 1.2 and 1.3, we recover the Dolbeault theorem

$$H^q(M,\Omega^p)\simeq H^{p,q}_{\bar\partial}(M).$$

We define the "cup product"

$$A^{p,q}(\mathcal{U})\times A^{p',q'}(\mathcal{U})\longrightarrow A^{p+p',q+q'}(\mathcal{U})$$

by assigning to σ in $A^{p,q}(\mathcal{U})$ and τ in $A^{p',q'}(\mathcal{U})$ the element $\sigma\smile\tau$ in $A^{p+p',q+q'}(\mathcal{U})$ given by

$$(\sigma\smile\tau)_{\alpha_0\dots\alpha_r}=\sum_{\nu=0}^{r}(-1)^{(p+q-\nu)(r-\nu)}\sigma_{\alpha_0\dots\alpha_\nu}\wedge\tau_{\alpha_\nu\dots\alpha_r}.$$

Then $\sigma\smile\tau$ is linear in σ and τ and we have

$$\bar D(\sigma\smile\tau)=\bar D\sigma\smile\tau+(-1)^{p+q}\sigma\smile\bar D\tau.$$

Thus it induces the cup product

$$H^{p,q}_{\bar D}(\mathcal{U})\times H^{p',q'}_{\bar D}(\mathcal{U})\longrightarrow H^{p+p',q+q'}_{\bar D}(\mathcal{U})$$

compatible, via the isomorphism of Theorem 1.2, with the product in the Dolbeault cohomology induced by the exterior product of forms.

Now we define the integration on the Čech–Dolbeault cohomology. Let M and $\mathcal{U}=\{U_\alpha\}_{\alpha\in I}$ be as above and $\{R_\alpha\}_{\alpha\in I}$ a system of honeycomb cells adapted to $\mathcal{U}$ ([11], see also [14]).

Suppose M is compact, then each R_α is compact and we may define the integration

$$\int_M : A^{n,n}(\mathcal{U})\longrightarrow\mathbb{C}$$

by the sum

$$\int_M\sigma=\sum_{r=0}^{n}\left(\sum_{(\alpha_0,\dots,\alpha_r)\in I^{(r)}}\int_{R_{\alpha_0\dots\alpha_r}}\sigma_{\alpha_0\dots\alpha_r}\right)$$

for σ in $A^{n,n}(\mathcal{U})$. Then we see that

(i) if $\bar D\sigma = 0$, then the sum does not depend on the choice of $\{R_\alpha\}$,
(ii) if $\sigma = \bar D\tau$, then $\int_M \sigma = 0$.
Hence it induces the integration on the cohomology

$$\int_M : H^{n,n}_{\bar D}(\mathcal{U}) \longrightarrow \mathbb{C},$$

which is compatible, via the isomorphism of Theorem 1.2, with the usual integration on the Dolbeault cohomology $H^{n,n}_{\bar\partial}(M)$. Also the bilinear pairing

$$A^{p,q}(\mathcal{U}) \times A^{n-p,n-q}(\mathcal{U}) \longrightarrow A^{n,n}(\mathcal{U}) \longrightarrow \mathbb{C}$$

defined as the composition of the cup product and the integration induces the Kodaira–Serre duality

$$(1.4)\quad KS : H^{p,q}_{\bar\partial}(M) \simeq H^{p,q}_{\bar D}(\mathcal{U}) \xrightarrow{\sim} H^{n-p,n-q}_{\bar D}(\mathcal{U})^* \simeq H^{n-p,n-q}_{\bar\partial}(M)^*.$$

If V is an analytic subvariety of codimension p in M, by integration on V, V defines a class, denoted by $[V]$, in $H^{n-p,n-p}_{\bar\partial}(M)^*$. The corresponding class in $H^{p,p}_{\bar\partial}(M)$ is called the Dolbeault class of V.

§2. Relative Čech–Dolbeault cohomology

Now let S be a closed set in M. Let $U_0 = M \setminus S$ and U_1 a neighborhood of S in M and consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M with the order $0 < 1$. We have

$$A^{p,q}(\mathcal{U}) = C^0(\mathcal{U}, A^{p,q}) \oplus C^1(\mathcal{U}, A^{p,q-1}).$$

Thus a cochain σ in $A^{p,q}(\mathcal{U})$ is written as

$$\sigma = (\sigma_0, \sigma_1, \sigma_{01}),$$

where σ_i is a (p,q)-forms on U_i, $i = 0, 1$, and σ_{01} is a $(p, q-1)$-form on U_{01}. We denote by $A^{p,q}(\mathcal{U}, U_0)$ the subspace of $A^{p,q}(\mathcal{U})$ consisting of elements σ with $\sigma_0 = 0$ so that we have the exact sequence

$$0 \longrightarrow A^{p,q}(\mathcal{U}, U_0) \longrightarrow A^{p,q}(\mathcal{U}) \longrightarrow A^{p,q}(U_0) \longrightarrow 0.$$

We see that $\bar D$ maps $A^{p,q}(\mathcal{U}, U_0)$ into $A^{p,q+1}(\mathcal{U}, U_0)$. Denoting by $H^{p,q}_{\bar D}(\mathcal{U}, U_0)$ the q-th cohomology of the complex $(A^{p,\bullet}(\mathcal{U}, U_0), \bar D)$, we have the long exact sequence

$$\cdots \to H^{p,q-1}_{\bar\partial}(U_0) \to H^{p,q}_{\bar D}(\mathcal{U}, U_0) \to H^{p,q}_{\bar D}(\mathcal{U}) \to H^{p,q}_{\bar\partial}(U_0) \to \cdots .$$

In view of the fact that $H^{p,q}_{\bar D}(\mathcal{U}) \simeq H^{p,q}_{\bar\partial}(M)$, we set

$$H^{p,q}_{\bar\partial}(M, M \setminus S) = H^{p,q}_{\bar D}(\mathcal{U}, U_0).$$

Suppose S is compact (M may not be) and let $\{R_0, R_1\}$ be a system of honey-comb cells adapted to $\mathcal{U}$. Then we may assume that R_1 is compact and we have the integration on $A^{n,n}(\mathcal{U}, U_0)$ given by

$$\int_M \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}.$$

This again induces the integration on the cohomology

$$\int_M : H^{n,n}_{\bar D}(\mathcal{U}, U_0) \longrightarrow \mathbb{C}.$$

The cup product $A^{p,q}(\mathcal{U}) \times A^{n-p,n-q}(\mathcal{U}) \to A^{n,n}(\mathcal{U})$ induces a pairing $A^{p,q}(\mathcal{U}, U_0) \times A^{n-p,n-q}(U_1) \to A^{n,n}(\mathcal{U}, U_0)$, which, followed by the integration, gives a bilinear pairing

$$A^{p,q}(\mathcal{U}, U_0) \times A^{n-p,n-q}(U_1) \longrightarrow \mathbb{C}.$$

This induces a homomorphism

$$\bar A : H^{p,q}_{\bar\partial}(M, M \setminus S) = H^{p,q}_{\bar D}(\mathcal{U}, U_0) \longrightarrow H^{n-p,n-q}_{\bar\partial}(U_1)^*, \tag{2.1}$$

which we call the $\bar\partial$-Alexander homomorphism.

If M is compact, the following diagram is commutative :

$$\begin{array}{ccc} H^{p,q}_{\bar\partial}(M, M \setminus S) & \xrightarrow{j^*} & H^{p,q}_{\bar\partial}(M) \\ {\scriptstyle \bar A}\downarrow & & \wr\downarrow{\scriptstyle KS} \\ H^{n-p,n-q}_{\bar\partial}(U_1)^* & \longrightarrow & H^{n-p,n-q}_{\bar\partial}(M)^*. \end{array}$$

Suppose that S is a compact complex submanifold of M and that there exists a holomorphic retraction $r : U_1 \to S$, i.e., a holomorphic map with $r \circ i = 1_S$, where $i : S \hookrightarrow U_1$ is the embedding. Then we have the exact sequence

$$0 \to \operatorname{Ker} r_* \to H^{n-p,n-q}_{\bar\partial}(U_1)^* \xrightarrow{r_*} H^{n-p,n-q}_{\bar\partial}(S)^* \to 0$$

and $i_* : H^{n-p,n-q}_{\bar\partial}(S)^* \to H^{n-p,n-q}_{\bar\partial}(U_1)^*$ gives a splitting of the above, where r_* and i_* denote, respectively, the transposed of the pull-backs $r*$: $H^{n-p,n-q}_{\bar\partial}(S) \to H^{n-p,n-q}_{\bar\partial}(U_1)$ and $i^* : H^{n-p,n-q}_{\bar\partial}(U_1) \to H^{n-p,n-q}_{\bar\partial}(S)$.

We consider this situation more in detail in Section 7.

§3. Homomorphism induced by a holomorphic map

Let $f : M' \longrightarrow M$ be a holomorphic map of complex manifolds and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ and $\mathcal{V} = \{V_\lambda\}_{\lambda \in J}$ be, respectively, open coverings of M and M'. We assume that there is a map $\iota : J \to I$ such that $f(V_\lambda) \subset U_{\iota(\lambda)}$ for all λ in J. We define a $\mathbb{C}$-linear map

$$f_\iota^* : A^{p,q}(\mathcal{U}) \longrightarrow A^{p,q}(\mathcal{V})$$

by assigning to σ in $A^{p,q}(\mathcal{U})$ the element $f_\iota^*(\sigma)$ in $A^{p,q}(\mathcal{V})$ given by

$$f_\iota^*(\sigma)_{\lambda_0 \dots \lambda_p} = f^* \sigma_{\iota(\lambda_0) \dots \iota(\lambda_p)}.$$

Then the map f_ι^* induces a $\mathbb{C}$-linear map

$$f^* : H_{\bar{D}}^{p,q}(\mathcal{U}) \longrightarrow H_{\bar{D}}^{p,q}(\mathcal{V}),$$

which is independent of ι and is compatible, via the isomorphism of Theorem 1.2, with the one induced from the pull-back for the Dolbeault cohomology.

Next we consider the relative case. Thus let M and M' be complex manifolds and S and S' closed sets in M and M', respectively. Let $f : M' \longrightarrow M$ be a holomorphic map with $f(S') \subset S$ and $f(M' \setminus S') \subset M \setminus S$. Set $U_0 = M \setminus S$ and $V_0 = M' \setminus S'$ and let U_1 and V_1 be open neighborhoods of S and S', respectively, such that $f(V_1) \subset U_1$. With these, set $\mathcal{U} = \{U_0, U_1\}$ and $\mathcal{V} = \{V_0, V_1\}$. We have a $\mathbb{C}$-linear map

$$f^* : A^{p,q}(\mathcal{U}, U_0) \longrightarrow A^{p,q}(\mathcal{V}, V_0),$$

which induces a $\mathbb{C}$-linear map

$$f^* : H_{\bar{\partial}}^{p,q}(M, M \setminus S) \longrightarrow H_{\bar{\partial}}^{p,q}(M', M' \setminus S').$$

§4. $\bar{\partial}$-integration along the fiber

Let M be a complex manifold of dimension n and $\pi : E \to M$ a holomorphic vector bundle of rank ℓ. We identify M with the image of the zero section of E and set $W_0 = E \setminus M$. Let W_1 be a neighborhood of M in E and consider the Čech–Dolbeault cohomology with respect to the covering $\mathcal{W} = \{W_0, W_1\}$ of E. Choosing a Hermitian metric on E, we may reduce the structure group to the unitary group so that we have a disk bundle $T_1 \to M$ in W_1. We set $T_0 = E \setminus \operatorname{Int} T_1$. Then $\{T_0, T_1\}$ is a system of honey-comb cells adapted to $\mathcal{W}$.

We denote by π_1 and π_{01}, respectively, the restrictions of π to T_1 and T_{01}. Thus $\pi_1 : T_1 \to M$ is a closed (complex) ℓ-disk bundle and

$\pi_{01} : T_{01} \to M$ a $(2\ell-1)$-sphere bundle. Note that the orientation of T_{01} is opposite to that of the boundary ∂T_1 of T_1.

In this situation, we have the usual integration along the fiber (e.g., [14, Ch.II, 4]):

$$(\pi_1)_* : A^r(W_1) \to A^{r-2\ell}(M) \quad \text{and} \quad (\pi_{01})_* : A^{r-1}(W_{01}) \to A^{r-2\ell}(M).$$

Note that $(\pi_1)_*$ sends a (p,q)-form to a $(p-\ell, q-\ell)$-form:

$$(\pi_1)_* : A^{p,q}(W_1) \longrightarrow A^{p-\ell,q-\ell}(M).$$

For a $(p, q-1)$-form σ_{01} on $W_{01} = W_0 \cap W_1$, the form $(\pi_{01})_*(\sigma_{01})$ consists of $(p-\ell, q-\ell)$ and $(p-\ell+1, q-\ell-1)$-components. We denote by $(\bar{\pi}_{01})_*(\sigma_{01})$, the $(p-\ell, q-\ell)$-component of $(\pi_{01})_*(\sigma_{01})$ so that we have the map

$$(\bar{\pi}_{01})_* : A^{p,q-1}(W_{01}) \longrightarrow A^{p-\ell,q-\ell}(M).$$

Remark 4.1. If $p = n+\ell$ or $q = \ell$, $(\bar{\pi}_{01})_* = (\pi_{01})_*$.

With these, we define

$$\bar{\pi}_* : A^{p,q}(\mathcal{W}, W_0) \longrightarrow A^{p-\ell,q-\ell}(M)$$

by assigning to $\sigma = (0, \sigma_1, \sigma_{01})$ the $(p-\ell, q-\ell)$-form $(\pi_1)_*\sigma_1 + (\bar{\pi}_{01})_*\sigma_{01}$ on M.

From [14, Ch.II, Proposition 5.1], we have the following :

Proposition 4.2. *(Projection formula)*
(1) *For σ in $A^{p,q}(\mathcal{W}, W_0)$ and θ in $A^{p',q'}(M)$,*

$$\bar{\pi}_*(\sigma \smile \pi^*\theta) = \bar{\pi}_*\sigma \wedge \theta,$$

where $\pi^*\theta$ is considered as an element in $A^{p',q'}(W_1)$.
(2) *If M is compact, then for σ in $A^{p,q}(\mathcal{W}, W_0)$ and θ in $A^{(n+\ell-p, n+\ell-q)}(M)$,*

$$\int_E \sigma \smile \pi^*\theta = \int_M \bar{\pi}_*\sigma \wedge \theta.$$

Note that, in (2) above, $\bar{\pi}_*$ is in fact equal to π_* of [14] (cf. Remark 4.1). Also, from [14, Ch.II, Proposition 5.2], we have

Proposition 4.3. *We have*

$$\bar{\pi}_* \circ \bar{D} = \bar{\partial} \circ \bar{\pi}_*.$$

By the above proposition, $\bar\pi_*$ induces a homomorphism

$$\bar\pi_* : H^{p,q}_{\bar\partial}(E, E \setminus M) \longrightarrow H^{p-\ell,q-\ell}_{\bar\partial}(M).$$

From the projection formula, we have

Theorem 4.4. *If M is compact, the following diagram is commutative :*

$$\begin{array}{ccc} H^{p,q}_{\bar\partial}(E, E \setminus M) & \xrightarrow{\bar A} & H^{n+\ell-p,n+\ell-q}_{\bar\partial}(W_1)^* \\ \Big\downarrow{\scriptstyle \bar\pi_*} & & \Big\downarrow{\scriptstyle \pi_*} \\ H^{p-\ell,q-\ell}_{\bar\partial}(M) & \underset{KS}{\xrightarrow{\sim}} & H^{n+\ell-p,n+\ell-q}_{\bar\partial}(M)^*, \end{array}$$

where π_* on the second downarrow denotes the dual of the pull-back π^*.

§5. Atiyah classes

The "Atiyah class" of a holomorphic vector bundle was introduced in [2] as the obstruction to the existence of complex analytic connections for the bundle. Here we reconstruct the theory using C^∞ connections of type $(1,0)$. This point of view, which appears in [2] in the framework of principal bundles, allows us to compare directly the Atiyah forms and the Chern forms and is particularly suited for developing the localization theory, using a Chern–Weil type theory adapted to the Čech–Dolbeault cohomology. For details we refer to [1]. Also, as to the Chern–Weil theory, we refer to, e.g., [3], [4], [13], [14]. Here we use the notation in [14] (with connection and curvature matrices transposed and r and ℓ interchanged).

Throughout this section, we let M be a complex manifold of dimension n and E a holomorphic vector bundle of rank ℓ over M.

For an open set U in M, we denote by $A^r(U)$ the complex vector space of complex valued C^∞ r-forms on U. Also, we let $A^r(U, E)$ be the vector space of "E-valued r-forms" on U, i.e., C^∞ sections of the bundle $\bigwedge^r (T^{\mathbb{C}}_{\mathbb{R}} M)^* \otimes E$ on U, where $(T^{\mathbb{C}}_{\mathbb{R}} M)^*$ denotes the dual of the complexification of the real tangent bundle $T_{\mathbb{R}} M$ of M. Thus $A^0(U)$ is the ring of C^∞ functions and $A^0(U, E)$ is the $A^0(U)$-module of C^∞ sections of E on U.

Recall that a *connection* for E is a $\mathbb{C}$-linear map

$$\nabla : A^0(M, E) \longrightarrow A^1(M, E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla(s) \quad \text{for} \quad f \in A^0(M) \text{ and } s \in A^0(M, E).$$

Note that a connection ∇ is a local operator. Thus, if $s^{(\ell)} = (s_1, \ldots, s_\ell)$ is a frame (ℓ C^∞ sections linearly independent everywhere) of E on an open set U, we have the connection matrix $\theta = (\theta_{ij})$ with entries θ_{ij} 1-forms on U defined by

$$\nabla(s_i) = \sum_{j=1}^{\ell} \theta_{ji} \otimes s_j.$$

Definition 5.1. A connection ∇ for E is of type $(1,0)$ if the entries of the connection matrix with respect to a holomorphic frame are forms of type $(1,0)$. In this case, we also say that ∇ is a $(1,0)$-connection.

Note that the above property of ∇ does not depend on the choice of a holomorphic frame and that a holomorphic vector bundle always admits a $(1,0)$-connection.

If ∇ is a connection for E, it induces a $\mathbb{C}$-linear map

$$\nabla : A^1(M,E) \longrightarrow A^2(M,E)$$

satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s) \quad \text{for} \quad \omega \in A^1(M) \text{ and } s \in A^0(M,E).$$

The composition

$$K = \nabla \circ \nabla : A^0(M,E) \longrightarrow A^2(M,E)$$

is called the *curvature* of ∇. It is not difficult to see that K is $A^0(M)$-linear so that we may think of it as a C^∞ 2-form with coefficients in the bundle $H = \mathrm{Hom}(E,E)$.

If ∇ is a $(1,0)$-connection for E, we may write

$$K = K^{2,0} + K^{1,1}$$

with $K^{2,0}$ and $K^{1,1}$, respectively, a $(2,0)$-form and a $(1,1)$-form with coefficients in H.

In fact, let θ be the connection matrix of ∇ with respect to a (local) holomorphic frame of E. Then the curvature matrix κ is given by $\kappa = d\theta + \theta \wedge \theta$. We have the decomposition $\kappa = \kappa^{2,0} + \kappa^{1,1}$ with

$$\kappa^{2,0} = \partial\theta + \theta \wedge \theta \qquad \text{and} \qquad \kappa^{1,1} = \bar\partial\theta.$$

The components $K^{2,0}$ and $K^{1,1}$ are locally represented by $\kappa^{2,0}$ and $\kappa^{1,1}$, respectively. Thus $K^{1,1}$ is a $\bar\partial$-closed $(1,1)$-form with coefficients in H, since it is locally $\bar\partial$-exact. By a change of (holomorphic) frame, we get a curvature matrix similar to $\kappa^{1,1}$. Thus we may define, for each elementary symmetric polynomial σ_p ($p = 1, 2, \dots$), a $\bar\partial$-closed C^∞ (p,p)-form $\sigma_p(K^{1,1})$ on M. Locally it is given by $\sigma_p(\kappa^{1,1})$ which is the coefficient of t^p in the expansion of $\det(I + t\kappa^{1,1})$.

Definition 5.2. We set $a^p(\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^p \sigma_p(K^{1,1})$ and call it the Atiyah form of type (p,p) associated to ∇. It is a $\bar\partial$-closed (p,p)-form on M.

Remark 5.3. The p-th Chern form $c^p(\nabla)$ of ∇ is defined by $c^p(\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^p \sigma_p(K)$, which is a closed $2p$-form having $(2p,0),\dots,(p,p)$ components. The Atiyah form $a^p(\nabla)$ is the (p,p)-component of $c^p(\nabla)$. In particular, $a^n(\nabla) = c^n(\nabla)$.

The following is proved using the construction of the Chern difference forms (cf. [4], [14]).

Proposition 5.4. *Suppose we have $r+1$ $(1,0)$-connections $\nabla_0, \dots \nabla_r$ for E. Then there exists a $(p, p-r)$-form $a^p(\nabla_0,\dots,\nabla_r)$, alternating in the $r+1$ entries and satisfying*

$$\sum_{\nu=0}^{r}(-1)^\nu a^p(\nabla_0,\dots,\widehat{\nabla_\nu},\dots,\nabla_r) + (-1)^r \bar\partial a^p(\nabla_0,\dots,\nabla_r) = 0.$$

Proof. Recall that there exists a $(2p-r)$-form $c^p(\nabla_0,\dots,\nabla_r)$, alternating in the $r+1$ entries and satisfying

$$\sum_{\nu=0}^{r}(-1)^\nu c^p(\nabla_0,\dots,\widehat{\nabla_\nu},\dots,\nabla_r) + (-1)^r d c^p(\nabla_0,\dots,\nabla_r) = 0.$$

In fact the form $c^p(\nabla_0,\dots,\nabla_r)$ is constructed as follows (cf. [4, p. 65], here we use the sign convention of [14, p. 69]). We consider the vector bundle $E\times\mathbb{R}^r \to M\times\mathbb{R}^r$ and let $\tilde\nabla$ be the connection for it given by

$$\tilde\nabla = \left(1 - \sum_{\nu=1}^{r} t_\nu\right)\nabla_0 + \sum_{\nu=1}^{r} t_\nu \nabla_\nu,$$

where $(t_1,\dots,t_r)$ denotes a coordinate system on $\mathbb{R}^r$. Denoting by Δ^r the standard r-simplex in $\mathbb{R}^r$ and by $\pi : M\times\Delta^r \to M$ the projection, we have the integration along the fiber

$$\pi_* : A^{2p}(M\times\Delta^p) \longrightarrow A^{2p-r}(M).$$

Then we have $c^p(\nabla_0,\dots,\nabla_r) = \pi_*(c^p(\tilde{\nabla}))$. Thus $c^p(\nabla_0,\dots,\nabla_r)$ is a $(2p-r)$-form having $(2p-r,0),\dots,(p,p-r)$ components. Let $a^p(\nabla_0,\dots,\nabla_r)$ be the $(p,p-r)$-component of $c^p(\nabla_0,\dots,\nabla_r)$. Q.E.D.

In particular, if $r=1$, we have

$$a^p(\nabla_1) - a^p(\nabla_0) = \bar{\partial} a^p(\nabla_0,\nabla_1).$$

Thus, if ∇ is a $(1,0)$-connection for E, the class of $a^p(\nabla)$ in $H^{p,p}_{\bar{\partial}}(M)$ does not depend on the choice of ∇.

Definition 5.5. The Atiyah class of E of type (p,p) is the class of $a^p(\nabla)$ in $H^{p,p}_{\bar{\partial}}(M)$, where ∇ is a $(1,0)$-connection for E.

§6. Localization of Atiyah classes

In this section, we first define the Atiyah class in the Čech–Dolbeault cohomology and then discuss a special type of localization of Atiyah classes, namely the localization of the top Atiyah class by a section. We refer to [1] for other types of localization, including localization by frames and localization by Bott type vanishing.

Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M as in Section 1. Also, let E be a holomorphic vector bundle over M. For each α, we choose a $(1,0)$-connection ∇_α for E on U_α, and for the collection $\nabla_* = (\nabla_\alpha)_\alpha$, we define the element $a^p(\nabla_*)$ in $A^{p,p}(\mathcal{U}) = \oplus_{r=0}^p C^r(\mathcal{U}, A^{p,p-r})$ by

$$a^p(\nabla_*)_{\alpha_0\dots\alpha_r} = a^p(\nabla_{\alpha_0},\dots,\nabla_{\alpha_r}).$$

Then we have $\bar{D}a^p(\nabla_*) = 0$ by the identity in Proposition 5.4.

Proposition 6.1. *The class $[a^p(\nabla_*)]$ in $H^{p,p}_{\bar{D}}(\mathcal{U})$ does not depend on the choice of the collection of $(1,0)$-connections ∇_* and corresponds to the Atiyah class $a^p(E)$ by the isomorphism of Theorem 1.2.*

Proof. Let $\nabla'_* = (\nabla'_\alpha)_\alpha$ be another collection of $(1,0)$-connections for E. We define an element σ in $A^{p,p-1}(\mathcal{U})$ by

$$\sigma_{\alpha_0\dots\alpha_r} = \sum_{\nu=0}^{r} (-1)^\nu a^p(\nabla_{\alpha_0},\dots,\nabla_{\alpha_\nu},\nabla'_{\alpha_\nu},\dots,\nabla'_{\alpha_r}).$$

Then we have

$$a^p(\nabla'_*) - a^p(\nabla_*) = \bar{D}\sigma$$

so that $[a^p(\nabla'_*)] = [a^p(\nabla_*)]$ in $H^{p,p}_{\bar{D}}(\mathcal{U})$.

Moreover, comparing with the class defined by a global $(1,0)$-connection, we see that the class $[a^p(\nabla_*)]$ corresponds to the class $a^p(E)$ in $H^{p,p}_{\bar{\partial}}(M)$ under the isomorphism of Theorem 1.2. Q.E.D.

The following proposition follows from the corresponding vanishing theorem for Chern forms (cf., e.g., [14, Ch.II, Proposition 9.1]).

Proposition 6.2. *Let s be a non-vanishing holomorphic section of E on an open set U in M and ∇ an s-trivial $(1,0)$-connection for E on U, then, on U,*

$$a^\ell(\nabla) \equiv 0.$$

Let M be a complex manifold of dimension n and S a closed set in M. Also let E be a holomorphic vector bundle over M and s a non-vanishing holomorphic section of E on $M \setminus S$. Let $U_0 = M \setminus S$ and U_1 a neighborhood of S and consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. Let ∇_0 be an s-trivial $(1,0)$-connection for E on U_0 and ∇_1 an arbitrary $(1,0)$-connection for E on U_1. The top Atiyah class $a^\ell(E)$ is represented by the Čech-Dolbeault cocycle

$$a^\ell(\nabla_*) = (a^\ell(\nabla_0), a^\ell(\nabla_1), a^\ell(\nabla_0, \nabla_1)).$$

By Proposition 6.2, $a^\ell(\nabla_0) = 0$ and it is in fact in $A^{\ell,\ell}(\mathcal{U}, U_0)$ and determines a class $a^\ell(E, s)$ in $H^{\ell,\ell}_{\bar\partial}(M, M \setminus S)$. We call it the localization of $a^\ell(E)$ by s.

Remark 6.3. Recall that, in the above situation, the localization $c^\ell(E, s)$ in $H^{2\ell}(M, M \setminus S)$ of the top Chern class is represented by the Čech–de Rham cocycle $(0, c^\ell(\nabla_1), c^\ell(\nabla_0, \nabla_1))$. The form $a^\ell(\nabla_1)$ is the (ℓ,ℓ)-component of $c^\ell(\nabla_1)$ and $a^\ell(\nabla_0, \nabla_1)$ the $(\ell, \ell-1)$-component of $c^\ell(\nabla_0, \nabla_1)$.

§7. The $\bar\partial$-Thom class

Let M be a complex manifold of dimension n and $\pi : E \to M$ a holomorphic vector bundle of rank ℓ over M. We have the diagonal section s_Δ of the bundle $\pi^*E \to E$. We identify the (image of) zero section of E with M so that the zero set of s_Δ is M.

Please note that, in this section, π^*E, E and M play the roles of E, M and S, respectively, in the previous sections.

Definition 7.1. The *$\bar\partial$-Thom class* $\bar\Psi_E$ of E is the localization of the top Atiyah class of the bundle π^*E by s_Δ; $\bar\Psi_E = a^\ell(\pi^*E, s_\Delta)$, which is in $H^{\ell,\ell}_{\bar\partial}(E, E \setminus M)$.

As a fundamental example, we consider the case where M consists of a point p. Thus $\pi : E = \mathbb{C}^\ell \to \{p\}$ and $\pi^*E = \mathbb{C}^\ell \times \mathbb{C}^\ell$, which is thought of as a vector bundle with projection $\varpi : \pi^*E \to \mathbb{C}^\ell$ onto the first factor. Let $W_0 = \mathbb{C}^\ell \setminus \{0\}$ and W_1 a neighborhood of 0 in $\mathbb{C}^\ell$ and consider the covering $\mathcal{W} = \{W_0, W_1\}$ of $\mathbb{C}^\ell$ (the base space of π^*E).

Theorem 7.2. *In the above situation, the $\bar\partial$-Thom class $\bar\Psi_{\mathbb{C}^\ell}$ of $E = \mathbb{C}^\ell$ is represented by a cocycle in $A^{\ell,\ell}(\mathcal{W}, W_0)$ of the form*

$$(0, 0, -\beta_\ell),$$

where β_ℓ denotes the Bochner–Martinelli kernel on $\mathbb{C}^\ell$.

Proof. Let ∇_0 be an s_Δ-trivial $(1,0)$-connection on W_0 and ∇_1 an arbitrary $(1,0)$-connection on W_1. We have $a^\ell(\nabla_0) = 0$ and the $\bar\partial$-Thom class $\bar\Psi_{\mathbb{C}^\ell}$ is represented by the cocycle

$$(0, a^\ell(\nabla_1), a^\ell(\nabla_0, \nabla_1))$$

in $A^{\ell,\ell}(\mathcal{W}, W_0)$.

Now we can compute $a^\ell(\nabla_0, \nabla_1)$ for particular choices of ∇_0 and ∇_1 (cf. the proof of [14, Ch.III, Theorem 4.4]). We denote by $z = (z_1, \dots, z_\ell)$ a coordinate system on the first factor (the base) and by $\zeta = (\zeta_1, \dots, \zeta_\ell)$ a coordinate system on the second factor (the fiber). Let $e^{(\ell)} = (e_1, \dots, e_\ell)$ denote the frame of π^*E given by $e_1(z) = (z; 1, 0, \dots, 0)$, $\dots, e_\ell(z) = (z; 0, \dots, 0, 1)$. The diagonal section s_Δ is given by $s_\Delta(z) = (z; z) = \sum_{i=1}^{\ell} z_i e_i(z)$. Consider the covering $\mathcal{U} = \{U_i\}_{i=1}^{\ell}$ of W_0 given by $U_i = \{\, z \mid z_i \neq 0 \,\}$. On U_i, $e_i^{(\ell)} = (e_1, \dots, e_{i-1}, s_\Delta, e_{i+1}, \dots, e_\ell)$ is a frame of π^*E. Let D_i be the $(1,0)$-connection for π^*E on U_i trivial with respect to $e_i^{(\ell)}$. Set $\rho_i = |z_i|^2/\|z\|^2$, $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_\ell|^2}$. Then $\nabla_0 = \sum_{i=1}^{\ell} \rho_i D_i$ is a $(1,0)$-connection for π^*E on W_0. Note that it is s_Δ-trivial, since each D_i is. Let ∇_1 be the $(1,0)$-connection on W_1 trivial with respect to $e^{(\ell)}$.

The matrix A_i of frame change from $e^{(\ell)}$ to $e_i^{(\ell)}$ is obtained from the identity matrix by replacing the i-th column by $(z_1, \dots, z_\ell)$. The connection matrix θ_0 of ∇_0 with respect to the frame $e^{(\ell)}$ is given by

$$\theta_0 = \sum_{i=1}^{\ell} \rho_i\, A_i^{-1} \cdot dA_i,$$

while the connection matrix θ_1 of ∇_1 with respect to $e^{(\ell)}$ is zero. Thus $a^\ell(\nabla_1) = 0$.

By definition, $a^\ell(\nabla_0, \nabla_1)$ is the $(\ell, \ell-1)$-component of $c^\ell(\nabla_0, \nabla_1)$. To find $c^\ell(\nabla_0, \nabla_1)$, let $\tilde\nabla = (1-t)\nabla_0 + t\nabla_1$. Then connection matrix $\tilde\theta$ of $\tilde\nabla$ with respect to the frame $e^{(\ell)}$ is given by $\tilde\theta = (1-t)\theta_0$ and the corresponding curvature matrix $\tilde\kappa$ is $\tilde\kappa = d\tilde\theta + \tilde\theta \wedge \tilde\theta = -dt \wedge \theta_0 + (1-t)\, d\theta_0 - (1-t)^2 \theta_0 \wedge \theta_0$. Thus $c^\ell(\nabla_0, \nabla_1)$ is a $(2\ell-1)$-form on W_{01} with $(2\ell-1, 0), \dots, (\ell, \ell-1)$ components. By dimension reason, it has only

$(\ell,\ell-1)$-component, which coincides with $-\beta_\ell$, where β_ℓ denotes the Bochner–Martinelli kernel on $\mathbb{C}^\ell$. Therefore, we have

$$a^\ell(\nabla_0,\nabla_1)=c^\ell(\nabla_0,\nabla_1)=-\beta_\ell.$$

Q.E.D.

Now we come back to the general situation and let E be a holomorphic vector bundle of rank ℓ over a complex manifold M of dimension n. Recall that we have the $\bar\partial$-integration along the fiber

$$\bar\pi_*:H^{\ell,\ell}_{\bar\partial}(E,E\setminus M)\longrightarrow H^{0,0}_{\bar\partial}(M)=H^0(M,\mathcal{O})$$

and the $\bar\partial$-Thom class $\bar\Psi_E$ of E in $H^{\ell,\ell}_{\bar\partial}(E,E\backslash M)$. The following theorem is proved as [14, Ch.III, Theorem 4.4], see also Theorem 7.2.

Theorem 7.3. *In the above situation,*

$$\bar\pi_*\bar\Psi_E=1.$$

Remark 7.4. The $\bar\partial$-Thom class $\bar\Psi_E$ is represented by a cocycle in $A^{\ell,\ell}(\mathcal{W},W_0)$ of the form

$$(0,\pi^*a^\ell(\nabla),-\bar\psi),$$

where ∇ is a $(1,0)$-connection for E on M and $\bar\psi$ an $(\ell,\ell-1)$-form on W_{01} with $\bar\partial\bar\psi=\pi^*a^\ell(\nabla)$ and $-(\bar\pi_{01})_*\bar\psi=1$.

We define the $\bar\partial$-Thom homomorphism

$$\bar T_E:H^{p,q}_{\bar\partial}(M)\longrightarrow H^{p+\ell,q+\ell}_{\bar\partial}(E,E\setminus M)$$

by

$$\bar T_E(a)=\bar\Psi_E\smile\pi^*a\qquad\text{for}\quad a\in H^{p,q}_{\bar\partial}(M).$$

Thus, if $\bar\Psi_E$ is represented by a cocycle $(0,\psi_1,\psi_{01})\in A^{p,q}(\mathcal{W},W_0)$, $\bar T_E$ is induced in cohomology by the map $\theta\mapsto(0,\psi_1\wedge\pi^*\theta,\psi_{01}\wedge\pi^*\theta)$ from $A^{p,q}(M)$ into $A^{p+\ell,q+\ell}(\mathcal{W},W_0)$.

From the projection formula and Theorem 7.3, we have

$$\bar\pi_*\circ\bar T_E=1.$$

Thus $\bar T_E$ gives a splitting of the exact sequence

$$0\longrightarrow\operatorname{Ker}\bar\pi_*\longrightarrow H^{p+\ell,q+\ell}_{\bar\partial}(E,E\setminus M)\xrightarrow{\bar\pi_*}H^{p,q}_{\bar\partial}(M)\longrightarrow 0$$

If we set

$$\hat H^{p+\ell,q+\ell}_{\bar\partial}(E,E\setminus M)=\operatorname{Im}\bar T_E\ (=\operatorname{Ker}\bar\pi_*),$$

we have

Theorem 7.5. *($\bar\partial$-Thom isomorphism) We have an isomorphism*

$$\bar T_E : H^{p,q}_{\bar\partial}(M) \xrightarrow{\sim} \hat H^{p+\ell,q+\ell}_{\bar\partial}(E, E \setminus M)$$

whose inverse is given by $\bar\pi_*$.

Now suppose M is compact. Then we have the commutative diagram:

$$\begin{array}{ccc} H^{p+\ell,q+\ell}_{\bar\partial}(E, E\setminus M) & \xrightarrow{\bar\pi_*} & H^{p,q}_{\bar\partial}(M) \\ \bar A \big\downarrow & & \wr \big\downarrow KS \\ H^{n-p,n-q}_{\bar\partial}(W_1)^* & \xrightarrow{\pi_*} & H^{n-p,n-q}_{\bar\partial}(M)^*. \end{array} \tag{7.6}$$

This induces the commutative diagram:

$$\begin{array}{ccc} \hat H^{p+\ell,q+\ell}_{\bar\partial}(E, E\setminus M) & \underset{\sim}{\xrightarrow{\bar\pi_*}} & H^{p,q}_{\bar\partial}(M) \\ \bar A \big\downarrow \wr & & \wr \big\downarrow KS \\ H^{n-p,n-q}_{\bar\partial}(W_1)^*/\operatorname{Ker}\pi_* & \underset{\sim}{\xrightarrow{\pi_*}} & H^{n-p,n-q}_{\bar\partial}(M)^*. \end{array}$$

In particular, when $p = q = 0$, $\bar\pi_*\bar\Psi_E = [1]$ and $KS([1]) = [M]$.

Remark 7.7. Let $\pi : E \to M$ be as above. The (usual) Thom class Ψ_E of E can be defined as the localization $c^\ell(\pi^*E, s_\Delta)$ of $c^\ell(\pi^*E)$ by s_Δ [14, Ch.III, Theorem 4.4] ;

$$\Psi_E = c^\ell(\pi^*E, s_\Delta),$$

which is in $H^{2\ell}(E, E \setminus M)$. In the case $M = \{p\}$ and thus $E = \mathbb{C}^\ell$, Ψ_E is represented by a cocycle of the same form as $\bar\Psi_E$ (cf. Remark 6.3).

Example 7.8. Let $\mathcal{W} = \{W_0.W_1\}$ be the covering of $\mathbb{C}^\ell$ as above. If $\ell = 1$, since $H^{1,0}_{\bar D}(\mathcal{W}, W_0) = 0$, we have the exact sequence

$$0 \to H^{1,0}_{\bar D}(\mathcal{W}) \to H^{1,0}_{\bar\partial}(W_0) \xrightarrow{\delta} H^{1,1}_{\bar D}(\mathcal{W}, W_0) \to H^{1,1}_{\bar D}(\mathcal{W}) \to \cdots.$$

Notice that we have $H^{1,0}_{\bar D}(\mathcal{W}) \simeq H^{1,0}_{\bar\partial}(\mathbb{C}) = H^0(\mathbb{C}, \Omega^1)$, $H^{1,0}_{\bar\partial}(W_0) = H^0(\mathbb{C} \setminus \{0\}, \Omega^1)$ and $H^{1,1}_{\bar D}(\mathcal{W}) = H^{1,1}_{\bar\partial}(\mathbb{C}) \simeq H^1(\mathbb{C}, \Omega^1) = 0$. Hence we have the exact sequence

$$0 \to H^0(\mathbb{C}, \Omega^1) \to H^0(\mathbb{C} \setminus \{0\}, \Omega^1) \xrightarrow{\delta} H^{1,1}_{\bar D}(\mathcal{W}, W_0) \to 0.$$

The coboundary operator δ sends a holomorphic 1-form ω on $\mathbb{C}\setminus\{0\}$ to the class of $(0,0,-\omega)$ in $H^{1,1}_{\bar D}(\mathcal{W},W_0)=H^{1,1}_{\bar\partial}(\mathbb{C},\mathbb{C}\setminus\{0\})$. Also, the $\bar\partial$-Alexander homomorphism

$$\bar A : H^{1,1}_{\bar\partial}(\mathbb{C},\mathbb{C}\setminus\{0\}) \longrightarrow H^{0,0}_{\bar\partial}(W_1)^* = H^0(W_1,\mathcal{O})^*$$

sends the class of $(0,0,-\omega)$ to the functional assigning to a holomorphic function f on W_1 the value $-\int_{R_{01}} f\omega = \int_{\partial R_1} f\omega$, where $\{R_0,R_1\}$ is a system of honeycomb cells adapted to $\mathcal{W}$. Note that in this case, we may assume that $\partial R_1=\gamma$ is a small circle around 0 oriented counterclockwise.

Now suppose $\ell>1$. We have the exact sequence

$$\to H^{\ell,\ell-1}_{\bar D}(\mathcal{W}) \to H^{\ell,\ell-1}_{\bar\partial}(W_0) \xrightarrow{\delta} H^{\ell,\ell}_{\bar D}(\mathcal{W},W_0) \to H^{\ell,\ell}_{\bar D}(\mathcal{W}) \to \cdots .$$

We have $H^{\ell,\ell-1}_{\bar D}(\mathcal{W}) \simeq H^{\ell,\ell-1}_{\bar\partial}(\mathbb{C}^\ell) = H^{\ell-1}(\mathbb{C}^\ell,\Omega^\ell)=0$ and $H^{\ell,\ell}_{\bar D}(\mathcal{W}) = H^{\ell,\ell}_{\bar\partial}(\mathbb{C}^\ell) \simeq H^\ell(\mathbb{C}^\ell,\Omega^\ell)=0$. Hence we have the exact sequence

$$0 \longrightarrow H^{\ell,\ell-1}(\mathbb{C}^\ell\setminus\{0\}) \xrightarrow{\delta} H^{\ell,\ell}_{\bar D}(\mathcal{W},W_0) \longrightarrow 0.$$

The coboundary operator δ sends the class of a $\bar\partial$-closed $(\ell,\ell-1)$-form ω on $\mathbb{C}^\ell\setminus\{0\}$ to the class of $(0,0,-\omega)$ in $H^{\ell,\ell}_{\bar D}(\mathcal{W},W_0) = H^{\ell,\ell}_{\bar\partial}(\mathbb{C}^\ell,\mathbb{C}^\ell\setminus\{0\})$. Also, the $\bar\partial$-Alexander homomorphism

$$\bar A : H^{\ell,\ell}_{\bar\partial}(\mathbb{C}^\ell,\mathbb{C}^\ell\setminus\{0\}) \longrightarrow H^{0,0}_{\bar\partial}(W_1)^* = H^0(W_1,\mathcal{O})^*$$

sends the class of $(0,0,-\omega)$ to the functional assigning to a holomorphic function f on W_1 the value $-\int_{R_{01}} f\omega = \int_{\partial R_1} f\omega$, where $\{R_0,R_1\}$ is a system of honeycomb cells adapted to $\mathcal{W}$. We may take a small ball of dimension 2ℓ around the origin as R_1.

For $\ell\geq 1$, in the commutative diagram

$$\begin{array}{ccc}
H^{\ell,\ell}_{\bar\partial}(\mathbb{C}^\ell,\mathbb{C}^\ell\setminus\{0\}) & \xrightarrow{\bar\pi_*} & H^{0,0}_{\bar\partial}(\{0\})=\mathbb{C} \\
{\scriptstyle \bar A}\downarrow & & \wr\downarrow{\scriptstyle KS} \\
H^{0,0}_{\bar\partial}(W_1)^* & \xrightarrow{\pi_*} & H^{0,0}_{\bar\partial}(\{0\})^*=\mathbb{C},
\end{array}$$

$\bar\pi_*$ sends the class of $(0,0,-\omega)$ to the number $\int_{\partial R_1}\omega$.

We also have the commutative diagram:

$$\begin{array}{ccc}\hat{H}^{\ell,\ell}_{\bar\partial}(\mathbb{C}^\ell,\mathbb{C}^\ell\setminus\{0\}) & \xrightarrow[\sim]{\bar\pi_*} & H^{0,0}_{\bar\partial}(\{0\})=\mathbb{C}\\ \bar{A}\big\downarrow\wr & & \wr\big\downarrow KS\\ H^{0,0}_{\bar\partial}(W_1)^*/\operatorname{Ker}\pi_* & \xrightarrow[\sim]{\pi_*} & H^{0,0}_{\bar\partial}(\{0\})^*=\mathbb{C}.\end{array}$$

Note that the class of $(0,0,-\beta_\ell)$ corresponds to 1 by $\bar\pi_*$. Note also that $\operatorname{Ker}\pi_*$ consists of functionals on $H^{0,0}_{\bar\partial}(W_1)$ that are zero on the constants.

Remark 7.9. The arguments in this section, as well as those in Section 4, also work in the case M is a compact complex submanifold of a complex manifold W and there is a holomorphic retraction $r:W\to M$ (cf. the end of Section 2), replacing E by W and π by r.

§8. Deformation to the normal bundle

Let W be a complex manifold of dimension m and M a compact complex submanifold of W of dimension n. In this situation, we have the $\bar\partial$-Alexander homomorphism (cf. Section 2)

$$H^{p,q}_{\bar\partial}(W,W\setminus M)\longrightarrow H^{m-p,m-q}_{\bar\partial}(W_1)^*,$$

where W_1 is a neighborhood of M in W. As it is, the receiving group depends on the choice of W_1. In this section, we indicate an idea to make the receiving group the "homology" (the dual of cohomology) of M, as in the case of usual Alexander homomorphism. For this purpose, as mentioned in the introduction, we use "deformation to normal bundle", which we recall below, following the description as given in [7, Ch. 5].

Let $\mathbb{D}$ be a complex one dimensional disk . Then we have a complex manifold $W^\star$ of dimension $m+1$ together with a closed embedding $M\times\mathbb{D}\hookrightarrow W^\star$ and a flat holomorphic map $\rho:W^\star\to\mathbb{D}$ such that

(1) The diagram

$$\begin{array}{ccc}M\times\mathbb{D} & \xrightarrow{\hookrightarrow} & W^\star\\ p_2\big\downarrow & & \big\downarrow\rho\\ \mathbb{D} & = & \mathbb{D}\end{array}$$

commutes, where p_2 denotes the projection onto the second factor.

(2) Over $\mathbb{D}^*=\mathbb{D}\setminus\{0\}$, $\rho^{-1}(\mathbb{D}^*)=W\times\mathbb{D}^*$ and the embedding is the trivial embedding : $M\times\mathbb{D}^*\hookrightarrow W\times\mathbb{D}^*$.

(3) Over 0, $\rho^{-1}(0) = N_M$, (the total space of) the normal bundle of M in W, and $M \times \mathbb{D}$ intersects with N_M in its zero section.

In fact, let $\widetilde{W \times \mathbb{D}}$ be the blow-up of $W \times \mathbb{D}$ along the submanifold $M \times \{0\}$. Then $W^\star$ is obtained by removing the blow-up of $W \times \{0\}$ along $M \times \{0\}$ from $\widetilde{W \times \mathbb{D}}$.

We denote by α the composition of holomorphic maps

$$(N_M, M) \xrightarrow{\iota} (W^\star, M \times \mathbb{D}) \xrightarrow{\mu} (W \times \mathbb{D}, M \times \mathbb{D}) \xrightarrow{p_1} (W, M),$$

where ι denotes the embedding, μ the restriction to $W^\star$ of the map $\widetilde{W \times \mathbb{D}} \longrightarrow W \times \mathbb{D}$ and p_1 the projection onto the first factor.

Then we have

$$H^{p,q}_{\bar\partial}(W, W \setminus M) \xrightarrow{\alpha^*} H^{p,q}_{\bar\partial}(N_M, N_M \setminus M) \longrightarrow H^{m-p,m-q}_{\bar\partial}(M)^*,$$

where the second homomorphism is $\pi_* \circ \bar{A} = KS \circ \bar\pi_*$ (see (7.6)).

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Department of Mathematics
Hokkaido University
Sapporo 060-0810
Japan

E-mail address: `tsuwa@sci.hokudai.ac.jp`