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A plane sextic with finite fundamental group

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Abstract.

We analyze irreducible plane sextics whose fundamental group factors to $\mathbb{D}_{14}$. We produce explicit equations for all curves and show that, in the simplest case of the set of singularities $3\mathbf{A}_6$, the group is $\mathbb{D}_{14}\times\mathbb{Z}_3$.

§1. Introduction

1.1. Motivation and principal results

In this paper, we use the term $\mathbb{D}_{2n}$*-sextic* for an irreducible plane sextic $B \subset \mathbb{P}^2$ whose fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus B)$ factors to the dihedral group $\mathbb{D}_{2n}$, $n \geqslant 3$. The $\mathbb{D}_{2n}$-sextics are classified in [3] and [4] (the case of non-simple singularities). The integer n can take values 3, 5, and 7. All $\mathbb{D}_6$-sextics are of torus type (*i.e.*, they are given by equations of the form $p^3 + q^2 = 0$); in particular, their fundamental groups are infinite. The $\mathbb{D}_{10}$-sextics form 13 equisingular deformation families, and their fundamental groups are known, see [5] and [7]; with one exception, they are all finite. Finally, the $\mathbb{D}_{14}$-sextics form two equisingular families, see Proposition 2.1.1 below, and their groups are not known. In this paper, we compute one of the two groups. Our principal result is the following statement.

Theorem 1.1.1. *The fundamental group of the complement of a* $\mathbb{D}_{14}$*-sextic with the set of singularities* $3\mathbf{A}_6$ *is* $\mathbb{D}_{14} \times \mathbb{Z}_3$.

Theorem 1.1.1 is proved in Section 3.3.

Our result can be regarded as another attempt to substantiate a modified version [5] of Oka's conjecture [6] on the fundamental group of an irreducible plane sextic, stating that the group of an irreducible sextic with simple singularities that is not of torus type is finite. (Note that

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the finiteness of the group is sufficient to conclude that the Alexander polynomial of the curve is trivial, see, *e.g.*, [10].)

1.2. Contents of the paper

In Section 2 we analyze the geometric properties of $\mathbb{D}_{14}$-sextics, whose existence was proved in [3] purely arithmetically. We use the theory of $K3$-surfaces to show that any $\mathbb{D}_{14}$-sextic admits a $\mathbb{Z}_3$-symmetry, see Theorem 2.1.2, and we use this symmetry to obtain explicit equations defining all $\mathbb{D}_{14}$-sextics, see Theorem 2.1.3. The curves form a dimension one family, depending on one parameter $t \in \mathbb{C}$, $t^3 \neq 1$. Most calculations involving polynomials were done using `Maple`.

The heart of the paper is Section 3. We use a particular value $t = 5/6$ of the parameter (close to $t = 1$, where the curve degenerates to a triple cubic) and analyze the real part of the curve obtained. With respect to an appropriately chosen real pencil of lines, it has sufficiently many real critical values, and we apply van Kampen's method (ignoring all non-real critical values) to produce an 'upper estimate' on the fundamental group, see Theorem 3.1.1. Comparing the latter with the known 'lower estimate' (the fact that the curve is known to be a $\mathbb{D}_{14}$-sextic), we prove Theorem 1.1.1.

§2. The construction

2.1. Statements

A $\mathbb{Z}_3$-action on $\mathbb{P}^2$ is called *regular* if it lifts to a regular representation $\mathbb{Z}_3 \to GL(3, \mathbb{C})$. An order 3 element $c \in PGL(3, \mathbb{C})$ is called *regular* if it generates a regular $\mathbb{Z}_3$-action. Any regular order 3 automorphism of $\mathbb{P}^2$ has three isolated fixed points (and no other fixed points). Conversely, any order 3 automorphism c of $\mathbb{P}^2$ with isolated fixed points only is regular (as isolated fixed points correspond to dimension one eigenspaces of the lift of c to $\mathbb{C}^3$).

The following statement is proved in [3] (see also [4], where sextics with non-simple singular points are ruled out).

Proposition 2.1.1. *All $\mathbb{D}_{14}$-sextics form two equisingular deformation families, one family for each of the sets of singularities* $3\mathbf{A}_6$ *and* $3\mathbf{A}_6 \oplus \mathbf{A}_1$.

The principal results of this section are the following two theorems.

Theorem 2.1.2. *Any $\mathbb{D}_{14}$-sextic B is invariant under a certain regular order* 3 *automorphism $c\colon \mathbb{P}^2 \to \mathbb{P}^2$ acting on the three type* $\mathbf{A}_6$ *singular points of B by a cyclic permutation.*

Theorem 2.1.3. *Up to projective transformation, the $\mathbb{D}_{14}$-sextics form a connected one parameter family $B(t)$; in appropriate homogeneous coordinates, they are given by the polynomial*

$$
\begin{gathered}
2t(t^3-1)(z_0^4z_1z_2+z_1^4z_2z_0+z_2^4z_0z_1)\\
+(t^3-1)(z_0^4z_1^2+z_1^4z_2^2+z_2^4z_0^2)\\
+t^2(t^3-1)(z_0^4z_2^2+z_1^4z_0^2+z_2^4z_1^2)\\
+2t(t^3+1)(z_0^3z_1^3+z_1^3z_2^3+z_2^3z_0^3)\\
+4t^2(t^3+2)(z_0^3z_1^2z_2+z_1^3z_2^2z_0+z_2^3z_0^2z_1)\\
+2(t^6+4t^3+1)(z_0^3z_1z_2^2+z_1^3z_2z_0^2+z_2^3z_0z_1^2)\\
+t(t^6+13t^3+10)z_0^2z_1^2z_2^2,
\end{gathered}
\tag{2.1}
$$

where $t\in\mathbb{C}$ and $t^3\neq 1$. The restriction of $B(t)$ to the subset $t^3\neq 1,-27$ is an equisingular deformation, all curves having the set of singularities $3\mathbf{A}_6$. The three curves with $t^3=-27$ are extra singular; their sets of singularities are $3\mathbf{A}_6\oplus\mathbf{A}_1$.

Remark 2.1.4. We do not assert that all curves $B(t)$ are pairwise distinct. In fact, one can observe that the substitution $t\mapsto\epsilon t$, $\epsilon^3=1$, results in an equivalent curve, the corresponding change of coordinates being $(z_0:z_1:z_2)\mapsto(z_0:\epsilon^2z_1:\epsilon z_2)$. In particular, all three extra singular curves are equivalent.

Theorems 2.1.2 and 2.1.3 are proved, respectively, in Sections 2.3 and 2.8 below.

2.2. Discriminant forms

An (*integral*) *lattice* is a finitely generated free abelian group L supplied with a symmetric bilinear form $b\colon L\otimes L\to\mathbb{Z}$. We abbreviate $b(x,y)=x\cdot y$ and $b(x,x)=x^2$. A lattice L is *even* if $x^2=0 \bmod 2$ for all $x\in L$. As the transition matrix between two integral bases has determinant ±1, the determinant $\det L\in\mathbb{Z}$ (*i.e.*, the determinant of the Gram matrix of b in any basis of L) is well defined. A lattice L is called *nondegenerate* if $\det L\neq 0$; it is called *unimodular* if $\det L=\pm1$.

Given a lattice L, the bilinear form extends to $L\otimes\mathbb{Q}$ by linearity. If L is nondegenerate, the dual group $L^*=\operatorname{Hom}(L,\mathbb{Z})$ can be identified with the subgroup

$$\bigl\{x\in L\otimes\mathbb{Q}\bigm| x\cdot y\in\mathbb{Z}\text{ for all }x\in L\bigr\}.$$

In particular, $L\subset L^*$. The quotient L^*/L is a finite group; it is called the *discriminant group* of L and is denoted by $\operatorname{discr} L$ or $\mathcal{L}$. The discriminant group $\mathcal{L}$ inherits from $L \otimes \mathbb{Q}$ a symmetric bilinear form $\mathcal{L} \otimes \mathcal{L} \to \mathbb{Q}/\mathbb{Z}$, called the *discriminant form*, and, if L is even, its quadratic extension $\mathcal{L} \to \mathbb{Q}/2\mathbb{Z}$. When speaking about the discriminant groups, their (anti-) isomorphisms, etc., we always assume that the discriminant form (and its quadratic extension if the lattice is even) is taken into account. One has $\#\mathcal{L} = |\det L|$; in particular, $\mathcal{L} = 0$ if and only if L is unimodular.

An *extension* of a lattice L is another lattice M containing L, so that the form on L is the restriction of that on M. An *isomorphism* between two extensions $M_1 \supset L$ and $M_2 \supset L$ is an isometry $M_1 \to M_2$ whose restriction to L is the identity. In what follows, we are only interested in the case when both L and M are even and $[M : L] < \infty$. Next two theorems are found in V. V. Nikulin [9].

Theorem 2.2.1. *Given a nondegenerate even lattice L, there is a canonical one-to-one correspondence between the set of isomorphism classes of finite index extensions $M \supset L$ (by even lattices) and the set of isotropic subgroups $\mathcal{K} \subset \mathcal{L}$. Under this correspondence, one has $M = \{x \in L^* \mid x \bmod L \in \mathcal{K}\}$ and* $\operatorname{discr} M = \mathcal{K}^\perp/\mathcal{K}$.

The isotropic subgroup $\mathcal{K} \subset \mathcal{L}$ as in Theorem 2.2.1 is called the *kernel* of the extension $M \supset L$. It can be defined as the image of M/L under the homomorphism induced by the natural inclusion $M \hookrightarrow L^*$.

Theorem 2.2.2. *Let $M \supset L$ be a finite index extension of a non-degenerate even lattice L (by an even lattice M), and let $\mathcal{K} \subset \mathcal{L}$ be its kernel. Then, an auto-isometry $L \to L$ extends to M if and only if the induced automorphism of $\mathcal{L}$ preserves $\mathcal{K}$.*

2.3. Proof of Theorem 2.1.2

Fix a $\mathbb{D}_{14}$-sextic $B \subset \mathbb{P}^2$ and consider the double covering $X \to \mathbb{P}^2$ ramified at B and its minimal resolution $\tilde{X}$. Since all singular points of B are simple, see Proposition 2.1.1, $\tilde{X}$ is a $K3$-surface. For a singular point P of B, denote by D_P the set of exceptional divisors in $\tilde{X}$ over P, as well as its incidence graph, which is the Dynkin graph of the same name **A**–**D**–**E** as P. Let $\Sigma_P \subset H_2(\tilde{X})$ be the sublattice spanned by D_P. (Here, $H_2(\tilde{X})$ is regarded as a lattice *via* the intersection index form.) Let, further, $\Sigma' = \bigoplus_P \Sigma_P$, the summation running over all type $\mathbf{A}_6$ singular points P of B (see Proposition 2.1.1), and let $\tilde{\Sigma}' \supset \Sigma'$ be the primitive hull of Σ' in $H_2(\tilde{X})$, *i.e.*, $\tilde{\Sigma}' = (\Sigma' \otimes \mathbb{Q}) \cap H_2(\tilde{X})$. It is a finite index extension; denote by $\mathcal{K} \subset \operatorname{discr} \Sigma'$ its kernel.

Let P_0, P_1, P_2 be the type $\mathbf{A}_6$ points. For each point $P = P_i$, $i = 0, 1, 2$, fix an orientation of its (linear) graph D_P and let $e_{i1}, \ldots, e_{i6}$ be the elements of D_P numbered consecutively according to the chosen orientation. Denote by $e_{i1}^*, \ldots, e_{i6}^*$ the dual basis for Σ_P^*. The discriminant group $\operatorname{discr} \Sigma_P \cong \mathbb{Z}_7$ is generated by $e_{i1}^* \bmod \Sigma_P$, and, for each $k = 1, \ldots, 6$, one has $e_{ik}^* = k e_{i1}^* \bmod \Sigma_P$. Let

$$(2.2)\quad \gamma_0 = e_{04}^* + e_{12}^* + e_{21}^*, \quad \gamma_1 = e_{01}^* + e_{14}^* + e_{22}^*, \quad \gamma_2 = e_{02}^* + e_{11}^* + e_{24}^*.$$

According to [3], under an appropriate numbering of the type $\mathbf{A}_6$ singular points of B and appropriate orientation of their Dynkin graphs, the kernel $\mathcal{K} \cong \mathbb{Z}_7$ is generated by the residue $\gamma_0 \bmod \Sigma'$. (For the convenience of the further exposition, we use an indexing slightly different from that used in [3].) Observe that each of the residues $\gamma_1 = 2\gamma_0 \bmod \Sigma'$ and $\gamma_2 = 4\gamma_0 \bmod \Sigma'$ also generates $\mathcal{K}$.

Define an isometry $c_\Sigma \colon \Sigma' \to \Sigma'$ *via* $e_{0k} \mapsto e_{1k} \mapsto e_{2k} \mapsto e_{0k}$, $k = 1, \ldots 6$. Clearly, $c_\Sigma^3 = \mathrm{id}$ and the induced action on $\operatorname{discr} \Sigma'$ is a regular representation of $\mathbb{Z}_3$ over $\mathbb{F}_7$. Hence, $\operatorname{discr} \Sigma'$ splits into direct (not orthogonal) sum of 1-dimensional eigenspaces, $\operatorname{discr} \Sigma' = V_1 \oplus V_2 \oplus V_4$, corresponding to the three cubic roots of unity $1, 2, 4 \in \mathbb{F}_7$. Since $c_\Sigma(\gamma_0) = \gamma_1 = 2\gamma_0 \bmod \Sigma'$, see above, one has $\mathcal{K} = V_2$ and it is immediate that $\mathcal{K}^\perp/\mathcal{K}$ can be identified with V_1. Hence, c_Σ extends to an auto-isometry $\tilde{c}_\Sigma \colon \tilde{\Sigma}' \to \tilde{\Sigma}'$, see Theorem 2.2.2, and the induced action on $\operatorname{discr} \tilde{\Sigma} = V_1$, see Theorem 2.2.1, is trivial. Applying Theorem 2.2.2 to the finite index extension $H_2(\tilde{X}) \supset \tilde{\Sigma}' \oplus (\tilde{\Sigma}')^\perp$, one concludes that the direct sum $\tilde{c}_\Sigma \oplus \mathrm{id}$ extends to an order 3 auto-isometry $\tilde{c}_* \colon H_2(\tilde{X}) \to H_2(\tilde{X})$.

By construction, $\tilde{c}_*$ preserves the class h of the pull-back of a generic line in $\mathbb{P}^2$ and the class ω of a holomorphic 2-form on $\tilde{X}$ (as clearly both $h, \omega \in (\tilde{\Sigma}')^\perp$). Furthermore, $\tilde{c}_*$ preserve the positive cone V^+ of $\tilde{X}$. (We recall that the positive cone of $\tilde{X}$ is an open fundamental polyhedron $V^+ \subset (\operatorname{Pic} X) \otimes \mathbb{R}$ of the group generated by reflections defined by vectors $x \in \operatorname{Pic} X$ with $x^2 = -2$; in the case under consideration, it is uniquely characterized by the requirement that $V^+ \cdot e > 0$ for any exceptional divisor e over a singular point of B and that the closure of V^+ should contain h.) The usual averaging argument shows that $\tilde{X}$ has a Kähler metric with $\tilde{c}_*$-invariant fundamental class $\rho \in V^+$. The pair ($\omega \bmod \mathbb{C}^*, \rho \bmod \mathbb{R}^*$) represents a point in the fine period space of marked quasipolarized $K3$-surfaces, see A. Beauville [1], and, since this point is fixed by $\tilde{c}_*$, there is a unique automorphism $\tilde{c} \colon \tilde{X} \to \tilde{X}$ inducing $\tilde{c}_*$ in the homology. It is of order 3 (as the only automorphism inducing $\tilde{c}_*^3 = \mathrm{id}$ is the identity), symplectic (*i.e.*, preserving holomorphic 2-forms), and commutes with the deck translation of the ramified covering $\tilde{X} \to \mathbb{P}^2$ (as the map $\tilde{X} \to \mathbb{P}^2$ is defined by the linear system $h \in \operatorname{Pic} \tilde{X}$ preserved by $\tilde{c}_*$). Thus, $\tilde{c}$ descends to an order 3 automorphism $c \colon \mathbb{P}^2 \to \mathbb{P}^2$. The latter preserves B (as it lifts to $\tilde{X}$) and has isolated fixed points only (as so does its lift $\tilde{c}$, as any symplectic automorphism of a $K3$-surface); in particular, c is regular. Q.E.D.

2.4. Geometric properties of $\mathbb{D}_{14}$-sextics

The following geometric characterization of $\mathbb{D}_{14}$-sextics can be found in [3].

Proposition 2.4.1. *The three type $\mathbf{A}_6$ singular points P_0, P_1, P_2 of a $\mathbb{D}_{14}$-sextic B can be ordered so that there are three conics Q_0, Q_1, Q_2 such that each Q_i, $i = 0, 1, 2$, intersects B at P_{i-k}, $k = 1, 2, 3$, with multiplicity $2k$.*

Remark 2.4.2. Here and below, to shorten the notation, we use the cyclic indexing $P_{i+3s} = P_i$ and $Q_{i+3s} = Q_i$ for $s \in \mathbb{Z}$. In fact, the points should be ordered as explained in Section 2.3; then Q_i is the projection to $\mathbb{P}^2$ of the rational curve realizing the (-2)-class $\gamma_i + h \in \operatorname{Pic} \tilde{X}$, where γ_i is given by (2.2).

Lemma 2.4.3. *The automorphism c given by Theorem 2.1.2 acts on the set of conics $\{Q_0, Q_1, Q_2\}$ as in Proposition 2.4.1 by a cyclic permutation.*

Proof. For each $i = 0, 1, 2$, the incidence conditions described above define at most one conic Q_i (as otherwise two conics would intersect at six points). Since c permutes the singular points of B, it must also permute the conics. Q.E.D.

Lemma 2.4.4. *Let Q_0, Q_1, Q_2 be the conics as described in Proposition 2.4.1. Then, either all Q_i, $i = 0, 1, 2$, are irreducible or else, for each $i = 0, 1, 2$, one has a splitting $Q_i = (P_{i-1}P_i) + (P_iP_{i+1})$. In the latter case, B is tangent to $(P_{i-1}P_i)$ at P_i.*

Proof. Due to Lemma 2.4.3, if one of Q_i is reducible, so are the others. Assume that Q_0 splits into two lines, $Q_0 = L_0' + L_0''$. If the intersection point $L_0' \cap L_0''$ is a singular point of B, one immediately concludes that $Q_0 = (P_2P_0) + (P_0P_1)$ and extends this splitting to the other conics *via* c.

Otherwise, assume that it is L_0' that intersects B at P_0 with multiplicity 6. Then the component $L_2'' = c^2(L_0'')$ of $Q_2 = c^2(Q_0)$ is tangent to B at P_0 (we assume that c acts *via* $P_0 \mapsto P_1 \mapsto P_2 \mapsto P_0$); hence, $L_2'' = L_0'$ and this line cannot pass through P_1. (Neither can the other component $L_2' = c^2(L_0')$, as it intersects B at P_2 with the maximal multiplicity 6.) Q.E.D.

2.5. Theorem 2.1.3: the generic case

Fix a $\mathbb{D}_{14}$-sextic B and denote by P_0, P_1, P_2 its three type $\mathbf{A}_6$ singular points, ordered as explained above. Let Q_0, Q_1, Q_2 be the conics as in Proposition 2.4.1, and let $c\colon \mathbb{P}^2 \to \mathbb{P}^2$ be the order 3 automorphism given by Theorem 2.1.2. In this section, we assume that Q_0, Q_1, Q_2 are irreducible, see Lemma 2.4.4.

Perform the triangular transformation centered at P_0, P_1, P_2, *i.e.*, blow up the three points and blow down the proper transforms of the lines (P_iP_j), $0 \leqslant i < j \leqslant 2$. Denote by bars the (proper) images of the curves and points involved. The curve $\bar{B}$ has three type $\mathbf{A}_4$ singular points $\bar{P}_i$, $i = 0, 1, 2$. The transforms $\bar{Q}_i$, $i = 0, 1, 2$, are lines, so that each $\bar{Q}_i$ passes through $\bar{P}_{i+1}$ and is tangent to $\bar{B}$ at $\bar{P}_i$. Besides, $\bar{B}$ has three (at least) nodes $\bar{S}_i$, $i = 0, 1, 2$, located at the blow-up centers of the inverse triangular transformation. Note that (under appropriate indexing) the line $(\bar{S}_i\bar{S}_{i+1})$ contains $\bar{P}_i$, $i = 0, 1, 2$.

Since the blow-up centers form a single orbit of c, the transformation commutes with c and the new configuration is still $\mathbb{Z}_3$-symmetric.

Choose homogeneous coordinates $(u_0 : u_1 : u_2)$ in $\mathbb{P}^2$ so that $\bar{P}_0 = (1:0:0)$, $\bar{P}_1 = (0:1:0)$, $\bar{P}_2 = (0:0:1)$, and $(1:1:1)$ is one of the fixed points of c. Then c acts *via* a cyclic permutation of the coordinates, and its three fixed points are $(1:\epsilon:\epsilon^2)$, $\epsilon^3 = 1$. The condition that $\bar{P}_i \in (\bar{S}_i\bar{S}_{i+1})$ and that the triple $\bar{S}_0$, $\bar{S}_1$, $\bar{S}_2$ is c-invariant translates as follows: there is a parameter $t \in \mathbb{C}$ such that $\bar{S}_0 = (1:t:t^2)$, $\bar{S}_1 = (t^2:1:t)$, and $\bar{S}_2 = (t:t^2:1)$. In order to get three distinct points other than $\bar{P}_0$, $\bar{P}_1$, $\bar{P}_2$, one must have $t \neq 0$, $t^3 \neq 1$.

In the chosen coordinates, $\bar{B}$ has three type $\mathbf{A}_4$ singular points located at the vertices of the coordinate triangle and tangent to its edges. Since $\bar{B}$ is also preserved by c, it must be given by a polynomial $F(u_0, u_1, u_2)$ of the form

$$\begin{aligned} a(u_0^4u_2^2 + u_1^4u_0^2 + u_2^4u_1^2) + b(u_0^3u_1^2u_2 + u_1^3u_2^2u_0 + u_2^3u_0^2u_1)& \\ + c(u_0^3u_1u_2^2 + u_1^3u_2u_0^2 + u_2^3u_0u_1^2) + du_0^2u_1^2u_2^2& \end{aligned} \tag{2.3}$$

for some $a, b, c, d \in \mathbb{C}$. Conversely, any curve $\bar{B}$ given by a polynomial as above is preserved by c and has three singular points adjacent to $\mathbf{A}_3$ and situated in the prescribed way with respect to the coordinate lines $\bar{Q}_i$, $i = 0, 1, 2$. Due to the symmetry, it suffices to make sure that $\bar{B}$ is singular at $\bar{S}_0$ and that its singularity at $\bar{P}_0$ is adjacent to $\mathbf{A}_4$. The former condition results in the linear system

$$\begin{gathered} 6at^4 + (3t^4 + 3t^7)b + (5t^5 + t^8)c + 2dt^6 = 0, \\ (4t^3 + 2t^9)a + (2t^3 + 4t^6)b + (4t^4 + 2t^7)c + 2dt^5 = 0, \end{gathered}$$

$$(4t^8+2t^2)a+(t^2+5t^5)b+(3t^3+3t^6)c+2dt^4=0,$$

and the latter condition is equivalent to $b=2a$ or $b=-2a$. In both cases, the solution space of the linear system has dimension one; it is spanned by

$$(a,b,c,d)=(t^2,2t^2,-2t(t^3+2),(t^3+2)^2)$$

and

$$(a,b,c,d)=(1,-2,-2t^2,t(t^3+8)),$$

respectively. The first solution, with $b=2a$, results in a reducible polynomial

$$(tu_1^2u_0-2u_1u_2u_0-u_1u_2u_0t^3+tu_0^2u_2+tu_1u_2^2)^2;$$

hence, it should be disregarded. For the second solution, the substitution $u_0=1$, $u_1=x$, $u_2=y+x^2$ results in a polynomial with the principal part $y^2-4t^2x^5$, *i.e.*, the singularity of $\bar{B}$ at $\bar{P}_0$ (and, due to the symmetry, at $\bar{P}_1$ and $\bar{P}_2$ as well) is of type $\mathbf{A}_4$ exactly. In particular, the curve is irreducible. (Indeed, the only possible splitting would be into an irreducible quintic and a line, but in this case all nodes of $\bar{B}$ would have to be collinear.)

To obtain the original curve B, one should perform the substitution

$$u_0=v_0+t^2v_1+tv_2,\quad u_1=tv_0+v_1+t^2v_2,\quad u_2=t^2v_0+tv_1+v_2$$

(passing to an invariant coordinate triangle with the vertices at the points $\bar{S}_i$, $i=0,1,2$) followed by the inverse triangular transformation $v_0=z_1z_2$, $v_1=z_2z_0$, $v_2=z_0z_1$. The resulting polynomial is the one given by (2.1) with $t\neq 0$.

Counting the genus and taking into account the symmetry, one concludes that the singularities of $\bar{B}$ at $\bar{S}_i$, $i=0,1,2$, are either all nodes or all cusps, the latter possibility corresponding to $t^3=-3$ or $t^3=-1/3$. Note that the cusps of $\bar{B}$ merely mean that the original curve B is tangent to the lines (P_iP_j), $0\leqslant i<j\leqslant 2$; these curves are still in the same equisingular deformation family.

Remark 2.5.1. If $t^3=1$, the polynomial (2.1) becomes reducible. For example, if $t=1$, it turns into $4(z_0z_1+z_1z_2+z_2z_0)^3$.

2.6. Theorem 2.1.3: the case of reducible conics

Now, assume that the conics Q_0, Q_1, Q_2 are reducible, see Lemma 2.4.4. In this case, we can start directly from (2.3), placing the singular points so that $P_0=(1:0:0)$, $P_1=(0:0:1)$, and $P_2=(0:1:0)$. Note that $a\neq 0$; hence, we can let $a=1$.

As above, in view of the symmetry it suffices to analyze the singularity at P_0.

The condition that the singularity is adjacent to $\mathbf{A}_4$ is equivalent to $b = \pm 2$. If $b = 2$, the substitution $u_0 = 1$, $u_1 = x$, $u_2 = y - x^2 + cx^3/2$ produces a polynomial in (x, y) with the principal part $y^2 + (d - c^2/4)x^6$. Hence, $d = c^2/4$. However, in this case the original polynomial F is reducible:

$$F = \frac{1}{4}(2u_1^2u_0 + 2u_2^2u_1 + 2u_0^2u_2 + cu_0u_1u_2)^2.$$

Let $b = -2$. Then, substituting $u_0 = 1$, $u_1 = x$, $u_2 = y + x^2$, one obtains

$$y^2 + 3cyx^3 + 2cx^5 + dx^6 - 4x^7 + (\text{higher order terms}).$$

Hence, $c = d = 0$, and in this case the singularity at the origin is exactly $\mathbf{A}_6$. The curve is irreducible (as any sextic with three type $\mathbf{A}_6$ singular points) and, after the coordinate change $(u_0 : u_1 : u_2) \mapsto (z_0 : z_2 : z_1)$, the resulting equation is (2.1) with $t = 0$.

2.7. Extra singular $\mathbb{D}_{14}$-sextics

Since the total Milnor number of a plane sextic does not exceed 19, a curve $B = B(t)$ can have at most one extra singular point, which must be of type $\mathbf{A}_1$. Since, in addition, B is preserved by c, this extra singular point must be fixed by c, *i.e.*, it must be of the form $(1 : \epsilon : \epsilon^2)$, $\epsilon^3 = 1$. Solving the corresponding linear system shows that B is singular at $(1 : \epsilon : \epsilon^2)$, $\epsilon^3 = 1$, whenever it passes through this point, and this is the case when $t = -3/\epsilon$. In conclusion, B has an extra node (the set of singularities $3\mathbf{A}_6 \oplus \mathbf{A}_1$) if and only if $t^3 = -27$.

2.8. Proof of Theorem 2.1.3

As shown in Sections 2.5 and 2.6, any $\mathbb{D}_{14}$-sextic belongs to the connected family $B(t)$, $t^3 \neq 1$. According to Section 2.7, this family represents two equisingular deformation classes: the restriction to the connected subset $t^3 \neq 1, -27$ (the set of singularities $3\mathbf{A}_6$) and three equivalent isolated curves corresponding to $t^3 = -27$ (the set of singularities $3\mathbf{A}_6 \oplus \mathbf{A}_1$; the equivalence is given by the coordinate change $(z_0 : z_1 : z_2) \mapsto (z_0 : \epsilon z_1 : \epsilon^2 z_2)$, $\epsilon^3 = 1$, *cf.* Remark 2.1.4). Comparing this result with Proposition 2.1.1, one concludes that any curve given by (2.1) with $t^3 \neq 1$ is a $\mathbb{D}_{14}$-sextic. Q.E.D.

§3. The fundamental group

3.1. Calculation of the group

For the calculation, we choose a real curve $B(t)$ given by Theorem 2.1.3 and close to the triple conic $B(1)$, see Remark 2.5.1.

Theorem 3.1.1. *For the curve $B = B(5/6)$, there is an epimorphism*

$$G := \langle \omega, \xi \mid \xi^2 = e,\ \omega^2 = \xi\omega^5\xi \rangle \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus B).$$

Proof. Let $B = B(5/6)$ and make the following change of coordinates:

$$z_0 = 1/3\,Z - 1/3\,Y + 1/3\,X,\ z_1 = -1/3\,X + 2/3\,Z - 5/3\,Y,\ z_2 = Y.$$

Then, in the affine space $\mathbb{C}^2 = \mathbb{P}^2 \smallsetminus \{Z = 0\}$, B is defined by $g(x, y) = 0$, where

$$\begin{aligned} g(x,y) \;=\;& \frac{3568}{177147} + \frac{716}{19683}x + \frac{17872}{177147}y - \frac{8137}{472392}x^2 - \frac{11503}{708588}xy \\ &- \frac{449027}{1417176}y^2 - \frac{57539}{1417176}x^3 - \frac{722513}{2834352}x^2y - \frac{356093}{354294}xy^2 \\ &- \frac{4427549}{2834352}y^3 + \frac{2243}{209952}x^4 + \frac{322559}{5668704}x^3y + \frac{2726579}{7558272}x^2y^2 \\ &+ \frac{1092623}{1259712}xy^3 + \frac{56261293}{22674816}y^4 + \frac{12505}{944784}x^5 + \frac{397175}{2834352}x^4y \\ &+ \frac{9868757}{11337408}x^3y^2 + \frac{11718893}{3779136}x^2y^3 + \frac{77768419}{11337408}xy^4 \\ &+ \frac{81485377}{11337408}y^5 - \frac{26011}{5668704}x^6 - \frac{309307}{5668704}x^5y \\ &- \frac{9030539}{22674816}x^4y^2 - \frac{9923629}{5668704}x^3y^3 - \frac{61362175}{11337408}x^2y^4 \\ &- \frac{28582655}{2834352}xy^5 - \frac{255219619}{22674816}y^6. \end{aligned}$$

Its three singularities are located at

$$P_0 = (-1, 0), \quad P_1 = (2, 0), \quad P_2 = (-1/2, 1/2),$$

and the graph in the real plane is given in Figure 1.

For the calculation, we apply van Kampen's method [8] to the horizontal pencil $L_\eta = \{y = \eta\}$. The singular pencils corresponds to the

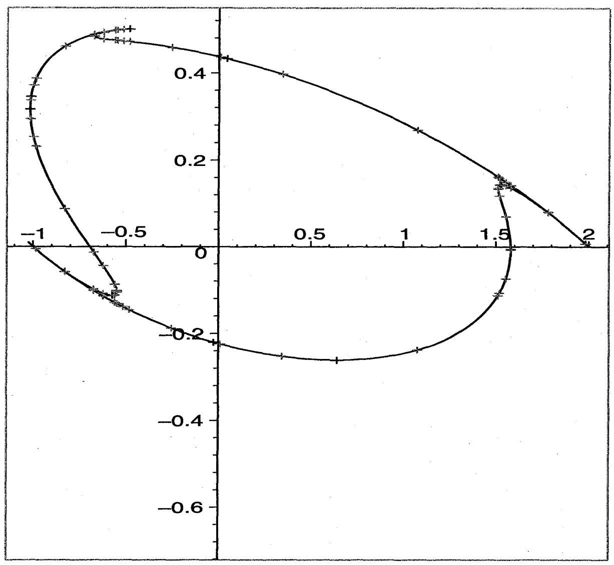

Fig. 1. The graph of B

roots of

$$\begin{aligned}&(90617210907008\,y^9 - 60741238168704\,y^8 - 52338630572904\,y^7\\&+38781803208839\,y^6 + 8841431367018\,y^5 - 8143800845364\,y^4\\&-176669916264\,y^3 + 512733413664\,y^2 - 7789219200\,y\\&-6298560000)y^{14}\,(2\,y-1)^7 = 0.\end{aligned}$$

Note that we have five real singular pencil lines

$$L_\eta, \quad \eta = \eta_i,\ i = 1, 2, \ldots, 5,$$
$$\eta_1 \approx -0.26, \quad \eta_2 \approx -0.11, \quad \eta_3 = 0, \quad \eta_4 \approx 0.14, \quad \eta_5 = 1/2,$$

where L_{η_i}, $i = 1, 2, 4$ are tangent to B, which come from the first factor of degree 9. There also are three pairs of complex conjugate singular fibers, but we do not use them; that is why we only assert that the map constructed below is an epimorphism, not an isomorphism. We take the base point at infinity $b = (1 : 0 : 0)$, and we fix generators $\rho_1, \ldots, \rho_6$ on the regular pencil line $y = -\varepsilon$ (where ε is a sufficiently small positive real number) as in Figure 2. The bullets are lassos, which are counterclockwise oriented loops going around a point of B.

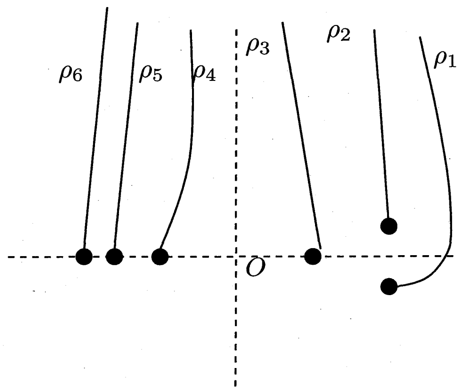

Fig. 2. Generators in the fiber $y = -\varepsilon$

The monodromy relations at $y = \eta_2$, η_1 are tangent relations, they are given as

$$(R_1): \qquad \rho_4 = \rho_5, \quad \rho_3 = \rho_4^{-1}\rho_6\rho_4$$

Thus, hereafter we eliminate the generator ρ_5.

The monodromy relations at $y = 0$ are two $\mathbf{A}_6$-cusp relations, they are given as

$$(R_2): \qquad \begin{cases} \omega^3\rho_6 = \rho_4\omega^3, & \omega = \rho_6\rho_5, \\ \tau^3\rho_2 = \rho_1\tau^3, & \tau = \rho_2\rho_1. \end{cases}$$

To see the relations at $y = \eta_4$, η_5 effectively, we take new elements ρ_6', ρ_5', ρ_2', ρ_1' as in Figure 3. The new elements are defined as

$$(R_3): \ \rho_6' = \omega^{-2}\rho_6\omega^2, \quad \rho_4' = \omega^{-1}\rho_4\omega, \quad \rho_2' = \tau^{-1}\rho_1\tau, \quad \rho_1' = \tau^{-2}\rho_2\tau^2.$$

Note that they satisfy the relations

$$\rho_4'\rho_6' = \omega, \quad \rho_2'\rho_1' = \tau.$$

Now, the monodromy relation at $y = \eta_4$ is given as

$$(R_4): \qquad \rho_3 = \rho_2' \quad \text{or} \quad \rho_3 = \tau^{-1}\rho_1\tau.$$

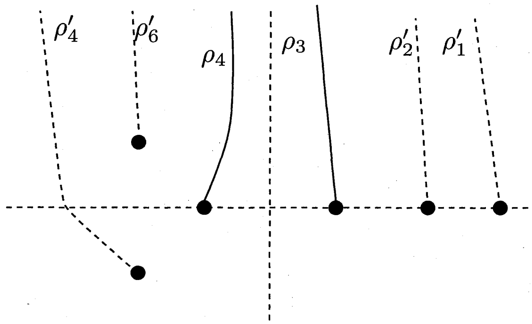

Fig. 3. Generators in the fiber $y = \varepsilon$

The relation at $y = 1/2$ is an $\mathbf{A}_6$-cusp relation, which is given as

$$(R_5): \qquad (\rho_4\rho_2'\rho_1'\rho_2'^{-1})^3\rho_4 = (\rho_2'\rho_1'\rho_2'^{-1})(\rho_4\rho_2'\rho_1'\rho_2'^{-1})^3.$$

Finally, the vanishing relation at infinity is given as

$$(R_\infty): \qquad \omega^2\tau = e.$$

We eliminate the generator ρ_3 using (R_1). Then (R_4) is translated into the following relation:

$$(R_4'): \qquad \rho_4^{-1}\rho_5\rho_4 = \tau^{-1}\rho_1\tau.$$

The relation (R_∞) can be rewritten as

$$(R_\infty'): \qquad \tau = \omega^{-2}.$$

From (R_∞') and (R_4'), we get

$$\rho_1 = (\tau\rho_4^{-1})\rho_6(\rho_4\tau^{-1}) = \omega^{-2}\rho_4^{-1}\rho_6\rho_4\omega^2 \underset{R_2}{=} \rho_6\rho_4\rho_6^{-1}.$$

As $\rho_2 = \tau\rho_1^{-1}$, this implies

$$(R_4''): \qquad \rho_1 = \rho_6\rho_4\rho_6^{-1}, \quad \rho_2 = \omega^{-1}\rho_4^{-1}\omega^{-1}.$$

We can rewrite ρ_2', using the above relations, as follows:

$$\rho_2' = \omega^3\rho_6^{-1}\omega^{-2} \underset{R_2}{=} \rho_4^{-1}\rho_6\rho_4.$$

Thus, $\rho_2'\rho_1'\rho_2'^{-1} = \omega^{-3}\rho_4$, and (R_5) can be rewritten in ρ_6, ρ_5 as follows:

$$(R_5') : \qquad (\rho_4\rho_6\omega^{-3})^3\rho_4 = (\rho_6\omega^{-3})(\rho_4\rho_6\omega^{-3})^3.$$

We have to rewrite the relations in the words of ρ_4, ρ_6. The relation $\tau^3\rho_2 = \rho_1\tau^3$ gives

$$\omega^{-6}(\omega^{-1}\rho_4^{-1}\omega^{-1}) = (\rho_6\rho_4\rho_6^{-1})\omega^{-2},$$

which reduces to $\omega^5\rho_6 = \rho_4\omega^8$. Using the relation $\omega^3\rho_6 = \rho_4\omega^3$ several times, we get $\rho_4 = \omega\rho_6\omega^2$. Now, we eliminate ρ_4 using $\omega = \rho_6\rho_4$ to obtain

$$(R_6) : \qquad (\omega\rho_6)^2 = e.$$

Replace the generator ρ_6 by $\xi = \omega\rho_6$, so that the new generators are ω, ξ; then ρ_4, ρ_6 are expressed as $\rho_6 = \omega^{-1}\xi$ and $\rho_4 = \xi^{-1}\omega^2$, and (R_6) is written as $\xi^2 = e$. The relation $\omega^3\rho_6 = \rho_4\omega^3$ reduces to

$$(R_7) : \qquad \omega^2 = \xi\omega^5\xi.$$

Thus, we have shown that $\pi_1(\mathbb{P}^2 \setminus B)$ is generated by two elements ω, ξ, which are subject to the relations

$$\xi^2 = e, \quad \omega^2 = \xi\omega^5\xi.$$

This establishes the required epimorphism. Q.E.D.

3.2. The group structure of G

Below, we analyze the group G obtained in Theorem 3.1.1 and show that it is isomorphic to

$$\mathbb{D}_{14} \times \mathbb{Z}_3 = \langle a, b, \xi \mid \xi^2 = e,\ a^3 = e,\ b^7 = e,\ \xi b\xi = b^6,\ [a, b] = [a, \xi] = e\rangle,$$

where $[a, b] = aba^{-1}b^{-1}$ is the commutator.

Lemma 3.2.1. *The map $\xi \mapsto \xi$, $\omega \mapsto ab$ establishes an isomorphism $G \cong \mathbb{D}_{14} \times \mathbb{Z}_3$.*

Proof. Putting $c = ab$, we see that $a = c^7$, $b = c^{15}$. Thus we can use two generators ξ, c, and

$$\mathbb{D}_{14} \times \mathbb{Z}_3 = \langle \xi, c \mid \xi^2 = e,\ c^{21} = e,\ \xi c^{15}\xi = c^6,\ [c^7, \xi] = e\rangle.$$

Now we consider our group G:

$$G = \langle \omega, \xi \mid \xi^2 = e,\ \xi\omega^5\xi = \omega^2\rangle.$$

First we see that $\xi\,\omega^{25}\,\xi = \omega^{10} = (\xi\,\omega^2\,\xi)^2 = \xi\omega^4\xi$. Thus we get $\omega^{21} = e$. We assert that ω^7 is in the center of G. Indeed, $\xi\omega^{14}\xi = (\xi\,\omega^2\,\xi)^7 = \omega^{35} = \omega^{14}$. Thus, ω^{14} is in the center, and so is $\omega^7 = (\omega^{14})^2$. Observe that $\xi\omega^{15}\xi = (\omega^2)^3 = \omega^6$. Thus, we have another presentation of G,

$$G = \langle \omega,\, \xi \,|\, \xi^2 = e,\ \omega^{21} = e,\ \xi\omega^{15}\xi = \omega^6,\ [\omega^7, \xi] = e \rangle,$$

which coincides with that of $\mathbb{D}_{14} \times \mathbb{Z}_3$. (The original relation $\xi\omega^2\xi = \omega^5$ is recovered by squaring the relation $\xi\omega^{15}\xi = \omega^6$, taking into account $\omega^{21} = e$, and cancelling the central element ω^7.) Q.E.D.

3.3. Proof of Theorem 1.1.1

According to Theorem 2.1.3, any curve $B = B(t)$, $t^3 \neq 1$, is a $\mathbb{D}_{14}$-sextic, *i.e.*, its fundamental group $\pi = \pi_1(\mathbb{P}^2 \smallsetminus B)$ factors to $\mathbb{D}_{14}$. On the other hand, $\pi/[\pi, \pi] = \mathbb{Z}_6$. The smallest group with these properties is $\mathbb{D}_{14} \times \mathbb{Z}_3$, *i.e.*, one has $\operatorname{ord}\pi \geqslant \operatorname{ord}(\mathbb{D}_{14} \times \mathbb{Z}_3)$. In view of Theorem 3.1.1 and Lemma 3.2.1, there is an epimorphism $\mathbb{D}_{14} \times \mathbb{Z}_3 \twoheadrightarrow \pi$; comparing the orders, one concludes that it is an isomorphism. Q.E.D.

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