Advanced Studies in Pure Mathematics 56, 2009
Singularities — Niigata–Toyama 2007
pp. 1–34

A divisor on the moduli space of curves associated to the signature of fibered surfaces (with an appendix by Kazuhiro Konno)

Dedicated to Professor Mutsuo Oka on his sixtieth birthday

Tadashi Ashikaga and Ken-Ichi Yoshikawa

Abstract.

In this note, we study the problem of localizing the signature of a fibered surface, i.e., a compact complex surface equipped with the structure of a fiber space over a compact Riemann surface. As applications, we give an estimate for the number of critical points and a formula for the Horikawa index for a certain class of fibered surfaces.

CONTENTS

§1. Introduction

Let $f : S \to B$ be a proper surjective holomorphic map from a compact complex surface S to a compact Riemann surface B, whose general fiber is a compact Riemann surface of genus g. We call f a *fibered*

Received March 31, 2008.
Revised November 8, 2008.
Key words and phrases. Signature, fibered surface, moduli space of curves, localization.

surface of genus g. For $b \in B$, the fiber germ induced by f is denoted by $(f, F_b := f^{-1}(b))$. Let $\mathrm{Sign}(S) \in \mathbf{Z}$ be the signature of the intersection form on $H^2(S, \mathbf{Q})$. If there exists a map $\sigma : B \ni b \to \sigma(f, F_b) \in \mathbf{Q}$ supported at certain finite subset of B such that

$$\mathrm{Sign}(S) = \sum_{b \in B} \sigma(f, F_b),$$

then we say that $\mathrm{Sign}(S)$ is localized and we call σ a *local signature.* (See Section 3 for more precise definition.) As far as we know, this concept has its origin in Matsumoto [32], [33], Ueno [44], Atiyah [7], Xiao [47]... (See [16], [34], [1], [4], [22], [30] etc. for recent related results).

The aim of this paper is to give a construction of a local signature, together with its applications. We mainly study the local signature for *stable* fibered surfaces, i.e., fibered surfaces whose every fiber is a stable curve. (For the local signature for unstable fibered surfaces, see [3].)

Let $\overline{\mathcal{M}_g}$ be the Deligne–Mumford compactification of the moduli space of compact Riemann surfaces of genus g. For a stable fibered surface $f : S \to B$ of genus g, let $\mu_f : B \to \overline{\mathcal{M}_g}$ be the map sending $b \in B$ to the isomorphism class of F_b. A $\mathbf{Q}$-divisor $\mathcal{D}_{\mathrm{sign}}$ on $\overline{\mathcal{M}_g}$ with the following property is called a *signature divisor:*

$$\mathrm{Sign}(S) = \int_B \mu_f^* c_1(\mathcal{D}_{\mathrm{sign}}).$$

By the Grothendieck–Riemann–Roch theorem and the Hirzebruch signature theorem, a signature divisor $\mathcal{D}_{\mathrm{sign}}$ should be $\mathbf{Q}$-linearly equivalent to $4\lambda - \delta$, where λ is the class of Hodge bundle and δ is the class of boundary divisor on $\overline{\mathcal{M}_g}$. We seek after an "explicit" divisor in the class $4\lambda - \delta$ whose pull back via μ_f is expressible in terms of certain geometric properties of the fiber germs of f.

First we assume that f is *Harris–Mumford general* (HM-general for short). Namely, the general fiber F of f is a compact Riemann surface of odd genus $g = 2k - 1 \geq 3$ with maximal gonality, i.e., $(k+1)$-gonal. This condition means that $\mu_f(B)$ is not contained in the support of the Harris–Mumford divisor $\overline{\mathcal{D}}_{\mathrm{HM}}$, i.e., the closure of the locus of smooth curves with gonality $\leq k$. By the Harris–Mumford formula [19], $4\lambda - \delta$ is linearly equivalent to an explicit linear combination of $\overline{\mathcal{D}}_{\mathrm{HM}}$ and the irreducible components of δ. As a result, $\mathrm{Sign}(S)$ is localized at the germs of singular fibers and smooth fibers with gonality $\leq k$, so that the resulting local signature σ_{HM} (Harris–Mumford local signature) for a stable fibered surface f is written as a linear combination of the local intersection numbers $\mu_f \cdot \delta$ and $\mu_f \cdot \overline{\mathcal{D}}_{\mathrm{HM}}$.

When f is *unstable* and HM-general, a correction term called *the local signature defect* comes into the formula for the local signature (Theorem 4.2), whose explicit formula using the monodromy data is obtained by the first author [3]. Together with Theorem 4.2 below and the main result of [3], we have an explicit formula for the local signature of HM-general fiber germs without the assumption of stability.

As applications of the construction of a local signature, we study two other related topics for HM-general fibered surfaces.

The one is the problem of estimating the number of critical points of a fibered surface $f : S \to B$. This problem was posed by Szpiro, and remarkable results have been obtained by Beauville [10], Tan [41] etc. (For recent topological and gauge-theoretic approaches, see e.g. [39].) In this paper, we estimate the number of the critical points of a HM-general fibered surface $f : S \to B$ using $\mathrm{Sign}(S)$ and the genus of F (Theorem 5.1). By a result of Green–Lazarsfeld [18], our estimate is sharp; there are some examples attaining the equality in the estimate.

The other is a formula for the Horikawa index Ind_{HM}, i.e., the invariant measuring the contribution of the fiber germ in the geographical lower bound of slope. Note that the notion of Horikawa index originated from the work of Horikawa [21] for genus 2 fibrations, and recently several works have been done in more general situations (see [6]). We study the Horikawa index appearing in Konno's slope bound [28]. For semi-stable fibered surfaces, we give an expression of the Horikawa index Ind_{HM} using again the local intersection numbers $\mu_f \cdot \delta$ and $\mu_f \cdot \overline{\mathcal{D}}_{\mathrm{HM}}$ (cf. (6.4)). We propose a conjectural formula for Konno's Horikawa index [28] for a certain class of fibered surfaces (Conjecture 6.6).

Replacing the Harris–Mumford divisor by the Eisenbud–Harris divisor, we study similar problems for fibered surfaces of genus 4. Using the Eisenbud–Harris formula [15], we construct a local signature σ_{EH} for those fibered surfaces of genus 4 whose general fiber has mutually distinct trigonal structures (Theorem 7.2). In those cases, σ_{EH} is supported at the germs of singular fibers, smooth fibers with exactly one trigonal structure and smooth hyperelliptic fibers. Moreover, in the semi-stable case, we construct a Horikawa index Ind_{EH}, which measures the local contributions of fiber germs in Chen–Konno's slope bound [11], [27]. It is very likely that, when $g+1$ is composite, most of the results in this note hold by replacing the Harris–Mumford divisor by the Brill–Noether divisor [20, Theorem 6.62].

This paper is organized as follows. In Section 2, we recall the results of Harris–Mumford and Eisenbud–Harris. In Section 3, we introduce the notion of local signature. In Section 4, we give an explicit formula for the Harris–Mumford local signature. In Section 5, we estimate the number of critical points for HM-general stable fibered surfaces. In Section 6, we recall Konno's slope inequality and give its simple proof for HM-general stable fibered surfaces. In Section 7, we introduce Eisenbud–Harris local signature. In Section 8, we study some examples.

Acknowledgements. The authors thank Professor Kazuhiro Konno for useful discussions and also for a valuable appendix to our paper in this volume. They also thank the referee for pointing out many errors in the earlier versions and for suggesting important improvements; Propositions 5.2 and 5.4, Lemma 7.1, Example 7.6 and their proofs are due to the referee. The first author is partially supported by the Grants-in-Aid for Scientific Research (C) 19540047. The second author is partially supported by the Grants-in-Aid for Scientific Research (B) 19340016 and (S) 17104001, JSPS.

§2. Signature divisor on the moduli space of stable curves

A connected, reduced projective curve C is a *stable curve* of genus g if the following conditions are satisfied:

(1) The singular set of C consists of (possibly empty) nodes;
(2) if an irreducible component Γ of C is a smooth rational curve, then Γ meets other components at three points or more;
(3) $h^1(C, \mathcal{O}_C) = g$.

By Deligne–Mumford [14], there exists a coarse moduli space of stable curves of genus g. Let $\overline{\mathcal{M}}_g$ be the coarse moduli space of stable curves of genus g. Let $\mathcal{M}_g$ be the Zariski open subset of $\overline{\mathcal{M}}_g$ consisting of the isomorphism classes of *smooth* stable curves of genus g. Then $\mathcal{M}_g$ is the coarse moduli space of compact Riemann surfaces of genus g. For a stable curve C of genus g, the isomorphism class of C is denoted by $[C] \in \overline{\mathcal{M}}_g$. For $[C] \in \overline{\mathcal{M}}_g$, let $(\mathrm{Def}(C), [C])$ denote the Kuranishi space of C and let $\Gamma_{[C]} := \mathrm{Aut}(C)$ denote the group of automorphisms of C. One has the isomorphisms of germs of complex spaces $(\mathrm{Def}(C), [C]) \cong (\mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C), 0)$ and $(\overline{\mathcal{M}}_g, [C]) \cong (\mathrm{Def}(C)/\mathrm{Aut}(C), [C]) \cong (\mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)/\Gamma_{[C]}, 0)$ (cf. [14, §1]). Hence $\overline{\mathcal{M}}_g$ is a complex orbifold.

There exist irreducible Weil divisors $\Delta_0, \dots, \Delta_{[g/2]} \subset \overline{\mathcal{M}}_g$ such that $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g = \bigcup_{i=0}^{[g/2]} \Delta_i$. There exists a Zariski open subset $\Delta_i^o \subset \Delta_i$ with the following property: If $[C] \in \Delta_0^o$, then C is an irreducible stable curve of genus g with a unique node. If $i > 0$ and $[C] \in \Delta_i^o$, then C is a reducible stable curve of genus g with a unique node whose irreducible components consist of a smooth curves of genus i and a smooth curves of genus $g-i$. Following [19, p.51], [20, Corollary 3.95], we define the $\mathbf{Q}$-divisors $\delta_0, \ldots, \delta_{[g/2]}$ and $\delta \subset \overline{\mathcal{M}}_g$ as

$$(2.1) \qquad \delta_i := \begin{cases} \Delta_i & (i \neq 1) \\ \Delta_1/2 & (i = 1) \end{cases}, \qquad \delta := \sum_{i=0}^{[g/2]} \delta_i.$$

We often identify δ_i and δ with the corresponding $\mathbf{Q}$-line bundles over $\overline{\mathcal{M}}_g$.

Let X be a connected complex space, let B be a complex space, and let $f\colon X \to B$ be a proper surjective flat holomorphic map. Then $f\colon X \to B$ is called a *stable curve of genus g over B* if every fiber of f is a stable curve of genus g. For a stable curve $f\colon X \to B$ of genus g over B, the *induced map* $\mu_f\colon B \to \overline{\mathcal{M}}_g$ is defined by

$$\mu_f(b) := [f^{-1}(b)], \qquad b \in B.$$

In the rest of this section, we consider a stable curve $f\colon S \to B$ of genus g such that the base space B is a Riemann surface and S is a normal complex space. The surface S has at most rational double points of type A and the dualizing sheaf ω_S of S is locally free. Let $\omega_{S/B} := \omega_S \otimes f^*\omega_B^{-1}$ denote the relative dualizing sheaf of the family f and set

$$\lambda(S/B) := (\det f_*\mathcal{O}_S) \otimes (\det R^1 f_*\mathcal{O}_S)^\vee = \det f_*\omega_{S/B},$$

which is a holomorphic line bundle over B. Let $\mathrm{CH}^1(B)$ denote the divisor class group of B. There exists a unique divisor class $\lambda_g \in \mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbf{Q}$ such that for every stable curve $f\colon S \to B$ of genus g over a *compact* Riemann surface B,

(2.2)

$$\mu_f^*\lambda_g = \frac{1}{\nu}[\mathrm{div}(s)] \in \mathrm{CH}^1(B)\otimes\mathbf{Q}, \qquad \forall\, s \in H^0(B, \lambda(S/B)^{\otimes\nu}), \quad \nu \gg 0.$$

The divisor class λ_g is called the *Hodge class.* The critical locus of $f\colon S \to B$ is the subset of S defined as

$$\Sigma_f := \{x \in S \setminus \mathrm{Sing}\, S;\ df(x) = 0\} \cup \mathrm{Sing}\, S.$$

The *discriminant divisor* of $f\colon S \to B$ is defined as

$$\mathfrak{D}_f := \sum_{p \in \Sigma_f} \mu_S(p)\, f(p) \in \mathrm{Div}(B),$$

where $\mu_S(p)$ is the Milnor number of the point $(p, \mathcal{O}_{S,p})$. If p is an A_n-singularity of S, then $\mu_S(p) = n+1$. The line bundle defined by $\mathfrak{D}_f$ is denoted by $[\mathfrak{D}_f]$.

Lemma 2.1. *The following identities of cohomology classes of B hold:*

$$c_1(\lambda(S/B)) = \mu_f^* c_1(\lambda_g), \qquad c_1([\mathfrak{D}_f]) = \mu_f^* c_1(\delta).$$

Proof. The first assertion follows from (2.2) and the second one follows from [20, Proposition 3.92 and 3.93, Lemma 3.94, Corollary 3.95]. Q.E.D.

In this note, a $\mathbf{Q}$-divisor D of $\overline{\mathcal{M}}_g$ is called a *signature divisor* if the following identity holds in $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbf{Q}$:

$$D \equiv 4\,\lambda_g - \delta. \tag{2.3}$$

Let $\pi\colon S' \to S$ be the minimal resolution of the singularities of S. Then $f' := f \circ \pi\colon S' \to B$ is a *semi-stable curve* over B, i.e., every fiber of f' is reduced, has at most normal crossing singularities and contains no (-1)-curves. The fibration f' is called the *semi-stable model* of f. We denote by $\mathrm{Sign}(S')$ the signature of the intersection form on $H^2(S', \mathbf{Q})$. The following is the functorial property of the signature divisor.

Proposition 2.2. *Let $\mathcal{D} \subset \overline{\mathcal{M}}_g$ be a signature divisor. Let $f\colon S \to B$ be a stable curve of genus g over a compact Riemann surface. Then*

$$\mathrm{Sign}\,(S') = \int_B \mu_f^*\, c_1(\mathcal{D}),$$

where S' is the total space of the semi-stable model of f.

Proof. By the Grothendieck–Riemann–Roch formula, Mumford formula and the Hirzebruch signature formula, the following identity holds (cf. Smith [40]):

$$\mathrm{Sign}(S') = 4 \deg \lambda(S'/B) - \deg \mathfrak{D}_{f'}.$$

Since S has at most rational double points, we have $\lambda(S'/B) \simeq \lambda(S/B)$. Since $\deg \mathfrak{D}_{f'} = \deg \mathfrak{D}_f$ by the definition of $\mathfrak{D}_f$, the assertion follows from Lemma 2.1 Q.E.D.

In this note, we mainly use two types of explicit signature divisors which come form the results of Harris–Mumford [19] and Eisenbud–Harris [15], respectively.

First, for a compact Riemann surface C, let $\mathbf{C}(C)$ denote the field of meromorphic functions on C. The *gonality of C* is the integer defined by

$$\mathrm{gon}(C) := \min\{\deg f;\ f \in \mathbf{C}(C),\ f \text{ is not constant}\}.$$

Then $\mathrm{gon}(C) \le [(g(C)+3)/2]$. When $g = 2k-1$, define the locus $\mathcal{D}_{\mathrm{HM}}$ of $\mathcal{M}_g$ by

$$\mathcal{D}_{\mathrm{HM}} := \{[C] \in \mathcal{M}_g;\ \mathrm{gon}(C) \le k = (g+1)/2\} \subset \mathcal{M}_g.$$

When $g = 2k-1$, we define the *Harris–Mumford divisor* of $\overline{\mathcal{M}}_g$ as the closure $\overline{\mathcal{D}}_{\mathrm{HM}}$ of $\mathcal{D}_{\mathrm{HM}}$ in $\overline{\mathcal{M}}_g$, which is a Weil divisor. By [19, p.62 Thorem 5], one has $\overline{\mathcal{D}}_{\mathrm{HM}} \cap \mathcal{M}_g = \mathcal{D}_{\mathrm{HM}}$. See [19, §4] for more details about the Harris–Mumford divisor.

Theorem 2.3 (Harris–Mumford). *When $g = 2k-1$, the following identity holds in $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbf{Q}$:*

$$(2.4)\quad \overline{\mathcal{D}}_{\mathrm{HM}} \equiv \frac{(2k-4)!}{k!(k-2)!}\left\{6(k+1)\lambda_g - k\,\delta_0 - \sum_{i=1}^{k-1} 3i(2k-1-i)\,\delta_i\right\}.$$

In particular, the following $\mathbf{Q}$-divisor is a signature divisor of $\overline{\mathcal{M}}_g$ when $g = 2k-1$:

$$\mathcal{D}_{\mathrm{signHM}} := \frac{2\cdot k!(k-2)!}{3(k+1)(2k-4)!}\,\overline{\mathcal{D}}_{\mathrm{HM}} - \frac{k+3}{3(k+1)}\,\delta_0 + \sum_{i=1}^{k-1} \frac{2i(2k-1-i)-(k+1)}{k+1}\,\delta_i.$$

Proof. See [19, §6]. Q.E.D.

For the related notion, we also recall the following which will be used afterward.

Definition 2.4. Let C be a compact Riemann surface of genus $g \ge 2$. If $g \ge 4$, define the Clifford index $\mathrm{Cliff}(C)$ of C as

$$\mathrm{Cliff}(C) := \min\{\deg L - 2\dim|L|;\ L \in \mathrm{Pic}(C),\ h^0(L) > 1,\ h^1(L) > 1\}.$$

If $g = 2$, set $\mathrm{Cliff}(C) = 0$. If $g = 3$, set $\mathrm{Cliff}(C) = 0$ or 1 according to whether C is hyperelliptic or not.

For the properties of Clifford index, see e.g. [13]. Assume that g is odd and $g \ge 3$. Then the inequality $0 \le \mathrm{Cliff}(C) \le \frac{g-1}{2}$ is well-known. We put

$$\mathcal{D}_{\mathrm{Cliff}} := \{[C] \in \mathcal{M}_g;\ \mathrm{Cliff}(C) < \frac{g-1}{2}\} \subset \mathcal{M}_g.$$

Since the inequality $\mathrm{gon}(C) \ge \mathrm{Cliff}(C) + 2$ holds for every compact Riemann surface C (cf. [13, p.199]), there is an inclusion

$$\mathcal{D}_{\mathrm{HM}} \subset \mathcal{D}_{\mathrm{Cliff}}.$$

Next, put $g = 2(k-1)$ with $k \geq 3$. Let E_k^1 be the locus of the Riemann surfaces $[C]$ of genus g possessing a linear pencil V in a complete linear system $|L|$ of degree k with "violating the Petri condition", i.e., the product map

$$V \otimes H^0(K_C \otimes L^{-1}) \to H^0(C, K_C)$$

is not injective. Let $\overline{E}_k^1$ be its closure in $\overline{\mathcal{M}}_g$, which is a Weil divisor.

Theorem 2.5 (Eisenbud–Harris). *When $g = 2(k-1)$, the following identity holds in* $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbf{Q}$*:*

$$\overline{E}_k^1 = 2\frac{(2k-4)!}{k!(k-2)!}\left\{(6k^2+k-6)\lambda - \sum_{i=0}^{k-1} a_i\delta_i\right\},$$

where $a_0 = k(k-1)$, $a_1 = (2k-3)(3k-2)$, $a_2 = 3(k-2)(4k-3)$. In particular, the following $\mathbf{Q}$-divisor is a signature divisor of $\overline{\mathcal{M}}_g$ when $g = 2(k-1)$:

$$\mathcal{D}_{\mathrm{signEH}} := 2\frac{k!(k-2)!}{(2k-4)!(6k^2+k-6)}\overline{E}_k^1 + \sum_{i=0}^{k-1}\left(\frac{4a_i}{6k^2+k-6} - 1\right)\delta_i.$$

Proof. See [15, Theorem 2]. Q.E.D.

Note that Eisenbud–Harris also determined the coefficients a_i $(3 \leq i \leq k-1)$ explicitly [15, §5].

§3. Local signature

By a *fibered surface* $f\colon S \to B$, we mean that f is a proper surjective holomorphic map from a compact connected nonsingular complex surface S to a compact Riemann surface B such that f is *relatively minimal*, i.e., any fiber of f contains no (-1)-curve. The genus of a fibered surface is defined as the genus of its general fiber. Although f is an unstable curve in general, since B is complex one-dimensional, the induced map from the complement of the critical locus $B \setminus \Sigma_f$ to $\mathcal{M}_g$ extends to a holomorphic map from B to $\overline{\mathcal{M}}_g$ by [23] or by the valuative criterion. This extended map is again written as $\mu_f\colon B \to \overline{\mathcal{M}}_g$ and is called the induced map.

We consider the infinitesimal neighborhood of a fiber of a fibered surface. Let $f_{\mathrm{loc}}\colon (S, S_0) \to (\Delta, 0)$ be a relatively minimal one-parameter deformation germ of a curve S_0 of arithmetic genus g, whose total space S is assumed to be smooth. For simplicity, we write (f, S_0) $(= (f_{\mathrm{loc}}, S_0))$

for $f_{\mathrm{loc}}\colon (S, S_0) \to (\Delta, 0)$ and call it a *fiber germ*. We often identify a fiber germ with its representative.

Definition 3.1. Let $\mathcal{A}$ be a subset of $\mathcal{M}_g$.

(1) A fiber germ $f\colon (S, S_0) \to (\Delta, 0)$ of genus g is said to be $\mathcal{A}$-*general* if the induced map satisfies $\mu_f(\Delta \setminus \{0\}) \subset \mathcal{A}$. A fibered surface $f\colon S \to B$ of genus g is $\mathcal{A}$-general if the fiber germ $f\colon (S, S_b) \to (B, b)$ is $\mathcal{A}$-general for all $b \in B$. The set of all $\mathcal{A}$-general fiber germs of genus g is denoted by $\mathbf{Germ}(\mathcal{A})$.

(2) When $\mathcal{A} = \mathcal{M}_g \setminus \operatorname{Supp} \mathcal{D}_{\mathrm{HM}}$, an $\mathcal{A}$-general fibered surface of genus g is said to be *Harris–Mumford general* (HM-general for short). An *Eisenbud–Harris general* (EH-general for short) fibered surface is defined in the same manner by putting $\mathcal{A} = \mathcal{M}_g \setminus \operatorname{Supp} E^1_k$.

We introduce the notion of local signature as follows.

Definition 3.2. Let $\mathcal{A}$ be a subset of $\mathcal{M}_g$. A function $\sigma_{\mathcal{A}}\colon \mathbf{Germ}(\mathcal{A}) \to \mathbf{Q}$ is called a *local signature with respect to* $\mathcal{A}$ if the following hold:

(1) If $[S_0] \in \mathcal{A}$ and if $f\colon (S, S_0) \to (\Delta, 0)$ is an $\mathcal{A}$-general fiber germ, then
$$\sigma_{\mathcal{A}}(f, S_0) = 0.$$

(2) For every $\mathcal{A}$-general fibered surface $f\colon S \to B$ of genus g,
$$\operatorname{Sign}(S) = \sum_{b \in B} \sigma_{\mathcal{A}}(f, S_b),$$
which is a finite sum by (1).

Once a signature divisor $\mathcal{D} \subset \overline{\mathcal{M}}_g$ is given, we can associate the corresponding local signature with respect to $\mathcal{M}_g \setminus \operatorname{Supp} \mathcal{D}$. To do this, we need the notion of local signature defect, which was introduced by the first author [3].

Let $f\colon (S, S_0) \to (\Delta, 0)$ be a fiber germ of genus g. The topological monodromy around S_0 belongs to the conjugacy class of the pseudo-periodic map of negative twist in the mapping class group of genus g (e.g. [31]). Let N_0 be the (minimal) pseudo-period of the monodromy, i.e., N_0 is the smallest positive integer so that the N_0-th power of the monodromy map is isotopic to the identity on the complement of the admissible system of cut curves of the pseudo-periodic map.

Let $\rho\colon \Delta' \to \Delta$ be the cyclic cover of degree N_0 totally ramified at 0. Then there exists a unique semi-stable curve $f'\colon S' \to \Delta'$ so that S' is birational to the fiber product of $f^{-1}(\Delta)$ and Δ' over Δ. The fiber germ $(f', S'_{0'})$ over $0' = \rho^{-1}(0)$ is called the germ of the *minimal semi-stable* *reduction* of (f, S_0). (See [14], [3, §2].) For the fiber germ (f, S_0), the point $\mu_f(0) \in \overline{\mathcal{M}}_g$ is the isomorphism class of the stable curve obtained by contracting the (-2)-curves in $S'_{0'}$.

Take the closure $\overline{f}\colon \overline{S} \to \overline{\Delta}$ of f in the complex topology, where $\overline{\Delta}$ is a closed disk. Let $\widehat{f}\colon \widehat{S} \to \overline{\Delta}$ be the *normally minimal model* of $\overline{f}$. By definition, $\widehat{f}$ is the unique minimal element in the birational equivalence class of $\overline{f}$ so that the reduced scheme of the central fiber is normal crossing, i.e., any (-1)-curve in the component of the fiber intersects other components of the fiber at least three points (and therefore the germ after contracting it cannot have the normal crossing property).

Let $h_{\partial\overline{S}}$ be a Riemannian metric on $\partial\overline{S}$. Since we have the natural identification $\partial\widehat{S} = \partial\overline{S}$, $h_{\partial\overline{S}}$ is regarded as a Riemannian metric on $\partial\widehat{S}$. The η-invariant of $(\partial\widehat{S}, h_{\partial\overline{S}})$ is denoted by $\eta(\partial\widehat{S}, h_{\partial\overline{S}})$. See [8] for the definition of the η-invariant.

On the other hand, let $\overline{f'}\colon \overline{S}' \to \overline{\Delta}$ be the closure of the minimal semi-stable reduction of f. If $p : \overline{S}' \setminus S'_0 \to \overline{S} \setminus S_0$ denotes the projection induced from $\mathrm{pr}_1\colon f^{-1}(\Delta) \times_\Delta \Delta' \to f^{-1}(\Delta)$, then $p|_{\partial\overline{S'}} : \partial\overline{S}' \to \partial\overline{S} = \partial\overline{S'}/(\mathbf{Z}/N_0\mathbf{Z})$ is an étale covering of degree N_0 and $p^*h_{\partial\overline{S}}$ is a Riemannian metric on $\partial\overline{S}'$. Let $\eta(\partial\overline{S'}, p^*h_{\partial\overline{S}})$ denote the η-invariant of $(\partial\overline{S'}, p^*h_{\partial\overline{S}})$.

Definition 3.3. The *local signature defect* of the fiber germ (f, S_0) is defined as

$$\mathrm{Lsd}(f, S_0) := \left(\mathrm{Sign}(\widehat{S}) - \eta(\partial\widehat{S}, h_{\partial\overline{S}})\right) - \frac{1}{N_0}\left(\mathrm{Sign}(\overline{S}') - \eta(\partial\overline{S'}, p^*h_{\partial\overline{S}})\right),$$

where $\mathrm{Sign}(\widehat{S})$ (resp. $\mathrm{Sign}(\overline{S}')$) is the signature of the cup-product on $H^2(\widehat{S}, \partial\widehat{S}; \mathbf{Q})$ (resp. $H^2(\overline{S}', \partial\overline{S}'; \mathbf{Q})$).

Since the difference $\eta(\partial\widehat{S}, h_{\partial\overline{S}}) - \frac{1}{N_0}\eta(\partial\overline{S'}, p^*h_{\partial\overline{S}})$ is independent of the choice of the metric $h_{\partial\overline{S}}$ by [8, II, Theorem 2.4], so is $\mathrm{Lsd}(f, S_0)$. Hence $\mathrm{Lsd}(f, S_0)$ is an invariant of the fiber germ (f, S_0). More precisely, $\mathrm{Lsd}(f, S_0)$ is explicitly written in terms of the monodromy data. See §8.[1]

By definition, $\mathrm{Lsd}(f, S_0) = 0$ for any *stable* fiber germ (f, S_0). Finally, we define

$$\widehat{\mathrm{Lsd}}(f, S_0) := \mathrm{Lsd}(f, S_0) + \#\mathrm{Bd}_{NR}.$$

[1]For the global formalism of this type of argument in another viewpoint, see [42].

Here $\#\mathrm{Bd}_{NR}$ is the number of contracted (-1)-curves of $\widehat{S}$ to obtain the relatively minimal model $\overline{f}_{\mathrm{loc}}$ from the normally minimal model $\widehat{f}_{\mathrm{loc}}$. Note that

$$\mathrm{Sign}(\overline{S}) = \mathrm{Sign}(\widehat{S}) + \#\mathrm{Bd}_{NR}.$$

Theorem 3.4. *Let $\mathcal{D}_{\mathrm{sign}}$ be a signature divisor on $\overline{\mathcal{M}}_g$. Then*

$$\sigma_{\mathcal{A}}(f, S_0) := \frac{1}{N_0}\mathrm{mult}_{t=0}\,[\mu_{f'}^{*}\mathcal{D}_{\mathrm{sign}}] + \widehat{\mathrm{Lsd}}(f, S_0)$$

is a local signature with respect to $\mathcal{A} = \mathcal{M}_g \setminus \mathrm{Supp}(\mathcal{D}_{\mathrm{sign}})$, where $\mathrm{mult}_{t=0}\,[\mu_{f'}^{*}\mathcal{D}_{\mathrm{sign}}]$ *is the multiplicity of the divisor $\mu_{f'}^{*}\mathcal{D}_{\mathrm{sign}}$ at the origin of the base Δ' of the minimal semi-stable reduction.*

Proof. Let $f\colon S \to B$ be a $\mathcal{A}$-general fibered surface. Let $T := \{b_1, \cdots, b_\ell, b_{\ell+1}, \cdots, b_n, b'_1, \cdots, b'_k\}$ be the set of all points in B such that

(i) S_{b_i} $(1 \le i \le n)$ are not semi-stable;
(ii) $\mu_f(b_1), \cdots, \mu_f(b_\ell), \mu_f(b'_1), \cdots, \mu_f(b'_k) \in \mathrm{Supp}\ \mathcal{D}_{\mathrm{sign}}$.

Let N_{b_i} be the minimal pseudo-period of the topological local monodromy around the singular fiber S_{b_i}. Write N for the least common multiple of $N_{b_1}, \cdots, N_{b_n}$. Adding one more generic point $b_{n+1} \in B \setminus T$ if necessary, we can construct a Galois covering $\rho\colon \widetilde{B} \to B$ of degree N branched exactly at $b_1, \cdots, b_n, (b_{n+1})$ so that $\rho^{-1}(b_i)$ $(1 \le i \le n)$ consists of N/N_{b_i} points with ramification index N_{b_i}. (Note that the presence of b_{n+1} does not affect the calculation below.) Then there exists a semi-stable curve $\widetilde{f}\colon \widetilde{S} \to \widetilde{B}$ birational to the fiber product of S and $\widetilde{B}$ over B (cf. [9, p.95]). Now a result in [3] says that

$$\mathrm{Sign}(S) = \frac{1}{N} \cdot \mathrm{Sign}(\widetilde{S}) + \sum_{i=1}^{n} \widehat{\mathrm{Lsd}}(f, S_{b_i}).$$

On the other hand, since $\widetilde{f}$ is semi-stable, it follows from Proposition 2.2 that

$$\mathrm{Sign}(\widetilde{S}) = \int_{\widetilde{B}} \mu_{\widetilde{f}}^{*}\, c_1(\mathcal{D}_{\mathrm{sign}}).$$

The fiber germ of $\widetilde{f}$ over $\rho^{-1}(b_i)$ $(1 \le i \le n)$ is N/N_{b_i} copies of (f, S_{b_i}). The fiber germ of $\widetilde{f}$ over $\rho^{-1}(b)$ $(b \in B \setminus \{b_1, \ldots, b_{n+1}\})$ is N copies of (f, S_b). Therefore, by choosing a point $\widetilde{b_i}$ (resp. $\widetilde{b_j'}$) on $\widetilde{B}$ such that

$\rho(\widetilde{b_i}) = b_i$ (resp. $\rho(\widetilde{b'_j}) = b'_j$), we obtain

$$\begin{aligned}
\mathrm{Sign}(S) &= \frac{1}{N}\left\{\sum_{i=1}^{\ell}\frac{N}{N_{b_i}}\mathrm{mult}_{\widetilde{b_i}}[\mu_{\widetilde{f}}^{*}\mathcal{D}_{\mathrm{sign}}] + N\sum_{j=1}^{k}\mathrm{mult}_{\widetilde{b'_j}}[\mu_{\widetilde{f}}^{*}\mathcal{D}_{\mathrm{sign}}]\right\} \\
&\qquad + \sum_{i=1}^{n}\widehat{\mathrm{L}}\mathrm{sd}(f, S_{b_i}) \\
&= \sum_{i=1}^{\ell}\left(\frac{1}{N_{b_i}}\mathrm{mult}_{\widetilde{b_i}}[\mu_{\widetilde{f}}^{*}\mathcal{D}_{\mathrm{sign}}] + \widehat{\mathrm{L}}\mathrm{sd}(f, S_{b_i})\right) \\
&\qquad + \sum_{i=\ell+1}^{n}\widehat{\mathrm{L}}\mathrm{sd}(f, S_{b_i}) + \sum_{j=1}^{k}\mathrm{mult}_{\widetilde{b'_j}}[\mu_{\widetilde{f}}^{*}\mathcal{D}_{\mathrm{sign}}] \\
&= \sum_{i=1}^{n}\sigma_{\mathcal{A}}(f, S_{b_i}) + \sum_{j=1}^{k}\sigma_{\mathcal{A}}(f, S_{b'_j}).
\end{aligned}$$

Hence the assertion follows. Q.E.D.

§4. A local signature for Harris–Mumford general fibered surfaces

In this section, we assume that $f\colon S \to B$ is a HM-general fibered surface of genus $g = 2k - 1 \geq 3$.

Definition 4.1. The *Harris–Mumford local signature* is defined as

$$\begin{aligned}
\sigma_{\mathrm{HM}}(f, S_b) &= \frac{1}{N_b}\cdot \mathrm{mult}_{b'}[\mu_{f'}^{*}\mathcal{D}_{\mathrm{signHM}}] + \widehat{\mathrm{L}}\mathrm{sd}(f, S_b) \\
&= \frac{1}{N_b}\left\{\frac{2\cdot k!(k-2)!}{3(k+1)(2k-4)!}\mathrm{mult}_{b'}[\mu_{f'}^{*}\overline{\mathcal{D}}_{\mathrm{HM}}] - \frac{k+3}{3(k+1)}\mathrm{mult}_{b'}[\mu_{f'}^{*}\delta_0]\right. \\
&\quad \left. + \sum_{i=1}^{k-1}\frac{2i(2k-1-i)-(k+1)}{k+1}\mathrm{mult}_{b'}[\mu_{f'}^{*}\delta_i]\right\} + \widehat{\mathrm{L}}\mathrm{sd}(f, S_b).
\end{aligned}$$

Theorem 4.2. *Let $f\colon S \to B$ be a HM-general fibered surface. Then*

$$\mathrm{Sign}(S) = \sum_{b\in B}\sigma_{\mathrm{HM}}(f, S_b).$$

Proof. The assertion follows from Theorems 2.3 and 3.4. Q.E.D.

Corollary 4.3. *Let $f\colon S \to B$ be a non-hyperelliptic fibered surface of genus* 3. *Then*

$$\operatorname{Sign}(S) = \sum_{b \in B} \sigma_{\mathrm{HM}}(f, S_b) = \sum_{b \in B} \left\{ \frac{1}{N_b} \left(\frac{4}{9} \operatorname{mult}_{b'} [\mu_{f'}^* \overline{\mathcal{D}}_{\mathrm{hyper}}] - \frac{5}{9} \operatorname{mult}_{b'} [\mu_{f'}^* \delta_0] + \frac{1}{3} \operatorname{mult}_{b'} [\mu_{f'}^* \delta_1] \right) + \widehat{\operatorname{Lsd}}(f, S_b) \right\},$$

where $\overline{\mathcal{D}}_{\mathrm{hyper}}$ is the divisor defined as the closure of the hyperelliptic locus in $\overline{\mathcal{M}}_3$.

Proof. The result follows from the fact that $\overline{\mathcal{D}}_{\mathrm{HM}} = \overline{\mathcal{D}}_{\mathrm{hyper}}$ when $g = 3$. Q.E.D.

Example 4.4. If (f, S_b) is a germ of a generic smooth hyperelliptic curve (resp. a generic non-separated Lefschetz fiber, resp. a generic separated Lefschetz fiber) in a non-hyperelliptic fibered surface of genus 3, then $\sigma_{\mathrm{HM}}(f, S_b) = 4/9$ (resp. $-5/9$, resp. $1/3$). Therefore σ_{HM} coincide with Kuno's local signature [30] in these cases.

§5. The number of critical points and signature

In this section, we prove the following result, which seems to be closely related to [17, §2].

Theorem 5.1. *Let $f\colon S \to B$ be a stable fibered surface of genus $g = 2k - 1 > 1$.*

(1) *If $f\colon S \to B$ is HM-general, then*

$$\#\Sigma_f \geq 3\,\frac{g+3}{g+7}\,(-\operatorname{Sign}(S)). \tag{5.1}$$

In particular, if f has no critical points, then $\operatorname{Sign}(S) \geq 0$.

(2) *If the equality holds in* (5.1), *then $\mu_f(\mathfrak{D}_f) \subset \operatorname{Supp}(\delta_0 \setminus \bigcup_{i \geq 1} \delta_i)$ and every regular fiber of $f\colon S \to B$ has the maximal gonality $k + 1$.*

(3) *The inequality* (5.1) *is sharp. Namely, for every odd $g > 1$, there exists a HM-general Lefschetz-fibration $f\colon S \to \mathbf{P}^1$ of genus g with*

$$\#\Sigma_f = 3\,\frac{g+3}{g+7}\,(-\operatorname{Sign}(S)).$$

For the proof of Theorem 5.1 (3), we need the following two results.

Proposition 5.2. *Let g be a positive integer with $g \geq 3$. Then there exists a polarized $K3$ surface (S, H) such that every smooth member $C \in |H|$ has maximal gonality $[(g+3)/2]$.*

Proof. By [18], the Clifford index is constant on all smooth curves in a fixed complete linear system on a $K3$ surface. Let (S, H) be a polarized $K3$ surface with $\rho(S) = 1$ and $H^2 = 2g-2$. Then $|H|$ is free from base points. Let $C \in |H|$ be a non-singular member. If $\mathrm{Cliff}(C) < [(g-1)/2]$, then it also follows from [18] that there exists a line bundle M on S such that $M|_C$ computes the Clifford index of C. Since $\rho(S) = 1$, we have $M = nH$ for some positive integer n. However, it is impossible, because $H|_C$ is the canonical bundle of C and $M|_C$ should be a special line bundle with $h^1(M|_C) > 1$. The contradiction shows $\mathrm{Cliff}(C) = [(g-1)/2]$ and, therefore, $\mathrm{gon}(C) = [(g+3)/2]$. Note that this also implies the very ampleness of H for $g \geq 3$. Q.E.D.

Proposition 5.3. *Let X be a smooth algebraic surface and let H be a very ample line bundle on X. Let $f\colon \widetilde{X} \to \mathbf{P}^1$ be a generic Lefschetz pencil of the complete linear system $|H|$. Let g be the genus of a general fiber of $f\colon \widetilde{X} \to \mathbf{P}^1$. Then*

$$
\begin{aligned}
\frac{\#\Sigma_f}{-\mathrm{Sign}(\widetilde{X})} - 3\,\frac{g+3}{g+7} &= \frac{2\,H^2(K_X \cdot H)}{(H^2 - \mathrm{Sign}(X))(H^2 + K_X \cdot H + 16)} \\
&\quad + \frac{(\chi(X) + 3\,\mathrm{Sign}(X) + 24)H^2 + 2(K_X \cdot H)^2}{(H^2 - \mathrm{Sign}(X))(H^2 + K_X \cdot H + 16)} \\
&\quad + \frac{(\chi(X) + 3\,\mathrm{Sign}(X) + 32)K_X \cdot H}{(H^2 - \mathrm{Sign}(X))(H^2 + K_X \cdot H + 16)} \\
&\quad + \frac{16\,\chi(X) + 24\,\mathrm{Sign}(X)}{(H^2 - \mathrm{Sign}(X))(H^2 + K_X \cdot H + 16)},
\end{aligned} \tag{5.2}
$$

where $\chi(X)$ is the topological Euler number of X. In particular, when X is a $K3$ surface, the following identity holds:

$$
\frac{\#\Sigma_f}{-\mathrm{Sign}(\widetilde{X})} = 3\,\frac{g+3}{g+7}. \tag{5.3}
$$

Proof. Since H is very ample, a generic pencil of H is Lefschetz by [25]. In particular, $\#\Sigma_f$ is equal to the number of the singular fibers of $f\colon \widetilde{X} \to \mathbf{P}^1$. Let $L \subset |H|$ be the line corresponding to the pencil $f\colon \widetilde{X} \to \mathbf{P}^1$. Let $\Phi_{|H|}(X)^\vee \subset |H|$ be the projective dual variety of the projective embedding $\Phi_{|H|}(X)$. The number of the singular fibers of $f\colon \widetilde{X} \to \mathbf{P}^1$ is given by the intersection number of the line L with the divisor $\Phi_{|H|}(X)^\vee$, i.e.,

$$\#\Sigma_f = \#(L \cap \Phi_{|H|}(X)^\vee) = \deg \Phi_{|H|}(X)^\vee. \tag{5.4}$$

Let $c(X) = 1 + c_1(X) + c_2(X)$ be the total Chern class of X. By a formula of N. Katz [25], we have

$$\deg \Phi_{|H|}(X)^\vee = (-1)^2 \int_X \frac{c(X)}{(1+H)^2} = 3\,H^2 + 2\,K_X \cdot H + \chi(X). \tag{5.5}$$

By (5.4), (5.5), we have

$$\#\Sigma_f = 3\,H^2 + 2\,K_X \cdot H + \chi(X). \tag{5.6}$$

If $C \in |H|$ is a smooth member, we get by the adjunction formula

$$g = g(C) = \frac{C^2 + C \cdot K_X}{2} + 1 = \frac{H^2 + K_X \cdot H + 2}{2}. \tag{5.7}$$

Since $\widetilde{X}$ is the blow-up of X at $(H)^2$ points, we have

$$\mathrm{Sign}(\widetilde{X}) = \mathrm{Sign}(X) - H^2. \tag{5.8}$$

Equation (5.2) follows from (5.6), (5.7), (5.8).

Assume that X is an algebraic $K3$ surface. Since $K_X = 0$, $\chi(X) = 24$, and $\mathrm{Sign}(X) = -16$, the right hand side of (5.2) vanishes, which implies (5.3). Q.E.D.

It is pointed out by the referee that the degree of the dual variety is known as the *class* of the surface and that (5.5) (or (5.6)) is classically known.

Proof of Theorem 5.1. By the definition of the discriminant divisor, we have

$$\#\Sigma_f = \deg \mathfrak{D}_f = \deg \mu_f^* \delta = \sum_{i=0}^{k} \deg \mu_f^* \delta_i.$$

Since $f : S \to B$ is a stable fibered surface, we get

$$\widehat{\mathrm{Lsd}}(f, S_b) = 0, \qquad N_b = 1$$

for all $b \in B$ in Definition 4.1. By Theorem 2.3 and the positivity of the coefficients of $\overline{\mathcal{D}}_{\mathrm{HM}}$, $\delta_1, \dots, \delta_k$ in the expression of $\mathcal{D}_{\mathrm{signHM}}$, we have

$$\begin{aligned}
\mathrm{Sign}(S) = \deg \mu_f^* \mathcal{D}_{\mathrm{signHM}} &= \frac{2 \cdot k!(k-2)!}{3(k+1)(2k-4)!} \deg \mu_f^* \overline{\mathcal{D}}_{\mathrm{HM}} \\
&- \frac{k+3}{3(k+1)} \deg \mu_f^* \delta_0 + \sum_{i=1}^{k-1} \frac{2i(2k-1-i) - (k+1)}{k+1} \deg \mu_f^* \delta_i \\
&\geq - \frac{k+3}{3(k+1)} \deg \mu_f^* \delta_0 \\
&= - \frac{k+3}{3(k+1)} \# \Sigma_\pi .
\end{aligned}$$

Substituting $k = (g+1)/2$ into this inequality, we get (5.1). This proves (1).

If the equality holds in (5.1), then we have

$$\deg \mu_f^* \overline{\mathcal{D}}_{\mathrm{HM}} = \deg \mu_f^* \delta_1 = \cdots = \deg \mu_f^* \delta_{k-1} = 0.$$

The equality $\deg \mu_f^* \delta_1 = \cdots = \deg \mu_f^* \delta_{k-1} = 0$ implies that

$$\mu_f(\mathfrak{D}_f) \subset \delta_0 \setminus \bigcup_{i \geq 0} \delta_i .$$

Since $\deg \mu_f^* \overline{\mathcal{D}}_{\mathrm{HM}} = 0$, we get $\mu_f(B \setminus \mathfrak{D}_f) \subset \mathcal{M}_g \setminus \mathcal{D}_{\mathrm{HM}}$. Thus $\mathrm{gon}(S_b) = k+1$ for $b \in B \setminus \mathfrak{D}_f$. This proves (2). By Propositions 5.2 and 5.3 Equation (5.3), we get (3). Q.E.D.

An extension of Theorem 5.1 shall be given in the end of the next section. A simple way of constructing a stable fibered surface is to take a Lefschetz pencil of curves on a surface. Unfortunately, it is rather rare for a Lefschetz pencil to be HM-general by the following result due to the referee.

Proposition 5.4. *Let H be a nef and big line bundle on a smooth projective surface S. If $h^0(S,H) \geq 2$, then every smooth member of $|mH|$ is not HM-general for $m \geq 5$.*

Proof. Let H be a nef and big line bundle on a surface S with $h^0(S,H) \geq 2$. We put $g(H) = \frac{K_S H + H^2}{2} + 1$. Since $H \in |H|$ is 1-connected, we have $g(H) = h^1(H, \mathcal{O}_H)$. Recall that we have $g(H) \geq q(S)$, an inequality immediately verified with

$$0 \to \mathcal{O}_S(-H) \to \mathcal{O}_S \to \mathcal{O}_H \to 0$$

and Ramanujam's vanishing theorem $H^1(S, -H) = 0$. Let m be a positive integer such that there is a smooth member $C \in |mH|$.

We claim that $h^0(C, H_C) > 1$ and $h^1(C, H_C) > 1$ hold when $m \geq 5$, where H_C denotes the restriction of H to C.

The first inequality is easy: We get $h^0(C, H_C) = h^0(S, H)$ for $m \geq 2$ using

$$0 \to \mathcal{O}_S(-(m-1)H) \to \mathcal{O}_S(H) \to \mathcal{O}_C(H_C) \to 0.$$

Since we have assumed that $h^0(S, H) \geq 2$, we get $h^0(C, H_C) \geq 2$.

The second one can be shown as follows. We have $h^1(C, H_C) = h^0(C, K_C - H_C)$ by the Serre duality. By the Riemann–Roch theorem and the Kawamata–Viehweg vanishing theorem, we have

$$\begin{aligned} h^0(S, K_S + (m-1)H) &= \frac{m-1}{2}(K_S H + (m-1)H^2) + \chi(\mathcal{O}_S) \\ &= g(H) - q(S) + p_g(S) + \frac{m-2}{2}(K_S H + mH^2) \\ &= g(H) - q(S) + p_g(S) + \frac{m-2}{2}(2g(H) + (m-1)H^2). \end{aligned}$$

The cohomology long exact sequence for

$$0 \to \mathcal{O}_S(K_S - H) \to \mathcal{O}_S(K_S + (m-1)H) \to \mathcal{O}_C(K_C - H_C) \to 0$$

yields

$$h^0(C, K_C - H_C) \geq h^0(S, K_S + (m-1)H) - h^0(S, K_S - H).$$

We have $h^0(S, K_S - H) \leq p_g(S)$. Hence

$$h^0(C, K_C - H_C) \geq g(H) - q(S) + \frac{m-2}{2}(2g(H) - 2 + (m-1)H^2) \geq 2$$

for $m \geq 5$. In sum, we have shown the claim and see that H_C contributes to the Clifford index of C.

Using

$$2g(C) - 2 = m(K_S H + mH^2), \qquad \mathrm{Cliff}(H_C) = mH^2 - 2h^0(C, H_C) + 2,$$

one gets

$$\begin{aligned} 2g(C) - 6 - 4\mathrm{Cliff}(H_C) &= (m^2 - 4m)H^2 + mK_S H + 8h^0(S, H) - 12 \\ &= m(m-5)H^2 + 2m(g(H) - 1) + 8h^0(S, H) - 12. \end{aligned}$$

When $g(H) \geq 1$ and $m \geq 5$, we have $\mathrm{Cliff}(H_C) < (g-3)/2$. Assume that $g(H) = 0$. Then $q(S) = 0$ and $h^0(H, H|_H) = H^2 + 1$. Hence $h^0(S, H) = H^2 + 2$ and it follows

$$2g(C) - 6 - 4\mathrm{Cliff}(H_C) = (m^2 - 5m + 8)H^2 - 2m + 4,$$

which is positive if $m \geq 5$.

We have shown $\mathrm{Cliff}(C) \leq \mathrm{Cliff}(H_C) < (g-3)/2$ for $m \geq 5$. This implies that C is not HM-general when $g(C)$ is odd, because the gonality of C can not exceed $\mathrm{Cliff}(C) + 3$ by a result of Coppens–Martens [13]. Q.E.D.

§6. Konno's slope (in)equality

For a fibered surface $f\colon S \to B$ of genus g, we set

$$K^2_{S/B} := \int_S c_1(\omega_{S/B})^2 = \int_S c_1(S)^2 - 2c_1(S)f^*c_1(B), \quad \chi_f := \deg f_*\omega_{S/B}.$$

The ratio

$$\lambda_f := K^2_{S/B}/\chi_f$$

is called the *slope* of f. The slope inequality of Xiao [46] says that

$$\frac{4(g-1)}{g} \leq \lambda_f \leq 12.$$

If f belongs to a certain restricted class of fibered surfaces, a sharper inequality and also the existence of a local invariant are expected.

Definition 6.1. Let $\mathcal{A}$ be a subset of $\mathcal{M}_g$, and let $\lambda \in \mathbf{Q}$ be a constant with $4 - 4/g \leq \lambda \leq 12$. A *non-negative* function $\mathrm{Ind}_{\mathcal{A}}\colon \mathbf{Germ}(\mathcal{A}) \to \mathbf{Q}_{\geq 0}$ is called a *Horikawa index* with respect to the pair $(\mathcal{A}, \lambda)$ if the following hold:

(1) If $[S_0] \in \mathcal{A}$ and if (f, S_0) is an $\mathcal{A}$-general fiber germ of genus g, then

$$\mathrm{Ind}_{\mathcal{A}}(f, S_0) = 0.$$

(2) For every $\mathcal{A}$-general fibered surface $f\colon S \to B$ of genus g, the following *slope equality* holds:

$$K^2_{S/B} = \lambda \chi_f + \sum_{b \in B} \mathrm{Ind}_{\mathcal{A}}(f, S_b).$$

If there exists a Horikawa index $\mathrm{Ind}_{\mathcal{A}}(\cdot)$ with respect to $(\mathcal{A}, \lambda)$, then

$$\sigma_{\mathcal{A}}(\cdot) := \frac{4}{12-\lambda}\mathrm{Ind}_{\mathcal{A}}(\cdot) - \frac{8-\lambda}{12-\lambda}\epsilon(\cdot) \tag{6.1}$$

is a local signature with respect to $\mathcal{A}$, where the function $\epsilon\colon \mathbf{Germ}(\mathcal{A}) \to \mathbf{Z}$ is the topological Euler contribution

$$\epsilon(f, S_b) := \chi_{\mathrm{top}}(S_b) - (2 - 2g).$$

See [4], [6] for more about Horikawa index.

In the rest of this section, we assume that g is odd and $g \geq 3$. A fibered surface $f\colon S \to B$ of genus g is *Clifford-general* if it is $\mathcal{M}_g \setminus \mathcal{D}_{\mathrm{Cliff}}$-general, i.e., a general fiber of $f\colon S \to B$ has the maximal Clifford index $(g-1)/2$. Note that the Clifford-generality of a fibered surface $f\colon S \to B$ implies its HM-generality.

Theorem 6.2 (Konno). *With respect to the pair* $(\mathcal{M}_g \setminus \mathcal{D}_{\mathrm{Cliff}}, 6(g-1)/(g+1))$, *there exists a Horikawa index. In particular, the following slope inequality holds for every Clifford-general fibered surface* $f\colon S \to B$ *of odd genus* g*:*

$$\lambda_f \geq \frac{6(g-1)}{g+1}.$$

Proof. See [28]. Q.E.D.

Notice that the stability of f is not assumed in Theorem 6.2. The proof of Konno's theorem is involved and relies on the solution of the Green conjecture [45].

Now we restrict our consideration of the above slope equality problem to (semi-)stable curves over compact Riemann surfaces. Namely, fibered surfaces in Definition 6.1 are assumed to be (semi-)stable. In this restricted setting, we give a simpler proof of Konno's theorem as an application of the Harris–Mumford formula:

Proposition 6.3. *With respect to the pair* $(\mathcal{M}_g \setminus \mathcal{D}_{\mathrm{HM}}, 6(g-1)/(g+1))$, *there exists a Horikawa index for (semi-)stable curves over compact Riemann surfaces. In particular, the following slope inequality holds for every HM-general (semi-)stable curve* $f\colon S \to B$ *of genus* $g = 2k-1 \geq 3$ *over a compact Riemann surface* B*:*

$$\lambda_f \geq \frac{6(g-1)}{g+1}.$$

Proof. We have $K_{S/B}^2 = 12\,\chi_f - \deg \mathfrak{D}_f$ by the first Mumford relation [20, Equation (3.110)], so that

$$\begin{aligned} K_{S/B}^2 - \frac{6(g-1)}{g+1}\chi_f &= \left\{12 - \frac{6(g-1)}{g+1}\right\}\chi_f - \deg \mathfrak{D}_f \\ &= \deg \mu_f^* \left[\left\{12 - \frac{6(g-1)}{g+1}\right\}\lambda_g - \delta\right] \qquad (6.2) \\ &= \deg \mu_f^* \left\{\frac{6(k+1)}{k}\lambda_g - \delta\right\}. \end{aligned}$$

We prove that the $\mathbf{Q}$-divisor $6(k+1)/k\,\lambda_g - \delta$ is effective. Since

$$\lambda_g \equiv \frac{k!(k-2)!}{6(k+1)(2k-4)!}\overline{\mathcal{D}}_{\mathrm{HM}} + \frac{k}{6(k+1)}\delta_0 + \sum_{i=1}^{k-1}\frac{i(2k-1-i)}{2(k+1)}\delta_i$$

by the Harris–Mumford formula, we get

(6.3)
$$\frac{6(k+1)}{k}\lambda_g - \delta \equiv \frac{(k-1)!(k-2)!}{(2k-4)!}\overline{\mathcal{D}}_{\mathrm{HM}} + \sum_{i=1}^{k-1}\frac{3i(2k-1-i)-k}{k}\delta_i.$$

Since $k \geq 2$ by the assumption $g = 2k-1 \geq 2$, we get

$$3i(2k-1-i)-k \geq 3(2k-2)-k = 5(k-\frac{6}{5}) > 0,$$

which implies the effectivity of $6(k+1)/k\,\lambda_g - \delta$.

For a fiber germ $f\colon (S, S_0) \to (\Delta, 0)$ belonging to $\mathbf{Germ}(\mathcal{M}_g \backslash \mathcal{D}_{\mathrm{HM}})$, we define the Horikawa index $\mathrm{Ind}_{\mathrm{HM}}(\cdot)$ by

(6.4)
$$\mathrm{Ind}_{\mathrm{HM}}(f, S_0)$$
$$:= \frac{(k-1)!(k-2)!}{(2k-4)!}\deg \mu_f^*\overline{\mathcal{D}}_{\mathrm{HM}} + \sum_{i=1}^{k-1}\frac{3i(2k-1-i)-k}{k}\deg \mu_f^*\delta_i \geq 0.$$

To get the non-negativity, we used the effectivity of $6(k+1)/k\lambda_g - \delta$ and the fact that μ_f intersects $\mathcal{D}_{\mathrm{HM}}$ properly. By (6.2), (6.3), (6.4), we get the slope equality

$$K_{S/B}^2 = \frac{6(g-1)}{g+1}\chi_f + \sum_{b\in B}\mathrm{Ind}_{\mathrm{HM}}(f, S_b).$$

This completes the proof. Q.E.D.

Corollary 6.4. *Let $f\colon S \to B$ be a nonhyperelliptic (semi-)stable curve of genus* 3*. Then*

$$K_{S/B}^2 - 3\chi_f = \sum_{b\in B}\mathrm{Ind}_{\mathrm{HM}}(f, S_b)$$
$$= \sum_{b\in B}\left\{\mathrm{mult}_b\left(\mu_f^*\overline{\mathcal{D}}_{\mathrm{hyper}}\right) + 2\cdot \mathrm{mult}_b\left(\mu_f^*\delta_1\right)\right\}.$$

Especially if (f, S_0) is a generic deformation germ of the following curve S_0, then the following holds:

$$\mathrm{Ind}_{\mathrm{HM}}(f, S_0) = \begin{cases} 0 \\ 1 \\ 2 \end{cases} \quad \textit{if } S_0 \textit{ is} \quad \begin{cases} \textit{an irreducible Lefschetz curve,} \\ \textit{a smooth hyperelliptic curve,} \\ \textit{a reducible Lefschetz curve.} \end{cases}$$

Proof. The result follows from (6.4). Q.E.D.

Remark 6.5. The result in the latter part of Corollary 6.4 coincides with the ones obtained by Reid [38] and Chen–Tan [12].

In view of the definition of the Harris–Mumford local signature and the relation (6.1), it seems to be natural to propose the following:

Conjecture 6.6. For a Clifford-general fiber germ $f\colon (S, S_0) \to (\Delta, 0)$ of genus $g = 2k - 1 \geq 3$, let $\mathrm{ind}_{\mathrm{K}}(f, S_0)$ be Konno's Horikawa index in Theorem 6.2 ([28, §3]). Then the following equality holds:

(6.5)

$$\begin{aligned} \mathrm{ind}_{\mathrm{K}}(f, S_0) =& \frac{1}{N_0} \left\{ \frac{(k-1)!(k-2)!}{(2k-4)!} \deg \mu_{f'}^* \overline{\mathcal{D}}_{\mathrm{HM}} - \frac{k+3}{2k} \deg \mu_{f'}^* \delta_0 \right. \\ & \left. + \sum_{i=1}^{k-1} \frac{6i(2k-1-i) - 3(k+1)}{2k} \deg \mu_{f'}^* \delta_i \right\} \\ & + \frac{3(k+1)}{2k} \widehat{\mathrm{Lsd}}(f, S_0) + \frac{k+3}{2k} \epsilon(f, S_0). \end{aligned}$$

In particular, the right hand side of (6.5) is non-negative.

We extend Theorem 5.1 using Konno's slope inequality.

Theorem 6.7. *Let $f\colon S \to B$ be a Clifford-general fibered surface of genus $g = 2k - 1 > 1$. If $\#\Sigma_f < +\infty$, then the following inequality holds:*

$$\sum_{p \in \Sigma_f} \mu_f(p) \geq 3\, \frac{g+3}{g+7} \,(-\mathrm{Sign}(S)), \tag{6.6}$$

where $\mu_f(p)$ is the Milnor number of the critical point $p \in \Sigma_f$.

Proof. Set

$$e_f := \sum_{b \in B} \epsilon(f, S_b) = \chi_{\mathrm{top}}(X) - \chi_{\mathrm{top}}(F)\chi_{\mathrm{top}}(S).$$

By Noether' formula and Hirzebruch's formula, we get

$$12\chi_f = K_{S/B}^2 + e_f, \qquad \mathrm{Sign}(S) = K_{S/B}^2 - 8\chi_f = 4\chi_f - e_f.$$

Substituting these equalities into the slope inequality in Theorem 6.2, we get

$$e_f \geq 3\frac{g+3}{g+7}(-\mathrm{Sign}(S)). \tag{6.7}$$

Since $\#\Sigma_f < \infty$, we deduce from e.g. [24, (5.5)] that

$$e_f = \chi_{\mathrm{top}}(X) - \chi_{\mathrm{top}}(F)\chi_{\mathrm{top}}(S) = \sum_{p\in\Sigma_f} \mu(f,p). \tag{6.8}$$

The result follows from (6.7) and (6.8). Q.E.D.

Question 6.8. Does inequality (6.6) remain valid if we replace the condition of Clifford-generality by that of HM-generality in Theorem 6.7?

§7. Eisenbud–Harris general fibered surfaces of genus 4

In this section, we discuss the local signature and the Horikawa index of an EH-general fibered surface $f\colon S \to B$ of genus 4. We start with the following Lemma which was pointed out to us by the referee. (See also [37], [35].)

Lemma 7.1. *A hyperelliptic curve of genus* 4 *is in* E_3^1.

Proof. Take a hyperelliptic curve C of genus 4 and a point $x \in C$. Put $L = g_2^1 + x$. Then $|L|$ is a pencil of degree 3 with a base point x. Note that any g_3^1 on a hyperelliptic curve is obtained in this way. If x' is the conjugate to x, that is, $x + x' \in g_2^1$, then $K_C - L = g_2^1 + x'$. Hence $|K_C - L|$ is a pencil of degree 3 with a base point x'. Recall that $|K_C|$ is free from base points. This implies that the multiplication map $\mu : H^0(C, L) \otimes H^0(C, K_C - L) \to H^0(C, K_C)$ cannot be surjective, because the image consists of those sections vanishing at $x + x'$. Since $h^0(C, L) = h^0(C, K_C - L) = 2$ and $h^0(C, K_C) = 4$, one also sees that μ fails to be injective. Q.E.D.

Therefore the isomorphism class of a Riemann surface C of genus 4 is contained in E_3^1 of $\mathcal{M}_4$ if and only if C is hyperelliptic or non-hyperelliptic with a unique trigonal structure g_3^1, i.e. the canonical image of C is contained in a singular quadric. Note that a generic Riemann surface of genus 4 is non-hyperelliptic with two mutually distinct g_3^1, i.e. its canonical image is contained in a smooth quadric.

We remark that $E_3^1 \subset \mathcal{M}_4$ coincides with the locus of vanishing even theta constants. Namely, the product of all even theta constants

$\prod_{(a,b)\text{ even}} \theta_{a,b}$ is regarded as a holomorphic section of a holomorphic line bundle on $\overline{\mathcal{M}}_4$ (cf. [43, p.542]), and one has the equation of divisors $\operatorname{div}(\prod_{(a,b)\text{ even}} \theta_{a,b}) = \overline{E}_3^1$ on $\overline{\mathcal{M}}_4$.

Now Theorem 2.5 says that

$$\mathcal{D}_{\mathrm{signEH}} := \frac{2}{17}\overline{E}_3^1 - \frac{9}{17}\delta_0 + \frac{11}{17}\delta_1 + \frac{19}{17}\delta_2$$

is a signature divisor of $\overline{\mathcal{M}}_4$. We define the *Eisenbud–Harris local signature* of a fiber germ (f, S_b) of f by

$$\sigma_{\mathrm{EH}}(f, S_b) = \frac{1}{N_b}\left(\frac{2}{17}\mathrm{mult}_{b'}[\mu_{f'}^*\overline{E}_3^1] - \frac{9}{17}\mathrm{mult}_{b'}[\mu_{f'}^*\delta_0] + \frac{11}{17}\mathrm{mult}_{b'}[\mu_{f'}^*\delta_1] \right.$$
$$\left. + \frac{19}{17}\mathrm{mult}_{b'}[\mu_{f'}^*\delta_2]\right) + \widehat{\mathrm{L}}\mathrm{sd}(f, S_b).$$

Theorem 7.2. *Let $f\colon S \to B$ be an EH-general fibered surface of genus 4. Then*

$$\mathrm{Sign}(S) = \sum_{b\in B} \sigma_{\mathrm{EH}}(f, S_b).$$

Proof. The result follows from Theorems 2.5 and 3.4. Q.E.D.

Furthermore, if $f\colon (S, S_0) \to (\Delta, 0)$ is *stable* and EH-general, the Horikawa index is defined as

$$\mathrm{Ind}_{\mathrm{EH}}(f, S_0) := \frac{1}{4}\deg \mu_f^*\overline{E}_3^1 + \frac{5}{2}\deg \mu_f^*\delta_1 + \frac{7}{2}\deg \mu_f^*\delta_2, \tag{7.1}$$

which is a non-negative rational number.

Proposition 7.3. *Let $f\colon S \to B$ be a stable EH-general curve of genus 4. Then*

$$K_{S/B}^2 = \frac{7}{2}\chi_f + \sum_{b\in B} \mathrm{Ind}_{\mathrm{EH}}(f, S_b).$$

Proof. From the first Mumford relation and Theorem 2.5, the assertion follows. Q.E.D.

Remark 7.4. Under the same assumption as in Theorem 7.2, Chen [11] and Konno [27] proved the following slope inequality

$$K_{S/B}^2 \geq \frac{7}{2}\chi_f.$$

Konno [29] also defined the Horikawa index of unstable EH-general fiber germs of genus 4 from another viewpoint.

Example 7.5. Let $L = \mathcal{O}_{\mathbf{P}^3}(1)$ be the hyperplane bundle over $\mathbf{P}^3$. Let $q_0, q_1 \in H^0(\mathbf{P}^3, 2L)$ be generic quadrics. Then $Q_0 := \mathrm{div}(q_0)$ and $Q_1 := \mathrm{div}(q_1)$ are generic hyperquadrics of $\mathbf{P}^3$. Let $\{Q_t\}_{t\in\mathbf{P}^1}$ be the pencil of hyperquadrics of $\mathbf{P}^3$ generated by Q_0 and Q_1, where $Q_t = \mathrm{div}(t_0 q_0 + t_1 q_1)$, $t = (t_0 : t_1) \in \mathbf{P}^1$.

Let X be a generic cubic hypersurface of $\mathbf{P}^3$. Then $\{X \cap Q_t\}_{t\in\mathbf{P}^1}$ is a Lefschetz pencil of curves of genus 4 with base locus $B = X \cap Q_0 \cap Q_1$, where $\#B = 3 \cdot 2 \cdot 2 = 12$ by Bezout's theorem. Let $\pi\colon S \to X$ be the blow-up of X at B. Then S has the structure of a stable fibered surface $f\colon S \to \mathbf{P}^1$ of genus 4 over $\mathbf{P}^1$ such that $f^{-1}(t) = X \cap Q_t$. Since $f\colon S \to \mathbf{P}^1$ is a Lefschetz pencil, every fiber of f has at most one node. Moreover, every fiber of f is irreducible, and every smooth fiber is a non-hyperelliptic curve since its canonical map is clearly an embedding. By the definition of the Eisenbud–Harris local signature, $\sigma_{\mathrm{HM}}(f, S_t) \neq 0$ if and only if one of the following is satisfied:

(i) S_t is singular. In this case, $\mu_f(t) \in \delta_0$;

(ii) S_t is smooth and Q_t is singular. In this case, $\mu_f(t) \in E_3^1$.

Case (i) Since $f\colon S \to \mathbf{P}^1$ is a Lefschetz pencil, we get

$$\begin{aligned}\#\{t \in \mathbf{P}^1;\, \mu_f(t) \in \delta_0\} &= \#(\text{Singular Fibers of } f)\\ &= \deg(\Phi_{|2L|_X|}(X), \mathcal{O}_{\mathbf{P}}(1))^\vee,\end{aligned}$$

where $\mathbf{P} = \mathbf{P}(H^0(X, 2L|_X)^\vee)$ and $\deg(\Phi_{|2L|_X|}(X), \mathcal{O}_{\mathbf{P}}(1))^\vee$ denotes the degree of the projective dual variety of $\Phi_{|2L|_X|}(X) \subset \mathbf{P}$. Setting $H = 2L|_X$ in (5.5), we get

$$\#\{t \in \mathbf{P}^1;\, \mu_f(t) \in \delta_0\} = (-1)^2 \int_X \frac{c(X)}{(1 + 2L|_X)^2} = 33.$$

Since the total space of the germ (f, S_b) is smooth, μ_f intersects δ_0 transversally at b when $\mathrm{Sing}\, S_b \neq \emptyset$. Namely, $\mathrm{mult}_b[\mu_f^*\delta_0] = 1$ when $\mathrm{Sing}\, S_b \neq \emptyset$.

Case (ii) Choosing a generic $X \in |3L|$, we may assume $X \cap \bigcup_{t\in\mathbf{P}^1} \mathrm{Sing}\, Q_t = \emptyset$. Hence S_t is smooth when Q_t is singular. Since $\{Q_t\}_{t\in\mathbf{P}^1} \subset |2L|$ is a Lefschetz pencil, we deduce from [25] that

$$\begin{aligned}\#\{t \in \mathbf{P}^1;\, \mu_f(t) \in E_3^1\} &= \#\{t \in \mathbf{P}^1;\, \mathrm{Sing}\, Q_t \neq \emptyset\}\\ &= (-1)^3 \int_{\mathbf{P}^3} \frac{c(\mathbf{P}^3)}{(1 + 2L)^2} = 4.\end{aligned}$$

Let $\{t \in \mathbf{P}^1;\, \mu_f(t) \in E_3^1\} = \{a_1, a_2, a_3, a_4\}$ and set $\nu_i := \mathrm{mult}_{a_i}[\mu_f^* E_3^1] \in \mathbf{Z}_{\geq 1}$.

By Cases (i), (ii) and Theorem 7.2, we get

$$\begin{aligned}\mathrm{Sign}(S) &= \frac{2}{17} \sum_{t\in\mathbf{P}^1;\,\mu_f(t)\in E_3^1} \mathrm{mult}_t[\mu_f^* E_3^1] - \frac{9}{17} \sum_{t\in\mathbf{P}^1;\,\mu_f(t)\in\delta_0} \mathrm{mult}_t[\mu_f^*\delta_0] \\ &= \frac{2}{17}\cdot\sum_{i=1}^{4}\nu_i - \frac{9}{17}\cdot 33 = \frac{2(\nu_1+\nu_2+\nu_3+\nu_4)-297}{17}.\end{aligned}$$

Since S is the blow-up of a cubic hypersurface $X \subset \mathbf{P}^3$ at B, S is isomorphic to the blow-up of $\mathbf{P}^2$ at 18 points in total, so that

$$\mathrm{Sign}(S) = 1 - 18 = -17.$$

Comparing these two formulae for $\mathrm{Sign}(S)$, we get $\nu_1+\nu_2+\nu_3+\nu_4 = 4$. Since $\nu_i \geq 1$, we get $\nu_i = 1$ for all i. Hence we get the formula for $\sigma_{\mathrm{EH}}(f, S_b)$ as follows:

$$\sigma_{\mathrm{EH}}(f,S_b) = \begin{cases} 2/17 & (b\in\mu_f^{-1}(E_3^1)), \\ -9/17 & (b\in\mu_f^{-1}(\delta_0)), \\ 0 & (b\in\mathbf{P}^1\setminus\mu_f^{-1}(E_3^1\cup\delta_0)). \end{cases}$$

By (7.1), the formula for the Horikawa index $\mathrm{Ind}_{\mathrm{EH}}(f, S_b)$ is given as follows:

$$\mathrm{Ind}_{\mathrm{EH}}(f,S_b) = \begin{cases} 1/4 & (b\in\mu_f^{-1}(E_3^1)), \\ 0 & (b\notin\mu_f^{-1}(E_3^1)). \end{cases}$$

Example 7.6. We study the local signature and Horikawa index for hyperelliptic curves of genus 4. A hyperelliptic curve is said to be *generic* if its automorphism group is generated by the hyperelliptic involution. Let $\mathcal{H}_4 \subset \mathcal{M}_4$ denote the hyperelliptic locus of genus 4.

Let C be a generic hyperelliptic curve of genus 4 with hyperelliptic involution ι, and let ι^* be the induced action on $\mathrm{Def}(C) \simeq H^1(C,\Theta_C)$. Then $(\mathcal{M}_4,[C]) \cong (\mathrm{Def}(C)/\iota^*,[C]) \cong (H^1(C,\Theta_C)/\iota^*,[0])$. Let $H^1(C,\Theta_C)_+$ (resp. $H^1(C,\Theta_C)_-$) be the invariant (resp. anti-invariant) subspace of $H^1(C,\Theta_C)$ with respect to the ι^*-action. Since ι is hyperelliptic, it follows from the Lefschetz fixed point formula that $\mathrm{Trace}\,\iota^*|_{H^1(C,\Theta_C)} = 10\cdot\frac{1}{2} = 5$, i.e. $\dim H^1(C,\Theta_C)_+ = 7$ and $H^1(C,\Theta_C)_- = 2$. Hence we get an isomorphism $(\mathcal{M}_4,[C]) \cong (\mathbf{C}^7\times(\mathbf{C}^2/\{\pm1\}),0)$.

Let p: $(\mathrm{Def}(C),[C]) \to (\mathcal{M}_4,[C]) = (\mathrm{Def}(C)/\iota^*,[C])$ be the projection. By Pringsheim [37] and Tsuyumine [43, p.561 l.20–p.562 l.9, Theorem 4], there exist invariant functions $\tau_1,\dots,\tau_7 \in \mathcal{O}_{\mathrm{Def}(C),[C]}$ and anti-invariant functions $f_1,\dots,f_{10} \in \mathcal{O}_{\mathrm{Def}(C),[C]}$ with respect to the ι^*-action on $\mathcal{O}_{\mathrm{Def}(C),[C]}$ satisfying the following:

(i) For any $i \neq j$, $(\tau_1, \dots, \tau_7, f_i, f_j)$ is a system of coordinates of $\mathrm{Def}(C)$ such that $\mathrm{Def}(C) \cap p^{-1}(\mathcal{H}_4) = \{f_i = f_j = 0\}$. In particular, there is an isomorphism of local rings $\mathcal{O}_{\mathcal{M}_4,[C]} \cong \mathbf{C}\{\tau_1, \dots, \tau_7, f_i^2, f_j^2, f_i f_j\}$.

(ii) There exist exactly 10 even theta constants $\{\theta_{a_i,b_i}\}_{1 \leq i \leq 10}$ vanishing at $[C]$. Moreover, f_i $(1 \leq i \leq 10)$ is the square root of θ_{a_i,b_i}, i.e., $f_i = \sqrt{\theta_{a_i,b_i}}$.

By (i), (ii) and the equation of divisors $p^{-1}(\overline{E}_3^1) = \mathrm{div}[p^*(\prod_{(a,b)\,\mathrm{even}} \theta_{a,b})]$ on $\mathrm{Def}(C)$, the germ of analytic subset $p^{-1}(\overline{E}_3^1) \subset \mathrm{Def}(C)$ is defined by the ideal

$$\mathcal{I}_{p^{-1}(\overline{E}_3^1),[C]} := \mathcal{O}_{\mathrm{Def}(C),[C]}(\prod_{i=1}^{10} f_i)^2.$$

Let $\gamma\colon (\Delta, 0) \to (\mathcal{M}_4, [C])$ be a curve and $c\colon (\Delta, 0) \to (\mathrm{Def}(C), [C])$ be a lifting of the curve γ such that $\gamma = p \circ c$. Then γ is said to be *generic* if its lifting c intersects the divisor $\{f_i = 0\} \subset (\mathrm{Def}(C), [C])$ transversally at $[C]$ for all $1 \leq i \leq 10$. If γ is generic, then

$$\mathrm{mult}_{t=0}\gamma^*\overline{E}_3^1 = \frac{1}{\deg(p)}\mathrm{mult}_{t=0}c^*p^*\overline{E}_3^1 = \frac{1}{2}\mathrm{mult}_{t=0}\prod_{i=1}^{10} f_i(c(t))^2 = 10.$$

Let $f\colon (S, S_0) \to (\Delta, 0)$ be a stable EH-general fibered surface of genus 4 such that S_0 is hyperelliptic. The deformation germ (f, S_0) is said to be *generic* if S_0 is generic and if $\mu_f\colon (\Delta, 0) \to (\mathcal{M}_4, [S_0])$ is generic. For a generic deformation germ (f, S_0) of a generic hyperelliptic curve S_0 of genus 4, we have

$$\sigma_{\mathrm{EH}}(f, S_0) = \frac{2}{17}\mathrm{mult}_{t=0}\mu_f^*\overline{E}_3^1 = \frac{20}{17},$$
$$\mathrm{Ind}_{\mathrm{EH}}(f, S_0) = \frac{1}{4}\,\mathrm{mult}_{t=0}\mu_f^*\overline{E}_3^1 = \frac{5}{2}.$$

§8. Examples of the local signature of genus 3

In this section, we give two examples of fiber germs of genus 3, whose Harris–Mumford local signatures and hence local signature defects are calculated explicitly. For the terminology about monodromy maps, see [31], [5] etc.

For our purpose, we first recall the main result of [3]. Let $\rho_f : \mathcal{R}_g \to \mathcal{R}_g$ be the pseudo-periodic map of a Riemann surface $\mathcal{R}_g$ of genus g, which is the monodromy map of $f_{\mathrm{loc}} : (S, S_0) \to (\Delta, 0)$ as in §3. Let

$$\mathcal{A}(\mathcal{C}) = \coprod_j \mathcal{A}(C_j) \coprod_k \mathcal{A}'(C_k)$$

be the decomposition of the annular neighborhood $\mathcal{A}(\mathcal{C})$ of the admissible system of cut curves $\mathcal{C}$ for ρ_f, where $\mathcal{A}(C_j)$ is a non-amphidrome annulus and $\mathcal{A}'(C_k)$ is an amphidrome annulus. For each $\mathcal{A}(C_j)$, let $\sigma_j^{(1)}/\lambda_j^{(1)}$ and $\sigma_j^{(2)}/\lambda_j^{(2)}$ be the valency at both banks of $\mathcal{A}(C_j)$, and let $\delta_j^{(1)}/\lambda_j^{(1)}$ and $\delta_j^{(2)}/\lambda_j^{(2)}$ be their covalencies, respectively. By definition, the following relations are satisfied:

$$\sigma_j^{(i)}\delta_j^{(i)} \equiv 1(\mathrm{mod}\lambda_j^{(i)}), \qquad 1 \leq \sigma_j^{(i)} < \lambda_j^{(i)}, \qquad 1 \leq \delta_j^{(i)} < \lambda_j^{(i)} \quad (i = 1, 2).$$

Let $\mathbf{K}(C_j)$ be the integer greater than or equal to -1 defined by

$$\mathbf{K}(C_j) = -\frac{\delta_j^{(1)}}{\lambda_j^{(1)}} - \frac{\delta_j^{(2)}}{\lambda_j^{(2)}} - \mathbf{s}(C_j),$$

where $\mathbf{s}(C_j)$ is Nielsen's screw number ([36]). For each $\mathcal{A}'(C_k)$, let δ_k/λ_k be the covalency at both banks of $\mathcal{A}'(C_k)$.

Let $\mathbf{B} := \mathcal{R}_g \backslash \mathcal{A}(\mathcal{C}) = \coprod_i \mathcal{B}_i$ be the decomposition into the connected components. Each $\mathcal{B}_i$ is a Riemann surface of genus $\leq g$ with boundary. Let $\left\{\sigma_\alpha^{(i)}/\lambda_\alpha^{(i)}\right\}_\alpha$ and $\left\{\delta_\alpha^{(i)}/\lambda_\alpha^{(i)}\right\}_\alpha$ be the set of valencies and covalencies for ρ_f attached to the multiple points and the boundary curves on $\mathcal{B}_i$, respectively. Let

$$\frac{\lambda_\alpha^{(i)}}{\sigma_\alpha^{(i)}} = K_1(\alpha, i) - \cfrac{1}{K_2(\alpha, i) - \cfrac{1}{\cdots - \cfrac{1}{K_r(\alpha, i)}}}$$

be the continued linear fraction.

Let $\mathbf{B}/\sim = \coprod [\mathcal{B}_i]$ be the orbit decomposition with respect to the cyclic action generated by ρ_f. Here $\mathcal{B}_{i_1} \sim \mathcal{B}_{i_2}$ if and only if $\mathcal{B}_{i_2} = (\rho_f)^n(\mathcal{B}_{i_1})$ for some integer n, and $[\mathcal{B}_i]$ denotes the equivalence class of $\mathcal{B}_i$. Similarly, for the set $\mathbf{A}'$ of non-amphidrome annuli (resp. the set $\mathbf{A}''$ of amphidrome annuli), we put $\mathbf{A}'/\sim = \coprod [\mathcal{A}(C_j)]$ (resp. $\mathbf{A}''/\sim = \coprod [\widetilde{\mathcal{A}}(C_k)]$).

Theorem 8.1. ([3]) *The local signature defect of* (f, S_0) *is given by*

$$\mathrm{Lsd}(f, S_0) = -\frac{1}{3}\sum_{[\mathcal{B}_i]}\sum_{\alpha}\left\{\frac{\sigma_\alpha^{(i)} + \delta_\alpha^{(i)}}{\lambda_\alpha^{(i)}} + \sum_{\beta=1}^{r} K_\beta(\alpha, i)\right\} + \sum_{[\mathcal{A}(C_j)]}\left(\frac{\delta_j^{(1)}}{\lambda_j^{(1)}} + \frac{\delta_j^{(2)}}{\lambda_j^{(2)}} + \epsilon_j\right) + \sum_{[\mathcal{A}'(C_k)]}\left(\frac{\delta_k}{\lambda_k} - 2\right),$$

where ϵ_j *is the rational number defined in* [3]. *If* $\mathbf{K}(C_j) \geq 0$, *then* $\epsilon_j = 0$.

Example 8.2. Let (ξ, x, t) be a system of coordinates of $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \Delta$, where ξ, x are the inhomogeneous coordinates of the first and the second projective lines, respectively. Define the divisor S' on X by

$$S' := \{(\xi, x, t) \in X;\ \xi^4 + x^4 + t^2 = 0\}.$$

The surface S' has a simple elliptic singularity of type $\widetilde{E}_7$ at $(0,0,0)$. Let $\pi\colon S \to S'$ be the minimal resolution and let $f : S \to \Delta$ be the morphism defined as $f := \mathrm{pr}_3|_X \circ \pi$. By the same argument as in [2, pp.67–68], any fiber of f except for the central fiber $F = f^{-1}(0)$ is a smooth non-hyperelliptic curve of genus 3, and the irreducible decomposition of F is expressed as $F = 2E + \sum_{i=1}^4 F_i$, where E is a nonsingular elliptic curve with $E^2 = -2$ and F_i $(1 \leq i \leq 4)$ are (-2)-curves with $EF_i = 1$ and $F_{i_1}F_{i_2} = 0$ $(i_1 \neq i_2)$. (See [2, Fig.1 in p.68].) We compute the local signature $\sigma_{HM}(f, F)$.

In the classification [5], the topological monodromy map of f is the periodic map of order 2 with the total valency $1/2 + 1/2 + 1/2 + 1/2$, i.e., the list (iii) (12) in [5, p.199]. Since (f, F) is normally minimal, Theorem 8.1 implies that

$$\widehat{\mathrm{Lsd}}(f, F) = \mathrm{Lsd}(f, F) = -\frac{1}{3}\left(\frac{1+1}{2} + 2\right)\cdot 4 = -4.$$

On the other hand, the minimal semi-stable reduction of f is given as follows: Let $B_0 = \{x^4 + t^2 = 0\}$ be the branch divisor of the 4-fold cover $S' \to W = \mathbf{P}^1 \times \Delta$. Let $\widetilde{\Delta} \to \Delta$ be the double cover defined by $u \mapsto t = u^2$. Then the pull back B_0' of B_0 on $W' = \mathbf{P}^1 \times_\Delta \widetilde{\Delta}$ is defined by the equation $x^4 + u^4 = 0$.

By the elementary transformation at the center $(x, u) = (0, 0)$, the $\mathbf{P}^1$-bundle W' is transformed to a new $\mathbf{P}^1$-bundle $\widetilde{W}$ so that the proper image $\widetilde{B_0}$ of B_0' is a smooth curve meeting any fiber of $\pi\colon \widetilde{W} \to \Delta$ transversally at four points. (See Figure 1.)

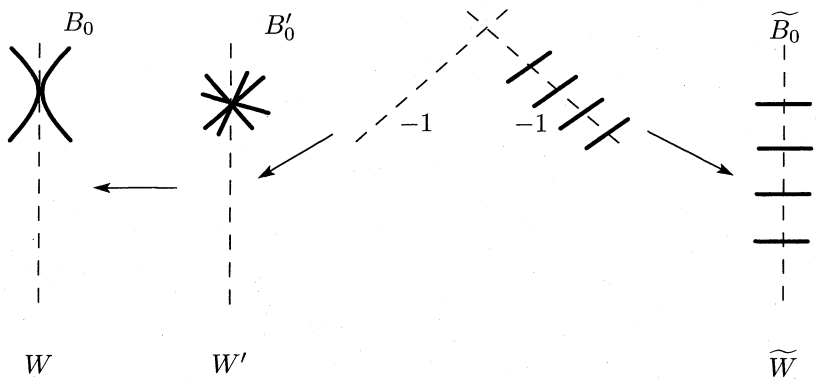

Figure 1

Let $\widetilde{S}$ be the divisor on $\mathbf{P}^1 \times \widetilde{W}$ defined by the equation

$$\xi^4 + \widetilde{b}(y,u) = 0,$$

where $\widetilde{b}(y,u) = 0$ is the equation defining $\widetilde{B_0}$. Then $\widetilde{f} := \widetilde{\pi} \circ \mathrm{pr}_3|_{\widetilde{S}} \colon \widetilde{S} \to \widetilde{\Delta}$ is a smooth family of non-hyperelliptic curves of genus 3. By the uniqueness of the minimal semi-stable reduction, the germ $(\widetilde{f}, \widetilde{F} = \widetilde{f}^{-1}(0))$ is the minimal semi-stable reduction of (f, F).

Since $\widetilde{F}$ is non-hyperelliptic and hence $\sigma_{\mathrm{HM}}(\widetilde{f}, \widetilde{F}) = 0$, we get by Theorem 3.4

$$\sigma_{\mathrm{HM}}(f,F) = \frac{1}{2}\sigma_{\mathrm{HM}}(\widetilde{f}, \widetilde{F}) + \mathrm{Lsd}(f,F) = -4.$$

Since $\epsilon(f,F) = 8$, the "conjectural Konno's Horikawa index" of genus 3 in (6.5) is

$$\text{“}\mathrm{ind}_{\mathrm{K}}(f,F)\text{”} = 1,$$

as is expected (cf. [26, §9.7] and [2]).

Example 8.3. Let X be the same as in Example 8.2, and let S be a smooth divisor on X defined by the equation

$$\xi^4 + (x^2 + t)(x+a)(x+b) = 0, \qquad (a \neq b,\ a \neq 0,\ b \neq 0).$$

Set $f := \mathrm{pr}_3|_S$. Then $f : S \to \Delta$ is a non-hyperelliptic fibered surface of genus 3. The central fiber $F = f^{-1}(0)$ is an irreducible singular curve with a tacnode, so that the normalization of F is an elliptic curve. We compute $\sigma_{\mathrm{HM}}(f,F)$.

The central fiber $\hat{F}$ of the normally minimal model $\hat{f}: \hat{S} \to \Delta$ of f is written as

$$\hat{F} = E + 4F_1 + 2F_2,$$

where E is an elliptic curve with $E^2 = -8$ and F_1, F_2 are rational curves with $F_1^2 = -1$, $F_2^2 = -2$, $EF_1 = 2$, $F_1F_2 = 1$ and $EF_2 = 0$. We obtain S from $\hat{S}$ by blowing down (-1)-curves two times. The admissible system of cut curves of the monodromy map consists of two simple closed curves of type (E) in [5, p.221,Table 2]. The monodromy map acts trivially on one body, and acts on another body with the total valency $1/4 + 1/4 + 1/2$. The screw numbers on both annular neighborhoods of the cut curves are $-1/4$. It follows from Theorem 8.1 that

$$\mathrm{Lsd}(f, F) = -\frac{1}{3}\left(\frac{1+1}{4} \cdot 2 + 4 \cdot 2 + \frac{1+1}{2} + 2\right) + \frac{1}{4} \cdot 2 = -\frac{7}{2}.$$

Thus $\widehat{\mathrm{Lsd}}(f, F) = -3/2$.

The minimal semi-stable reduction of $f: S \to \Delta$ is described similarly as in Example 8.2. Namely, by the cyclic base change of degree 4, the pull back B_0' of the branch curve of the 4-fold cover consists of three connected components; the one component has a tacnode whose tangent line is transversal to the fiber and the other two components meet the fiber transversally. The 4-fold cyclic cover S' branched along B_0' is a normal surface with a simple elliptic singularity of type $\widetilde{E}_7$. The minimal resolution $\widetilde{S}$ of S' has the structure of fibered surface $\widetilde{f}: \widetilde{S} \to \widetilde{\Delta}$ induced from $f: S \to \widetilde{\Delta}$, whose central fiber $\widetilde{F}$ is a stable curve consisting two elliptic components with two nodes. The germ $(\widetilde{f}, \widetilde{F})$ is the minimal semi-stable reduction of (f, F).

The locus of stable curves consisting two elliptic components with two nodes is of codimension 2 in $\overline{\mathcal{M}}_3$, and is contained in the closure of the hyperelliptic locus, i.e. the support of $\overline{\mathcal{D}}_{\mathrm{HM}}$. We can see that the curve $\mu_{\widetilde{f}}(\widetilde{\Delta})$ intersects $\overline{\mathcal{D}}_{\mathrm{HM}}$ transversally. (For instance, this is verified from the "global method" as in Example 7.5, which is omitted.) Namely,

$$\mathrm{mult}_{t=0}(\mu_{\widetilde{f}}^{*}\overline{\mathcal{D}}_{\mathrm{HM}}) = 1.$$

On the other hand, the germ $(\widetilde{f}, \widetilde{F})$ has a splitting deformation in the sense of [6, §4] into two deformation germs of an irreducible Lefschetz fiber, each of which is stable under deformation (i.e. atomic) by the smoothness of $\widetilde{S}$ at the nodes of $\widetilde{F}$. Thus

$$\mathrm{mult}_{t=0}(\mu_{\widetilde{f}}^{*}\delta_0) = 2, \qquad \mathrm{mult}_{t=0}(\mu_{\widetilde{f}}^{*}\delta_1) = 0.$$

Hence

$$\sigma_{\mathrm{HM}}(\widetilde{f},\widetilde{F}) = \frac{4}{9}\mathrm{mult}_{t=0}(\mu_{\widetilde{f}}^*\overline{\mathcal{D}}_{\mathrm{HM}}) - \frac{5}{9}\mathrm{mult}_{t=0}(\mu_{\widetilde{f}}^*\delta_0) + \frac{1}{3}\mathrm{mult}_{t=0}(\mu_{\widetilde{f}}^*\delta_1)$$
$$= -\frac{2}{3}.$$

By Theorem 3.4, we obtain

$$\sigma_{\mathrm{HM}}(f,F) = \frac{1}{4}\sigma_{\mathrm{HM}}(\widetilde{f},\widetilde{F}) + \widehat{\mathrm{L}}\mathrm{sd}(f,F) = -\frac{5}{3}.$$

From the above calculation and (6.5), the conjectural Konno's Horikawa index is

$$\text{“}\mathrm{ind}_{\mathrm{K}}(f,F)\text{”} = 0,$$

as is expected.[2]

References

[1] T. Arakawa and T. Ashikaga, Local splitting families of hyperelliptic pencils I, Tohoku Math. J., **53** (2001), 369–394.

[2] T. Ashikaga, A remark on the geography of surfaces with birational canonical morphisms, Math. Ann., **290** (1991), 63–76.

[3] ———, Local signature defect of fibered complex surfaces via monodromy and stable reduction, to appear in Comment. Math. Helv.

[4] T. Ashikaga and H. Endo, Various aspects of degenerate families of Riemann surfaces, Sugaku Expositions, **19** (2006), 171–196.

[5] T. Ashikaga and M. Ishizaka, Classification of degenerations of curves of genus 3 via Matsumoto–Montesinos' theorem, Tohoku Math. J., **53** (2001), 369–394.

[6] T. Ashikaga and K. Konno, Global and local properties of pencils of algebraic curves, In: Algebraic Geometry 2000 Azumino, (eds. by S. Usui et al.), Adv. Stud. Pure Math., **36**, Math. Soc. Japan, 2002, pp. 1–49.

[7] M. Atiyah, The logarithm of Dedekind η-function, Math. Ann., **278** (1987), 335–380.

[8] M. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry, I, Math. Proc. Camb. Phil. Soc., **77** (1975), 43–69; II, ibid., **78** (1975), 405–432.

[9] W. Barth, C. Peters and A. Van de Ven, Compact Complex Surfaces, Springer, 1984.

---

[2]Indeed, since this germ is relatively minimal in the original fibration of quartic curves, it does not contribute the “distance” from the Castelnuovo line, which is nothing but the geographical lower bound of non-hyperelliptic fibrations of genus 3 (See [2], [6]).

[10] A. Beauville, The Szpiro inequality for higher genus fibrations, In: Algebraic Geometry, A volume in memory of Paolo Francia, (eds. M. C. Beltrametti et al), Walter de Gruyter, 2002, pp. 61–63.
[11] Z. Chen, On the lower bound of the slope of a nonhyperelliptic fibration of genus 4, Internat. J. Math., **4** (1993), 367–378.
[12] Z. Chen and S. L. Tan, Upper bound on the slope of a genus 3 fibration, Contemp. Math., **400** (2006), 65–87.
[13] M. Coppens and G. Martens, Secant spaces and Clifford's theorem Compositio Math., **78** (1991), 193–212.
[14] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, Inst. Hautes Études Sci. Publ. Math., **36** (1969), 75–110.
[15] D. Eisenbud and J. Harris, The Kodaira dimension of the moduli space of curves of genus ≥ 23, Invent. Math., **90** (1987), 359–387.
[16] H. Endo, Meyer's signature cocycle and hyperelliptic fibrations (with Appendix written by T. Terasoma), Math. Ann., **316** (2000), 237–257.
[17] G. Farkas and M. Popa, Effective divisors on $\overline{\mathcal{M}}_g$, curves on $K3$ surfaces, and the slope conjecture, J. Algebraic Geom., **14** (2005), 241–267.
[18] M. Green and R. Lazarsfeld, Special divisors on curves on a $K3$ surface, Invent. Math., **89** (1987), 357–370.
[19] J. Harris and D. Mumford, On the Kodaira dimension of the moduli space of curves, Invent. Math., **67** (1982), 23–86.
[20] J. Harris and I. Morrison, Moduli of Curves, Springer-Verlag, Berlin, 1998.
[21] E. Horikawa, On algebraic surfaces with pencils of curves of genus 2, In: Complex Analysis and Algebraic Geometry, a volume dedicated to K. Kodaira, Iwanami Shoten Publishers and Cambridge Univ. Press, Tokyo and Cambridge, 1977, pp. 79–90.
[22] S. Iida, Adiabatic limits of η-invariants and the Meyer functions, to appear in Math. Ann., DOI 10.1007/s00208-009-0412-y.
[23] Y. Imayoshi, Holomorphic families of Riemann surfaces and Teichmüller spaces, In: Riemann surfaces and related topics, Ann. of Math. Stud., **97**, Princeton Univ. Press, 1981, pp. 277–300.
[24] B. Iversen, Critical points of an algebraic function, Invent. Math., **12** (1971), 210–224
[25] N. Katz, Pinceaux de Lefschetz: théorème d'existence, Lecture Notes in Math. **340**, Springer-Verlag, 1973, pp. 212–253.
[26] K. Konno, Algebraic surfaces of general type with $c_1^2 = 3p_g - 6$, Math. Ann., **290** (1991), 77–107.
[27] ———, Nonhyperelliptic fibrations of small genus and certain irregular canonical surfaces, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) , **20** (1993), 575–595.
[28] ———, Clifford index and the slope of fibered surfaces, J. Algebraic Geom., **8** (1999), 207–220.
[29] ———, Appendix to this paper, in this volume.
[30] Y. Kuno, The mapping class group and the Meyer function for plane curves, Math. Ann., **342** (2008), 923–949.

[31] Y. Matsumoto and J. M. Montesinos-Amilibia, Pseudo-periodic maps and degeneration of Riemann surfaces, I, II, preprints, Univ. of Tokyo and Univ. Complutense de Madrid, 1991/1994 (revised in 2006).

[32] Y. Matsumoto, On 4-manifolds fibered by tori, I, Proc. Japan Acad. Ser. A Math. Sci., **58** (1982), 298–301; II, ibid., **59** (1983), 100–103.

[33] ———, Lefschetz fibrations of genus two — a topological approach, In: Proceedings of the 37th Taniguchi Symposium on Topology and Teichmüller Spaces, World Scientific, Singapore, 1996, pp. 123–148.

[34] T. Morifuji, On Meyer's function of hyperelliptic mapping class groups, J. Math. Soc. Japan, **55** (2003), 117–129.

[35] D. Mumford, Tata Lectures on Theta II, Progr. Math., **43**, Birkhäuser, Boston, 1984.

[36] J. Nielsen, Surface transformation classes of algebraically finite type, Danske Vid. Selsk. Math.-Phys. Medd., **21** (1944), 3–89; English translation, Collected Papers 2, Birkhäuser, 1986.

[37] A. Pringsheim, Zur Theorie der hyperelliptischen Funktionen, insbesondere derjenigen dritter Ordnung ($\rho = 4$), Math. Ann., **12** (1877), 435–475.

[38] M. Reid, Problems on pencils of small genus, preprint (in his home page), 1990.

[39] Y. Sato, 2-spheres of square -1 and the geography of genus 2 Lefschetz fibrations, preprint.

[40] I. Smith, Lefschetz fibrations and the Hodge bundle, Geom. Topol., **3** (1999), 211–233.

[41] S. L. Tan, The minimal number of singular fibers of a semistable curve over $\mathbf{P}^1$, J. Algebraic Geom., **4** (1995), 591–596.

[42] ———, On the invariants of base changes of pencils of curves II, Math. Z., **222** (1996), 655–676.

[43] S. Tsuyumine, Thetanullwerte on a moduli space of curves and hyperelliptic loci, Math. Z., **207** (1991), 539–568.

[44] K. Ueno, Discriminants of curves of genus 2 and arithmetic surfaces, In: Algebraic Geometry and Commutative Algebra in honor of M. Nagata, Kinokuniya, Tokyo, 1987, pp. 749–770.

[45] C. Voisin, Green's canonical syzygy conjecture for generic curves of odd genus, Compos. Math., **141** (2005), 1163–1190.

[46] G. Xiao, Fibered algebraic surfaces with low slope, Math. Ann., **276** (1987), 449–466.

[47] ———, The Fibrations of Algebraic Surfaces, Shanghai Scientific and Technical Publishers, 1992, in Chinese.

Tadashi Ashikaga
Faculty of Engineering
Tohoku-Gakuin University
Tagajo, Miyagi, 985-8537
Japan

Ken-Ichi Yoshikawa
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Tokyo 153-8914
Japan

E-mail address: tashikaga@tjcc.tohoku-gakuin.ac.jp
yosikawa@ms.u-tokyo.ac.jp