Advanced Studies in Pure Mathematics 55, 2009
Noncommutativity and Singularities
pp. 335–343

Ends of metric measure spaces with nonnegative Ricci curvature

Masayoshi Watanabe

Abstract.

We prove that metric measure spaces with nonnegative Ricci curvature have at most two ends.

§1. Introduction

We show a result of our previous paper [19], which states that measured length spaces with nonnegative Ricci curvature have at most two ends.

Lott–Villani [10], [11], [18] and Sturm [15], [16] introduced the concept of lower Ricci curvature bounds for measured length spaces. We use the definition in [18]: the weak curvature-dimension $\mathrm{CD}(K,N)$ condition ($K \in \mathbb{R}, N \in [1,\infty]$). Ohta [12] and Sturm [16] also gave a definition of lower Ricci curvature bounds: the measure contraction property $\mathrm{MCP}(K,N)$ ($K \in \mathbb{R}, N \in [1,\infty)$).

The parameters K and N play roles of lower Ricci curvature bound and dimension respectively. In fact, given a complete Riemannian manifold (M,g) with Riemannian distance d_g and measure ν_g, the measured length space (M,d_g,ν_g) satisfies the weak $\mathrm{CD}(K,N)$ condition if and only if $\mathrm{Ric}_M \geq K$ and $\dim(M) \leq N$. The property $\mathrm{MCP}(K,\dim(M))$ implies $\mathrm{Ric}_M \geq K$; however, $\mathrm{MCP}(K,N)$ does not imply $\mathrm{Ric}_M \geq K$ for $N > \dim(M)$.

If a measured length space X satisfies the weak $\mathrm{CD}(K,N)$ condition and if all geodesics in X do not branch, then X satisfies $\mathrm{MCP}(K,N)$; see [16]. Both the weak $\mathrm{CD}(K,N)$ condition and $\mathrm{MCP}(K,N)$ are preserved under measured Gromov–Hausdorff limits.

Received August 2, 2007.
2000 *Mathematics Subject Classification.* 53C21, 53C23.
Key words and phrases. Ricci curvature, metric measure space, end.
Partially supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

This paper is concerned with the case that $K = 0$ and $N < \infty$ (i.e. the weak $\mathrm{CD}(0, N)$ condition or $\mathrm{MCP}(0, N)$). Our main theorem is as follows:

Theorem 1.1. *Let (X, d) be a complete, locally compact, separable length space equipped with a nonnegative Radon measure ν. Given $N \in [1, \infty)$, we assume one of the following (i) and (ii):*

(i) *The measured length space (X, d, ν) satisfies the weak* $\mathrm{CD}(0, N)$ *condition.*
(ii) *The measured length space (X, d, ν) satisfies* $\mathrm{MCP}(0, N)$.

Then X has at most two ends.

In the case of Riemannian manifolds with nonnegative Ricci curvature, the Cheeger–Gromoll splitting theorem [5] implies Theorem 1.1. The splitting theorem states that if a complete Riemannian manifold M of nonnegative Ricci curvature contains a straight line, then M is isometric to the product $\mathbb{R} \times N$ for some Riemannian manifold N. Cheeger and Colding [2] extended this to the Gromov–Hausdorff limits of a sequence of Riemannian manifolds M_i with $\mathrm{Ric}_{M_i} \geq -\delta_i$, where $\delta_i \to 0$. Unfortunately, the splitting theorem does not hold under the assumption of Theorem 1.1. In fact, any finite-dimensional, say n-dimensional, normed linear space with Lebesgue measure satisfies the weak $\mathrm{CD}(0, n)$ condition and $\mathrm{MCP}(0, n)$ [18]. Theorem 1.1 is proved without the splitting theorem.

For many recent results on this area, see [8], [9], [13], and [14].

This paper is organized as follows: In Section 2, we recall basic definitions: length spaces, the (pointed, measured) Gromov–Hausdorff convergence and the Wasserstein distance. In Section 3, we give the definition of the weak $\mathrm{CD}(0, N)$ condition and then summarize some basic properties. We prove Theorem 1.1 in Section 4.

Acknowledgements. The author thanks Professor Takashi Shioya for his constant encouragement and helpful discussions.

§2. Preliminaries

Length spaces. Let (X, d) be a metric space. Given $x \in X$ and $r > 0$, we denote by $B_r(x)$ and $\overline{B}_r(x)$ the open and closed ball of radius r and centered at x, respectively. The sphere of radius r and centered at x is denoted by $S_r(x)$.

A path $\gamma : [0, l] \to X$ is called a *geodesic* if it is locally minimizing and has a constant speed. We say that (X, d) is a *length space* if $d(x, y) =$

$\inf_\gamma \mathrm{Length}(\gamma)$ for all $x, y \in X$, where the infimum is taken over all paths joining x and y. If X is a complete, locally compact length space, then all two points in X are joined by a minimal geodesic.

Gromov–Hausdorff convergence. Let (X, d_X), (Y, d_Y) be metric spaces. We say that, for $\epsilon > 0$, a map $\varphi : X \to Y$ is an ϵ-*approximation* if

(i) $|d_X(x,y) - d_Y(\varphi(x), \varphi(y))| < \epsilon$ holds for all $x, y \in X$, and
(ii) the ϵ-neighborhood of $\varphi(X)$ coincides with Y.

Let (X_i, x_i) for $i = 1, 2, \dots$ and (X, x) be pointed metric spaces. We say that $\{(X_i, x_i)\}$ *pointed Gromov–Hausdorff converges to* (X, x) if for each $R > 0$ there exist $R_i \searrow R$, $\epsilon_i \searrow 0$, and ϵ_i-approximations $\varphi_i : B_{R_i}(x_i) \to B_R(x)$ with $\varphi_i(x_i) = x$. The Gromov–Hausdorff limits of a sequence of length spaces are also length spaces. See [1] and [7] for further information.

Measured Gromov–Hausdorff convergence. A *metric measure space* is a triple (X, d, ν) where (X, d) is a metric space and ν is a nonnegative Radon measure on X. Let (X_i, x_i, ν_i) for $i = 1, 2, \dots$ and (X, x, ν) be pointed metric measure spaces. We say that $\{(X_i, x_i, \nu_i)\}$ *pointed measured Gromov–Hausdorff converges to* (X, x, ν) if there exist $\epsilon_i \searrow 0$, $R_i \to \infty$, and Borel measurable ϵ_i-approximations $\varphi_i : B_{R_i}(x_i) \to B_{R_i}(x)$ such that the sequence of push-forward measures $\{(\varphi_i)_* \nu_i\}$ converges vaguely to ν, that is, $\lim_{i\to\infty} \int_{X_i} f \circ \varphi_i \, d\nu_i = \int_X f \, d\nu$ holds for all continuous functions $f : X \to \mathbb{R}$ with compact support. We refer to [6] for details.

Wasserstein distance. Let (X, d, x) be a complete, locally compact, separable, pointed length space. We denote by $P(X)$ the set of Borel probability measures on X. Let $P_2(X)$ be the set of Borel probability measures on X with finite second moment:

$$P_2(X) = \left\{ \mu \in P(X) \;\middle|\; \int_X d(x,y)^2 d\mu(y) < \infty \right\},$$

which is independent of the choice of x. Given $\mu_0, \mu_1 \in P_2(X)$, a probability measure $\pi \in P(X \times X)$ is called a *transference plan* between μ_0 and μ_1 if $\pi(A \times X) = \mu_0(A)$ and $\pi(X \times A) = \mu_1(A)$ hold for all measurable sets $A \subset X$. For example, the product measure $\mu_0 \times \mu_1$ is a transference plan between μ_0 and μ_1. We define the *Wasserstein* *distance* W_2 *of order* 2 between μ_0 and μ_1 by

$$W_2(\mu_0,\mu_1)^2 = \inf_{\pi} \left\{ \int_{X\times X} d(x_0,x_1)^2 d\pi(x_0,x_1) \right\},$$

where the infimum (in fact, minimum) is taken over all transference plans π between μ_0 and μ_1. Then W_2 defines a metric on $P_2(X)$; moreover, $(P_2(X), W_2)$ is a complete, separable length space. We refer to the book [17] of Villani for details.

§3. Weak curvature-dimension condition

In this section we first define a suitable set of "entropy" functions in order to give the definition of the weak $\mathrm{CD}(0,N)$ condition. We then give some basic properties of measured length spaces with the weak $\mathrm{CD}(0,N)$ condition.

Results in this section come from [10], [11], [15], [16], or [18].

Let (X,d) be a complete, locally compact, separable length space equipped with a nonnegative Radon measure ν.

3.1. Definition of the weak CD(0, N) condition

Let $U : [0,\infty) \to \mathbb{R}$ be a continuous, convex function with $U(0) = 0$. Given a compactly supported Borel measure μ on X, the *relative entropy function* U_ν is defined by

$$U_\nu(\mu) = \int_X U(\rho(x))d\nu(x) + U'(\infty)\mu_s(X),$$

where $\mu = \rho\nu + \mu_s$ is the Lebesgue decomposition of μ with respect to ν, and $U'(\infty) := \lim_{r\to\infty}(U(r)/r) \in \mathbb{R}\cup\{\infty\}$. For $N \in [1,\infty)$, we define the *displacement convexity class* $\mathcal{DC}_N$ *of order* N by

$$\mathcal{DC}_N = \big\{U : [0,\infty) \to \mathbb{R} \bigm| U \text{ is a continuous, convex function with } U(0) = 0 \text{ such that } (0,\infty) \ni \lambda \mapsto \lambda^N U(\lambda^{-N}) \text{ is convex}\big\}.$$

If $N' \geq N$, then $\mathcal{DC}_{N'} \subset \mathcal{DC}_N$. Put

$$U_N(r) = \begin{cases} Nr(1 - r^{-1/N}) & \text{if } 1 < N < \infty, \\ r & \text{if } N = 1. \end{cases}$$

Then $U_N \in \mathcal{DC}_N$.

Definition 3.1. Let $N \in [1, \infty)$. We say that (X, d, ν) satisfies the *weak* $\mathrm{CD}(0, N)$ *condition* if for any compactly supported probability measures μ_0 and μ_1 with $\mathrm{supp}(\mu_0)$, $\mathrm{supp}(\mu_1) \subset \mathrm{supp}(\nu)$ there exists a geodesic $\{\mu_t\}_{t\in[0,1]}$ in $(P_2(X), W_2)$ from μ_0 to μ_1 such that

$$U_\nu(\mu_t) \le (1-t)\, U_\nu(\mu_0) + t\, U_\nu(\mu_1)$$

holds for all $U \in \mathcal{DC}_N$ and all $t \in [0, 1]$.

If (X, d, ν) satisfies the weak $\mathrm{CD}(0, N)$ condition for some $N \in [1, \infty)$, then it satisfies the weak $\mathrm{CD}(0, N')$ condition for $N' \ge N$.

3.2. Basic properties

A subset A in X is said to be *totally convex* if for any two points $x, y \in A$, all minimal geodesics between x and y are contained in A.

Proposition 3.2. *Assume that (X, d, ν) satisfies the weak* $\mathrm{CD}(0, N)$ *condition for some $N \in [1, \infty)$.*

(1) *Let A be a totally convex, closed subset of X. Then $(A, d, \nu|_A)$ also satisfies the weak* $\mathrm{CD}(0, N)$ *condition.*
(2) *Given $\epsilon, \delta > 0$, the measured length space $(X, \epsilon d, \delta\nu)$ also satisfies the weak* $\mathrm{CD}(0, N)$ *condition.*
(3) *The measure ν either is a delta function or is non-atomic.*

For a point $x \in X$, a subset $A \subset X$, and $t \in [0, 1]$, we put

$$[x, A]_t = \{\gamma(t) \mid \gamma : [0,1] \to X \text{ is a minimal geodesic with } \gamma(0) = x \text{ and } \gamma(1) \in A\}$$

(that is, $[x, A]_t$ is the set of all t-barycenters of x and each point in A). The proof of Theorem 30.11 in [18] (Theorem 5.31 in [10]) gives a directionally restricted version of the Bishop–Gromov inequality:

Proposition 3.3. *Assume that (X, d, ν) satisfies the weak* $\mathrm{CD}(0, N)$ *condition for some $N \in [1, \infty)$. Then for any $x \in \mathrm{supp}(\nu)$ and any Borel set $A \subset X$,*

$$t^N \nu(A) \le \nu([x, A]_t) \tag{3.1}$$

holds for all $t \in [0, 1]$.

Theorem 3.4. *Let $\{(X_i, x_i, \nu_i)\}_{i=1}^\infty$ be a sequence of pointed measured length spaces satisfying the weak* $\mathrm{CD}(0, N)$ *condition for some $N \in [1, \infty)$ with $\mathrm{supp}(\nu_i) = X_i$ and $\nu_i(B_1(x_i)) = 1$.*

Then there exists a subsequence $\{j\} \subset \{i\}$ such that $\{(X_j, x_j, \nu_j)\}$ pointed measured Gromov–Hausdorff converges to some pointed measured length space $(X_\infty, x_\infty, \nu_\infty)$. Moreover, the limit space (X_∞, ν_∞) satisfies the weak $\mathrm{CD}(0, N)$ *condition.*

Remark 3.5 (On the measure contraction property). Given $N \in [1, \infty)$, Proposition 3.2, Proposition 3.3, and Theorem 3.4 hold for measured length spaces with $\mathrm{MCP}(0, N)$. See [12] and [16, Section 5] for details.

§4. Proof of Theorem 1.1

We start with the definition of ends. Let X be a complete length space. A path $\gamma : [0, \infty) \to X$ is a *ray* if each finite geodesic segment is minimal. Let $\gamma_1, \gamma_2 : [0, \infty) \to X$ be rays from the base point x. Two rays γ_1 and γ_2 are said to be *cofinal* if $\gamma_1(t)$ and $\gamma_2(t)$ lie in the same connected component of $X \setminus B_r(x)$ for all $t, r > 0$ with $t \geq r$. An equivalence class of cofinal rays is called an *end* of X.

Example 4.1. Let $\| \cdot \|$ be any norm on $\mathbb{R}^n$. Consider the set $A := \mathbb{R} \times [0, 1]^{n-1}$, which is totally convex in $(\mathbb{R}^n, \| \cdot \|)$. The length space $(A, \| \cdot \|)$ has two ends. Since $(\mathbb{R}^n, \| \cdot \|, \mathcal{L}^n)$ satisfies the weak $\mathrm{CD}(0, n)$ condition and $\mathrm{MCP}(0, n)$ as mentioned in the introduction, it follows from Proposition 3.2(1) that the measured length space $(A, \| \cdot \|, \mathcal{L}^n|_A)$ also satisfies the properties.

To prove Theorem 1.1, we first study the local structure of measured length spaces with the weak $\mathrm{CD}(0, N)$ condition or with $\mathrm{MCP}(0, N)$.

Definition 4.2 ([7, 3.32]). We say that a point $x \in X$ is a *local cut point* if $V \setminus \{x\}$ is disconnected for some connected neighborhood V of x. The *degree* of x, denoted by $\deg(x)$, is defined as the supremum of the number of connected components of $V \setminus \{x\}$ for all connected neighborhoods V of x.

If x is a local cut point, then $V \setminus \{x\}$ is disconnected for *every* sufficiently small neighborhood V of x. We have $\deg(x) \geq 2$ for each local cut point x. The end points in a graph (one-dimensional space) are *not* local cut points.

An interior point in a graph is not always a local cut point; consider the length space $\{(x, 0) \,|\, 0 \leq x \leq 1\} \cup \{(0, y) \,|\, 0 \leq y \leq 1\} \cup \left(\bigcup_{i=0}^{\infty} \{(x, -x + 2^{-i}) \,|\, 0 \leq x \leq 2^{-i})\} \right) \subset \mathbb{R}^2$. The origin is *not* a local cut point.

On the basis of an idea in the proof of Theorem 5.1 in [4], we have

Lemma 4.3. *Let (X, d) be a complete, locally compact, separable length space equipped with a nonnegative Radon measure ν. Given $N \in [1, \infty)$, we assume one of the following (i) and (ii):*

(i) *The measured length space* (X,d,ν) *satisfies the weak* $\mathrm{CD}(0,N)$ *condition.*

(ii) *The measured length space* (X,d,ν) *satisfies* $\mathrm{MCP}(0,N)$.

If there exists a local cut point $x \in X$*, then* $\deg(x)=2$.

Proof. Assume the condition (i). The proof is by contradiction: suppose that $\deg(x) \geq 3$. We may assume that $\mathrm{supp}(\nu) = X$; see [18, Theorem 30.2]. For a sufficiently small $r>0$, we can take three connected components O_1, O_2, O_3 of $\overline{B}_r(x) \setminus \{x\}$ such that $O_i \cap S_r(x)$ is nonempty for all $i=1,2,3$. Fix $0<l \leq r/2$. For each $i=1,2,3$, we choose a point $x_i \in O_i$ with $d(x,x_i)=l$. See Figure 1.

We put $A = B_\epsilon(x) \cap O_1$ for $0<\epsilon<l$. Then every minimal geodesic between any point in A and x_i for $i=2,3$ passes through the local cut point x.

We now use Proposition 3.3, the Bishop–Gromov inequality (3.1), for x_i $(i=2,3)$, A, and $t = l/(l+\epsilon)$:

$$\left(\frac{l}{l+\epsilon}\right)^N \nu(A) \leq \nu\big([x_i, A]_{l/(l+\epsilon)}\big). \tag{4.1}$$

Put $A_i = [x_i, A]_{l/(l+\epsilon)}$ for $i=2,3$ and $A' = B_\epsilon(x) \cap (O_2 \cup O_3)$. We remark that $A_2, A_3 \subset A'$ and $A_2 \cap A_3 = \emptyset$; hence, $\nu(A_2)+\nu(A_3) \leq \nu(A')$. From (4.1) for $i=2,3$, we have

$$2\left(\frac{l}{l+\epsilon}\right)^N \leq \frac{\nu(A_2)+\nu(A_3)}{\nu(A)} \leq \frac{\nu(A')}{\nu(A)}. \tag{4.2}$$

Next, we use the inequality (3.1) for x_1, A', and $t = l/(l+\epsilon)$:

$$\left(\frac{l}{l+\epsilon}\right)^N \nu(A') \leq \nu\big([x_1, A']_{l/(l+\epsilon)}\big). \tag{4.3}$$

Since $[x_1, A']_{l/(l+\epsilon)} \subset A$, it follows from (4.2) and (4.3) that

$$2\left(\frac{l}{l+\epsilon}\right)^N \leq \left(\frac{l+\epsilon}{l}\right)^N.$$

Taking $\epsilon \to 0$, we get a contradiction.

The proof in the case (ii) is the same as above. Q.E.D.

Remark 4.4. The assumption $K=0$ is not essential in Lemma 4.3: given $K \in \mathbb{R}$ and $N<\infty$, Lemma 4.3 (that is, $\deg(x)=2$) holds for measured length spaces with the weak $\mathrm{CD}(K,N)$ condition or with $\mathrm{MCP}(K,N)$; see [19].

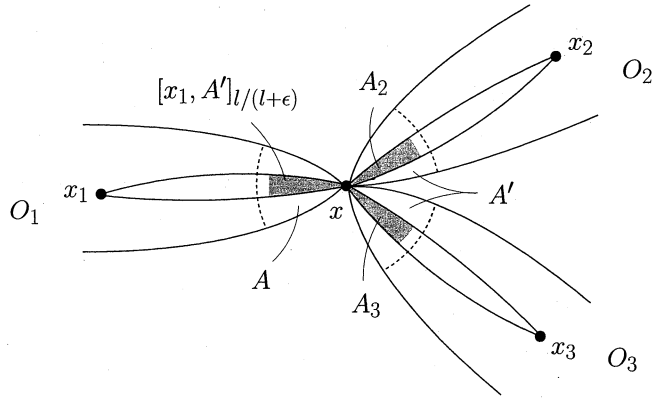

Fig. 1. Proof of Lemma 4.3

Proof of Theorem 1.1. Assume the condition (i). The proof is by contradiction: suppose that X has at least three ends. We assume that $\mathrm{supp}(\nu) = X$ as in Lemma 4.3. Given any sequence $\epsilon_i \to 0$ and any point $x \in X$, we put $(X_i, d_i, x_i) = (X, \epsilon_i d, x)$. Let ν_i be the push-forward of the measure $\nu(B_{1/\epsilon_i}(x))^{-1}\nu$ by the identity map from X to X_i. It follows from Proposition 3.2(2) that the measured length space (X_i, d_i, ν_i) satisfies the weak $\mathrm{CD}(0, N)$ condition. From Theorem 3.4 (or [3, Chapter 1]), there exists a subsequence $\{j\} \subset \{i\}$ such that $\{(X_j, d_j, x_j, \nu_j)\}$ pointed measured Gromov–Hausdorff converges to some pointed measured length space $(X_\infty, d_\infty, x_\infty, \nu_\infty)$; then, $(X_\infty, d_\infty, \nu_\infty)$ also satisfies the weak $\mathrm{CD}(0, N)$ condition. Since X has at least three ends, the point x_∞ is a local cut point. Then $\deg(x_\infty)$ is equal to the number of ends of X. This contradicts Lemma 4.3.

The proof in the case (ii) is the same as above. Q.E.D.

References

[1] D. Burago, Y. Burago and S. Ivanov, A Course in Metric Geometry, Grad. Stud. Math., **33**, Amer. Math. Soc., Providence, RI, 2001.

[2] J. Cheeger and T. H. Colding, Lower bounds on Ricci curvature and the almost rigidity of warped product, Ann. of Math. (2), **144** (1996), 189–237.

[3] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below. I, J. Differential Geom., **46** (1997), 406–480.

[4] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below. II, J. Differential Geom., **54** (2000), 13–35.
[5] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Differential Geom., **6** (1971/72), 119–128.
[6] K. Fukaya, Collapsing of Riemannian manifolds and eigenvalues of Laplace operator, Invent. Math., **87** (1987), 517–547.
[7] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Progress in Math., **152**, Birkhäuser Boston, Inc., Boston, MA, 1999.
[8] N. Juillet, Geometric inequalities and generalized Ricci bounds in the Heisenberg group, Int. Math. Res. Not., to appear.
[9] K. Kuwae and T. Shioya, Laplacian comparison for Alexandrov spaces, preprint, 2007.
[10] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Ann. of Math., **169** (2009), 903–991.
[11] J. Lott and C. Villani, Weak curvature conditions and functional inequalities, J. Funct. Anal., **245** (2007), 311–333.
[12] S.-i. Ohta, On the measure contraction property of metric measure spaces, Comment. Math. Helv., **82** (2007), 805–828.
[13] S.-i. Ohta, Products, cones, suspensions of spaces with the measure contraction property, J. London Math. Soc. (2), **76** (2007), 225–236.
[14] M.-K. von Renesse, On local Poincaré via transportation, Math. Z., **259** (2008), 21–31.
[15] K.-T. Sturm, On the geometry of metric measure spaces. I, Acta Math., **196** (2006), 65–131.
[16] K.-T. Sturm, On the geometry of metric measure spaces. II, Acta Math., **196** (2006), 133–177.
[17] C. Villani, Topics in Optimal Transportation, Grad. Stud. Math., **58**, Amer. Math. Soc., Providence, RI, 2003.
[18] C. Villani, Optimal transport, old and new, Grundlehren Math. Wiss., **338**, Springer-Verlag, Berlin, 2008.
[19] M. Watanabe, Local cut points and metric measure spaces with Ricci curvature bounded below, Pacific J. Math., **233** (2007), 229–256.

Mathematical Institute
Tohoku University
Sendai, 980-8578
Japan

E-mail address: sa3m33@math.tohoku.ac.jp