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# Generalized Q-functions and UC hierarchy of B-type

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**Abstract.**

We define a generalization of Schur's Q-function for an arbitrary pair of strict partitions, which is called the *generalized Q-function*. We prove that all the generalized Q-functions solve a series of non-linear differential equations called the *UC hierarchy of B-type* (*BUC hierarchy*). We furthermore investigate the BUC hierarchy from the viewpoint of representation theory. We consider the Fock representation of the algebra of neutral fermions and establish the boson-fermion correspondence. Using this, we discuss the relationship between the BUC hierarchy and a certain infinite dimensional Lie algebra.

## §1. Introduction

The *universal (rational) character* [7] is a generalization of Schur function, which plays a significant role in representation theory of the general linear groups. It is well known that the Schur function gives an irreducible character of any polynomial representation of $GL_n(\mathbb{C})$. On the other hand, any irreducible rational representation can be described by means of the universal character.

The Schur functions are known to satisfy the bilinear KP hierarchy (Kadomtsev–Petviashvili, [10]), which is one of the most fundamental example of infinite dimensional integrable systems. Recently, T. Tsuda proposed an extension of the KP hierarchy, called the *UC hierarchy* [12]. A remarkable result revealed in [12] is that all the universal characters are solutions of the UC hierarchy. A connection to an infinite dimensional Lie algebra (denoted by $\mathfrak{gl}(\infty) \oplus \mathfrak{gl}(\infty)$ in [12]) was also discussed by using the language of "charged free fermions".

From the viewpoint of infinite dimensional Lie algebras [5], the KP and UC hierarchies correspond to Lie algebras of A-type [4, 12]. From

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this point of view, the BKP hierarchy, which is one of the variants of the KP hierarchy, corresponds to a Lie algebra of B-type [1, 2, 4]. As was shown by Y.You [13], the bilinear BKP hierarchy has polynomial solutions called *Schur's Q-functions*, which originally arise in the study of projective representations of the symmetric and alternating groups [3, 11].

In [9], motivated by the facts mentioned above, we proposed a generalization of Schur's Q-function called *generalized Q-function.* The generalized Q-function is defined for any pair of "strict" partitions, while the Schur's Q-function is defined for any single strict partition; see §2. It was shown that all the generalized Q-functions are solutions of a series of non-linear differential equations. This system of differential equations is called the *UC hierarchy of B-type* (or *BUC hierarchy*) since it may be considered as a B-type analogue of the UC hierarchy.

This note is an exposition of the paper [9], in which we investigated the generalized Q-functions and the BUC hierarchy.

This note is organized as follows. In §2, we recall the definition of Schur's Q-functions and then define the generalized Q-functions in terms of Pfaffians. In §3, we express the generalized Q-function by means of *vertex operators* (Theorem 3). Using this expression, we prove that all the generalized Q-functions satisfy certain quadratic relations called *bilinear identities* (Theorem 5). The bilinear identities are transformed to an infinite number of Hirota bilinear equations of infinite order; it is this system we call the BUC hierarchy. In §4, we introduce neutral fermions and consider the Fock representation. This representation is given an explicit realization, so-called *boson-fermion correspondence*, in the polynomial algebra with infinite variables (Theorem 7). By making use of this correspondence, we see that a certain infinite dimensional Lie algebra acts on the whole space of polynomial solutions of the BUC hierarchy as infinitesimal transformations (Theorem 10).

## §2. The generalized Q-functions

### 2.1. Strict partitions

A sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of non-negative integers is called a *strict* (or *distinct*) *partition* if $\lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0$. We may write any strict partition as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2r})$ by adding $\lambda_{2r} = 0$ (if necessary). For example, $(1, 0)$, $(2, 0)$, $(3, 2, 1, 0)$ and so on.

### 2.2. Schur's Q-functions

We recall briefly the definition of Schur's Q-functions [3, 8]. Let $\boldsymbol{x} = (x_1, x_3, x_5, \ldots)$ be infinite variables and define the formal power series $\xi(\boldsymbol{x}, z)=\sum_{n\geq 1} x_{2n-1}z^{2n-1}$. We define *elementary Q-functions* $q_n(\boldsymbol{x})$ $(n \in \mathbb{Z})$ by the generating functional expression:

$$\sum_{n\in\mathbb{Z}} q_n(\boldsymbol{x})z^n = e^{\xi(\boldsymbol{x},z)}.$$

Explicitly, $q_0(\boldsymbol{x}) = 1$, $q_n(\boldsymbol{x}) = 0$ $(n < 0)$, and

$$q_n(\boldsymbol{x}) = \sum_{k_1+3k_3+5k_5+\cdots=n} \frac{x_1^{k_1}x_3^{k_3}x_5^{k_5}\cdots}{k_1!k_3!k_5!\cdots} \quad (n > 0).$$

For each $m, n \in \mathbb{Z}$, we define

$$q_{m,n}(\boldsymbol{x}) = q_m(\boldsymbol{x})q_n(\boldsymbol{x}) + 2\sum_{k\geq 1}(-1)^k q_{m+k}(\boldsymbol{x})q_{n-k}(\boldsymbol{x})$$

which satisfy $q_{m,n}(\boldsymbol{x}) + q_{n,m}(\boldsymbol{x}) = 2(-1)^m\delta_{m+n,0}$ for all $m, n \in \mathbb{Z}$.

For each strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2r})$, define the matrix $M_\lambda$ whose $(i, j)$-th entry is $q_{\lambda_i,\lambda_j}(\boldsymbol{x})$ if $i \neq j$ and 0 if $i = j$. By the relation for $q_{m,n}(\boldsymbol{x})$ just written above, $M_\lambda$ is a skew-symmetric matrix. The *Schur's Q-function* $Q_\lambda(\boldsymbol{x})$ associated with a strict partition $\lambda$ is defined by Pfaffian for $M_\lambda$ [3, 8]:

$$Q_\lambda(\boldsymbol{x}) = \mathrm{Pf}\left[M_\lambda\right]. \tag{1}$$

Recall that the Pfaffian of a skew-symmetric matrix $A = (a_{ij})_{1\leq i,j\leq 2r}$ is defined by

$$\mathrm{Pf}[A] = \sum_{\substack{i_1<i_3<\cdots<i_{2r-1}\\ i_1<i_2,\ldots,i_{2r-1}<i_{2r}}} \mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & 2r \\ i_1 & i_2 & \cdots & i_{2r}\end{pmatrix} a_{i_1i_2}a_{i_3i_4}\cdots a_{i_{2r-1}i_{2r}}.$$

For $\emptyset = (0)$, we put $Q_\emptyset(\boldsymbol{x}) = 1$. Note that $Q_{(\lambda_1,\lambda_2)}(\boldsymbol{x}) = q_{\lambda_1,\lambda_2}(\boldsymbol{x})$.

### 2.3. Definition of generalized Q-functions

Let us introduce more variables $\boldsymbol{y} = (y_1, y_3, y_5, \ldots)$ and put

$$r_{m,n}(\boldsymbol{x}, \boldsymbol{y}) = q_m(\boldsymbol{y})q_n(\boldsymbol{x}) + 2\sum_{k\geq 1}(-1)^k q_{m-k}(\boldsymbol{y})q_{n-k}(\boldsymbol{x})$$

which satisfy $r_{m,n}(\boldsymbol{x}, \boldsymbol{y}) = r_{n,m}(\boldsymbol{y}, \boldsymbol{x})$. For any pair of strict partitions $[\lambda, \mu] = [(\lambda_1, \ldots, \lambda_{2r}), (\mu_1, \ldots, \mu_{2s})]$, we define the matrix $N_{\lambda,\mu}$ with $r_{\mu_{2s-i+1},\lambda_j}(\boldsymbol{x}, \boldsymbol{y})$ on the $(i, j)$-th entry $(1 \leq i \leq 2s,\ 1 \leq j \leq 2r)$. Moreover, we put the matrix $\bar{M}_\mu$ whose $(i, j)$-th entry is $q_{\mu_{2s-j+1},\mu_{2s-i+1}}(\boldsymbol{y})$ if $i \neq j$ and 0 if $i = j$.

**Definition 1** ([9]). The generalized Q-function $Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ for any pair of strict partitions $[\lambda,\mu]$ is defined by Pfaffian of the form:

$$Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{Pf}\begin{bmatrix} \bar{M}_\mu & N_{\lambda,\mu} \\ -N^{\mathrm{T}}_{\lambda,\mu} & M_\lambda \end{bmatrix} \tag{2}$$

where $N^{\mathrm{T}}_{\lambda,\mu}$ denotes the transpose of $N_{\lambda,\mu}$.

We have the Schur's Q-function as a special case of (2):

$$Q_{[\lambda,\emptyset]}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{Pf}\,[M_\lambda] = Q_\lambda(\boldsymbol{x}).$$

Similarly, $Q_{[\emptyset,\mu]}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{Pf}\left[\bar{M}_\mu\right] = Q_\mu(\boldsymbol{y})$.

*Example* 1. If $[\lambda,\mu] = [(m,0),(n,0)]$, then

$$Q_{[(m,0),(n,0)]}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{Pf}\begin{bmatrix} 0 & \bar{q}_{n,0} & r_{0,m} & r_{0,0} \\ \bar{q}_{0,n} & 0 & r_{n,m} & r_{n,0} \\ -r_{0,m} & -r_{n,m} & 0 & q_{m,0} \\ -r_{0,0} & -r_{n,0} & q_{0,m} & 0 \end{bmatrix} = r_{n,m}$$

where we have denoted $q_{m,n} = q_{m,n}(\boldsymbol{x})$, $\bar{q}_{m,n} = q_{m,n}(\boldsymbol{y})$ and $r_{m,n} = r_{m,n}(\boldsymbol{x},\boldsymbol{y})$ for simplicity.

If we set the degree of each variables as $\deg x_n = n, \deg y_n = -n$, then $Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ has homogeneous degree $|\lambda| - |\mu|$, where $|\lambda| = \sum \lambda_i$.

*Example* 2. If $[\lambda,\mu] = [(2,1),(1,0)]$, then

$$Q_{[(2,1),(1,0)]}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{Pf}\begin{bmatrix} 0 & \bar{q}_{1,0} & r_{0,2} & r_{0,1} \\ \bar{q}_{0,1} & 0 & r_{1,2} & r_{1,1} \\ -r_{0,2} & -r_{1,2} & 0 & q_{2,1} \\ -r_{0,1} & -r_{1,1} & q_{1,2} & 0 \end{bmatrix}.$$

Since $r_{0,1} = x_1$, $r_{1,1} = x_1 y_1 - 2$, $r_{0,2} = x_1^2/2$ and $r_{1,2} = x_1^2 y_1/2 - 2x_1$, this Pfaffian yields $\left(x_1^3/6 - 2x_3\right) y_1 - x_1^2$, which has homogeneous degree $|\lambda| - |\mu| = 2$.

Another equivalent definition of $Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y})$ given below is sometimes convenient (see [9]).

**Theorem 2.** *The generalized Q-function has the following expression in terms of Schur's Q-functions:*

$$Q_{[\lambda,\mu]}(\boldsymbol{x},\boldsymbol{y}) = Q_\lambda(\boldsymbol{x} - 2\widetilde{\partial}_{\boldsymbol{y}})Q_\mu(\boldsymbol{y} - 2\widetilde{\partial}_{\boldsymbol{x}}) \cdot 1 \tag{3}$$

*where* $\widetilde{\partial}_{\boldsymbol{x}}$ *stands for* $(\partial_{x_1}, \partial_{x_3}/3, \partial_{x_5}/5, \dots)$ $(\partial_{x_n} = \partial/\partial_{x_n})$.

Notice that the formula (3) resembles a similar relation between the Schur function and the universal character (see [12], Lemma 4.7).

## §3. The UC hierarchy of B-type

In this section, we introduce a series of non-linear differential equations satisfied by the generalized Q-functions.

### 3.1. Vertex operators

We start with the following two types of linear differential operators:

$$X(z) = X(z; \boldsymbol{x}, \boldsymbol{y}, \partial_{\boldsymbol{x}}, \partial_{\boldsymbol{y}}) = e^{\xi(\boldsymbol{x}-2\tilde{\partial}_{\boldsymbol{y}}, z)} e^{-2\xi(\tilde{\partial}_{\boldsymbol{x}}, z^{-1})}$$
$$\bar{X}(z) = \bar{X}(z; \boldsymbol{x}, \boldsymbol{y}, \partial_{\boldsymbol{x}}, \partial_{\boldsymbol{y}}) = e^{\xi(\boldsymbol{y}-2\tilde{\partial}_{\boldsymbol{x}}, z)} e^{-2\xi(\tilde{\partial}_{\boldsymbol{y}}, z^{-1})}$$

with a non-zero complex number $z$. In physics, the operators of these types are called *vertex operators.* If we expand the vertex operators as

$$X(z) = \sum_{n\in\mathbb{Z}} X_n z^n \qquad \bar{X}(z) = \sum_{n\in\mathbb{Z}} \bar{X}_n z^n$$

then the coefficients $X_n = X_n(\boldsymbol{x}, \boldsymbol{y}, \partial_{\boldsymbol{x}}, \partial_{\boldsymbol{y}})$, $\bar{X}_n = \bar{X}_n(\boldsymbol{x}, \boldsymbol{y}, \partial_{\boldsymbol{x}}, \partial_{\boldsymbol{y}})$ $(n \in \mathbb{Z})$ are well defined operators on $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$. These operators have the following important properties; see [9] for the proofs.

**Theorem 3.** *Let* $\lambda = (\lambda_1, \ldots, \lambda_{2r})$, $\mu = (\mu_1, \ldots, \mu_{2s})$ *be arbitrary strict partitions. Then we have the formula*

$$Q_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = X_{\lambda_1} \cdots X_{\lambda_{2r}} \bar{X}_{\mu_1} \cdots \bar{X}_{\mu_{2s}} \cdot 1. \tag{4}$$

**Lemma 4.** *We have the relations*

$$X_0 \cdot 1 = \bar{X}_0 \cdot 1 = 1 \qquad X_n \cdot 1 = \bar{X}_n \cdot 1 = 0 \qquad n < 0;$$
$$[X_m, X_n]_+ = [\bar{X}_m, \bar{X}_n]_+ = 2(-1)^m \delta_{m+n,0} \qquad [X_m, \bar{X}_n] = 0.$$

### 3.2. The bilinear identities

Consider the bilinear relations for an unknown function $\tau = \tau(\boldsymbol{x}, \boldsymbol{y})$:

$$\sum_{n\in\mathbb{Z}} (-1)^n X_n \tau \otimes X_{-n} \tau = \sum_{n\in\mathbb{Z}} (-1)^n \bar{X}_n \tau \otimes \bar{X}_{-n} \tau = \tau \otimes \tau \tag{5}$$

or equivalently in terms of the vertex operators:

$$\oint X(z)\tau \otimes X(-z)\tau \frac{dz}{2\pi i z} = \oint \bar{X}(z)\tau \otimes \bar{X}(-z)\tau \frac{dz}{2\pi i z} = \tau \otimes \tau \tag{6}$$

where the contour integral means an algebraic operation $\oint z^n \frac{dz}{2\pi i z} = \delta_{n,0}$. Hereafter we call (5) or (6) *bilinear identities.*

We have now the following theorem.

**Theorem 5.** *The generalized Q-function for any pair of strict partitions satisfies the bilinear identities.*

*Proof.* This theorem is obtained by using Theorem 3, Lemma 4, along with a fact that a constant 1 satisfies the bilinear identities. Q.E.D.

### 3.3. Hirota bilinear equations

The bilinear identities can be converted to an infinite number of Hirota bilinear equations for $\tau$. Recall that for any polynomial $P(D)$ (possibly formal power series) in $D = (D_{x_1}, D_{x_3}, D_{x_5}, \dots, D_{y_1}, D_{y_3}, D_{y_5}, \dots)$, the Hirota bilinear equation $P(D)\tau \cdot \tau = 0$ is defined by setting

$$P(D)\tau \cdot \tau = P(\partial)\tau(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})\tau(\boldsymbol{x} - \boldsymbol{a}, \boldsymbol{y} - \boldsymbol{b})\big|_{\boldsymbol{a}=\boldsymbol{b}=0}$$

where $\partial = (\partial_{a_1}, \partial_{a_3}, \partial_{a_5}, \dots, \partial_{b_1}, \partial_{b_3}, \partial_{b_5}, \dots)$. By virtue of a calculus on "Hirota differentials" (cf.[5], Ch.14), the bilinear identities (6) can be transformed to

$$\begin{aligned}
&\sum_{n,m\geq 0} q_n(2\boldsymbol{a})q_{n+m}(-2\widetilde{D}_{\boldsymbol{x}})\, q_m(-2\widetilde{D}_{\boldsymbol{y}})e^{\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle + \langle \boldsymbol{b}, D_{\boldsymbol{y}}\rangle}\tau \cdot \tau \\
&\qquad = e^{\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle + \langle \boldsymbol{b}, D_{\boldsymbol{y}}\rangle}\tau \cdot \tau \\
&\sum_{n,m\geq 0} q_n(2\boldsymbol{b})q_m(-2\widetilde{D}_{\boldsymbol{x}})\, q_{n+m}(-2\widetilde{D}_{\boldsymbol{y}})\, e^{\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle + \langle \boldsymbol{b}, D_{\boldsymbol{y}}\rangle}\tau \cdot \tau \\
&\qquad = e^{\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle + \langle \boldsymbol{b}, D_{\boldsymbol{y}}\rangle}\tau \cdot \tau
\end{aligned} \tag{7}$$

where

$$\widetilde{D}_{\boldsymbol{x}} = (D_{x_1},\ D_{x_3}/3,\ D_{x_5}/5, \dots)$$

and

$$\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle = \sum_{n\geq 1} a_{2n-1} D_{x_{2n-1}}.$$

The equations (7) may be regarded as an extension of the bilinear BKP hierarchy. Indeed, if $\tau$ is independent of $\boldsymbol{y}$, then the first equation of (7) reduces to a bilinear form of the BKP hierarchy [1, 2]

$$\sum_{n\geq 1} q_n(2\boldsymbol{a})q_n(-2\widetilde{D}_{\boldsymbol{x}})\, e^{\langle \boldsymbol{a}, D_{\boldsymbol{x}}\rangle}\tau \cdot \tau = 0$$

while the second equation reduces to a trivial identity.

To obtain a single bilinear equation from (7), expand (7) as a multiple Taylor series with respect to $\boldsymbol{a}$ and $\boldsymbol{b}$. For example, from the coefficient of $\boldsymbol{a}^0\boldsymbol{b}^0$ of the first equation, one obtains

$$\sum_{n\geq 1} q_n(-2\widetilde{D}_{\boldsymbol{x}})q_n(-2\widetilde{D}_{\boldsymbol{y}})\tau\cdot\tau = 0$$

which is a Hirota bilinear equation of infinite order. All the equations obtained from (7) in such a way are, in fact, differential equations of infinite order, as in the case of the UC hierarchy [12].

**Definition 6.** A whole system of the Hirota bilinear equations included in (7) is called the *UC hierarchy of B-type* or the *BUC hierarchy.*

## §4. The BUC hierarchy and representation theory

In this section, we consider the Fock representation of the algebra of neutral fermions and establish the boson-fermion correspondence (Theorem 7, 8) without proofs. We then give a Lie algebraic description of the BUC hierarchy.

### 4.1. Neutral fermions and fermionic Fock space

Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$ generated by *neutral fermions* $\phi_m, \bar{\phi}_m$ $(m\in\mathbb{Z})$ with defining relations:

$$[\phi_m,\phi_n]_+ = [\bar{\phi}_m,\bar{\phi}_n]_+ = (-1)^m\delta_{m+n,0} \qquad [\phi_m,\bar{\phi}_n] = 0.$$

Note that $\phi_0^2 = \bar{\phi}_0^2 = 1/2$ and the latter relation is a "commutative" relation.

The *Fock representation* is an irreducible representation of $\mathcal{A}$ generated by the vacuum vector $|0\rangle$ satisfying

$$\phi_n|0\rangle = \bar{\phi}_n|0\rangle = 0 \qquad \text{for} \quad n<0.$$

The representation space denoted by $\mathcal{F}$ is called the *fermionic Fock space*, which is an infinite dimensional vector space spanned by the basis elements

$$\left\{\phi_{m_1}\cdots\phi_{m_r}\bar{\phi}_{n_1}\cdots\bar{\phi}_{n_s}|0\rangle \,|\, m_1>\cdots>m_r\geq 0, \quad n_1>\cdots>n_s\geq 0\right\}.$$

The dual Fock space $\mathcal{F}^*$ is defined in a parallel way, i.e., $\mathcal{F}^*$ is generated by the dual vacuum vector $\langle 0|$ satisfying

$$\langle 0|\phi_n = \langle 0|\bar{\phi}_n = 0 \qquad \text{for} \quad n>0.$$

There exists indeed a unique non-degenerate bilinear form $\langle\, , \rangle : \mathcal{F}^* \otimes \mathcal{F} \longrightarrow \mathbb{C}$ denoted by $\langle 0|a \otimes b|0\rangle \longmapsto \langle 0|a \cdot b|0\rangle \stackrel{\text{def}}{=} \langle ab\rangle$, such that $\langle 1\rangle = 1$, $\langle \phi_0 \rangle = \langle \bar{\phi}_0 \rangle = \langle \phi_0 \bar{\phi}_0 \rangle = 0$. The quantity $\langle a\rangle$ is said to be the *vacuum expectation value* of $a$.

### 4.2. The boson-fermion correspondence

The Fock representation has an explicit realization in the polynomial algebra with infinite variables. Let $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}, q, \bar{q}]$ be a polynomial algebra in $\boldsymbol{x}$, $\boldsymbol{y}$, $q$, $\bar{q}$, and $\mathcal{I}$ an ideal generated by $q^2 - 1/2$ and $\bar{q}^2 - 1/2$. The *bosonic Fock space* is defined by

$$\mathcal{B} = \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}, q, \bar{q}]/\mathcal{I} = \bigoplus_{i,j=0,1} \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}] q^i \bar{q}^j.$$

The "boson-fermion correspondence" states that the fermionic Fock space can be identified with the bosonic Fock space.

**Theorem 7.** *There exists a linear isomorphism $\sigma : \mathcal{F} \cong \mathcal{B}$.*

A concrete form of $\sigma$ can be constructed in the following way. For each $m \in 2\mathbb{Z} + 1$, we put

$$H_m = \frac{1}{2} \sum_{j \in \mathbb{Z}} (-1)^{j+1} \phi_j \phi_{-j-m} \qquad \bar{H}_m = \frac{1}{2} \sum_{j \in \mathbb{Z}} (-1)^{j+1} \bar{\phi}_j \bar{\phi}_{-j-m}.$$

It is straightforward to check that $[H_m, H_n] = [\bar{H}_m, \bar{H}_n] = m\delta_{m+n,0}/2$ and $H_n|0\rangle = \bar{H}_n|0\rangle = 0$ $(n > 0)$. We introduce the operator, called Hamiltonian, with variables $(\boldsymbol{x}, \boldsymbol{y})$:

$$H(\boldsymbol{x}, \boldsymbol{y}) = \sum_n \left\{ \left( x_n - \frac{2}{n} \frac{\partial}{\partial y_n} \right) H_n + \left( y_n - \frac{2}{n} \frac{\partial}{\partial x_n} \right) \bar{H}_n \right\}.$$

Multiplication of $H(\boldsymbol{x}, \boldsymbol{y})$, as well as $e^{H(\boldsymbol{x}, \boldsymbol{y})}$, on $\mathcal{F}$ is well-defined, so that we can define the linear map $\sigma : \mathcal{F} \rightarrow \mathcal{B}$ by

$$\sigma(|\nu\rangle) = \sum_{i,j=0,1} 2^{i+j} q^i \bar{q}^j \langle \phi_0^i \bar{\phi}_0^j e^{H(\boldsymbol{x}, \boldsymbol{y})} |\nu\rangle \cdot 1 \tag{8}$$

which yields the isomorphism of Theorem 7. An image of $\sigma$ can be calculated by means of a formula (11) given below.

We next describe the action of $\mathcal{A}$ on the bosonic Fock space. Define the generating sums of the neutral fermions

$$\phi(z) = \sum_{n \in \mathbb{Z}} \phi_n z^n \qquad \bar{\phi}(z) = \sum_{n \in \mathbb{Z}} \bar{\phi}_n z^n.$$

Then we have

**Theorem 8.** *Let $|\nu\rangle \in \mathcal{F}$. We have the following correspondence of operators:*

$$\sigma(\phi(z)|\nu\rangle) = qX(z)\sigma(|\nu\rangle) \qquad \sigma(\bar{\phi}(z)|\nu\rangle) = \bar{q}\bar{X}(z)\sigma(|\nu\rangle). \tag{9}$$

Let $|\nu\rangle = \phi_{\lambda_1} \cdots \phi_{\lambda_r}\bar{\phi}_{\mu_1} \cdots \bar{\phi}_{\mu_s}|0\rangle$ where $\lambda_1 > \cdots > \lambda_r \geq 0$ and $\mu_1 > \cdots > \mu_s \geq 0$. By virtue of Theorem 8, we have the formula

$$\sigma(|\nu\rangle) = q^r\bar{q}^s X_{\lambda_1} \cdots X_{\lambda_r}\bar{X}_{\mu_1} \cdots \bar{X}_{\mu_s} \cdot 1 \tag{10}$$

here notice that $\sigma(|0\rangle) = 1$. From Theorem 3, the right hand side can be written in terms of the generalized Q-functions as follows:

$$\sigma(|\nu\rangle) = q^r\bar{q}^s Q_{[(\lambda_1,\ldots,\lambda_r),(\mu_1,\ldots,\mu_s)]}(\boldsymbol{x}, \boldsymbol{y}). \tag{11}$$

If we notice that $\mathcal{F}_{0,0} = \sum \mathbb{C}\phi_{\lambda_1} \cdots \phi_{\lambda_{2r}}\bar{\phi}_{\mu_1} \cdots \bar{\phi}_{\mu_{2s}}|0\rangle$ (summed over all $\lambda_1 > \cdots > \lambda_{2r} \geq 0$, $\mu_1 > \cdots > \mu_{2s} \geq 0$), is a subspace of $\mathcal{F}$ isomorphic to $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$ by Theorem 7, we deduce from (11) the following corollary.

**Corollary 9.** *A whole set of the generalized Q-functions forms a linear basis of* $\mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$.

### 4.3. The bilinear identities as an orbit equation

Let us consider any element of the form

$$\sum_{i,j\in\mathbb{Z}} \left(a_{ij} :\phi_i\phi_{-j}: + \bar{a}_{ij} :\bar{\phi}_i\bar{\phi}_{-j}:\right) + c \qquad (c \in \mathbb{C})$$

where $a_{ij}$, $\bar{a}_{ij}$ are assumed to be subject to $a_{ij} = \bar{a}_{ij} = 0$ $(|i-j| \gg 0)$, and we have put $:\phi_m\phi_n: = \phi_m\phi_n - \langle\phi_m\phi_n\rangle$ (similarly for $\bar{\phi}$). It is easy to see that the bracket operation between such elements gives again an element of this form, i.e., the set of such elements forms a Lie algebra, which we denote by $\mathfrak{g}$.

We define the (formal) Lie group associated to $\mathfrak{g}$:

$$\boldsymbol{G} = \left\{e^{X_1} \cdots e^{X_k} | X_i \in \mathfrak{g} : \text{locally nilpotent}\right\}.$$

The Fock representation of $\mathfrak{g}$ gives rise to a representation of $\boldsymbol{G}$ on $\mathcal{F}$. Clearly, $\mathcal{F}_{0,0}$ (defined in the previous subsection) is an invariant subspace. Let us consider $\boldsymbol{G}|0\rangle \subset \mathcal{F}_{0,0}$, i.e., a $\boldsymbol{G}$-orbit of the vacuum vector. In general, a non-zero $|\nu\rangle \in \mathcal{F}_{0,0}$ lies in $\boldsymbol{G}|0\rangle$ if and only if $|\nu\rangle$ satisfies the following bilinear relations on $\mathcal{F}_{0,0} \otimes \mathcal{F}_{0,0}$:

$$\begin{cases} \displaystyle\sum_{n\in\mathbb{Z}}(-1)^n\phi_n|\nu\rangle \otimes \phi_{-n}|\nu\rangle = Q|\nu\rangle \otimes Q|\nu\rangle \\ \displaystyle\sum_{n\in\mathbb{Z}}(-1)^n\bar{\phi}_n|\nu\rangle \otimes \bar{\phi}_{-n}|\nu\rangle = \bar{Q}|\nu\rangle \otimes \bar{Q}|\nu\rangle \end{cases} \tag{12}$$

(cf.[6, 13] in case of the BKP hierarchy). Here $Q$, $\bar{Q}$ are linear operators on $\mathcal{F}$ defined via $\sigma Q \sigma^{-1} = q$, $\sigma \bar{Q} \sigma^{-1} = \bar{q}$, respectively, which satisfy the properties $Q|0\rangle = \phi_0|0\rangle$, $\bar{Q}|0\rangle = \bar{\phi}_0|0\rangle$ and $Q^2 = \bar{Q}^2 = 1/2$.

We are now in a position to state the following theorem.

**Theorem 10.** *Let $\tau \in \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]$. Then $\tau$ satisfies the bilinear identities* (5) *if and only if there exists a $g \in \boldsymbol{G}$ such that*

$$\tau = \sigma(g|0\rangle) = \langle e^{H(\boldsymbol{x},\boldsymbol{y})} g \rangle \cdot 1. \tag{13}$$

*Proof.* This theorem is obtained by noting that (12) is equivalent to the bilinear identities (5) by the correspondence (9). Q.E.D.

We have thus shown that a $\boldsymbol{G}$-orbit of the vacuum vector in the fermionic Fock space can be identified with a whole space of polynomial solutions of the BUC hierarchy, and in particular (13) gives a general formula for the polynomial solutions.

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