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Flop invariance of the topological vertex

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§1. Introduction

This is a summary of [6] which is a joint work with Yukiko Konishi. A toric Calabi–Yau (CY) threefold is a non-singular toric threefold with trivial canonical bundle[1] . The topological vertex is an algorithm which enables us to write down an explicit formula for the generating function of all genus Gromov–Witten (GW) invariants of toric CY threefolds. The formula takes a combinatorial form and is written in terms of skew-Schur functions. This method was developed in [1] based on the geometric transitions and the duality to the Chern–Simons theory. A mathematical theory (including a rigorous definition of GW invariants for toric CY threefolds) has been developed later in [8].

It was discovered [2, 13] that there is a kind of analytic continuation process on Kähler cones which links string theories on birationally equivalent CY manifolds. Motivated by these works, the transformation property of GW invariants of projective CY threefolds under flops was studied in [7] (see also [9]). In [6], the same problem was studied for general toric CY threefolds using the topological vertex. Some special cases were studied earlier in [5].

Now we state our main result. Let X be a toric CY threefold and $N_{g,\beta}(X) \in \mathbb{Q}$ be the GW invariant of X with the genus $g \geq 0$ and the degree $\beta \in H_2(X, \mathbb{Z})$[2].

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[1]Varieties which satisfy both the toric condition and the CY condition are necessary non-compact. They are even not quasi-projective in general.

[2]It is defined by the virtual counting of stable maps into X from the genus g curves and the prescribed homology class β of the images. See [8] for a precise definition of GW invariants of toric CY threefolds.

Theorem 1.1 ([6]). *Let X and X^+ be toric CY threefolds which are birationally equivalent under the flop $\phi : X \dashrightarrow X^+$ with respect to a $(-1,-1)$-curve $C \subset X$, i.e. a torus invariant (hence smooth rational) curve whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. For $\beta \in H_2(X, \mathbb{Z})$ which is not a multiple of $[C]$*[3]*, we have*

$$N_{g,\beta}(X) = N_{g,\phi_*(\beta)}(X^+) \ .$$

The above theorem boils down to a combinatorial identity on skew-Schur functions by virtue of the topological vertex, together with a local analysis of fans of toric CY threefolds. In the following, we outline the proof of Theorem 1.1. For details, we refer to [6].

§2. Topological vertex

2.1. Notations

Let μ be a partition, i.e. a non-increasing sequence of positive integers $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(\mu)} > 0)$. The number $l(\mu)$ is called the length of μ and $|\mu| := \mu_1 + \cdots + \mu_{l(\mu)}$ is called the weight of μ. Define the integer $\kappa(\mu)$ by

$$\kappa(\mu) := |\mu| + \sum_{i=1}^{l(\mu)} \mu_i(\mu_i - 2i), \tag{1}$$

which is an even number. It has the following important property:

$$\kappa(\mu^t) = -\kappa(\mu) \ , \tag{2}$$

where μ^t denotes the conjugate partition, i.e. the partition obtained by the transposition of the Young diagram of μ. We denote by $\mathcal{P}$ the set of partitions.

[3]For a multiple of flopping curve class $[C]$, we have

$$N_{g,d[C]}(X) = N_{g,d[C^+]}(X^+) = N_{g,d[\mathbb{P}^1]}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \ ,$$

where C^+ is the flopped curve. Here we regard the total space of the rank 2 vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$ as a toric CY threefold (a *local toric curve*).

2.2. Topological vertex

Definition 2.1 ([12]). Let $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{P}$. Define the topological vertex $C_{\lambda_1,\lambda_2,\lambda_3}(q) \in \mathbb{Q}(q^{\frac{1}{2}})$ to be

$$q^{\frac{1}{2}\kappa(\lambda_3)} s_{\lambda_2}(q^\rho) \sum_{\mu\in\mathcal{P}} s_{\lambda_1/\mu}(q^{\lambda_2^t+\rho}) s_{\lambda_3^t/\mu}(q^{\lambda_2+\rho}) \ , \tag{3}$$

where $s_{\mu/\nu}(q^{\mu+\rho})$ (resp. $s_\mu(q^\rho)$) is the skew-Schur function (cf. [10]) with the specialization of variables:

$$s_{\mu/\nu}(x_i = q^{\mu_i - i + \frac{1}{2}}) \qquad (\text{resp. } s_\mu(x_i = q^{-i+\frac{1}{2}})) \ .$$

2.3. An identity

Take four partitions $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathcal{P}$. Introduce:

$$Z_0'(q, Q_0) = \frac{Z_0(q, Q_0)}{Z_{(-1,-1)}(q, Q_0)} \ , \qquad Z_0^{+\prime}(q, Q_0^+) = \frac{Z_0^+(q, Q_0^+)}{Z_{(-1,-1)}(q, Q_0^+)} \ , \tag{4}$$

where

$$\begin{aligned} Z_0(q, Q_0) &= \sum_{\mu\in\mathcal{P}} (-Q_0)^{|\mu|} C_{\lambda_1,\lambda_2,\mu^t}(q) C_{\lambda_3,\lambda_4,\mu}(q) \ , \\ Z_0^+(q, Q_0^+) &= \sum_{\mu\in\mathcal{P}} (-Q_0^+)^{|\mu|} C_{\lambda_1,\mu^t,\lambda_4}(q) C_{\lambda_3,\mu,\lambda_2}(q) \ , \end{aligned}$$

and $Z_{(-1,-1)}(q, Q) = \prod_{k=1}^\infty (1 - Qq^k)^k$[4]. These are formal power series in Q_0 (resp. Q_0^+).

Theorem 2.2. *Under the identification $Q_0^+ = Q_0^{-1}$, we have*

$$Z_0^{+\prime}(q, Q_0^+) =$$
$$(-Q_0)^{-(|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4|)} q^{\frac{1}{2}(\kappa(\lambda_1)-\kappa(\lambda_2)+\kappa(\lambda_3)-\kappa(\lambda_4))} Z_0'(q, Q_0).$$

§3. Toric CY threefolds and partition functions

3.1. Toric CY threefolds

Let X be a toric CY threefold and Σ be its fan. See [4, 11] for basic facts about toric varieties. We assume that Σ is finite and satisfies the following conditions :

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[4] $Z_{(-1,-1)}(q, Q_0)$ is the partition function (cf.§3) of the local toric curve $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$. See [3, Theorem 3].

(i) the primitive generator $\vec{\omega}$ of every 1-cone satisfies $\vec{\omega} \cdot \vec{u} = 1$ where $\vec{u} = (0, 0, 1)$,
(ii) all maximal cones are three dimensional and
(iii) $|\Sigma| \cap \{z = 1\}$ is connected, where $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subset \mathbb{R}^3$ is the support of Σ and z is the third coordinate of $\mathbb{R}^3$.

The condition (i) is the CY condition. An important example is a so-called *local toric surface* which is the total space of the canonical bundle K_S of a non-singular complete toric surface S.

3.2. Toric graphs

Let $\Sigma_i(X)$ be the set of i-cones in Σ. Denote by $\Sigma_2'(X)$ the set of 2-cones which lie in the interior of $|\Sigma|$. We consider the following directed graph Γ_X (called *a toric graph*) with labels on edges of a certain type. The vertex set is

$$V(\Gamma_X) = V_3(\Gamma_X) \cup V_1(\Gamma_X),$$

where

$$V_3(\Gamma_X) = \{v_\sigma | \sigma \in \Sigma_3(X)\},\ V_1(\Gamma_X) = \{v_\tau | \tau \in \Sigma_2(X) \setminus \Sigma_2'(X)\}.$$

The edge set is

$$E(\Gamma_X) = E_3(\Gamma_X) \cup E_1(\Gamma_X),$$

where

$$E_3(\Gamma_X) = \{e_\tau | \tau \in \Sigma_2'(X)\},\ E_1(\Gamma_X) = \{e_\tau | \tau \in \Sigma_2(X) \setminus \Sigma_2'(X)\}.$$

An edge $e_\tau \in E_3(\Gamma_X)$ joins $v_\sigma, v_{\sigma'} \in V_3(\Gamma)$ if and only if $\tau = \sigma \cap \sigma'$ (see Figure 1) and an edge $e_\tau \in E_1(\Gamma)$ joins $v_\sigma \in V_3(\Gamma_X)$ and $v_\tau \in V_1(\Gamma_X)$ if and only if σ is a unique 3-cone such that τ is a face of σ. This defines a finite planner graph. Note that $V_3(\Gamma_X) \neq \emptyset$ by the condition (ii) on Σ. A vertex in $V_3(\Gamma_X)$ is trivalent and a vertex in $V_1(\Gamma_X)$ is univalent. A graph Γ_X is connected by the condition (iii) on Σ. The direction of edges can be taken arbitrarily. The label $n : E_3(\Gamma_X) \to \mathbb{Z}$, called the *framing*, is given as follows:

$$n(e_\tau) = \frac{D_{\rho_1}.C_\tau - D_{\rho_2}.C_\tau}{2},$$

where $\tau \in \Sigma_2'$ and $\rho_1, \rho_2 \in \Sigma_1$ are as shown in Figure 1. Here C_τ and D_{ρ_i} are the torus invariant curve and divisor on X corresponding to τ and ρ_i respectively. Note that C_τ is a $(-1,-1)$-curve if and only if $n(e_\tau) = 0$.

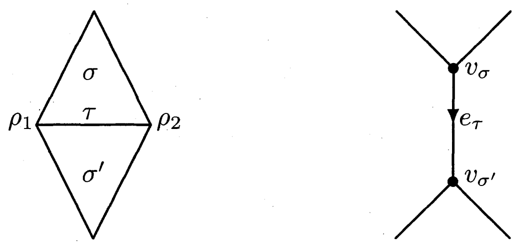

Fig. 1. Fan (section at $z = 1$) and toric graph

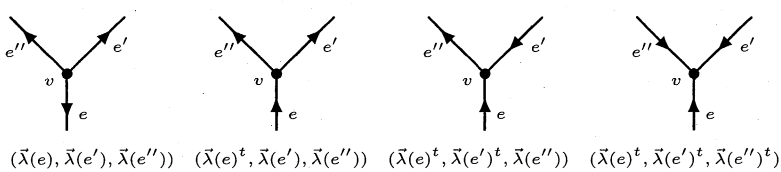

Fig. 2. $\vec{\lambda}$

3.3. Partition functions

Let $\mathcal{P}(\Gamma_X) := \{\vec{\lambda} : E_3(\Gamma) \to \mathcal{P}\}$. Take the set of formal variables $\vec{Q} = (Q_e)_{e \in E_3(\Gamma_X)}$ associated to $E_3(\Gamma_X)$.

We define the partition function $Z_X(q, \vec{Q})$ of X to be

$$(5) \quad \sum_{\vec{\lambda} \in \mathcal{P}(\Gamma_X)} \prod_{e \in E_3(\Gamma_X)} (-1)^{|\vec{\lambda}(e)|(n_e+1)} q^{\frac{\kappa(\vec{\lambda}(e))}{2} n(e)} Q_e^{|\vec{\lambda}(e)|} \prod_{v \in V_3(\Gamma_X)} C_{\vec{\lambda}_v}(q) \,,$$

which is a formal power series in $\vec{Q}$. Here $C_{\vec{\lambda}_v}(q)$ is the topological vertex defined in (3) and $\vec{\lambda}_v \in \mathcal{P}^3$ is defined in Figure 2 for $v \in V_3(\Gamma_X)$. We set $\vec{\lambda}(e) = \emptyset$ for $e \in E(\Gamma_X) \setminus E_3(\Gamma_X)$, where $\emptyset$ is the empty partition. We remark that $Z_X(q, \vec{Q})$ does not depend on the directions of edges since the framing changes the sign if one gives the opposite direction to an edge $e \in E_3(\Gamma_X)$ and it is compensated by (2) and the summation.

3.4. Partition functions and GW invariants

Now we explain how to obtain GW invariants $N_{g,\beta}(X)$ of X from the partition function $Z_X(q, \vec{Q})$.

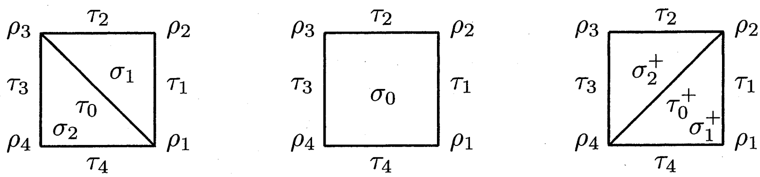

Fig. 3. Fans (sections at $z = 1$): Σ (left), Σ_0 (middle) and Σ^+ (right). The generators $\vec{\omega}_1, \dots, \vec{\omega}_4$ of $\rho_1, \dots, \rho_4$ satisfy the relation $\vec{\omega}_1 + \vec{\omega}_3 = \vec{\omega}_2 + \vec{\omega}_4$.

Theorem 3.1 ([1, 8]). *The GW invariants $N_{g,\beta}(X)$ are obtained form $Z_X(q, \vec{Q})$ as follows.*

$$\sum_{g\geq 0} N_{g,\beta}(X)\lambda^{2g-2} = \sum_{\substack{\vec{d}=(d_e)_{e\in E_3(\Gamma_X)},\\ \vec{d}[\vec{C}]=[\beta]}} F_{\vec{d}}(e^{\sqrt{-1}\lambda}), \tag{6}$$

where $[\vec{C}] = ([C_e])_{e\in E_3(\Gamma_X)}$, $C_e \subset X$ is the rational curve corresponding to e, $\vec{d}[\vec{C}] := \sum_{e\in E_3(\Gamma_X)} d_e[C_e]$, and $F_{\vec{d}}(q)$ is given by

$$\log Z_X(q, \vec{Q}) = \sum_{\vec{d}=(d_e)_{e\in E_3(\Gamma_X)}} F_{\vec{d}}(q)\vec{Q}^{\vec{d}},$$

where $\vec{Q}^{\vec{d}} = \prod_{e\in E_3(\Gamma_X)} Q_e^{d_e}$.

Remark 3.2. Precisely speaking, the partition function obtained in [8] has the expression almost same as (5) except that $C_{\vec{\lambda}_v}(q)$ is replaced by $\tilde{\mathcal{W}}_{\vec{\lambda}_v}(q)$. Here $\tilde{\mathcal{W}}_{\lambda_1,\lambda_2,\lambda_3}(q)$ is a rational function in $q^{\frac{1}{2}}$ similar to $C_{\lambda_1,\lambda_2,\lambda_3}(q)$ but has a slightly different expression. It is conjectured that $\tilde{\mathcal{W}}_{\lambda_1,\lambda_2,\lambda_3}(q) = C_{\lambda_1,\lambda_2,\lambda_3}(q)$ [8, Conjecture 8.3]. In the above theorem, we use $C_{\lambda_1,\lambda_2,\lambda_3}(q)$ assuming that the conjecture is true. The conjecture is true if at least one λ_i is empty.

§4. Transformations of partition functions

4.1. Flop invariance

We study the transformation property of the partition function of toric CY threefolds under a flop.

	X		X^+	
2-cone	$\tau_0, \tau_1, \ldots, \tau_4$	τ	$\tau_0^+, \tau_1, \ldots, \tau_4$	τ
curve	$C_0, C_1, \ldots, C_4$	C_τ	$C_0^+, C_1^+, \ldots, C_4^+$	C_τ^+
edge	$e_0, e_1, \ldots, e_4$	e_τ or e	$e_0^+, e_1^+, \ldots, e_4^+$	e_τ or e
variable	$Q_0, Q_1, \ldots, Q_4$	Q_e	$Q_0^+, Q_1^+, \ldots, Q_4^+$	Q_e

Table 1

4.1.1. *Flop.* Let X be a toric CY threefold and let Σ be its fan. Assume that X contains at least one $(-1,-1)$-curve C_0. Denote the corresponding 2-cone by τ_0. Near τ_0, the fan looks like the left diagram in Figure 3. We set

$$\Sigma_0 = (\Sigma \setminus \{\tau_0, \sigma_1, \sigma_2\}) \cup \{\sigma_0\}, \quad \Sigma^+ = (\Sigma \setminus \{\tau_0, \sigma_1, \sigma_2\}) \cup \{\tau_0^+, \sigma_1^+, \sigma_2^+\}$$

where $\tau_0, \sigma_1, \sigma_2, \sigma_0, \tau_0^+ \sigma_1^+, \sigma_2^+$ are cones shown in Figure 3. Let X_0 be the singular toric variety associated with the fan Σ_0 and X^+ be the toric CY threefold associated with the fan Σ^+. We denote by C_0^+ the $(-1,-1)$-curve on X corresponding to τ_0^+. Then associated to the evident maps $\Sigma \to \Sigma_0$ and $\Sigma^+ \to \Sigma_0$, there are the following birational maps:

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & X_0 & \end{tikzcd} .$$

The maps f and f^+ are small contractions whose exceptional sets are C_0, C_0^+ respectively. The birational map $\phi = (f^+)^{-1} \circ f$ is called the flop with respect to C_0. Note that ϕ is an isomorphism in codimension one. Under the flop $\phi : X \dashrightarrow X^+$, the curve classes transform as follows.

$$\begin{aligned} &\phi_*[C_0] = -[C_0^+], \quad \phi_*[C_i] = [C_i^+] + [C_0^+], \\ &\phi_*[C_\tau] = [C_\tau^+] \quad \text{for } \tau \in \Sigma_2'(X) \setminus \{\tau_0, \ldots, \tau_4\}. \end{aligned} \tag{7}$$

For the notations, see Table 1.

4.1.2. *Transformation of partition function.* We associate the same formal variables $\vec{Q} = (Q_e)$ to edges in $E_3(\Gamma_X) \setminus \{e_0, \ldots, e_4\}$ and those in $E_3(\Gamma_{X^+}) \setminus \{e_0^+, \ldots, e_4^+\}$ and write the partition functions of X and X^+ as $Z_X(q, \vec{Q}, Q_0, Q_1, Q_2, Q_3, Q_4)$ and $Z_{X^+}(q, \vec{Q}, Q_0^+, Q_1^+, Q_2^+, Q_3^+, Q_4^+)$ respectively. It is immediate to check that

$$\begin{aligned} Z_X(q, \vec{0}, Q_0, 0, 0, 0, 0) &= Z_{(-1,-1)}(q, Q_0), \\ Z_{X^+}(q, \vec{0}, Q_0^+, 0, 0, 0, 0) &= Z_{(-1,-1)}(q, Q_0^+). \end{aligned} \tag{8}$$

We set

$$Z'_X(q,\vec{Q},Q_0,Q_1,Q_2,Q_3,Q_4) \stackrel{\text{def.}}{=} \frac{Z_X(q,\vec{Q},Q_0,Q_1,Q_2,Q_3,Q_4)}{Z_X(q,\vec{0},Q_0,0,0,0,0)},$$

$$Z'_{X^+}(q,\vec{Q},Q_0^+,Q_1^+,Q_2^+,Q_3^+,Q_4^+) \stackrel{\text{def.}}{=} \frac{Z_{X^+}(q,\vec{Q},Q_0^+,Q_1^+,Q_2^+,Q_3^+,Q_4^+)}{Z_{X^+}(q,\vec{0},Q_0^+,0,0,0,0)}.$$

Theorem 4.1. *Under the identifications $Q_0 = (Q_0^+)^{-1}$ and $Q_i = Q_0^+Q_i^+$, we have*

$$Z'_X(q,\vec{Q},Q_0,Q_1,Q_2,Q_3,Q_4) = Z'_{X^+}(q,\vec{Q},Q_0^+,Q_1^+,Q_2^+,Q_3^+,Q_4^+).$$

This follows from Theorem 2.2. By (6) and (7), one can translate the statement in Theorem 4.1 into that in Theorem 1.1.

4.2. An application of Theorem 4.1

4.2.1. *Small modification of toric CY threefolds.* Let X be a toric CY threefold and Σ be its fan. Let $\sigma \in \Sigma_3$ be a 3-cone such that one of its three 2-dimensional faces τ_0 lies on the boundary of the support of the fan $|\Sigma|$. Let $\hat{\Sigma}$ be the following fan (see Figure 4):

$$\hat{\Sigma} = (\Sigma \setminus \{\tau_0,\sigma\}) \cup \{\rho_4,\hat{\tau}_0,\tau_3,\tau_4,\hat{\sigma}_1,\hat{\sigma}_2\},$$

and let $\hat{X}$ be the toric CY threefold associated with the fan Σ. We call $\hat{X}$ a *toric CY threefold obtained from X by a small modification.* We compare the partition function of $\hat{X}$ and that of X. We use the following notations in Table 2 for the rational curves, edges and formal variables. Note that the rational curve $\hat{C}_0$ corresponding to $\hat{\tau}_0$ is a $(-1,-1)$-curve.

	X		$\hat{X}$	
2-cone	τ_1,τ_2	τ	$\hat{\tau}_0,\tau_1,\tau_2$	τ
curve	C_1,C_2	C_τ	$\hat{C}_0,\hat{C}_1,\hat{C}_2$	$\hat{C}_\tau$
edge	e_1,e_2	e_τ or just e	$\hat{e}_0,\hat{e}_1,\hat{e}_2$	e_τ or just e
variable	Q_1,Q_2	Q_e	$\hat{Q}_0,\hat{Q}_1,\hat{Q}_2$	Q_e

Table 2

We study the transformation of the partition function under a small modification. We associate the same formal variables $\vec{Q} = (Q_e)$ to edges in $E_3(\Gamma_X)\backslash\{e_1, e_2\}$ and those in $E_3(\Gamma_{\hat{X}})\backslash\{\hat{e}_0, \hat{e}_1, \hat{e}_2\}$ and write the partition functions of X and $\hat{X}$ as $Z_X(q, \vec{Q}, Q_1, Q_2)$ and $Z_{\hat{X}}(q, \vec{Q}, \hat{Q}_0, \hat{Q}_1, \hat{Q}_2)$. It is immediate to check that

$$Z_{\hat{X}}(q, \vec{0}, \hat{Q}_0, 0, 0) = Z_{(-1,-1)}(q, \hat{Q}_0). \tag{9}$$

Define

$$Z'_{\hat{X}}(q, \vec{Q}, \hat{Q}_0, \hat{Q}_1, \hat{Q}_2) = \frac{Z_{\hat{X}}(q, \vec{Q}, \hat{Q}_0, \hat{Q}_1, \hat{Q}_2)}{Z_{\hat{X}}(q, \vec{0}, \hat{Q}_0, 0, 0)}.$$

Proposition 4.2. *We have*

$$Z_X(q, \vec{Q}, Q_1, Q_2) = Z'_{\hat{X}}(q, \vec{Q}, Q_0^{-1}, Q_1 Q_0, Q_2 Q_0)|_{Q_0 \to 0} \ .$$

Proof. Consider the toric CY threefold $\hat{X}^+$ obtained from $\hat{X}$ by the flop of $\hat{C}_0$. Let Q_0, $\hat{Q}_1^+$ and $\hat{Q}_2^+$ be the formal variables correspond to the flopped curves $\hat{C}_0^+$, $\hat{C}_1^+$ and $\hat{C}_2^+$ respectively. Let $\hat{\Sigma}^+$ be the fan of $\hat{X}^+$. A natural inclusion $\Sigma \hookrightarrow \hat{\Sigma}^+$ induces that of toric varieties $X \hookrightarrow \hat{X}^+$. Under this map, we identify $\hat{Q}_1^+$ and $\hat{Q}_2^+$ with Q_1 and Q_2 respectively. Then by Theorem 4.1, we have

$$Z'_{\hat{X}^+}(q, \vec{Q}, Q_0, Q_1, Q_2) = Z'_{\hat{X}}(q, \vec{Q}, Q_0^{-1}, Q_1 Q_0, Q_2 Q_0).$$

On the other hand, we have

$$\lim_{Q_0 \to 0} Z'_{\hat{X}^+}(q, \vec{Q}, Q_0, Q_1, Q_2) = Z_X(q, \vec{Q}, Q_1, Q_2).$$

Q.E.D.

4.2.2. *Toric surface and its blowup.* Let S be a complete smooth toric surface and $\hat{S}$ its blowup at a torus fixed point. The exceptional curve of $p : \hat{S} \to S$ is denoted by E. Let $X = K_S$ and $\hat{X} = K_{\hat{S}}$ be local toric surfaces. Then $\hat{X}$ is a small modification of X and E is a $(-1, -1)$-curve on $\hat{X}$ added by the small modification. Applying Proposition 4.2 to this case, we obtain the following

Corollary 4.3. *For $\beta \in H_2(\hat{S}, \mathbb{Z})$ satisfying $\beta.E = 0$,*

$$N_{g,\beta}(K_{\hat{S}}) = N_{g,p_*\beta}(K_S) \ .$$

Especially, the GW invariants of K_S are obtained from those of $K_{\hat{S}}$.

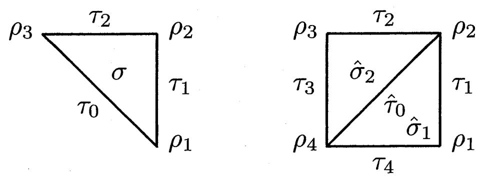

Fig. 4. Fans (sections at $z = 1$) : Σ (left) and $\hat{\Sigma}$ (right). The generators $\vec{\omega}_1, \ldots, \vec{\omega}_4$ of $\rho_1, \ldots, \rho_4$ satisfy the relation $\vec{\omega}_1 + \vec{\omega}_3 = \vec{\omega}_2 + \vec{\omega}_4$.

§5. Acknowledgments

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References

[1] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, The topological vertex, Comm. Math. Phys., **254** (2005), 425–478.

[2] P. S. Aspinwall, B. R. Greene and D. R. Morrison, Calabi–Yau moduli space, mirror manifolds and spacetime topology change in string theory, Nuclear Phys. B, **416** (1994), 414–480.

[3] C. Faber and R. Pandharipande, Hodge integrals and Gromov–Witten theory, Invent. Math., **139** (2000), 173–199.

[4] W. Fulton, Introduction to Toric Varieties, Ann. of Math. Stud., **131**, Princeton Univ. Press, 1993.

[5] A. Iqbal and A. K. Kashani-Poor, The vertex on a strip, Adv. Theor. Math. Phys., **10** (2006), 317–343.

[6] Y. Konishi and S. Minabe, Flop invariance of the topological vertex, Internat. J. Math., **19** (2008), 27–45.

[7] A.-M. Li and Y. Ruan, Symplectic surgery and Gromov–Witten invariants of Calabi–Yau 3-folds, Invent. Math., **145** (2001), 151–218.

[8] J. Li, C.-C. M. Liu, K. Liu and J. Zhou, A mathematical theory of the topological vertex, Geom. Topol., **13** (2009), 527–621.

[9] C.-H. Liu and S.-T. Yau, Transformation of algebraic Gromov–Witten invariants of three-folds under flops and small extremal transitions, with an appendix from the stringy and the symplectic viewpoint, preprint, arXiv:math.AG/0505084.

[10] I. G. Macdonald, Symmetric Functions and Hall Polynomials, Second ed., Oxford Math. Monogr., Oxford Sci. Publ., The Clarendon Press, Oxford Univ. Press, New York, 1995.
[11] T. Oda, Convex Bodies and Algebraic Geometry, Ergeb. Math. Grenzgeb. (3), **15**, Springer-Verlag, Berlin, 1988.
[12] A. Okounkov, N. Reshetikhin and C. Vafa, Quantum Calabi–Yau and classical crystals, In: The Unity of Mathematics, Progr. Math., **244**, Birkhäuser Boston, Boston, MA, 2006, pp. 597–618.
[13] E. Witten, Phases of $N = 2$ theories in two dimensions, Nuclear Phys. B, **403** (1993), 159–222.

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