Advanced Studies in Pure Mathematics 55, 2009
Noncommutativity and Singularities
pp. 209–221

An analogue of Serre fibrations for C^*-algebra bundles

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Abstract.

We study an analogue of Serre fibrations in the setting of C^*-algebra bundles. We derive in this framework a Leray–Serre type spectral sequence. We investigate a class of examples which generalise on one hand principal bundles with a n-torus as structural group and on the other hand non-commutative tori.

§1. Introduction

A Serre fibration in topology is a continuous map $p : Y \to X$ which satisfies the *Homotopy Lifting Property:* for any continuous map $h : Z \times \{0\} \to Y$ and any homotopy $H : Z \times [0,1] \to X$ such that $H \circ \iota = p \circ h$, where $\iota : Z \times \{0\} \hookrightarrow Z \times [0,1]$ is the inclusion, there exists a continuous map $\tilde{H} : Z \times [0,1] \to Y$ such that the following diagramme commutes:

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "h"] \arrow[d, "\iota"'] & Y \arrow[d, "p"] \\ Z \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & X \end{tikzcd}$$

If the space X is path connected, then all fibres $Y_x = p^{-1}(x)$ for x in X are homotopically equivalent and a Serre fibration behaves like a "locally trivial fibre bundle up to homotopy". The aim of this paper is

Received September 15, 2007.

2000 *Mathematics Subject Classification.* 14DXX, 19K35, 46L55, 46L80, 46L85, 55T10.

Key words and phrases. Non-commutative fibrations, K-theory, non-commutative tori, Leray–Serre spectral sequence.

This work was partially supported by the Deutsche Forschungsgemeinschaft (SFB 478) and by the EU-network MRTN-CT-2006-031962.

to investigate an analogue of Serre fibration in the setting of C^*-algebra bundles. Although the fibres of these bundles are quite irregular, they are locally trivial in a K-theoritical sense and we shall call these fibrations *K-fibrations.* As we shall see, numerous examples of K-fibrations are provided, using the Baum–Connes conjecture, by crossed products of C^*-algebra bundles over X by certain $C_0(X)$-linear actions of groups. Our main motivation for introducing K-fibrations was the study of *Non-Commutative Principal Torus (NCP) bundles.* For a path connected base space X, K-fibrations give rise to an action of the fundamental group $\pi_1(X)$ on the K-theory of the fibre and this action provides an invariant we shall fully describe for NCP $\mathbb{T}^2$-bundles. This invariant allows to detect which NCP $\mathbb{T}^n$-bundle are $\mathcal{R}$KK-equivalent to a "commutative" one, i.e to an algebra $C_0(Y)$ where Y is a principal $\mathbb{T}^n$-bundle over X. In the case $n = 2$, we obtain a classification of NCP $\mathbb{T}^2$-bundles, up to $\mathcal{R}$KK-equivalence and to a "twisting by a commutative principal torus bundle". As it is the case for usual Serre fibrations, we can derive from a K-fibration a Leray–Serre spectral sequence. Using this, we obtain a complete description of the NCP torus bundles which are $\mathcal{R}$KK-equivalent to a "trivial" one, i.e. to $C_0(X \times \mathbb{T}^n)$.

§2. Preliminaries on C^*-algebra bundles

By a C*-algebra bundle with base a locally compact space X, we shall mean a $C_0(X)$-algebra, i.e., a C*-algebra A which is equipped with a non-degenerate $*$-homomorphism $\Phi_A : C_0(X) \to ZM(A)$, where $ZM(A)$ denotes the centre of the multiplier algebra $M(A)$ of A. Throughout this paper we shall simply write $h \cdot a$ for $\Phi(h)a$ if $h \in C_0(X)$ and $a \in A$. The fibre A_x of A over $x \in X$ is then defined as $A_x = A/I_x$ with $I_x = \{h \cdot a; h \in C_0(X \setminus \{x\}) \text{ and } a \in A\}$. If $a \in A$, we set $a(x) := a + I_x \in A_x$. The elements $a \in A$ can be viewed in this way as sections of the bundle $(A_x)_{x\in X}$. The function $x \mapsto \|a(x)\|$ is then always upper semi-continuous and vanishes at infinity on X. Moreover, we have

$$\|a\| = \sup_{x\in X} \|a(x)\| \quad \text{for all } a \in A.$$

In what follows, we shall often write $A(X)$ for A, to indicate that we view A as a C^*-algebra bundle over X. If $A(X)$ and $B(X)$ are two C^*-algebra bundles, a morphism $\Psi : A(X) \to B(X)$ is said to be *fibre-wise* if it is $C_0(X)$-linear, i.e $\Psi(f \cdot a) = f \cdot \Psi(a)$ for all f in $C_0(X)$ and a in A. Notice that in this case Ψ induces $*$-homomorphisms $\Psi_x : A_x \to B_x$ such that $\Psi_x(a(x)) = \Psi(a)(x)$ for all a in $A(X)$. Let $\operatorname{Aut} A(X)$ be the set of fibre-wise automorphisms of $A(X)$. If G is a locally compact group, then a *fibre-wise* action of G on $A(X)$ is simply a group morphism $\alpha : G \to \operatorname{Aut} A(X)$. A fibre-wise action α induces actions α_x on the fibres A_x for all $x \in X$ and the full crossed product $A(X) \rtimes_\alpha G$ is again a C*-algebra bundle over X with structure map given by the composition $C_0(X) \overset{\Phi_A}{\to} ZM(A) \hookrightarrow ZM(A \rtimes G)$, and with fibres $A_x \rtimes_{\alpha_x} G$ (the same holds for reduced crossed products if G is exact).

If $A = A(X)$ is a C^*-algebra bundle, Y is a locally compact space and $f : Y \to X$ is a continuous map, then $C_0(X) \to C_b(Y) = M(C_0(Y))$; $g \mapsto g \circ f$ provides a C^*-algebra bundle structure of $C_0(Y)$ over X and the pull-back $f^*A = f^*A(Y)$ of $A(X)$ along f is defined as the balanced tensor product

$$f^*A := C_0(Y) \otimes_{C_0(X)} A.$$

The obvious inclusion of $C_0(Y)$ into $ZM(f^*A)$ turns f^*A into a C^*-algebra bundle over Y. In particular, if $Z \subseteq X$ is a locally compact subset of X [1] the pull-back $A(Z) := i_Z^*A$ of $A(X)$ along the inclusion map $i_Z : Z \to X$ becomes a $C_0(Z)$-algebra which we call the *restriction* of A to Z. If Y is a closed subset of X, then we have a short exact sequence

$$0 \to A(X \setminus Y) \to A(X) \to A(Y) \to 0.$$

Furthermore, if $A(X)$ and $B(X)$ are C^*-algebra bundles, then every fibre-wise morphism $\Psi : A(X) \to B(X)$ gives rise to a fibre-wise morphism $f^*\Psi : f^*A(Y) \to f^*B(Y)$ such that $f^*\Psi(h \otimes a) = h \otimes \Psi(a)$.

§3. Non-commutative principal torus bundles

We introduce in this section our toy example of K-fibrations which generalises classical principal $\mathbb{T}^n$ bundles to the non-commutative setting. Let $q : Y \to X$ be a principal $\mathbb{T}^n$-bundle with locally compact base space X. Then $C_0(Y)$ is a C^*-algebra bundle over X with fibres $C(q^{-1}(\{x\})) \cong C(\mathbb{T}^n)$. The given action of $\mathbb{T}^n$ on Y induces a fibre-wise action of $\mathbb{T}^n$ on $C_0(Y)$ and Green's theorem [6] implies that

$$C_0(Y) \rtimes \mathbb{T}^n \cong C_0(X, \mathcal{K}).$$

This leads to the following definition of NCP $\mathbb{T}^n$-bundles:

Definition 3.1. A (possibly) *non-commutative principal* $\mathbb{T}^n$*-bundle* (or *NCP* $\mathbb{T}^n$*-bundle*) with locally compact base space X is a separable

[1]Recall that a subset of a locally compact space is locally compact in the relative topology if and only if it is open in its closure.

C*-algebra bundle $A(X)$ equipped with a fibre-wise action $\alpha : \mathbb{T}^n \to \operatorname{Aut} A(X)$ such that

$$A(X) \rtimes_\alpha \mathbb{T}^n \cong C_0(X, \mathcal{K})$$

as C*-algebra bundles over X.

The most prominent example of a NCP torus bundle is certainly the Heisenberg bundle, i.e the C*-group algebra of the discrete Heisenberg group $H_2 = \left\{ \left(\begin{smallmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{smallmatrix} \right) ; n, m, k \in \mathbb{Z} \right\}$. The C*-algebra $C^*(H_2)$ is the universal C*-algebra generated by three unitaries U, V and W satisfying the relations $VU = WUV$, $UW = WU$ and $VW = WV$. Since the unitary W is central in $C^*(H_2)$, the isomorphism $C^*(W) \cong C(\mathbb{T})$ implements a canonical C^*-algebra bundle structure with base space $\mathbb{T}$ on $C^*(H_2)$. The fibre at $z = e^{2i\pi\theta}$ is canonically isomorphic to the non-commutative torus A_θ. It is straightforward to check that there is a fibre-wise action $\beta_2 : \mathbb{T}^2 \to \operatorname{Aut} C^*(H_2)(\mathbb{T})$ given on the generators by $\beta_2(z_1, z_2)U = z_1 U$, $\beta_2(z_1, z_2)V = z_2 V$ and $\beta_2(z_1, z_2)W = W$ for all $(z_1, z_2) \in \mathbb{T}^2$.

Proposition 3.2. *$C^*(H_2)$ together with the action β_2 is a NCP $\mathbb{T}^2$-bundle with base $\mathbb{T}$.*

Let us first remark that the non-commutative 2-tori A_θ differ substantially for different values of θ: they are simple for irrational θ and Morita equivalent to $C(\mathbb{T}^2)$ for rational θ. Hence the Heisenberg bundle is quite irregular in any classical sense. Nevertheless, all NCP $\mathbb{T}^n$-bundles are locally trivial in a K-theoretical sense. This shows up if we change from the category of C*-algebra bundles over X with fibre-wise $*$-homomorphisms to the category $\mathcal{R}\mathrm{KK}_X$ of C*-algebra bundles over X with morphisms given by the elements of Kasparov's group $\mathcal{R}\mathrm{KK}(X; A(X), B(X))$. We refer to [8] for the definition of the $\mathcal{R}$KK-group $\mathcal{R}\mathrm{KK}(X; A(X), B(X))$ for two C*-algebra bundles $A(X)$ and $B(X)$. We only recall here that the cycles are given by the usual cycles (E, ϕ, T) for Kasparov's bivariant K-theory group $\mathrm{KK}(A(X), B(X))$ with the extra requirement that the representation ϕ of $A(X)$ on the $B(X)$-Hilbert module E is $C_0(X)$-linear. Moreover $A(X)$ and $B(X)$ are said to be $\mathcal{R}$KK-equivalent if there exists an invertible class $\mathfrak{t}$ in $\mathcal{R}\mathrm{KK}(X; A(X), B(X))$. Isomorphic bundles $A(X)$ and $B(X)$ in this category are precisely the $\mathcal{R}$KK-equivalent bundles. The first observation we can make is the following

Theorem 3.3 ([4, Section 3]). *Any NCP $\mathbb{T}^n$-bundle $A(X)$ is locally $\mathcal{R}$KK-trivial. This means that for every $x \in X$ there is a neighbourhood*

U_x of x such that the restriction $A(U_x)$ of $A(X)$ to U_x is $\mathcal{R}$KK-equivalent to the trivial bundle $C_0(U_x, A_x)$. In particular $A(U_x)$ and $C_0(U_x, A_x)$ have the same K-theory.

Having this result, it is natural to ask the following questions:

Question 1: Suppose that $A(X)$ and $B(X)$ are two non-commutative $\mathbb{T}^n$-bundles with base X. Under what conditions is $A(X)$ $\mathcal{R}$KK-equivalent to $B(X)$?

Actually, in this paper we will only give a partial answer to the above question. But we shall give a complete answer, at least for (locally) path connected spaces X, to

Question 2: Which non-commutative principal $\mathbb{T}^n$-bundles are $\mathcal{R}$KK-equivalent to a "commutative" $\mathbb{T}^n$-bundle?

and

Question 3: Which non-commutative principal $\mathbb{T}^n$-bundles are $\mathcal{R}$KK-equivalent to a "trivial" $\mathbb{T}^n$-bundle?

These questions were the main motivations for introducing K-fibrations. As we shall see in Section 5, for each K-fibration $A(X)$ with X path connected, there is a canonical action of the fundamental group $\pi_1(X)$ on the K-theory of the fiber A_x. In the case of NCP torus bundles, this action detects the "non-commutativity" of the bundle. In particular, the action is trivial for all commutative principal $\mathbb{T}^n$-bundles. Together with earlier results of [2, 3], this allows to give the general answer to Question 2 and, in the case $n = 2$, to Question 1 up to a twisting with commutative principal $\mathbb{T}^n$-bundles. To go further, we shall derive in Section 6 from every K-fibration a Leray–Serre spectral which gives a new K-theoretical invariant to distinguish the total spaces of $\mathbb{T}^n$-bundles with a given base space X. As an application, we obtain the answer to Question 3.

§4. K-fibrations, KK-fibrations

The definition of a K-fibration is motivated from the following property of NCP torus bundles:

Proposition 4.1 ([4, Section 3]). *Let Δ be a contractible compact space and let $A(\Delta \times X)$ be a NCP $\mathbb{T}^n$-bundle over $\Delta \times X$. Then for any element z of Δ, the evaluation map*

$$e_z : A(\Delta \times X) \to A(\{z\} \times X)$$

gives an invertible class $[e_z] \in \mathcal{R}\mathrm{KK}(X; A(\Delta \times X), A(\{z\} \times X))$.

In particular the class of the evualation $e_z : A(\Delta \times X) \to A(\{z\} \times X)$ is invertible in $KK(A(\Delta \times X), A(\{z\} \times X))$. Since the pull-back of a NCP torus bundle is again a NCP torus bundle, applying this to $X = \{*\}$, we see that every NCP torus bundle is a KK-fibration in the sense of the following definition:

Definition 4.2. A C*-algebra bundle $A(X)$ is called a KK-*fibration* (resp. K-*fibration*) if for every compact contractible space Δ, any continuous map $f : \Delta \to X$ and any $z \in \Delta$, the evaluation map $e_z : f^*A(\Delta) \to A_{f(z)}$ is a KK-equivalence (resp. induces an isomorphism $K_*(f^*A(\Delta)) \cong K_*(A_{f(z)})$).

It is clear that every locally trivial C*-algebra bundle $A(X)$ is a KK-fibration. In particular, since being a KK-fibration is invariant under tensorising by the C^*-algebra $\mathcal{K}$ of compact operators on a separable Hilbert space, we see that a continuous trace algebra with spectrum X is a KK-fibration.

Numerous examples of K-fibrations and of KK-fibrations can be built from these elementary examples with help of the Baum–Connes conjecture (in its strong version) by taking crossed product by a fibre-wise action of a group. Recall that a group G satifies the strong Baum–Connes conjecture if there exists

- a locally compact space X equipped with a proper action of G and a C*-algebra bundle $A(X)$ with base X equipped with an action of G such that $g(\phi \cdot a) = g(\phi) \cdot g(a)$ for every g in G, ϕ in $C_0(X)$ and a in $A(X)$;
- an element x in $\mathrm{KK}^G(\mathbb{C}, A(X))$ and an element y in $\mathrm{KK}^G(A(X), \mathbb{C})$, such that $x \otimes_{A(X)} y = 1$ in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$.

This implies that the Baum–Connes assembly map

$$\mu_B : \mathrm{K}_*^{\mathrm{top}}(G; B) \to K_*(B \rtimes_r G)$$

from the topological K-theory of G with coefficient in any G-algebra B into the K-theory of the reduced crossed product $B \rtimes_r G$ is always an isomorphism. If B is a fixed G-algebra, then we say that G satisfies the Baum–Connes conjcture for B, if the assembly map is an isomorphism for this particular B. Then using the going-down technics developped in [1] together with [10, Theorem 9.3], we can prove

Proposition 4.3 ([5]). *Suppose that* $A(X)$ *is a separable* C^**-algebra bundle with base a locally compact space* X *and let* $\alpha : G \to \mathrm{Aut}\, A(X)$ *be a fibre-wise action of the second countable locally compact group* G

on A. Assume that for each compact subgroup K of G the C^*-algebra bundle $A(X) \rtimes K$ is a K-fibration. Then the following are true:

(i) *If G is exact and satisfies the Baum–Connes conjecture for $f^*(A)$ for all continuous $f : \Delta \to X$, where Δ is a compact contractible space (in particular, if G satisfies the strong Baum–Connes conjecture), then $A \rtimes_r G$ is a K-fibration.*

(ii) *If G satisfies the strong Baum–Connes conjecture, then $A \rtimes G$ is a K-fibration.*

(iii) *If G satisfies the strong Baum–Connes conjecture and $A \rtimes K$ is a KK-fibration for every compact subgroup $K \subseteq G$, then $A \rtimes G$ is a KK-fibration. If, in addition, G is exact, then $A \rtimes_r G$ is a KK-fibration, too.*

As a consequence, since amenable groups are exact and satisfy the strong Baum–Connes conjecture by [7], we get

Theorem 4.4 ([5]). *Suppose G is an amenable group which acts fibre-wise on the C^*-algebra bundle $A(X)$. Then: If $A(X) \rtimes K$ is a K-fibration (resp. KK-fibration) for all compact subgroups K of G, then $A(X) \rtimes G$ is a K-fibration (resp. KK-fibration).*

Corollary 4.5. *If $A(X)$ is a K-fibration (resp. KK-fibration) then the same is true for $A(X) \rtimes \mathbb{Z}^n$ or $A(X) \rtimes \mathbb{R}^n$ for every fibre-wise action $\alpha : \mathbb{Z}^n, \mathbb{R}^n \to \mathrm{Aut}(A(X))$.*

§5. The K-theory group bundle

If X is a locally compact space, an (abelian) *group bundle* with base space X is a family $\mathcal{G} = (G_x)_{x \in X}$ of abelian groups, together with group isomorphisms $c_\gamma : G_x \to G_y$ for each continuous path $\gamma : [0,1] \to X$ from x to y, such that

(i) If γ and γ' are homotopic paths from x to y, then $c_\gamma = c_{\gamma'}$.

(ii) If $\gamma_1 : [0,1] \to X$ and $\gamma_2 : [0,1] \to X$ are paths from y to z and from x to y, respectively, then

$$c_{\gamma_1 \circ \gamma_2} = c_{\gamma_1} \circ c_{\gamma_2},$$

where $\gamma_1 \circ \gamma_2 : [0,1] \to X$ is the usual composition of paths.

In particular, if X is path connected, then all groups G_x are isomorphic and we get a canonical action of the fundamental group $\pi_1(X)$ on each fibre G_x.

A morphism $\phi : \mathcal{G} \to \mathcal{G}'$ between two group bundles $\mathcal{G} = (G_x)_{x \in X}$ and $\mathcal{G}' = (G'_x)_{x \in X}$ with base X is a family $\phi = (\phi_x)_{x \in X}$ of group homomorphisms $\phi_x : G_x \to G'_x$ which commutes with the maps c_γ.

Proposition and Definition 5.1 ([4]). *Suppose that $A(X)$ is a K-fibration. For any path $\gamma : [0,1] \to X$ with starting point x and endpoint y, let $c_\gamma : K_*(A_x) \to K_*(A_y)$ denote the composition*

$$K_*(A_x) \xrightarrow[\cong]{\epsilon_{0,*}^{-1}} K_*(\gamma^* A) \xrightarrow[\cong]{\epsilon_{1,*}} K_*(A_y)$$

Then $\mathcal{K}_*(A) := (K_*(A_x))_{x\in X}$ together with the above defined maps c_γ is a group bundle over X which we call the K-theory group bundle associated to $A(X)$.

If $A(X)$ and $B(X)$ are two K-fibrations, then any class $\mathfrak{x} \in \mathcal{R}\mathrm{KK}(X, A(X), B(X))$ gives rise to a morphism of group bundles $\mathfrak{x}_* : \mathcal{K}_*(A) \to \mathcal{K}_*(B)$ given pointwise by right Kasparov product with the evaluation $\mathfrak{x}(x) \in \mathrm{KK}(A_x, B_x)$ of $\mathfrak{x}$ at x. Moreover if $\mathfrak{x}$ is invertible, then $\mathfrak{x}_*$ is an isomorphism of group bundles.

Let us describe the action of $\pi_1(\mathbb{T}) \cong \mathbb{Z}$ on the fiber at $z = 1$ of the K-theory group bundle associated to the Heisenberg bundle $C^*(H_2)(\mathbb{T})$. The fiber at $z = 1$ is isomorphic to $C(\mathbb{T}^2)$. Since the unitaries U and V are global sections of $C^*(H_2)$ and since $[U(1)]$ and $[V(1)]$ are generators for $K_1(C^*(H_2))_1 \cong K_1(C(\mathbb{T}^2))$, it turns out that the action of $\pi_1(\mathbb{T}) \cong \mathbb{Z}$ on $K_1(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$ is trivial, and hence that the group bundle $\mathcal{K}_1(C^*(H_2)) = (K_1(C^*(H_2)_z))_{z\in\mathbb{T}}$ is trivial. To describe the action on the even part, we equip $K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$ with the basis $([1], \beta)$, where β is the Bott element of $K_0(C(\mathbb{T}^2))$. It is then shown in [4] that the action of the generating loop $[0,1] \to \mathbb{T}; t \to e^{2i\pi t}$ of $\pi_1(\mathbb{T}) \cong \mathbb{Z}$ on $K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$ is given on this basis by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

§6. Leray–Serres spectral sequence

In this section, we want to explain the analogue for K-fibrations of the classical Leray–Serre spectral sequence. It is well known in algebraic topology, that one can use group bundles as local coefficients for simplicial cohomology on the base space. Our aim is then to show that a K-fibration $A(X)$ on a finite dimensional simplicial complex X gives rise to a spectral sequence with E_2-term isomorphic to the cohomology of X with local coefficients in the K-theory group bundle $\mathcal{K}(A)$.

Let X be a finite dimensional simplicial complex with skeleton $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$ and let $A(X)$ be a K-fibration with base space X. For p in $\mathbb{Z}$ we set $A_p := A(X_p)$ and $A_{p,p-1} = A(X_p \setminus X_{p-1})$ (where $X_p = \emptyset$ if $p < 0$ and $X_p = X$ if $p \geq n$). Then we have short exact sequences

$$0 \to A_{p,p-1} \to A_p \to A_{p-1} \to 0$$

with associated long exact sequences

$$\cdots \xrightarrow{\partial_{p,q-1}} K_q(A_{p,p-1}) \xrightarrow{\iota_{p,q}} K_q(A_p) \xrightarrow{j_{p,q}} K_q(A_{p-1}) \xrightarrow{\partial_{p,q}} K_{q+1}(A_{p,p-1}) \xrightarrow{\iota_{p,q+1}} \cdots$$

Let us define $\mathcal{A}^{p,q} = K_q(A_p)$ and $E_1^{p,q} = K_q(A_{p,p-1})$ for p and q in $\mathbb{Z}$. Then we get the exact couple

$$\begin{tikzcd}
\bigoplus_{(p,q)\in\mathbb{Z}^2} \mathcal{A}^{p,q} \arrow[rr, "j"] & & \bigoplus_{(p,q)\in\mathbb{Z}^2} \mathcal{A}^{p,q} \arrow[dl, "\partial"] \\
& \bigoplus_{(p,q)\in\mathbb{Z}^2} E_1^{p,q} \arrow[ul, "\iota"] &
\end{tikzcd} ,$$

where $\iota = \bigoplus_{(p,q)\in\mathbb{Z}^2} \iota_{p,q}$, $j = \bigoplus_{(p,q)\in\mathbb{Z}^2} j_{p,q}$ and $\partial = \bigoplus_{(p,q)\in\mathbb{Z}^2} \partial_{p,q}$. From this exact couple we can derive by the general procedure (which, for example, is explained in [9]) a spectral sequence $\{E_r^{p,q}, d_r\}$ such that

- we have $d_1 : E_1^{p,q} = K_q(A_{p,p-1}) \overset{\iota_{p,q}}{\to} K_q(A_p) \overset{\partial_{p+1,q}}{\to} K_{q+1}(A_{p+1,p}) = E_1^{p+1,q+1}$;
- the higher terms are derived from this iteratively by $E_{r+1}^{p,q} = (\ker d_r / \operatorname{im} d_r)_{p,q}$;
- this process stabilizes eventually with $E_\infty^{p,p-q} := F_p^q / F_{p+1}^q$ where

$$F_p^q := \ker\big(K_q(A(X)) \to K_q(A_p)\big).$$

Since $F_p^q = K_q(A)$ for $p < 0$ and $F_p^q = \{0\}$ for $p \geq n$, the spectral sequence converges to $K_q(A)$.

Theorem 6.1 ([5]). *Suppose that $A(X)$ is a K-fibration over the finite simplicial complex X. Then the E_2-term of the above described spectral sequence is given by $E_2^{p,q} \cong H^p(X, \mathcal{K}_q(A))$, the cohomology of X with local coefficients in the K-theory group bundle $\mathcal{K}_*(A(X))$.*

Remark 6.2.

- The case $A(X) = C(X)$ is the classical Atiyah–Hirzebruch spectral sequence for the K-theory of X.
- If $A(X)$ and $B(X)$ are $\mathcal{R}$KK-equivalent, then the spectral sequences associated to $A(X)$ and $B(X)$ are isomorphic and this isomorphism is given on the E_2-term by the isomorphism of the group bundles $\mathcal{K}(A) \cong \mathcal{K}(B)$ which is induced by the $\mathcal{R}$KK-equivalence. It follows that the spectral sequence is an invariant for $\mathcal{R}$KK-equivalence of K-fibrations.

§7. $\mathcal{R}$KK-equivalence for NCP $\mathbb{T}^n$-bundles

In this section we want to explain how we obtain answers to Questions 2 and 3 of Section 3 and a partial answer to Question 1 in the case $n = 2$ by using the invariants introduced in the previous sections. We start by giving an alternative definition of NCP torus bundles using Takesaki–Takai duality. If $A(X)$ is a NCP $\mathbb{T}^n$-bundle, then the crossed product $A(X) \rtimes_\alpha \mathbb{T}^n \cong C_0(X, \mathcal{K})$ is equipped with the dual action $\widehat{\alpha}$ of $\mathbb{Z}^n \cong \widehat{\mathbb{T}^n}$, and then the Takesaki–Takai duality theorem asserts that

$$A(X) \sim_M C_0(X, \mathcal{K}) \rtimes_{\widehat{\alpha}} \mathbb{Z}^n$$

as C*-algebra bundles over X, where $\sim_M$ stands for $\mathbb{T}^n$-equivariant $C_0(X)$-Morita equivalence, the dual action $\widehat{\alpha}$ being also fibre-wise. Conversely, if $\beta : \mathbb{Z}^n \to \operatorname{Aut} C_0(X, \mathcal{K})$ is any fibre-wise action, then the Takesaki–Takai duality theorem implies that

$$A(X) = C_0(X, \mathcal{K}) \rtimes_\beta \mathbb{Z}^n$$

together with the dual (fibre-wise) action $\widehat{\beta}$ of $\mathbb{T}^n \cong \widehat{\mathbb{Z}^n}$ is a NCP $\mathbb{T}^n$-bundle. In consequence, up to a suitable notion of Morita equivalence, the NCP $\mathbb{T}^n$-bundles are precisely the crossed products $C_0(X, \mathcal{K}) \rtimes_\beta \mathbb{Z}^n$ for some fibre-wise action β of $\mathbb{Z}^n$ on $C_0(X, \mathcal{K})$ and equipped with the dual $\mathbb{T}^n$-action.

Using this alternative definition, the results of [2, 3] provide a complete classification of NCP $\mathbb{T}^n$-bundles up to $\mathbb{T}^n$-equivariant Morita equivalence. To explain this classification we need to introduce the following higher dimensional generalisation of the Heisenberg bundle: let H_n be the group generated by $\{f_1, \ldots, f_n\}$ and $\{g_{ij} \,;\, 1 \le i < j \le n\}$ with relations $f_i f_j = g_{ij} f_j f_i$ and g_{ij} is central for all $1 \le i < j \le n$. Then $C^*(H_n)$ is the universal C^*-algebra generated by the unitaries $U_1, \ldots, U_n$ and $\{V_{i,j} \,;\, 1 \le i < j \le n\}$ with relations $U_i \cdot U_j = V_{i,j} \cdot U_j \cdot U_i$ and $V_{i,j}$ are central for $1 \le i < j \le n$. Since the central unitaries $\{V_{i,j} \,;\, 1 \le i < j \le n\}$ generate $\frac{n(n-1)}{2}$ copies of $\mathbb{T}$, the C^*-algebra $C^*(H_n)$ is equipped with a canonical C*-algebra bundle structure with base space $\mathbb{T}^{\frac{n(n-1)}{2}}$.

Consider the fibre-wise action $\beta_n : \mathbb{T}^n \to \operatorname{Aut} C^*(H_n)\left(\mathbb{T}^{\frac{n(n-1)}{2}}\right)$ given on generators by $\beta_n(z_1, \ldots, z_n)(U_i) = z_i U_i$, $\beta_n(z_1, \ldots, z_n)(V_{ij}) = V_{ij}$. As in the case $n = 2$, we get

Lemma 7.1. *$C^*(H_n)$ together with the action β_n is a NCP $\mathbb{T}^n$-bundle with base $\mathbb{T}^{\frac{n(n-1)}{2}}$.*

As we shall see, $C^*(H_n)$ can be viewed as the "universal" NCP-$\mathbb{T}^n$-bundle. In order to achieve the classification, we need to introduce an action of principal $\mathbb{T}^n$-bundles on NCP $\mathbb{T}^n$-bundle. If $q : Y \to X$ is a (commutative) principal $\mathbb{T}^n$-bundle and $A(X)$ is an NCP torus bundle, then we define $Y * A(X) = q^*A(Y)^{\mathbb{T}^n}$, where $q^*A(Y)^{\mathbb{T}^n}$ denotes the fixed point algebra of $q^*A(Y) = C_0(Y) \otimes_{C_0(X)} A(X)$ with respect to the diagonal action of $\mathbb{T}^n$, where we equip Y with the action $\mathbb{T}^n \times Y \to Y$; $(z, y) \mapsto z^{-1} \cdot y$. The action of $C_0(X)$ on either factor induces a C*-algebra bundle structure over X on $Y * A$ and one can check that the action $q^*\alpha : \mathbb{T}^n \to \operatorname{Aut} q^*A(Y)$ restricts to a well defined action $Y * \alpha$ on $Y * A(X)$ so that $Y * A(X)$ becomes also a NCP torus bundle. This action generalises the fibre-wise product of principal $\mathbb{T}^n$-bundles.

We are now able to state the classification result of [3] (see also [4, Section 2]).

Theorem 7.2. *Let X be a second countable locally compact space. Then the set of $\mathbb{T}^n$-equivariant Morita equivalence classes of NCP $\mathbb{T}^n$-bundles over X is classified by the set of all pairs $([q : Y \to X], f)$ with $[q : Y \to X]$ the isomorphism class of a (commutative) principal $\mathbb{T}^n$-bundle $q : Y \to X$ and $f : X \to \mathbb{T}^{\frac{n(n-1)}{2}}$ a continuous map. Given these data, the corresponding equivalence class of NCP $\mathbb{T}^n$-bundles is represented by the algebra*

$$A_{(Y,f)}(X) := Y * f^*\big(C^*(H_n)\big)(X).$$

It is worth to mention that the K-theory group bundle of a NCP torus bundle is invariant under the action of a principal $\mathbb{T}^n$-bundle [4, Lemma 4.5], i.e if $q : Y \to X$ is a principal $\mathbb{T}^n$-bundle and $A(X)$ is a NCP $\mathbb{T}^n$-bundle, then $\mathcal{K}(Y * A)$ and $\mathcal{K}(A)$ are canonicaly isomorphic (in particular, the K-theory group bundle of a "commutative" principal $\mathbb{T}^n$-bundle is trivial). Using this remark, we get

Lemma 7.3 ([4]). *Let $A(X)$ be a NCP $\mathbb{T}^n$-bundle with base X and classifying data $([q : Y \to X], f)$. Assume that x is an element of X such that $f(x) = 1$. Then the action of $\gamma \in \pi_1(X)$ on $K_1(C(\mathbb{T}^2))$ is trivial and the action on $K_0(C(\mathbb{T}^2))$ is given on the generators $[1], \beta$ by the matrix*

$$\begin{pmatrix} 1 & \langle f, \gamma\rangle \\ 0 & 1 \end{pmatrix},$$

where $\langle f, \gamma\rangle$ is the winding number of $f \circ g : \mathbb{T} \to \mathbb{T}$ for any representing loop $g : \mathbb{T} \to X$ of γ.

A similar (but more technical) result also holds for higher dimensional NCP torus bundles and as a consequence we obtain

Corollary 7.4. *The K-theory group bundle of a NCP torus bundle $A(X)$ with path connected base space X is trivial if and only if the associated continuous map f of the classification data is homotopic to a constant map.*

In view of Proposition 4.1, homotopies of NCP torus bundles induce $\mathcal{R}$KK-equivalences. Together with the above corollary, this gives the answer to Question 2.

Theorem 7.5 ([4]). *Let $A(X)$ be any NCP $\mathbb{T}^n$-bundle with path connected base space X. Then $A(X)$ is $\mathcal{R}$KK-equivalent to a "commutative bundle" $p : Y \to X$ (or rather $C_0(Y)(X)$) if and only if the K-theory bundle of $A(X)$ is trivial.*

For $n = 2$, we have a more accurate result.

Theorem 7.6 ([4]). *Let $A(X)$ and $B(X)$ be NCP $\mathbb{T}^n$-bundles with path connected base space X. Then the following assertions are equivalent :*

(i) *There exist principal $\mathbb{T}^n$-bundles $q_1 : Y_1 \to X$ and $q_2 : Y_2 \to X$ such that $Y_1 * A$ and $Y_2 * B$ are $\mathcal{R}$KK-equivalent.*

(ii) *The group bundles $\mathcal{K}(A)$ and $\mathcal{K}(B)$ are isomorphic.*

It is a well known fact in algebraic topology that a principal $\mathbb{T}^n$-bundle is trivial if and only if the differential d_2 of the E_2-term of the Leray–Serre spectral sequence vanishes. This result admits the following generalisation to NCP torus bundles:

Theorem 7.7 ([5]). *The NCP-torus bundle $A(X)$ is $\mathcal{R}$KK-equivalent to the trivial bundle $X \times \mathbb{T}^n$ if and only if the K-theory group bundle is trivial and the map*

$$d_2 : H^p(X, \mathcal{K}_q(A)) \to H^{p+2}(X, \mathcal{K}_{q+1}(A))$$

in the Leray–Serre spectral sequence associated the the K-fibation $A(X)$ is trivial for all p.

Let us remark that if the K-theory group bundle of $A(X)$ is trivial, then $H^p(X, \mathcal{K}_*(A))$ is canonically ismorphic to $H^p(X, K_*(A_x))$ for any x in X.

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