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On geometric analogues of Iwasawa main conjecture for a hyperbolic threefold

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Abstract.

We will discuss a relation between a special value of Ruelle–Selberg L-function of a unitary local system on a hyperbolic threefold of finite volume and Alexander invariant. A philosophy of our results are based on Iwasawa Main Conjecture in number theory.

§1. Introduction

This is a survey article of recent our progress of arithmetic topology developed in [20] and [21].

In order to understand properties of Riemann zeta function $\zeta(s)$ various geometric models have been considered. The most notable one will be *Hasse–Weil congruent zeta function* $Z(X, t)$ for a smooth projective variety X over a finite field $\mathbb{F}_q$;

$$Z(X, t) = \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}.$$

Here $|X|$ is the set of closed points of X and $\deg(x)$ is the degree of extension of the residue field F_x over $\mathbb{F}_q$. Taking its logarithmic derivative Grothendieck–Lefschetz trace formula implies

$$Z(X, t) = \prod_i \det(1 - \phi_q^* t \mid H^i_{et}(\overline{X}, \mathbb{Q}_l))^{(-1)^{i+1}}, \tag{1.1}$$

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where ϕ_q is q-th power of the geometric Frobenius and $\overline{X}$ is the base extension of X to the algebraic closure of $\mathbb{F}_q$. Since the RHS of (1.1) is an alternating product of characteristic polynomials of ϕ_q on étale cohomology groups we can relate $Z(X, t)$ to geometry of X. We expect that there exists a similar equation for various L-functions.

In fact, in topological category, Fried has obtained an analogous formula for a manifold with a dynamical system [3]. Deninger has considered such a problem for a manifold with a foliation of a certain type [1]. By Hodge theory for the leafwise Laplace operator he has computed a special value of his zeta function and proved a geometric analogue of Lichtenbaum's conjecture for such a manifold.

Now let us return to Riemann zeta function. Motivated by the theory of Hasse–Weil zeta function, Iwasawa has defined *Iwasawa module*, which is a finitely generated torsion module over $\mathbb{Z}_p[[s]]$. *Iwasawa polynomial* is defined as a generator of its characteristic ideal and this corresponds to the RHS of (1.1). Since this is a p-adic object, in order to have an analogous formula as (1.1) for $\zeta(s)$, we have to construct the p-adic analytic function which substitutes it. This is nothing but *a p-adic zeta function.* Then his main conjecture is that Iwasawa polynomial and the p-adic zeta function should generate the same ideal in $\mathbb{Z}_p[[s]]$. This conjecture was firstly solved by Mazur and Wiles ([12]) and Kolyvagin has given a much simpler proof than theirs. (Basic references for these subjects will be [9] or [22].)

About 40 years ago Mazur pointed out an analogy between the number theory and topology of threefolds. In particular he noticed that a similarity between Iwasawa polynomial and Alexander polynomial, which is the most well-known object in knot theory [11]. Morishita has observed certain analogies between primes and knots. In fact, based on similarities of the structure of a link group and a certain maximal pro l-Galois group, he has obtained invariants of a number field, which corresponds to Alexander module and Milnor invariants[16]. He has continued this line and has investigated the connection between his invariant and Massey product in Galois cohomology [17]. The theory which pursues these analogies is called *arithmetic topology* and is also developped by Reznikov and Kapranov and thier collaborators ([6][19]).

Because of these facts it may be natural to expect that an analogue of the Iwasawa main conjecture should exist for a topological threefold, which is a geometric object corresponding to the ring of integers of a number field by a viewpoint of étale homotopy. Let X be a smooth threefold. In order to define our L-fuction, *Ruelle–Selberg L-function* $R_X(z, \rho)$ (see §4 for its definition), we assume X admits a complete hyperbolic structure of finite volume. Also to define *Alexander invariant* we suppose that it adimits an infinite cyclic covering X_∞. Let us fix a generator g of $\mathrm{Gal}(X_\infty/X) \simeq \mathbb{Z}$ and let ρ be a unitary representation of $\pi_1(X)$ of finite rank. We assume that $H.(X_\infty, \mathbb{C})$ and $H.(X_\infty, \rho)$ are finite dimensional vector spaces over $\mathbb{C}$. This implies that $H^i(X_\infty, \rho)$ is also a finite dimensional vector space over $\mathbb{C}$ with a natural action of g. Then Alexander invariant $A_\rho(z)$ is defined to be an alternating product of its characteristic polynomials (see §3 for the definition). In general let V be a finite dimensional vector space over $\mathbb{C}$ with an action of g. If it does not have a submodule preserved by g other than 0 and itself we will refer to V as *simple.* A direct sum of a finite vector spaces of a simple g-action will be called *semisimple.* Suppose that $H^0(X_\infty, \rho) = 0$ and that the action of g on $H^1(X_\infty, \rho)$ is semisimple. Here is our main theorem.

Theorem 1.1. *Let X be a compact hyperbolic threefold or a complete hyperbolic threefold of finite volume. In the latter case we assume that ρ is* cuspidal*(see §4). Then*

$$(R_X(z, \rho)) = (A_\rho(z)^2) \quad in \quad \mathbb{C}[[z]].$$

Let $h^i(\rho)$ be the dimension of $H^i(X, \rho)$. We will show that their order at $z = 0$ is $2h^1(\rho)$. It can be also shown that if $h^1(\rho) = 0$ thier special value at the origin are essentially the square of Milnor–Reidemeister torsion.

Notice that Iwasawa main conjecture was formulated by ideals of $\mathbb{Z}_p[[s]]$ whose Krull dimension is two and it is neccessary to care about a p-adic integral structure of the p-adic zeta function. But in our case Krull dimension of $\mathbb{C}[[z]]$ is one and we do not have to worry about *an integrality* of $R_X(z, \rho)$. Thus our model is much simpler and easier than p-adic one. If ρ is not cuspidal the result of Park [18] shows that the orders $R_X(z, \rho)$ and $A_\rho(z)^2$ at the origin are different. Such a phenomenon also occurs for a p-adic L-function associated to an elliptic curve defined over $\mathbb{Q}$ which has a split multiplicative reduction at p [13]. It is quite surprising that although three of p-adic analysis, the arithmetic algebraic geometry over a finite field and the theory of hyperbolic threefolds are quite different in their feature, L-functions in each field have common properties.

A main difference between the model of Deninger or Fried and ours is that in their models there exist *a dynamical system* which corresponds to the geometric Frobenius but not in ours. Instead of the geometric Frobenius and Grotheidieck–Lefschetz trace formula, we will use the heat kernel of Laplacian and Selberg trace formula, respectively.

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§2. Iwasawa Main Conjecture

Since our motivation is based on Iwasawa Main Conjecture of number theory, we will briefly explain it. A basic reference of this section is [22].

In order to study arithmetic properties of a Dirichlet L-function, we will consider a p-adic L-function due to Kubota and Leopoldt. For simplicity we assume p is an odd prime. Let χ be a Dirichlet character whose conductor f_χ. It is known special values of a Dirichlet L-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

at nonpositive integers are given by

$$L(1-n, \chi) = -\frac{B_{n,\chi}}{n}, \quad 1 \leq n \in \mathbb{Z}. \tag{2.1}$$

Here $B_{n,\chi}$ is a generalized Bernoulli number defined by

$$\sum_{a=1}^{f_\chi} \frac{\chi(a) T e^{aT}}{e^{f_\chi T} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{T^n}{n!}.$$

By definition Kubota–Leopoldt L-function is a p-adic analytic function which interpolates special values of a Dirichlet L-function. More precisely let us fix a completion of the algebraic closure of $\mathbb{Q}_p$, which will be denoted by $\mathbb{C}_p$. Let $|\cdot|_p$ be a p-adic norm on $\mathbb{C}_p$ normalized

$$|p|_p = p^{-1}.$$

Fact 2.1. *For a non-trivial Dirichlet character* χ *(resp. the trivial character* $\mathbf{1}$*), there is the unique analytic function* $L_p(s, \chi)$ *(resp. meromorphic function* $L_p(s, \mathbf{1})$*) on a domain*

$$D = \{s \in \mathbb{C}_p \mid |s|_p < p^{-\frac{p-2}{p-1}}\},$$

which satisfies

$$(2.2) \qquad L_p(1-n, \chi) = -(1 - \chi\omega^{-n}(p)p^{n-1})\frac{B_{n,\chi\omega^{-n}}}{n}, \quad 1 \le n \in \mathbb{Z},$$

where ω *is Teichmüller character. Moreover* $L_p(s, \mathbf{1})$ *is analytic outside* $s = 1$ *and has a simple pole there whose residue is* $1 - p^{-1}$.

Let j be an integer such that $j \equiv n (\text{mod} \quad p-1)$, $0 \le j < p-1$. Then combining (2.1) and (2.2) we obtain the following identity of special values of these two functions:

$$L_p(1-n, \chi) = (1 - \chi\omega^{-j}(p)p^{n-1})L(1-n, \chi\omega^{-j}), \quad (1 \le n \in \mathbb{Z}).$$

Thus we may consider $L_p(s, \chi)$ as a p-adic analog of $L(s, \chi)$. Moreover it is known that, for an even integer such that $\omega^i \neq \mathbf{1}$, there is $f(t, \omega^i) \in \mathbb{Z}_p[[t]]$, which is called *Iwasawa power series*, satisfying

$$(2.3) \qquad f((1+p)^s - 1, \omega^i) = L_p(s, \omega^i), \quad s \in \mathbb{Z}_p.$$

In order to formulate Iwasawa Main Conjecture we need an algebraic object: *the characteristic ideal of Iwasawa module.* We will see in the next section that it is quite similar to Alexander invariant for a unitary local system over a threefold whose fundamental group has a infinite cyclic quotient.

We will fix a p^n-th root of unity ζ_{p^n} as

$$\zeta_{p^n} = \exp(\frac{2\pi i}{p^n}),$$

and let μ_{p^n} be the subgroup of $\mathbb{C}^\times$ generated by ζ_{p^n}. Since $\zeta_{p^n}^p = \zeta_{p^{(n-1)}}$ for any n, the invese limit with respect to the p-th power:

$$\zeta_{p^\infty} = \varprojlim \zeta_{p^n} \in \varprojlim \mu_{p^n}.$$

is defined.

There is a canonical decomposition of Galois group:

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}),$$

where $\mathbb{Q}_n$ is a finite abelian extension of $\mathbb{Q}$. In the decomposition, the former and the latter are isomorphic to $(\mathbb{Z}/(p))^\times$ and the kernel of the mod p reduction map:

$$\Gamma_n = \mathrm{Ker}[(\mathbb{Z}/(p^n))^\times \to (\mathbb{Z}/(p))^\times] \simeq \mathbb{Z}/(p^{n-1}),$$

respectively. Taking the inverse limit with respect to n, we have an infinite extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$ such that

$$\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) = \varprojlim \Gamma_n \overset{\kappa}{\simeq} (1+p\mathbb{Z}_p)^\times \overset{\log}{\simeq} \mathbb{Z}_p.$$

Here the cyclotomic character κ is defined as

$$\gamma(\zeta_{p^\infty}) = \zeta_{p^\infty}^{\kappa(\gamma)}, \quad \gamma \in \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}).$$

Then a topological ring

$$\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]] = \varprojlim \mathbb{Z}_p[\Gamma_n],$$

is referred as *Iwasawa algebra*. Choosing a topological generator γ_0 of $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ (e.g. $\kappa(\gamma_0) = 1+p$), Λ is isomorphic to a formal power series ring $\mathbb{Z}_p[[t]]$. Therefore

$$\begin{aligned}\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]] &\simeq \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})] \otimes_{\mathbb{Z}_p} \Lambda \\ &\simeq \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[t]],\end{aligned}$$

where $\mathbb{Q}(\zeta_{p^\infty})$ is the union of $\{\mathbb{Q}(\zeta_{p^n})\}_n$.

Let A_n be the p-primary part of the ideal class group of $\mathbb{Q}(\zeta_{p^n})$. Then *Iwasawa module* is defined to be

$$X_\infty = \varprojlim A_n.$$

Here the inverse limit is taken with respect to the norm map. Since $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$ acts on A_n, X_∞ becomes a $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]]$-module. For $i \in \mathbb{Z}/(p-1)$, let $X_{\infty,i}$ be its ω^i-component:

$$X_{\infty,i} = X_\infty \otimes_{\mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})]} \mathbb{Z}_p(\omega^i).$$

Here $\mathbb{Z}_p(\omega^i)$ is isomorphic to $\mathbb{Z}_p$ as an abstract module but $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ acts on it by the character ω^i. It is known $X_{\infty,i}$ is a finitely generated torsion Λ-module and let $char_\Lambda(X_{\infty,i})$ be its characteristic ideal.

Now let γ_0 be a topological generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ so that

$$\kappa(\gamma_0) = 1 + p,$$

and φ an isomorphism

$$\Lambda = \mathbb{Z}_p[[\Gamma_\infty]] \overset{\varphi}{\simeq} \mathbb{Z}_p[[t]], \quad \varphi(\gamma_0) = 1 + t.$$

By these identification a character κ^s ($s \in \mathbb{Z}_p$) induces a homomorphism of algebra

$$\mathbb{Z}_p[[t]] \overset{\kappa^s}{\to} \mathbb{Z}_p$$

which is

$$\kappa^s(t) = \kappa^s(\gamma_0) - 1 = (1+p)^s - 1.$$

In particular, by (2.3), we obtain

$$\kappa^s(f(t, \omega^i)) = f((1+p)^s - 1, \omega^i) = L_p(s, \omega^i).$$

for an even integer i such that ω^i is nontrivial. Now we formulate Iwasawa Main Conjecture.

Conjecture 2.1. *Let i be an odd integer such that $i \neq 1 \pmod{p-1}$. Then the characteristic ideal $\mathrm{char}_\Lambda(X_{\infty,i})$ should be generated by $f(t, \omega^{1-i})$.*

The conjecture is first proved by Mazur and Wiles([12]). Today there is a much simpler proof which uses Kolyvagin's Euler system (e.g. [22] Chapter 15).

§3. Alexander invariant

For convenience of a reader, we will give a brief review of the general theory of the torsion of a complex. For a complete treatment of the theory see [15] or [14].

Let $\Lambda_\infty = \mathbb{C}[t, t^{-1}]$ be a Laurent polynomial ring of complex coefficients. The following lemma is easy to see.

Lemma 3.1. *Let f and g be elements of Λ_∞ such that*

$$f = ug,$$

where u is a unit. Then their order at $t = 1$ are equal:

$$\mathrm{ord}_{t=1} f = \mathrm{ord}_{t=1} g.$$

Let $(C., \partial.)$ be a bounded complex of free Λ_∞-modules of finite rank whose homology groups are torsion Λ_∞-modules. Suppose that it is given a base $\mathbf{c}_i$ for each C_i. Such a complex will refered as *a based complex.* We set

$$C_{even} = \oplus_{i \equiv 0(2)} C_i, \quad C_{odd} = \oplus_{i \equiv 1(2)} C_i,$$

which are free Λ_∞-modules of finite rank with basis $\mathbf{c}_{even} = \oplus_{i \equiv 0(2)} \mathbf{c}_i$ and $\mathbf{c}_{odd} = \oplus_{i \equiv 1(2)} \mathbf{c}_i$ respectively. Choose a base $\mathbf{b}_{even}$ of a Λ_∞-submodule B_{even} of C_{even} (necessary free) which is the image of the differential and column vectors $\mathbf{x}_{odd}$ of C_{odd} so that

$$\partial \mathbf{x}_{odd} = \mathbf{b}_{even}.$$

Similarly we take $\mathbf{b}_{odd}$ and $\mathbf{x}_{even}$ satisfying

$$\partial \mathbf{x}_{even} = \mathbf{b}_{odd}.$$

Then $\mathbf{x}_{even}$ and $\mathbf{b}_{even}$ are expressed by a linear combination of $\mathbf{c}_{even}$:

$$\mathbf{x}_{even} = X_{even} \mathbf{c}_{even}, \quad \mathbf{b}_{even} = Y_{even} \mathbf{c}_{even},$$

and we obtain a square matrix

$$\begin{pmatrix} X_{even} \\ Y_{even} \end{pmatrix}.$$

Similarly equations

$$\mathbf{x}_{odd} = X_{odd} \mathbf{c}_{odd}, \quad \mathbf{b}_{odd} = Y_{odd} \mathbf{c}_{odd}$$

yield a square matrix

$$\begin{pmatrix} X_{odd} \\ Y_{odd} \end{pmatrix}.$$

Now the *Milnor–Reidemeister torsion* $\tau_{\Lambda_\infty}(C., \mathbf{c}.)$ of the based complex $\{C., \mathbf{c}.\}$ is defined as (up to a sign)

$$\tau_{\Lambda_\infty}(C., \mathbf{c}.) = \pm \frac{\det \begin{pmatrix} X_{even} \\ Y_{even} \end{pmatrix}}{\det \begin{pmatrix} X_{odd} \\ Y_{odd} \end{pmatrix}} \tag{3.1}$$

It is known $\tau_{\Lambda_\infty}(C., \mathbf{c}.)$ is independent of a choice of $\mathbf{b}.$.

Since $H.(C.)$ are torsion Λ_∞-modules, they are finite dimensional complex vector spaces. Let τ_{i*} be the action of t on $H_i(C.)$. Then

Alexander invariant is defined to be the alternating product of their characteristic polynomials:

$$A_{C.}(t) = \prod_i \det[t - \tau_{i*}]^{(-1)^i}. \tag{3.2}$$

Then **Assertion 7** of [15] shows that the fractional ideals generated by $\tau_{\Lambda_\infty}(C., \mathbf{c}.)$ and $A_{C.}(t)$ are equal:

$$(\tau_{\Lambda_\infty}(C., \mathbf{c}.)) = (A_{C.}(t)).$$

In particular **Lemma 3.1** implies

$$\mathrm{ord}_{t=1}\tau_{\Lambda_\infty}(C., \mathbf{c}.) = \mathrm{ord}_{t=1}A_{C.}(t), \tag{3.3}$$

and we know

$$\tau_{\Lambda_\infty}(C., \mathbf{c}.) = \delta \cdot t^k A_{C.}(t),$$

where δ is a non-zero complex number and k is an integer. δ will be referred as *the difference* of Alexander invariant and Milnor–Reidemeister torsion.

Let $\{\overline{C.}, \overline{\partial}\}$ be a bounded complex of a finite dimensional vector spaces over $\mathbb{C}$. Given basis $\overline{\mathbf{c}}_i$ and $\overline{\mathbf{h}}_i$ for each $\overline{C_i}$ and $H_i(\overline{C.})$ respectively, the Milnor–Reidemeister torsion $\tau_{\mathbb{C}}(\overline{C.}, \overline{\mathbf{c}.})$ is also defined ([14]). Such a complex will be called *a based complex* again. By definition, if the complex is acyclic, it coincides with (3.1). Let $(C., \mathbf{c}.)$ be a based bounded complex over Λ_∞ whose homology groups are torsion Λ_∞-modules. Suppose its annihilator $Ann_{\Lambda_\infty}(H_i(C.))$ does not contain $t-1$ for each i. Then

$$(\overline{C.}, \overline{\partial}) = (C., \mathbf{c}.) \otimes_{\Lambda_\infty} \Lambda_\infty/(t-1)$$

is a based acyclic complex over $\mathbb{C}$ with a preferred base $\overline{\mathbf{c}.}$ which is the reduction of $\mathbf{c}.$ modulo $(t-1)$. This observation shows the following proposition.

Proposition 3.1. *Let $(C., \mathbf{c}.)$ be a based bounded complex over Λ_∞ whose homology groups are torsion Λ_∞-modules. Suppose the annihilator $Ann_{\Lambda_\infty}(H_i(C.))$ does not contain $t-1$ for each i. Then we have*

$$\tau_{\Lambda_\infty}(C., \mathbf{c}.)|_{t=1} = \tau_{\mathbb{C}}(\overline{C.}, \overline{\mathbf{c}.})$$

For a later purpose we will consider these dual.

Let $\{C^{\cdot}, d\}$ be the dual complex of $\{C., \partial\}$:

$$(C^{\cdot}, d) = Hom_{\Lambda_\infty}((C., \partial), \Lambda_\infty).$$

By the universal coefficient theorem we have

$$H^q(C^{\cdot}, d) = Ext^1_{\Lambda_\infty}(H_{q-1}(C_{\cdot}, \partial), \Lambda_\infty)$$

and the cohomology groups are torsion Λ_∞-modules. Moreover the characteristic polynomial of $H^q(C^{\cdot}, d)$ is equal to one of $H_{q-1}(C_{\cdot}, \partial)$. Thus if we define *Alexander invariant* $A_{C^{\cdot}}(t)$ of $\{C^{\cdot}, d\}$ by the same way as (3.2), we have

$$A_{C^{\cdot}}(t) = A_{C_{\cdot}}(t)^{-1}. \tag{3.4}$$

Let us apply these arguments to a threefold. A detailed proof of theorems will be found in [20].

In general let X be a connected finite CW-complex and $\{c_{i,\alpha}\}_\alpha$ its i-dimensional cells. We will fix its base point x_0 and let Γ be the fundamental group of X. Let ρ be a unitary representation of finite rank and let V_ρ be its representation space. Suppose that there is a surjective homomorphism

$$\Gamma \xrightarrow{\epsilon} \mathbb{Z},$$

and let X_∞ be the infinite cyclic covering of X which corresponds to $\operatorname{Ker}\epsilon$ by the Galois theory. Finally let $\tilde{X}$ be the universal covering of X.

The chain complex $(C_{\cdot}(\tilde{X}), \partial)$ is a complex of free $\mathbf{C}[\Gamma]$-module of finite rank. We take a lift of $\mathbf{c}_i = \{c_{i,\alpha}\}_\alpha$ as a base of $C_i(\tilde{X})$, which will be also denoted by the same character. Note that such a choice of base has an ambiguity of the action of Γ.

Following [7] consider a complex over $\mathbb{C}$:

$$C_i(X, \rho) = C_i(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} V_\rho.$$

On the other hand, restricting ρ to $\operatorname{Ker}\epsilon$, we will make a chain complex

$$C_{\cdot}(X_\infty, \rho) = C_{\cdot}(\tilde{X}) \otimes_{\mathbb{C}[\operatorname{Ker}\epsilon]} V_\rho,$$

which has the following description. In the following we will fix an isomorphism between $\mathbb{C}[\mathbb{Z}]$ and Λ_∞ which sends the generator 1 of $\mathbb{Z}$ to t and will identify them. Let us regard $\mathbb{C}[\mathbb{Z}] \otimes_{\mathbb{C}} V_\rho$ as Γ-module by

$$\gamma(p \otimes v) = p \cdot t^{\epsilon(\gamma)} \otimes \rho(\gamma) \cdot v, \quad p \in \mathbb{C}[\mathbb{Z}],\, v \in V_\rho.$$

Then $C_{\cdot}(X_\infty, \rho)$ is isomorphic to a complex ([7] **Theorem 2.1**):

$$C_{\cdot}(X, V_\rho[\mathbb{Z}]) = C_{\cdot}(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} (\mathbb{C}[\mathbb{Z}] \otimes_{\mathbb{C}} V_\rho).$$

and we know $C_\cdot(X_\infty, \rho)$ is a bounded complex of free Λ_∞-modules of finite rank. We will fix a unitary base $\mathbf{v} = \{v_1, \cdots, v_m\}$ of V_ρ and make it a based complex with a preferred base $\mathbf{c}_\cdot \otimes \mathbf{v} = \{c_{i,\alpha} \otimes v_j\}_{\alpha,i,j}$.

By the surjection:

$$\Lambda_\infty \to \Lambda_\infty/(t-1) \simeq \mathbb{C},$$

$C_\cdot(X_\infty, \rho) \otimes_{\Lambda_\infty} \mathbb{C}$ is isomorphic to $C_\cdot(X, \rho)$. Moreover if we take $\mathbf{c}_\cdot \otimes \mathbf{v}$ as a base of the latter, they are isomorphic as based complexes.

Let $C^\cdot(\tilde{X})$ be the cochain complex of $\tilde{X}$:

$$C^\cdot(\tilde{X}) = Hom_{\mathbb{C}[\Gamma]}(C_\cdot(\tilde{X}), \mathbb{C}[\Gamma]),$$

which is a bounded complex of free $\mathbb{C}[\Gamma]$-module of finite rank. For each i we will take the dual $\mathbf{c}^i = \{c^i_\alpha\}_\alpha$ of $\mathbf{c}_i = \{c_{i,\alpha}\}_\alpha$ as a base of $C^i(\tilde{X})$. Thus $C^\cdot(\tilde{X})$ becomes a based complex with a preferred base $\mathbf{c}^\cdot = \{\mathbf{c}^i\}_i$. Since ρ is a unitary representation, it is easy to see that the dual complex of $C_\cdot(X_\infty, \rho)$ is isomorphic to

$$C^\cdot(X_\infty, \rho) = C^\cdot(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} (\Lambda_\infty \otimes_{\mathbb{C}} V_\rho),$$

if we twist its complex structure by the complex conjugation. Also we will make it a based complex by the base $\mathbf{c}^\cdot \otimes \mathbf{v} = \{c^i_\alpha \otimes v_j\}_{\alpha,i,j}$.

Dualizing the exact sequence

$$0 \to C_\cdot(X_\infty, \rho) \stackrel{t-1}{\to} C_\cdot(X_\infty, \rho) \to C_\cdot(X, \rho) \to 0$$

in the derived category of bounded complex of finitely generated Λ_∞-modules, we will obtain a distinguished triangle:

$$C^\cdot(X, \rho) \to C^\cdot(X_\infty, \rho) \stackrel{t-1}{\to} C^\cdot(X_\infty, \rho) \to C^\cdot(X, \rho)[1] \to . \tag{3.5}$$

Here we set

$$C^\cdot(X, \rho) = C^\cdot(\tilde{X}, \rho) \otimes_{\mathbb{C}[\Gamma]} V_\rho.$$

and for a bounded complex $C^\cdot$, $C^\cdot[n]$ denotes its *shift*, namely

$$C^i[n] = C^{i+n}.$$

Note that $C^\cdot(X, \rho)$ is isomorphic to the reduction of $C^\cdot(X_\infty, \rho)$ modulo $(t-1)$.

Let τ^* be the action of t on $H^{\cdot}(X_\infty, \rho)$. Then (3.5) induces an exact sequence:

$$(3.6) \quad \to H^q(X, \rho) \to H^q(X_\infty, \rho) \overset{\tau^*-1}{\to} H^q(X_\infty, \rho) \to H^{q+1}(X, \rho) \to .$$

In the following, we will assume that the dimension of X is three and that all $H_{\cdot}(X_\infty, \mathbb{C})$ and $H_{\cdot}(X_\infty, \rho)$ are finite dimensional vector spaces over $\mathbb{C}$. The arguments of §4 of [15] will show the following theorem.

Theorem 3.1. *([15])*

(1) *For $i \geq 3$, $H^i(X_\infty, \rho)$ vanishes.*

(2) *For $0 \leq i \leq 2$, $H^i(X_\infty, \rho)$ is a finite dimensional vector space over $\mathbb{C}$ and there is a perfect pairng:*

$$H^i(X_\infty, \rho) \times H^{2-i}(X_\infty, \rho) \to \mathbb{C}.$$

The perfect pairing will be referred as *Milnor duality.*

Let $A_\rho(t)$ and $\check{A}_\rho(t)$ be Alexander invariants of $C_{\cdot}(X_\infty, \rho)$ and $C^{\cdot}(X_\infty, \rho)$ respectively. Since the latter complex is the dual of the previous one, (3.4) implies

$$\check{A}_\rho(t) = A_\rho(t)^{-1}.$$

Let $\check{\tau}_{\Lambda_\infty}(X_\infty, \rho)$ be Milnor–Reidemeister torsion of $C^{\cdot}(X_\infty, \rho)$ with respect to a preferred base $\mathbf{c}^{\cdot} \otimes \mathbf{v}$. Because of an ambiguity of a choice of $\mathbf{c}^{\cdot}$ and $\mathbf{v}$, it is well-defined modulo

$$\{zt^n \mid z \in \mathbb{C}, \, |z| = 1, \, n \in \mathbb{Z}\}.$$

Let $\delta_\rho = |\check{A}_\rho(t) - \check{\tau}_{\Lambda_\infty}(X_\infty, \rho)|$ be the absolute value of the difference between $\check{A}_\rho(t)$ and $\check{\tau}_{\Lambda_\infty}(X_\infty, \rho)$. The previous discussion of the torsion of a complex implies the following theorem.

Theorem 3.2. *The order of $\check{\tau}_{\Lambda_\infty}(X_\infty, \rho)$, $\check{A}_\rho(t)$ and $A_\rho(t)^{-1}$ at $t = 1$ are equal. Let β be the order. Then we have*

$$\begin{aligned} \lim_{t\to 1} |(t-1)^{-\beta}\check{\tau}_{\Lambda_\infty}(X_\infty, \rho)| &= \delta_\rho \lim_{t\to 1} |(t-1)^{-\beta}\check{A}_\rho(t)| \\ &= \delta_\rho \lim_{t\to 1} |(t-1)^{-\beta}A_\rho(t)^{-1}|. \end{aligned}$$

By **Theorem 3.1** we see that Alexander invariant becomes

$$(3.7) \qquad \check{A}_\rho(t) = \frac{\det[t - \tau^* \mid H^0(X_\infty, \rho)] \cdot \det[t - \tau^* \mid H^2(X_\infty, \rho)]}{\det[t - \tau^* \mid H^1(X_\infty, \rho)]}.$$

Suppose $H^0(X_\infty,\, \rho)$ vanishes. Then Milnor duality implies

$$A_\rho(t) = \check{A}_\rho(t)^{-1} = \det[t - \tau^* \mid H^1(X_\infty,\, \rho)],$$

which is a generator of the characteristic ideal of $H^1(X_\infty,\, \rho)$. Thus if we think X_∞ corresponds to the $\mathbb{Z}_p$-extension of $\mathbb{Q}$, a similarity between the $char_\Lambda(X_{\infty,i})$ and the ideal generated by Alexander invariant is clear.

Suppose that $C_\cdot(X,\rho)$ is acyclic, which implies that so is $C^\cdot(X,\rho)$. Let $\tau(X,\rho)$ and $\check{\tau}(X,\rho)$ be the modulus of Milnor–Reidemeister torsion of complexes $C_\cdot(X,\rho)$ and $C^\cdot(X,\rho)$ with respect to the preferred basis, respectively. Note that these are independent of a choice of a unitary basis $\mathbf{v}$. By the universal coefficient theorem as before, we have

$$\check{\tau}(X,\rho) = \tau(X,\rho)^{-1}. \tag{3.8}$$

Choosing basis $\mathbf{h}_i$ and $\mathbf{h}^i$ of $H_i(X,\rho)$ and $H^i(X,\rho)$ respectively so that they are dual to each other, (3.8) also holds a non-acyclic complex. Let $h^i(\rho)$ be the dimension of $H^i(X,\, \rho)$. A standard argument shows the following theorem.

Theorem 3.3. *Suppose $H^0(X_\infty,\, \rho)$ vanishes. Then we have*

$$h^1(\rho) \leq \mathrm{ord}_{t=1} A_\rho(t) = -\mathrm{ord}_{t=1} \check{A}_\rho(t),$$

and the identity holds if the action of τ^* on $H^1(X_\infty,\, \rho)$ is semisimple.

Theorem 3.4. *Suppose $H^i(X,\, \rho)$ vanishes for all i. Then we have*

$$\tau(X,\, \rho)^{-1} = \check{\tau}(X,\, \rho) = \delta_\rho |\check{A}_\rho(1)| = \frac{\delta_\rho}{|A_\rho(1)|}.$$

Proof. The exact sequence (3.6) and the assumption implies $t-1$ is not contained in the annihilator of $H^\cdot(X_\infty,\, \rho)$. Now the theorem will follow from **Proposition 3.1** and **Theorem 3.2**. Q.E.D.

If X is a mapping torus, we obtain a finer information of the absolute value of the leading term of Alexander invariant. A proof of the following theorem is essentially contained in [1] or [3].

Theorem 3.5. *Let f be an automorphism of a connected finite CW-complex of dimension two S and X its mapping torus. Let ρ be a unitary representation of the fundamental group of X which satisfies $H^0(S,\, \rho) = 0$. Suppose that the surjective homomorphism*

$$\Gamma \xrightarrow{\epsilon} \mathbb{Z}$$

is induced by the structure map

$$X \to S^1,$$

and that the action of f^* *on* $H^1(S,\, \rho)$ *is semisimple. Then the order of* $A_\rho(t)$ *is* $h^1(\rho)$ *and*

$$\lim_{t\to 1} |(t-1)^{-h^1(\rho)} A_\rho(t)| = \tau(X,\rho).$$

Thus we know that $\tau(X,\rho)$ is determined by the homotopy class of f. As before without semisimplicity of f^*, we only have

$$\mathrm{ord}_{t=1} A_\rho(t) \geq h^1(\rho).$$

Let X is the complement of a knot K in S^3 and let ρ be a unitary representation of its fundamental group. Since, by Alexander duality, $H_1(X,\, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and X admits an infinite cyclic covering X_∞. Suppose $H_i(X_\infty,\, \rho)$ are finite dimensional complex vector spaces for all i. Then our twisted Alexander invariant $A_\rho(t)$ is essentially the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ defined by Kitano [8]. More precisely

Theorem 3.6. *Suppose* $H^0(X_\infty,\, \rho)$ *vanishes. Then we have*

$$\mathrm{ord}_{t=1}\Delta_{K,\rho}(t) = \mathrm{ord}_{t=1} A_\rho(t) \geq h^1(\rho),$$

and the identity holds if the action of τ^* *on* $H^1(X_\infty,\, \rho)$ *is semisimple. Moreover suppose* $H^i(X,\, \rho)$ *vanishes for all* i*. Then*

$$\tau(X,\, \rho) = |\Delta_{K,\rho}(1)|.$$

§4. Ruelle–Selberg L-function

In this section we will introduce Ruelle–Selberg L-function which corresponds to the p-adic zeta function in Iwasawa Main Conjecture.

Let X be a hyperbolic threefold of finite volume, which is a quotient of the Poincaré upper half space $\mathbb{H}^3$ by a torsion free discrete subgroup Γ_g of $PSL_2(\mathbb{C})$. By a well-known natural bijection between the set of closed geodesics and one of hyperbolic conjugacy classes $\Gamma_{g,conj}$, we will identify them. Using this identification, $l(\gamma)$ of $\gamma \in \Gamma_{g,conj}$ is defined to be the length of the corresponding closed geodesic.

Let ρ be a unitary representation of rank r. Then Ruelle–Selberg L-function is by definition

$$R_X(z,\rho) = \prod_{\gamma} \det[1 - \rho(\gamma)e^{-zl(\gamma)}]^{-1},$$

where γ runs over prime closed geodesics, i.e. not a positive multiple of another one. It is a general fact that Ruelle–Selberg L-function absolutely converges if $\mathrm{Re}\, z$ is greater one. Suppose X is compact. The following theorem is a special case of [4]**Theorem 3**.

Fact 4.1. *Ruelle–Selberg L-function is meromorphically continued to the whole plane and its order at $z = 0$ is*

$$e = 2h^1(\rho) - 4h^0(\rho).$$

Moreover if $h^0(\rho) = h^1(\rho) = 0$,

$$R_X(0,\rho) = \tau(X,\rho)^2.$$

Even though $e \neq 0$ he has also computed the leading coefficient of $R_X(z,\rho)$ to be the square of Ray–Singer analytic torsion. If the complex $C_\cdot(X,\rho)$ is acyclic Cheeger and Muller's theorem implies the above. Fried has shown his results for an orthogonal representation, but his proof is still valid for a unitary one.

We will generalize his result for a complete hyperbolic threefold with finite cusps $\{\infty_1, \cdots, \infty_h\}$. For a cusp ∞_ν there corresponds to a Borel subgroup B_ν of $\mathrm{P}SL_2(\mathbb{C})$. Let $\Gamma_{g,\nu}$ be the intersection of Γ with B_ν. Since Γ_g is torsion free $\Gamma_{g,\nu}$ is a free abelian group of rank two and therefore the restriction of ρ to $\Gamma_{g,\nu}$ becomes a direct sum of unitary characters, $\chi_{\nu,1}, \cdots, \chi_{\nu,r}$. We say that ρ is *cuspidal* if none of $\{\chi_{\nu,i}\}_{\nu,i}$ is trivial, which will be assumed in the following. Note that this implies $H^0(X,\rho) = 0$. The theorem below is proved in [21].

Theorem 4.1. *$R_X(z,\rho)$ is meromorphically continued to the whole plane and has a zero at the origin of order $2h^1(\rho)$. Moreover*

$$\lim_{z \to 0} z^{-2h^1(\rho)} R_X(z,\rho) = (\tau(X,\rho) \cdot \mathrm{Per}(X,\rho))^2.$$

Remark 4.1. *Using the result of Fried the argument of* [21] *shows the same equation holds for an arbitrary unitary local system on a compact hyperbolic threefold.*

Here $\mathrm{Per}(X,\rho)$ is the period of (X,ρ), roughly speaking, which measures a difference between unitary structures on $H^1(X,\rho)$ induced by analytic Hodge theory and combinatrical one. It is defined to be one if $H^1(X,\rho)$ vanishes. See [21] for precise definition. If $h^1(\rho)$ vanishes our theorem implies a following generalization of **Fact 4.1**.

Corollary 4.1. *Suppose* $h^1(\rho) = 0$. *Then*

$$R_X(0,\rho) = \tau(X,\rho)^2.$$

Here are some arguments on our results. Let

$$R_X(z,\rho) = c_0 z^h (1 + c_1 z + \cdots), \quad c_0 \neq 0$$

be Taylor expansion at the origin. **Theorem 4.1** implies

$$h = 2h^1(\rho),$$

and

$$c_0 = (\tau(X,\rho) \cdot \mathrm{Per}(X,\rho))^2.$$

In [21] we have computed

$$c_1 = -\frac{3r \cdot vol(X)}{\pi},$$

where $vol(X)$ is the volume of X. In particular if $h^1(\rho)$ vanishes we have

$$\log R_X(0,\rho) = 2\log\tau(X,\rho) \tag{4.1}$$

and

$$\frac{d}{dz}\log R_X(z,\rho)|_{z=0} = -\frac{3r \cdot vol(X)}{\pi}. \tag{4.2}$$

Both RHS can be interpreted as *a period* of a certain element of K-group of $\mathbb{C}$. In fact following Milnor (resp. Dupont–Sah[2] for a compact manifold and Goncharov[5] for a non-compact one) we can construct an element $\mu(X,\rho)$ (resp. γ_X) of $K_1(\mathbb{C})$ (resp. $K_3(\mathbb{C}) \otimes \mathbb{Q}$) whose image by Borel regulator map

$$K_{2n+1}(\mathbb{C}) \otimes \mathbb{Q} \overset{\gamma_{n+1}}{\to} \mathbb{R}$$

is $\frac{1}{2\pi}\log\tau(X,\rho)$ (resp. $vol(X)$). Now we want to mention that (4.1) and (4.2) may be compared to Lichtenbaum conjecture for a Dedekind zeta function $\zeta_F(s)$ of an algebraic number field F, which is a generalization of the class number formula [10]. Lichtenbaum observed that for an integer l greater than one the order of $\zeta_F(s)$ at $1-l$ is equal to the rank d_l of $K_{2l-1}(F)$. It leads him to conjecture that the limit

$$\lim_{s \to 1-l} (s+l-1)^{-d_l} \zeta_F(s)$$

should be equal to the covolume of a certain map

$$K_{2l-1}(F) \xrightarrow{r_l} \mathbb{R}^{d_l}.$$

Such a map has been constructed by Borel and is also referred as *Borel regulator.*

§5. A geometric analog of Iwasawa Main Conjecture

Let X be a complete hyperbolic threefold of finite volume which admits an infinite cyclic covering X_∞ and let g be a generator of the group $\mathrm{Gal}(X_\infty/X)$. Let ρ be a unitary representation of the fundamental group of X and we will always assume that the pair (X_∞, ρ) satisfies the assumption of Milnor duality.

Since $H^0(X, \rho)$ is a subspace of $H^0(X_\infty, \rho)$, **Theorem 3.3**, **Theorem 3.4**, **Theorem 3.5**, **Theorem 3.6**, **Fact 4.1** and **Corollary 4.1** imply the following theorem.

Theorem 5.1. *Suppose that $H^0(X_\infty, \rho)$ vanishes and that X and ρ satisfy one of the following conditions:*

(1) *X is compact.*
(2) *ρ is cuspidal.*

Then

$$\mathrm{ord}_{z=0} R_X(z, \rho) = 2h^1(\rho) \leq 2\mathrm{ord}_{t=1} A_\rho(t),$$

and the identity holds if the action of g on $H^1(X_\infty, \rho)$ is semisimple. If all $H^i(X, \rho)$ vanish, we have

$$R_X(0, \rho) = \delta_\rho^{-2} |A_\rho(1)|^2 = |\Delta_{K,\rho}(1)|^2 = \tau(X, \rho)^2.$$

Moreover **Theorem 4.1** and its remark imply the following.

Theorem 5.2. *Suppose that X is homeomorphic to a mapping torus of an automorphism of a CW-complex S of dimension two and that the surjective homomorphism from the fundamental group to $\mathrm{Gal}(X_\infty/X) \simeq \mathbb{Z}$ is induced by the structure map:*

$$X \to S^1.$$

Moreover suppose that $H^0(S,\rho)$ vanishes and that (X,ρ) satisfies one of the condition 1 or 2 in **Theorem 5.1**. *If the action of g on $H^1(S,\rho)$ is semisimple, we have*

$$\lim_{z\to 0} z^{-2h^1(\rho)}R_X(z,\rho)/\mathrm{Per}(X,\rho)^2 = \lim_{t\to 1}|(t-1)^{-h^1(\rho)}A_\rho(t)|^2 = \tau(X,\rho)^2.$$

Under an assumption that the action of g is semisimple, by a change of variables,

$$z = t - 1,$$

we have seen that ideals $(R_\rho(z))$ and $(A_\rho(z)^2)$ of $\mathbb{C}[[z]]$ are coincide if X is compact or if ρ is cuspidal. Thus we see that a geometric analog of Iwasawa Main Conjecture holds for a unitary representaion of the fundamental group. Notice that, as we have explained in the introduction, our analogue is much weaker than the original one. Moreover according to a computation due to Park [18] the two ideals are different for a non-cuspidal unitary local system. In number theory, it is known that such a phenominon also occurs for a p-adic L-function of an elliptic curve defined over $\mathbb{Q}$ which has split multiplicative reducion at p [13].

References

[1] C. Deninger, A dynamical systems analogue of Lichtenbaum's conjecture on special values of Hasse–Weil zeta functions, aeXiv:math/0605724v2, May 2007.

[2] J. L. Dupont, Scissors Congruence, Group Homology and Characteristic Classes, Nankai Tracts Math., **1**, World Scientific, 2001.

[3] D. Fried, Homological identities for closed orbits, Invent. Math., **71** (1983), 419–442.

[4] D. Fried, Analytic torsion and closed geodesics on hyperbolic manifolds, Invent. Math., **84** (1986), 523–540.

[5] A. Goncharov, Volume of hyperbolic manifolds and mixed Tate motives, J. Amer. Math. Soc., **12** (1999), 569–618.

[6] M. Kapranov, A. Reznikov and P. Moree, Arithmetic Topology, lecture at hauptseminar, Max-Planck-Institute, August 1996.

[7] P. Kirk and C. Livingston, Twisted Alexander invarinants, Reidemeister torsion, and Casson–Gordon invariants, Topology, **38** (1999), 635–661.

[8] T. Kitano, Twisted Alexander polynomial and Reidemeister torsion, Pacific J. Math., **174** (1996), 431–442.

[9] S. Lang, Cyclotomic fields I and II, Grad. Texts in Math., **121**, Springer-Verlag, 1990.

[10] S. Lichtenbaum, Value of zeta functions, étale cohomology and algebraic K-theory, In: Algebraic K-theory II, Lecture Notes in Math., **342**, Springer-Verlag, 1973, pp. 489–501.

[11] B. Mazur, E-mail to Morishita, on his unpublished mimeographed note (circa 1965), October 17 2000.

[12] B. Mazur and A. Wiles, Class fields of abelian extensions of Q, Invent. Math., **76** (1984), 179–330.

[13] B. Mazur, J. Tate and J.Teitelbaum, On p-adic analogues of the conjectures of Birch and Swinnerton–Dyer, Invent. Math., **84** (1986), 1–48.

[14] J. Milnor, Whitehead torsion, Bull. Amer. Math. Soc., **72** (1966), 358–426.

[15] J. Milnor, Infinite cyclic coverings, In: Conference on the Topology of Manifolds, (ed. J. G. Hocking), PWS Publishing Company, 1968, pp. 115–133.

[16] M. Morishita, On certain analogies between knots and primes, J. Reine Angew. Math., **550** (2002), 141–167.

[17] M. Morishita, Milnor invariants and Massey products for prime numbers, Compos. Math., **140** (2004), 69–83.

[18] J. Park, Analytic torsion and closed geodesics for hyperbolic manifolds with cusps, preprint, February 2008.

[19] A. Reznikov, Three-manifolds class field theory, Selecta Math. (N.S.), **3** (1997), 361–399.

[20] K. Sugiyama, An analog of the Iwasawa conjecture for a compact hyperbolic threefold, J. Reine Angew. Math., **613** (2008), 35–50.

[21] K. Sugiyama, The Taylor expansion of Ruelle L-function at the origin and the Borel regulator, preprint, April 2008.

[22] L. Washington, Introduction to cyclotomic fields, Grad. Texts in Math., **83**, Springer-Verlag, 1982.

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