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Iterating the hessian: a dynamical system on the moduli space of elliptic curves and dessins d'enfants

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Abstract.

Each elliptic curve can be embedded uniquely in the projective plane, up to projective equivalence. The hessian curve of the embedding is generically a new elliptic curve, whose isomorphism type depends only on that of the initial elliptic curve. One gets like this a rational map from the moduli space of elliptic curves to itself. We call it *the hessian dynamical system.* We compute it in terms of the j-invariant of elliptic curves. We deduce that, seen as a map from a projective line to itself, it has 3 critical values, which correspond to the point at infinity of the moduli space and to the two elliptic curves with special symmetries. Moreover, it sends the set of critical values into itself, which shows that all its iterates have the same set of critical values. One gets like this a sequence of dessins d'enfants. We describe an algorithm allowing to construct this sequence.

§1. Introduction

Consider a complex projective plane $\mathbb{P}^2$ with homogeneous coordinates $[x : y : z]$.

To each homogeneous polynomial $f \in \mathbb{C}[x, y, z]$ one can associate the corresponding *hessian polynomial* $\mathrm{Hess}(f)$, defined as the determinant:

$$\mathrm{Hess}(f) := \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}. \tag{1}$$

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If $C := Z(f)$ denotes the projective curve defined by the homogeneous polynomial f and $\mathrm{Hess}(C)$ denotes the curve defined by $\mathrm{Hess}(f)$, one knows that the intersection points of C and $\mathrm{Hess}(C)$ are exactly the inflection points of C, which shows that this set has an invariant meaning: it depends only on the couple $(\mathbb{P}^2, C)$, and not on the chosen defining homogeneous polynomial. But more is true: *the whole curve* $\mathrm{Hess}(C)$ *is invariantly attached to* $(\mathbb{P}^2, C)$. This remark was the starting point of the present study.

If $\deg(f) = n \geq 1$, then $\deg(\mathrm{Hess}(f)) = 3(n-2)$. This shows that C and $\mathrm{Hess}(C)$ have the same degree only when $n = 3$. Restricting to this case, we get a map from the space of projective plane cubic curves to itself, which is equivariant with respect to the action of the group of projectivities of $\mathbb{P}^2$. This shows that the map $C \to \mathrm{Hess}(C)$ descends to a rational map H from the quotient of the space of smooth cubic curves by the group of projectivities to itself. But this quotient space is *the coarse moduli space* $\mathcal{M}_1$ of elliptic curves (see Edidin [4]). The previous map extends to the compactification $\overline{\mathcal{M}_1} \simeq \mathbb{P}^1$. I propose:

Definition 1.1. *The algebraic map $H : \overline{\mathcal{M}_1} \to \overline{\mathcal{M}_1}$ which associates to each elliptic curve the isomorphism type of the hessian of a smooth plane cubic curve corresponding to it is called* ***the hessian dynamical system.***

The aim of this paper is to compute the hessian dynamical system and to start its dynamical study. I believe that considering it could bring new insights into the theory of elliptic curves. One could examine for example the relation between dynamically defined subsets of $\mathcal{M}_1$ and the various subsets with arithmetical meaning. More importantly, I believe that similar considerations related to higher dimensional classical invariant theory would allow to construct higher dimensional dynamical systems with special properties.

Let me describe briefly the content of the paper. In Section 2 are recalled various normal forms for plane cubic curves and for each one of them, the expression of the classical j-invariant. In Section 3 are computed the expression of the hessian dynamical system H in terms of the j-invariant (see Theorem 3.4). In Section 4 is showed that all the iterates of H have 3 critical values (see Proposition 4.5), which allows to introduce an associated sequence of dessins d'enfants $(\Gamma_n)_{n\geq 1}$. Finally, in Section 5, is given an algorithm which allows to construct up to topological conjugacy the sequence of preimages of the real axis by the iterates of H. In particular, one gets a sequence of graphs in which the sequence of dessins d'enfants introduced before embeds canonically, which gives us an algorithm for constructing the sequence $(\Gamma_n)_{n\geq 1}$ (see Proposition 5.1).

While this paper was refereed, I learned that Pilgrim [10] had studied in general the relation between complex dynamics in dimension one and dessins d'enfants and that Artebani & Dolgachev have surveyed in [1] the classical geometry of the Hesse pencil of cubics. In Remark 3.5 of this last paper, they notice that the map $\tilde{H}$ (see formula (6)) is an "interesting example of complex dynamics in one complex variable", and that it was studied from this view-point by Hollcroft [7]. See also Remark 5.2 for other relations with the litterature.

§2. The j-invariant of an elliptic curve

The classical j-invariant of an elliptic curve may be defined as follows (see Hartshorne [6, IV.4]):

Definition 2.1. *Denote by E_λ the elliptic curve birationally equivalent to the plane affine cubic with equation:*

$$y^2 = x(x-1)(x-\lambda).$$

*Then its j-**invariant** is defined by:*

$$j(E_\lambda) := 2^8 \cdot \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}.$$

The previous expression is adapted to the computation of the j-invariant of an elliptic curve seen as the double cover of a projective line $\mathbb{P}^1$, ramified over 4 distinct points, the cross-ratio of those points being λ. Indeed, E_λ is the total space of that covering over the complement of one of the 4 points of $\mathbb{P}^1$.

If an elliptic curve is presented in Weierstarss normal form, one has the following expression for its j-invariant (see Hartshorne [6, page 327]):

Proposition 2.2. *Denote by E_{g_2,g_3} the elliptic curve birationally equivalent to the smooth plane affine curve with equation:*

$$y^2 = 4x^3 - g_2 x - g_3,$$

where $g_2^3 - 27g_3^2 \neq 0$. Then its j-invariant is given by:

$$j(E_{g_2,g_3}) = 1728 \cdot \frac{g_2^3}{g_2^3 - 27g_3^2}.$$

In the sequel, we will work rather with the following normal form, for reasons explained in the next section:

$$X_0^3 + X_1^3 + X_2^3 - 3m\, X_0X_1X_2 = 0. \tag{2}$$

Denote by C_m the plane projective cubic curve defined by equation (2). As it is not easy to find a reference for the following proposition, we add an elementary proof.

Proposition 2.3. *The j-invariant of the elliptic curve C_m is given by:*

$$j(C_m) = 27 \cdot \left(\frac{m(m^3+8)}{m^3-1}\right)^3.$$

Proof. Define :

$$J(C_m) := \left(\frac{m(m^3+8)}{m^3-1}\right)^3. \tag{3}$$

By Brieskorn & Knörrer [3, page 302], we know that $J(C_m)$ and $j(C_m)$ are proportional, that is, there exists $t \in \mathbb{C}$ such that:

$$J(C_m) = t \cdot j(C_m). \tag{4}$$

In order to find t, it is enough to specialize (4) to a cubic curve C_m for which one knows how to compute both J and j. This is possible if one knows how to write C_m in Weierstrass normal form by a coordinate change.

As the projectivisation of an affine cubic in Weierstrass normal form has the property that the line at infinity is tangent to it at an inflection point, we naturally begin by choosing as line at infinity a tangent to C_m at an inflection point.

We choose the inflection point $(1 : -1 : 0) \in C_m$. The tangent to C_m at this point has the equation:

$$X_0 + X_1 + mX_2 = 0.$$

We make then the following change of projective coordinates:

$$\begin{cases} Z = X_0 + X_1 + mX_2 \\ U = \frac{1}{2}(X_0 + X_1) \\ V = \frac{1}{2}(X_0 - X_1) \end{cases} \Longleftrightarrow \begin{cases} X_0 = U + V \\ X_1 = U - V \\ X_2 = \frac{1}{m}(Z - 2U) \end{cases}.$$

The equation (2) is transformed in:

$$8U^3 + \frac{1}{m^3}(Z - 2U)^3 - 3(U^2 - V^2)Z = 0.$$

By passing to the affine coordinates $u = U/Z$, $v = V/Z$, we find the affine equation:

$$v^2 = \frac{8}{3}(\frac{1}{m^3} - 1)u^3 + (1 - \frac{4}{m^3})u^2 + \frac{2}{m^3}u - \frac{1}{3m^3}.$$

We specialize now to $m = \sqrt[3]{4}$. After the new change of variables $u = -2u_1$, $v = 2v_1$, the previous equation becomes:

$$v_1^2 = 4u_1^3 - \frac{1}{4}u_1 - \frac{1}{48}.$$

This shows that:

$$C_{\sqrt[3]{4}} \simeq E_{\frac{1}{4}, \frac{1}{48}}.$$

Combining this with relation (4), we get:

$$J(C_{\sqrt[3]{4}}) = t \cdot j(E_{\frac{1}{4}, \frac{1}{48}}).$$

From Proposition 2.2 and equation (3), we deduce that $t = \frac{1}{27}$. Q.E.D.

§3. Computation of the hessian dynamical system

Consider again the 1-dimensional linear system of plane projective cubics defined by the equation (2).

The parameter $m \in \mathbb{C}$ is seen as an affine coordinate of the projective line parametrizing the cubics of the pencil, the homogeneous coordinates being $[1 : m]$. To $m = \infty$ corresponds the cubic with equation $X_0X_1X_2 = 0$, the union of the edges of the fundamental triangle in the projective plane with the fixed homogeneous coordinates.

An immediate computation shows that :

$$\text{Hess}(C_m) = C_{\frac{4-m^3}{3m^2}}. \tag{5}$$

Remark 3.1. *Equation (5) shows that the hessian curve of a cubic expressed in the normal form (2) is again a cubic of the same normal form in the same system of homogeneous coordinates. If one had started instead from the expressions of Definition 2.1 or of Proposition 2.2, one wouldn't have got expressions of the same normal form. This is the reason why we preferred to work with the normal form (2).*

One gets like this a dynamical system $\tilde{H} : \mathbb{P}^1_{[1:m]} \to \mathbb{P}^1_{[1:m]}$, expressed by :

$$(6) \qquad \tilde{H}(m) = \frac{4 - m^3}{3m^2}$$

where $\mathbb{P}^1_{[1:m]} := \mathbb{A}^1_m \cup \{\infty\}$ denotes the projective line obtained by adding one point at ∞ to the affine line with coordinate m.

One has the following relation between the maps $\tilde{H} : \mathbb{P}^1_{[1:m]} \to \mathbb{P}^1_{[1:m]}$, $H : \mathbb{P}^1_{[1:j]} \to \mathbb{P}^1_{[1:j]}$, $j : \mathbb{P}^1_{[1:m]} \to \mathbb{P}^1_{[1:j]}$:

$$H \circ j = j \circ \tilde{H}.$$

Using Proposition 2.3, it can be rewritten more explicitly as :

$$(7) \qquad H\left(27\left(\frac{m(m^3+8)}{m^3-1}\right)^3\right) = 27\left(\frac{\tilde{H}(m)(\tilde{H}(m)^3+8)}{\tilde{H}(m)^3-1}\right)^3.$$

By specializing the previous equality at $m \in \{0, 1, 4^{\frac{1}{3}}\}$, we see that $H(0) = H(\infty) = \infty$ and $H(2^8 \cdot 3^3) = 0$. Moreover, when we see it as a holomorphic map from $\mathbb{P}^1_{[1:m]}$ to itself, $\tilde{H}$ is of degree 3, which implies that H is of degree 3. As an immediate consequence of these facts, we get:

Lemma 3.2. *The rational fraction $H^*(j)$ of the variable j is of the form:*

$$H^*(j) = \frac{(j - 2^8 \cdot 3^3)(j^2 + \alpha j + \beta)}{j(\gamma j + \delta)}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\beta \neq 0$ and $(\gamma, \delta) \neq (0, 0)$.

Remark 3.3. *The notation $H^*(j)$ means the pull-back of j seen as a function on $\overline{\mathcal{M}}^1$, by the algebraic morphism $H : \overline{\mathcal{M}}^1 \to \overline{\mathcal{M}}^1$. We prefer it instead of $H(j)$, in order not to get confused in the next section by a notation of the type $H(h)$, which is not simply obtained from $H(j)$ by replacing j with h.*

In order to find the unknown coefficients $\alpha, \beta, \gamma, \delta$, we look first for the order of the zero $j = 2^8 \cdot 3^3$ and of the pole $j = 0$ of $H(j)$. Writing $M := m^3$, relation (7) becomes :

$$(8) \quad H\left(27 \cdot \frac{M(M+8)^3}{(M-1)^3}\right) = \frac{(4-M)^3}{M^2} \cdot \left(\frac{(M-4)^3 - 6^3 M^2}{(M-4)^3 + 3^3 M^2}\right)^3.$$

By factoring $H^*(j) = \dfrac{(j-2^8\cdot 3^3)^k}{j^l}\cdot K(j)$, with $K(2^8\cdot 3^3)\in\mathbb{C}\setminus\{0,\infty\}$ and $K(0)\in\mathbb{C}\setminus\{0,\infty\}$, we deduce easily that $k=3$ and $l=2$. Combining this with Lemma 3.2, we see that:

$$H^*(j) = \frac{1}{\gamma}\cdot\frac{(j-2^8\cdot 3^3)^3}{j^2}.$$

By equating the dominating coefficients of both sides of (8), when $M\to\infty$, we deduce that $\gamma=-27$.

We have got like this the desired expression of the hessian dynamical system:

Theorem 3.4. *One has the following expression of the hessian dynamical system in terms of the parameter j, on the compactified modular curve $\overline{\mathcal{M}}_1$:*

$$H^*(j) = -\frac{1}{3^3}\cdot\frac{(j-2^8\cdot 3^3)^3}{j^2}.$$

§4. The associated sequence of dessins d'enfants

Let us make the following change of variable on the modular curve $\mathcal{M}_1$:

$$j = 2^6\cdot 3^3\cdot h \tag{9}$$

From Proposition 3.4, we deduce the following expression of the hessian dynamical system in terms of the variable h:

$$H^*(h) = -\frac{1}{27}\cdot\frac{(h-4)^3}{h^2} \tag{10}$$

which shows that:

$$\frac{dH^*(h)}{dh} = -\frac{1}{27}\cdot\frac{(h-4)^2(h+8)}{h^3}.$$

We deduce from this immediately:

Proposition 4.1. *Set-theoretically, the critical locus of $H:\mathbb{P}^1_{[1:h]}\to\mathbb{P}^1_{[1:h]}$ is equal to*

$$\mathrm{Crit}(H) = \{h=4, h=-8, h=0\}.$$

The critical image of H, also called its discriminant locus, is equal to:

$$\Delta(H) = \{h=0, h=1, h=\infty\}.$$

Seen as divisors, the critical fibers of H are:

$$\begin{aligned}
&\mathrm{div}(H^*(h)) = 3(h=4),\\
&\mathrm{div}(H^*(h-1)) = 2(h=-8)+(h=1),\\
&\mathrm{div}(H^*(1/h)) = 2(h=0)+(h=\infty),
\end{aligned}$$

where $(h=a)$ denotes the point of $\mathbb{P}^1_{[1:h]}$ where the rational function h takes the value $a \in \mathbb{C} \cup \{\infty\}$.

The previous proposition explains why we have chosen the change of variable (9): in order to get as critical image the set $\{0,1,\infty\}$ of values of the working parameter.

The elliptic curves corresponding to the critical values $h=0$ and $h=1$ of H inside $\mathcal{M}_1 = \mathbb{C}_h$ are exactly those with special symmetry, as shown by the following proposition (see Hartshorne [6, page 321]):

Proposition 4.2. *Let E be an elliptic curve over $\mathbb{C}$. Denote by G_E the group of automorphisms of E leaving a base point fixed. Then G_E is a finite group of order:*
- *2 if $j(E) \notin \{0, 1728\} \Leftrightarrow h \notin \{0,1\}$.*
- *4 if $j(E) = 1728 \Leftrightarrow h = 1$.*
- *6 if $j(E) = 0 \Leftrightarrow h = 0$.*

By Proposition 4.1, we see that the cardinal of the discriminant set $\Delta(H)$ of the hessian dynamical system is equal to 3 when we look at H as a ramified covering of $\mathbb{P}^1$. By the work [2] of Belyi, we know that ramified covers of $\mathbb{P}^1$ with 3 critical values are particularly important from the arithmetical viewpoint (see also Zapponi [11]). Following this last reference, let us recall the notion of *dessin d'enfant* associated to such a map, introduced initially at the suggestion of Grothendieck [5, section 3].

Let $\psi : C \to \mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$ be a holomorphic map from a compact Riemann surface C to $\mathbb{P}^1$. Denote by Γ the preimage $\psi^{-1}([0,1])$. Color the vertices of $\Gamma \cap \psi^{-1}(0)$ in *black* and those of $\Gamma \cap \psi^{-1}(1)$ in *white*. Moreover, order cyclically the germs of edges starting from each vertex of Γ as they occur when one turns positively with respect to the canonical orientation defined by the complex structure of C.

Definition 4.3. *The graph Γ with colored vertices and cyclically ordered germs of edges obtained as explained before is called the* ***dessin d'enfant*** *associated to ψ.*

The point of this definition is that this dessin (a purely *topological* object) encodes completely up to isomorphisms the map ψ (a *holomorphic* object).

Remark 4.4. *If one has a map $\psi : C \to P$ where P is isomorphic to $\mathbb{P}^1$ and the discriminant set has cardinal equal to 3, one has to choose which points between the 3 critical values are to be identified with 0 and 1 in order to define the dessin d'enfant associated to ψ. In our case, the point ∞ is determined geometrically as the point at infinity of the moduli space $\mathcal{M}_1$.*

Proposition 4.1 shows that $H(\Delta(H)) \subset \Delta(H)$. More precisely, $0 \to \infty, 1 \to 1, \infty \to \infty$. This implies:

Proposition 4.5. *All the iterates $H^{(n)} := \underbrace{H \circ \cdots \circ H}_{n \text{ times}}$ of H, where $n \geq 1$, are also ramified covers of $\overline{\mathcal{M}_1} := \mathbb{P}^1_{[1:h]}$, with ramification set $\{0, 1, \infty\}$.*

We deduce that each iterate has an associated dessin d'enfant Γ_n. As the map H is defined over $\mathbb{Q}$, all its iterates have the same property, which shows that all the dessins Γ_n are fixed under the natural action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

§5. An algorithm for constructing the sequence of dessins d'enfants

We want now to understand how evolves the sequence of dessins d'enfants $(\Gamma_n)_{n \geq 1}$. The graph Γ_n embeds into the preimage

$$(H^{(n)})^{-1}(\mathbb{R}_h \cup \{\infty\})$$

of the real projective line of the parameter h.

From the holomorphic view-point, the real projective line $\mathbb{R}_h \cup \{\infty\}$ is canonically determined by the dynamical system, as the unique circle C_Δ contained in the smooth projective rational curve $\overline{\mathcal{M}_1} = \mathbb{P}^1_{[1:h]}$, and which contains the discriminant set $\Delta(H)$.

Denote:

$$G_n := (H^{(n)})^{-1}(C_\Delta).$$

From the topological view-point, G_n is a graph embedded into $\overline{\mathcal{M}_1}$, its vertices being the preimages of $\{0, 1, \infty\} \in \mathbb{P}^1_{[1:h]} = \overline{\mathcal{M}_1}$. We decorate the edges of G_n in three different ways, according to the real interval $(\infty, 0), (0, 1), (1, \infty)$ which is their image under H. We orient them with the lift by H of the natural orientation of $\mathbb{R}_h$ from negative to positive numbers. The drawing convention we choose is indicated in Figure 1.

Write

$$x := \mathrm{Re}(h), \ y := \mathrm{Im}(h).$$

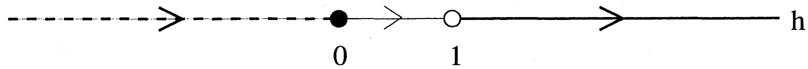

Fig. 1. The drawing convention

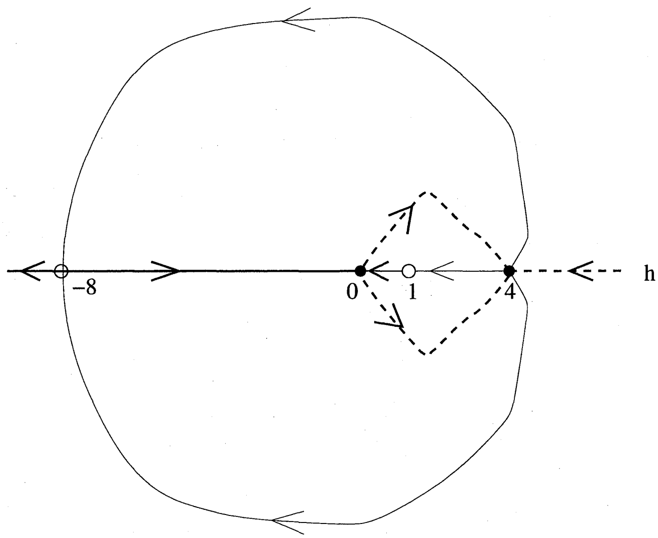

Fig. 2. The preimage by H of the real line

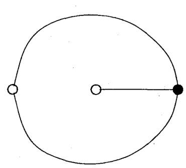

Fig. 3. The dessin d'enfant Γ_1 associated to H

Then the equation $\mathrm{Im}(H^*(h)) = 0$ of $H^{-1}(C_\Delta)$ becomes:

$$\mathrm{Im}\left[-\frac{1}{27} \cdot \frac{(x+iy-4)^3}{(x+iy)^2} \right] = 0.$$

After a few computations we get the equation:

$$(11)\quad y \cdot \Big[((x-4)^2+y^2)^2 + 16(x-4)((x-4)^2+y^2) + 16(3(x-4)^2-y^2) \Big] = 0.$$

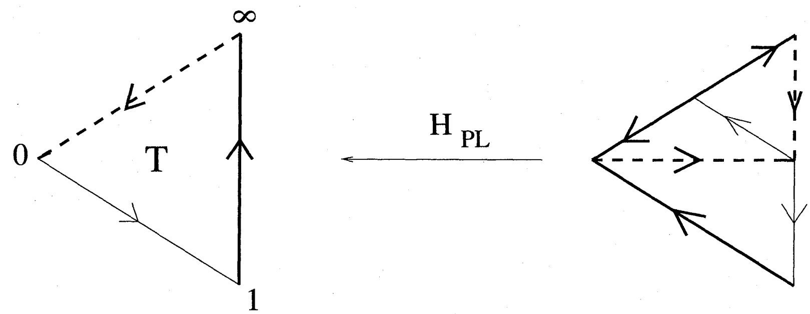

Fig. 4. The combinatorics of the triangulation $\mathcal{T}_1$

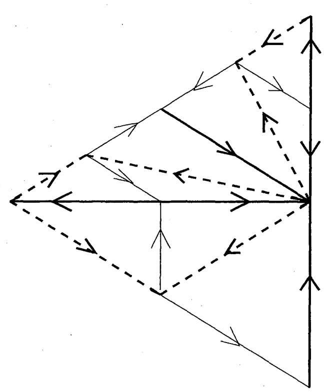

Fig. 5. The combinatorics of the triangulation $\mathcal{T}_2$

This shows that $G_1 = H^{-1}(C_\Delta)$ is the union of the real axis of the variable h and a singular quartic curve, which has as only singularity a real node at $h = 4$. The union of the two curves is drawn in Figure 2.

By looking inside Figure 2 at $H^{-1}([0,1])$, we deduce that the dessin d'enfant Γ_1 associated to H is as indicated in Figure 3.

Consider the map represented in Figure 4. In it, T is a compact affine triangle with the vertices denoted $0, 1, \infty$ and $H_{PL} : T \to T$ is a continuous piecewise-linear map, which is an affine homeomorphism onto T in restriction to each of the three closed triangles into which T is triangulated. Their vertices are midpoints of the edges of T. The distinct edges of the 1-skeleton at the source are decorated as their images by H_{PL}.

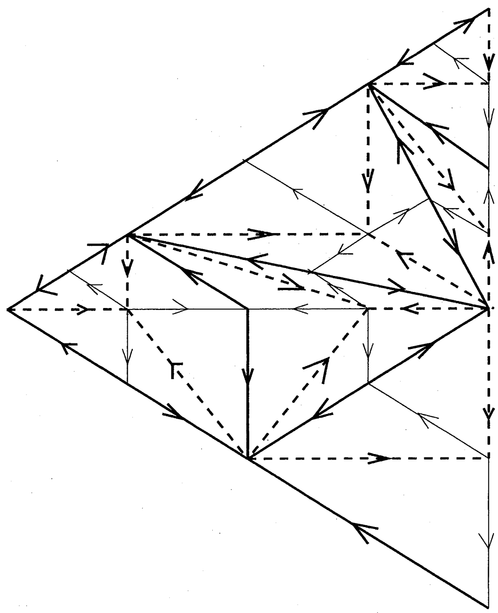

Fig. 6. The combinatorics of the triangulation $\mathcal{T}_3$

We see that the topology of the map is completely described by the triangulation of the source triangle and by the decorations of the edges. In the same way, we can describe the topology of $H_{PL}^{(n)}$, $\forall\, n \geq 1$, by a decorated triangulation:

$$\mathcal{T}_n := (H_{PL}^{(n)})^{-1}(\partial T).$$

The convention for the decoration of the edges of $\mathcal{T}_n$ is the same as the one used for G_n. We have the following algorithm for the construction of the sequence $(\mathcal{T}_n)_{n\geq 1}$:

• Start with H_{PL} and call the triangulation at the source $\mathcal{T}_1$.

• Given the triangulation $\mathcal{T}_n$, construct $\mathcal{T}_1$ inside each triangle of $\mathcal{T}_n$. Call the triangulation obtained like this $\mathcal{T}_{n+1}$.

Let Φ_n be the subgraph of the 1-skeleton $\mathcal{T}_n^1$ of the triangulation $\mathcal{T}_n$ obtained as the union of all edges whose image by $H_{PL}^{(n)}$ is the edge $[0,1]$ of T. Decorate the preimages of 0 in black and those of 1 in white. Then take the double $(T, \mathcal{T}_n^1, \Phi_n) \cup_{\partial T} (T, \mathcal{T}_n^1, \Phi_n)$ of the triple $(T, \mathcal{T}_n^1, \Phi_n)$ of topological spaces, that is glue two copies of it by the identity on ∂T. Orient arbitrarily the sphere $T \cup_{\partial T} T$ and orient cyclically the germs of

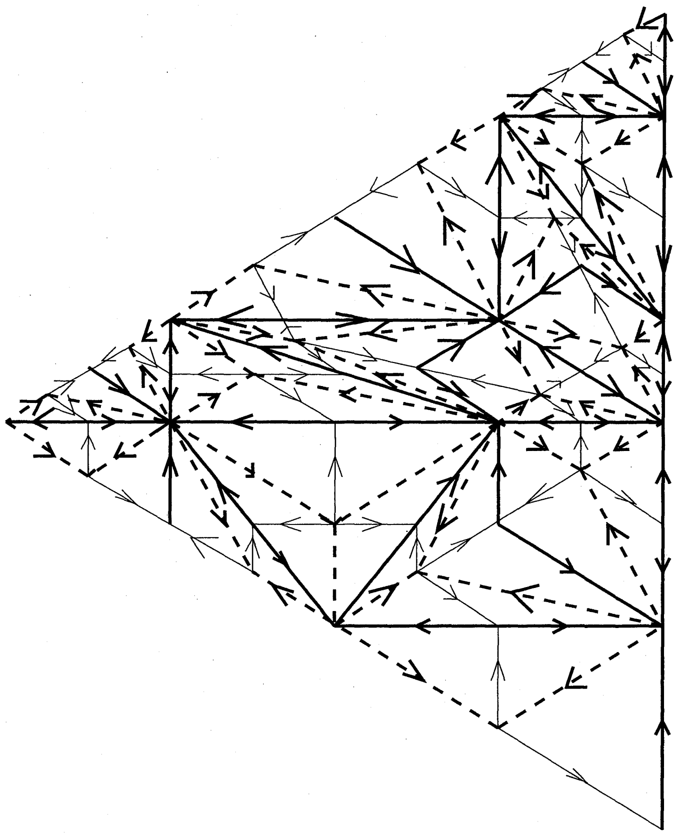

Fig. 7. The combinatorics of the triangulation $\mathcal{T}_4$

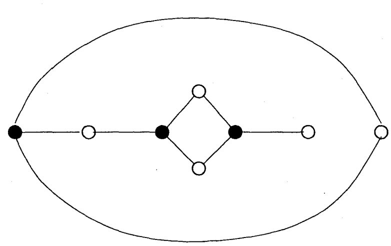

Fig. 8. The dessin d'enfant Γ_2 associated to $H^{(2)}$

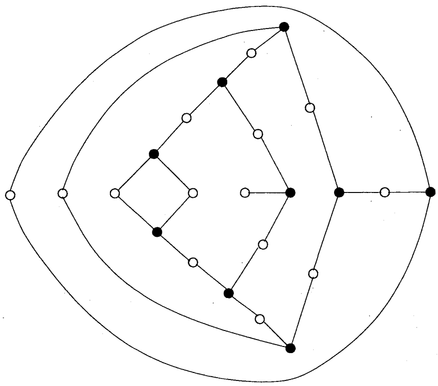

Fig. 9. The dessin d'enfant Γ_3 associated to $H^{(3)}$

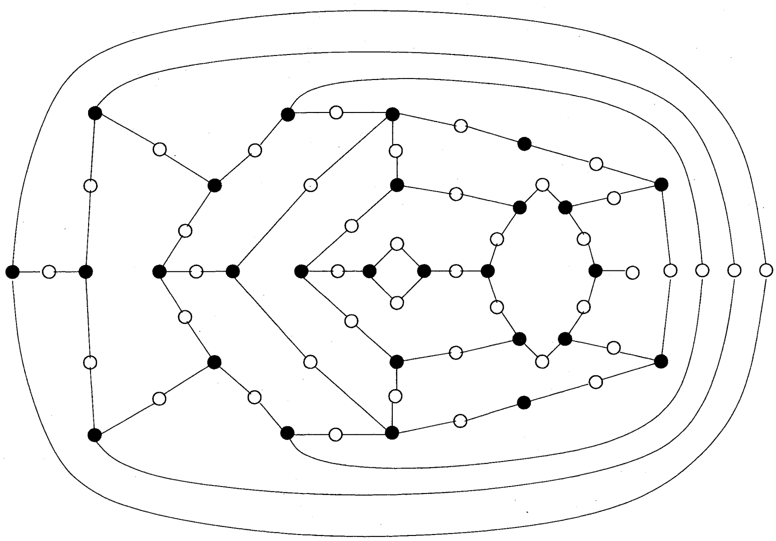

Fig. 10. The dessin d'enfant Γ_4 associated to $H^{(4)}$

edges of $\Phi_n \cup_{\partial T} \Phi_n$ around each vertex according to this orientation. Denote $\Sigma := T \cup_{\partial T} T$, $\tilde{\mathcal{T}}_n^1 := \mathcal{T}_n^1 \cup_{\partial T \cap \mathcal{T}_n^1} \mathcal{T}_n^1$, $\tilde{\Phi}_n := \Phi_n \cup_{\partial T \cap \Phi_n} \Phi_n$.

Proposition 5.1. *The decorated graph $\tilde{\Phi}_n$ is homeomorphic (respecting the decorations) to the dessin d'enfant Γ_n of $H^{(n)}$.*

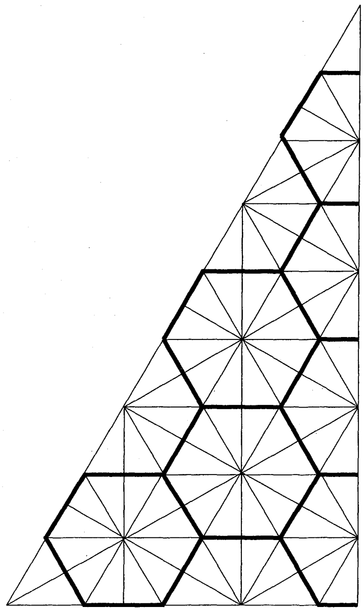

Fig. 11. Another picture of $\mathcal{T}_4$

In Figures 5, 6 and 7 we have drawn the decorated triangulations $\mathcal{T}_2, \mathcal{T}_3$ and $\mathcal{T}_4$. By Proposition 5.1, one can extract easily from them the dessins d'enfants Γ_2, Γ_3 and Γ_4, illustrated in Figures 8, 9 and 10 respectively. We have chosen a homeomorphism between T and a half-plane, sending the vertex ∞ of T to infinity, and then we have drawn the double $\tilde{\Phi}_n$ of Φ_n just by gluing the reflexion of Φ_n with respect to the border-line of the half-plane.

Remark 5.2. *(Added in proofs) After my exposition of the results of this paper in the University of Toulouse in December 2008, X. Buff and A. Chéritat proved easily using general theory that the map H is a so-called* Lattès map *(see* [8] *and* [9]*). More precisely, it is conjugate to the map induced on $\mathbb{C}/G$ by the multiplication $\cdot i\sqrt{3} : \mathbb{C} \to \mathbb{C}$. Here G denotes the group of automorphisms of $\mathbb{C}$ generated by the translations $z \to z+1$, $z \to z+\epsilon$ and the rotation $z \to \epsilon z$, where $\epsilon := e^{\frac{i\pi}{3}}$. This leads to a nice description of the sequences $(\mathcal{T}_n)_n$ and $(\Gamma_n)_n$ using euclidean geometry, in the spirit of Thurston's philosophy that topology is best understood using adapted geometric structures. The point here is that one should consider on $\mathbb{C}/G$ the structure of euclidean orbifold, quotient* *of the euclidean structure of* $\mathbb{C}$ *by* G*, which is a group of isometries for it. In this geometry, the figures of this paper appear in a way which is completely natural within the framework of euclidean geometry. Namely, the iterative operation drawn in Figure 4 should be replaced by one in which the triangle* T *is similar to each one of the 3 subtriangles created by the subdivision, the similarities respecting the decorations of the vertices by the symbols* $0, 1, \infty$*. This forces the angles of* T *to be of* $60°, 90°$ *and* $30°$ *respectively. In Figure 11 is drawn in this way* $\mathcal{T}_4$*, as well as the union of those edges allowing to construct* Γ_4 *by doubling. Compare with Figure 7 !*

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