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Singularities and self-similarity in gravitational collapse

Tomohiro Harada

Abstract.

Einstein's field equations in general relativity admit a variety of solutions with spacetime singularities. Numerical relativity has recently revealed the properties of somewhat generic spacetime singularities. It has been found that in a variety of systems self-similar solutions can describe asymptotic or intermediate behaviour of more general solutions in an approach to singularities. The typical example is the convergence to an attractor self-similar solution in gravitational collapse. This is closely related to the cosmic censorship violation in the spherically symmetric collapse of a perfect fluid. The self-similar solution also plays an important role in critical phenomena in gravitational collapse. The critical phenomena are understood as the intermediate behaviour around a critical self-similar solution. We see that the convergence and critical phenomena are understood in a unified manner in terms of attractors of codimension zero and one, respectively, in renormalisation group flow.

§1. The framework of general relativity

The essential assumption of general relativity is that the spacetime is given by a curved manifold with a metric $ds^2 = g_{ab}dx^a dx^b$ of the Lorentzian signature. g^{ab} denotes the inverse of g_{ab}. The curvature of the spacetime is given by the Riemann tensor $R^a{}_{bcd}$. The metric lifts and lowers the tensor indices. A vector is timelike, spacelike and null if $v^a v_a < 0$, $v^a v_a > 0$ and $v^a v_a = 0$, respectively. A hypersurface is called timelike, spacelike and null, if its normal vector is spacelike, timelike and null, respectively. We use the abstract index notation [36] in this article.

The field equation for the metric is given by Einstein's equations

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab}, \tag{1}$$

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where $R_{ab} \equiv R^{c}{}_{acb}$ is the Ricci tensor, $R \equiv R^{a}{}_{a}$ is the scalar curvature and T_{ab} is the stress-energy tensor of matter fields. We adopt the units in which $G = c = 1$. Einstein's equation was proposed so that it has the limit to Newtonian gravity in a weak-field and slow-motion regime. The conservation law

$$\nabla_b T^{b}{}_{a} = 0 \tag{2}$$

follows from the Bianchi identity, where ∇_b denotes the covariant derivative associated with g_{ab}. See [36, 18] for more complete description about the formulation of general relativity.

Because of the Lorentzian signature of the metric, the spacetime can admit a time function t and then "3+1" decomposition, i.e. foliation with spacelike hypersurfaces labelled by t. This is called time foliation or time slicing of the spacetime and this enables us to regard Einstein's equations as the combination of the evolution and constraint equations for the induced metric and the extrinsic curvature on the spacelike hypersurface. If the spacetime admits a timelike Killing vector, we can choose t so that the induced metric does not depend on t. Such a spacetime is called stationary.

General relativity is a self-consistent theory of gravity but not written in a completely closed form. We need to specify the physics of matter fields by giving the action or the stress-energy tensors of matter fields, which provide the source term on the right-hand side of Einstein's equations. Given the action of matter fields S_{m}, the stress-energy tensor is defined as a functional derivative in the following:

$$T_{ab} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}}{\delta g_{ab}}, \tag{3}$$

where g denotes the determinant of g_{ab}.

The simplest source term is vacuum, i.e. $T_{ab} = 0$. If $T_{ab} = -\Lambda g_{ab}$ for a constant Λ, this is called a cosmological constant. A perfect fluid is often used, which is given by

$$T_{ab} = \rho u_a u_b + p(u_a u_b + g_{ab}), \tag{4}$$

where u^a is the four-velocity of the fluid element, satisfying $u^a u_a = -1$. This provides a very good approximation for gases, liquids and solids in many circumstances. The equation which gives p is called the equation of state. If $p = 0$, the perfect fluid is called a dust. If $p = \rho/3$, it is called a radiation fluid. Another common example is a scalar field, for which the stress-energy tensor is given by

$$T^{ab} = \nabla^a \phi \nabla^b \phi - \frac{1}{2} g^{ab} \nabla_c \phi \nabla^c \phi - V(\phi) g^{ab}, \tag{5}$$

where $V(\phi)$ is called a potential. If $V(\phi) = 0$, the scalar field is called massless.

The matter fields evolve according to their equations of motion. If there is only a single matter component, the conservation law (2) gives the equation of motion. The metric is a solution of Einstein's equations with the matter fields as the source. Because of the conservation law, the distribution of matter fields in space and time cannot be put arbitrarily by any "external force" which does not contribute to the stress-energy tensor. In other words, we cannot determine the metric simply after we assume the distribution of matter fields as we often do to determine electric and magnetic fields in electromagnetism. We need to solve the metric and matter fields in a consistent manner. Moreover, Einstein's equations are highly nonlinear with respect to g_{ab}. These properties make it very difficult to get exact solutions in dynamical and generic situations with or without matter fields.

To obtain the general properties of spacetimes, it is important to know the general properties of matter fields. Among such conditions for matter fields are energy conditions, which impose the energy density being positive in some sense. The strong energy condition is one of them, assuming

$$R_{ab}\xi^a\xi^b = 8\pi(T_{ab}\xi^a\xi^b + T^a_{\ a}/2) \geq 0, \tag{6}$$

for any timelike vector ξ^a. This implies $\rho + p \geq 0$ and $\rho + 3p \geq 0$ for a perfect fluid.

Almost all known exact solutions to Einstein's equations have been obtained under strong assumption on symmetry. One very powerful method to obtain more or less general solutions is to establish numerical solutions. This is called numerical relativity in a broad sense. It has been achieving great success in recent days, not only in numerical simulations of relativistic astrophysical phenomena but also in the numerical experiments of phenomena with nonlinearly strong gravity.

§2. Singularities in general relativity

2.1. Examples of spacetime singularities

There are exact solutions to Einstein's equations which have spacetime singularities. See [36, 18, 7] for the precise notions of causal structure and spacetime singularities. Here we look at several examples. The first one is the Schwarzschild solution, in which the line element is written in the following form:

$$(7)\quad ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

This is the unique solution for spherically symmetric vacuum spacetimes. This describes a black hole for $M > 0$. In this case, $r = 2M$ is a coordinate singularity, corresponding to an event horizon, while $r = 0$ is a genuine spacetime singularity, towards which the scalar curvature polynomials tend to diverge. These features are well understood in the Penrose diagram or conformal diagram shown in Fig. 1. In this figure, the straight line with forty five degrees denotes a null ray and the physical spacetime is compactified through the conformal transformation. Suppose that an observer is in region I and she tries to send a signal to future null infinity $\mathcal{I}^+$ by emitting a light. We can see that the future-directed outgoing null rays earlier than $r = 2M$, i.e. in region I, can reach $\mathcal{I}^+$, while those later than $r = 2M$, i.e. in region II, cannot. $r = 0$ is described by the future and past boundaries of the spacetime, which are black hole and white hole singularities, respectively. No future-directed null ray can emanate from the black hole singularity.

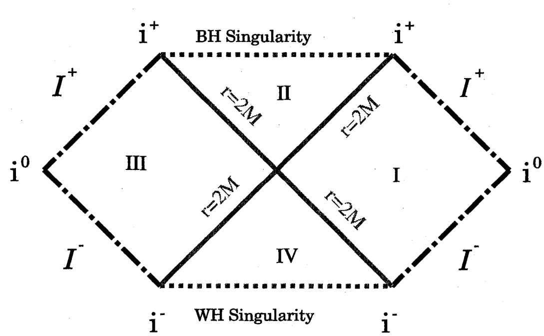

Fig. 1. The Penrose diagram of the Schwarzschild solution for $M > 0$.

The second example is the Friedmann solution which describes a homogeneous and isotropic universe. The line element is then given by

$$(8)\qquad ds^2 = -dt^2 + a^2(t)\left[\frac{1}{1-Kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],$$

where the scale factor $a = a(t)$ obeys the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{K}{a^2}, \tag{9}$$

the matter energy density ρ obeys the energy conservation law

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}, \tag{10}$$

and the dot denotes the derivative with respect to t. K denotes the curvature of the t =const spacelike hypersurface. Note that the source term for the Friedmann solution must be a perfect fluid. If the strong energy condition is satisfied, or $\rho + p \geq 0$ and $\rho + 3p \geq 0$ in this case, a then begins with 0 at $t = 0$, implying the divergence of scalar curvature polynomial where $\rho \to \infty$. This is a spacelike singularity and usually called big bang singularity or initial singularity. Figure 2 shows the evolution of the scale factor and the Penrose diagram.

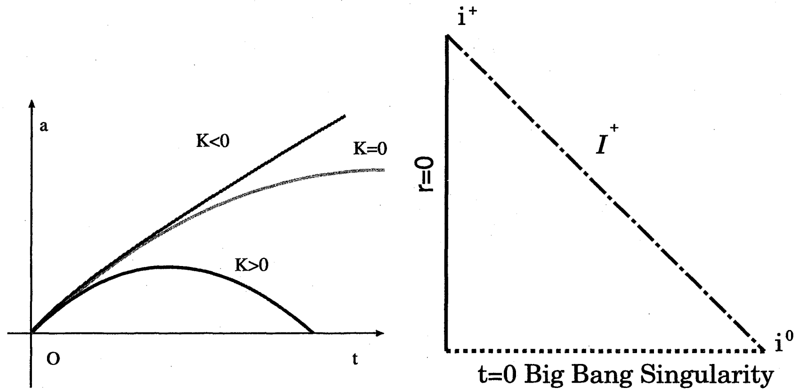

Fig. 2. The left and right panels show the evolution of the scale factor for different spatial curvatures and the Penrose diagram of the flat ($K = 0$) Friedmann solution, respectively.

The simplest model for gravitational collapse is the Oppenheimer–Snyder solution, which describes the complete collapse of a uniform dust ball. This is given by matching the interior and exterior solutions. The interior solution is the time-reversed Friedmann universe with a dust. The exterior is spherically symmetric and vacuum, which is given by the Schwarzschild solution. They are matched on a timelike hypersurface, which is generated by timelike radial geodesics. The Penrose diagram of the resultant spacetime is shown in Fig. 3, which is given by cutting and pasting those of the Friedmann solution and the Schwarzschild solution. We can see that the singularity is spacelike and hidden behind the event horizon from the external observer. There exists an achronal, i.e. spacelike or null, three-dimensional hypersurface such that the whole spacetime is the domain of dependence of this surface. This surface is called the Cauchy surface and the spacetime with the Cauchy surface is called globally hyperbolic.

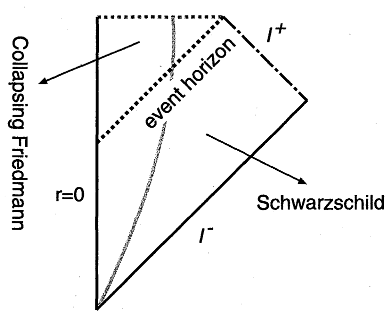

Fig. 3. The Penrose diagram of the Oppenheimer–Snyder solution.

2.2. Physical significance of singularities

The geodesic deviation equation implies

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}\xi^c\xi^d, \tag{11}$$

along a geodesic congruence, where ξ^a is the normalised tangent vector, τ is an affine parameter, and θ, σ_{ab} and ω_{ab} are respectively the expansion, shear and twist. This equation is called the Raychaudhuri equation. This implies that the timelike geodesic congruence tends to focus, in other words, θ tends to diverge to $-\infty$ if $R_{cd}\xi^c\xi^d \geq 0$ or matter fields satisfy the strong energy condition.

Based on the above properties, it was proved that there is at least one incomplete timelike or null geodesic in generic expanding universe and generic gravitational collapse [18]. This is called singularity theorems. The geodesic incompleteness implies the existence of spacetime singularities. This is a great achievement in the studies of classical general relativity. However, singularity theorems would not reveal the properties of generic singularities. Then, we see two important conjectures on this issue.

We first consider initial singularities. The Kasner solution is given by

$$ds^2 = -dt^2 + t^{p_1}dx^2 + t^{p_2}dy^2 + t^{p_3}dz^2, \tag{12}$$

where p_1, p_2 and p_3 are three indices, satisfying $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$ and $p_1 < 0 < p_2 < p_3$. This solution describes a homogeneous and anisotropic vacuum universe and has initial singularity at $t = 0$. The dynamics of the Bianchi type IX homogeneous universe was shown to be the successive series of Kasner regimes replacing the indices one another [1]. In each regime the solution is well approximated by the Kasner solution but the different regime has different indices (p_1, p_2, p_3). The transition is oscillatory and chaotic. Subsequently, it was conjectured that the process of approach to initial singularity in the general inhomogeneous case tends to be local, oscillatory and chaotic and the dynamics of nearby observers decouple from each other near the singularity [2]. This conjecture is called the Belinski–Khalatnikov–Lifshitz conjecture and has been recently strongly supported by numerical relativity experiment of initial singularity with no symmetry [9].

As for spacetime singularities formed in gravitational collapse, the cosmic censorship conjecture was proposed by Penrose. For the weak version, he conjectured, "A system which evolves, according to classical general relativity with reasonable equations of state, from generic non-singular initial data on a suitable Cauchy-hypersurface, does not develop any spacetime singularity which is visible from infinity" [31]. For the strong version, he conjectured, "$\cdots$ a physically reasonable classical spacetime M ought to have the property $\cdots$ M is globally hyperbolic $\cdots$" [32]. A singularity which is censored by this conjecture is called a naked singularity. In other words, the cosmic censorship claims that there is no naked singularity in physical spacetimes.

Obviously, there are undefined terms in this conjecture, such as reasonable equations of state and generic initial data. The cosmic censorship is a basic assumption to prove theorems on the properties of black holes, such as no bifurcation, area increase and the existence of an event horizon outside or coinciding to an apparent horizon [18]. The proof for the cosmic censorship is still very limited. On the other hand, it has been revealed that there are a lot of solutions of Einstein's equations, which satisfy energy conditions and have naked singularities. We do not know how to apply known physics at spacetime singularities. Hence, if there is a naked singularity, it would spoil the the predictability for our future within classical theory. On the other hand, naked singularities may be regarded as a window into physics beyond general relativity [17]. See [20, 15, 13] for naked-singular solutions in gravitational collapse and possible physical processes in naked singularity formation.

§3. Gravitational collapse and self-similar solutions

3.1. Self-similar solutions

General relativity has no characteristic scale of its own, which implies the existence of self-similar solutions. Self-similar solutions are easier to obtain than more general solutions because partial differential equations reduce to ordinary differential equations for spherically symmetric self-similar spacetimes. For self-similar solutions, the energy density ρ, for example, is written as $\rho(t,r) = t^{-2}f(r/t)$ with appropriate time and radial coordinates t and r. It is found that in some spatially homogeneous models self-similar solutions can describe the asymptotic behaviour of more general solutions [35]. It was also conjectured that spherically symmetric fluctuations might naturally evolve via Einstein's equations from complex initial conditions to a self-similar form [4]. See [5] for a recent review of self-similar solutions in general relativity.

More precisely, if a vector field ξ satisfies

$$\mathcal{L}_\xi g_{ab} = 2g_{ab}, \tag{13}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative along ξ, this is said to be a homothetic Killing vector. If there exists a homothetic Killing vector in a spacetime, this spacetime is said to be self-similar or homothetic. For a spherically symmetric self-similar spacetime, introducing coordinates (t,r) such that

$$\xi = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \tag{14}$$

a nondimensional quantity Q satisfies

$$Q(t,r) = Q(at, ar), \tag{15}$$

for any $a > 0$ and hence $Q = Q(r/t)$. The line element is then given by

$$ds^2 = -e^{\sigma(z)}dt^2 + e^{\omega(z)}dr^2 + r^2S^2(z)(d\theta^2 + \sin^2\theta d\phi^2), \tag{16}$$

where $z \equiv \ln|r/(-t)|$. On the other hand, if there exists a positive Δ such that $Q(t,r) = Q(e^{n\Delta}t, e^{n\Delta}r)$ holds only for $n = 0, \pm1, \pm2, \cdots$, this is called discretely self-similar.

To have self-similar spacetimes, matter fields are strongly restricted. Both a perfect fluid with the equation of state $p = k\rho$ and a massless scalar field ϕ are still compatible with self-similar spacetimes. For self-similar spacetimes, Einstein's equations reduce to a set of ordinary differential equations. Note that a perfect fluid with $p = k\rho$ $(0 < k < 1)$ has a sound wave at the speed $\sqrt{k}$, while a massless scalar field has a scalar wave at the speed of light. This property results in a critical point of the ordinary differential equations. This is called a sonic point for the perfect fluid case. Critical points are classified through dynamical systems theory.

3.2. Global attractor and cosmic censorship

We focus on spherically symmetric self-similar solutions with a perfect fluid with $p = k\rho$ $(0 < k < 1)$. Because of the critical point in the ordinary differential equations, the smoothness condition strongly restricts the class of solutions. We assume analytic initial data for self-similar perfect fluid solutions. Analyticity in the present context means the Taylor-series expandability of the energy density with respect to the Riemannian normal coordinates. In particular, we impose analyticity both at the centre and at the sonic point. Then we have only a discrete set of solutions. They are the flat Friedmann solution, general relativistic Larson–Penston (GRLP) solution, general relativistic Hunter (a) (GRHA) solution and so on. These are obtained numerically except for the flat Friedmann solution. For these self-similar solutions, a set of analytic initial data is prepared at $t = t_0 < 0$ and singularity appears at $t = 0$. It is found that, except for the flat Friedmann solution, the singularity appears at $t = 0$ only at the centre $r = 0$ and we can analytically extend the solution beyond $t = 0$ to positive t for $r > 0$. If it is extended, the GRLP solution describes the formation of naked singularity from analytic and therefore regular initial data for $0 < k < 0.0105$ [28, 30]. See Fig. 4 for the Penrose diagram of this spacetime. We can see that the singularity is not completely hidden behind the event horizon.

Since the cosmic censorship conjecture censors the generic occurrence of naked singularity, it is important whether the naked-singular solution is stable or not. The self-similar solution is given representatively as $H_{\rm ss}$. We assume that a linearly perturbed solution from this is given in the following form:

$$h(\tau, z) = H_{\rm ss}(z) + \epsilon e^{\lambda\tau} F(z), \tag{17}$$

where $\tau = -\ln(-t)$ and $|\epsilon| \ll 1$. We can get equations for linear perturbation from Einstein's equation. We impose regularity condition both at the centre and the sonic point. Then, we can determine the value of λ as

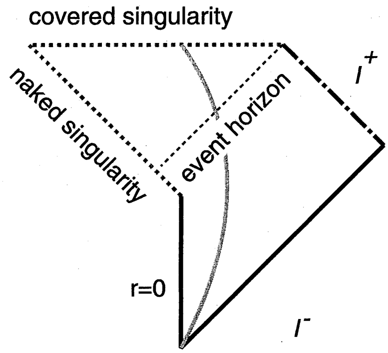

Fig. 4. The Penrose diagram of the GRLP solution for $0 < k < 0.0105$. The dashed line denotes the event horizon.

an eigenvalue problem. Then it is found that the GRLP has no unstable mode, the GRHA has one unstable mode and other solutions except the flat Friedmann have more than one unstable modes [21, 22, 25, 14, 3, 34].

Since the sonic point is a one-way membrane for sound waves, there can appear a different kind of instability [12]. If we inject density gradient discontinuity at the sonic point, we can find this discontinuity evolves locally at the sonic point. This perturbation mode is called a kink mode. The stability against this discontinuity is completely determined by the class to which the sonic point belongs as a critical point. Then, it is found that, against this specific mode, the flat Friedmann solution is unstable for $0 < k \leq 1/3$ and stable for $1/3 < k \leq 1$, the GRLP is stable for $0 < k < 0.036$ and unstable for $0.036 \leq k < 1/3$, and the GRHA is stable for $0 < k < 0.89$ and unstable for $0.89 \leq k \leq 1$.

As seen above, we have a self-similar solution which has no unstable mode. This is the GRLP solution for $0 < k < 0.036$. In fact, a numerical relativity experiment strongly suggests that this is a global attractor [14]. In the numerical simulation, the collapse ends in singularity formation at the centre for a certain subset of initial data sets, where the initial data sets were prepared without fine-tuning. Then, the profile of the density profile tends to evolve in a self-similar manner and agree very well with the GRLP solution. It was confirmed that this convergence to the self-similar attractor solution does not depend on the detailed choice of the initial density profile. The above is the results of numerical simulations for $0 < k \leq 0.03$. Although the first numerical simulation was done by the simple Misner–Sharp scheme, this result has been recently confirmed by a much more elaborated numerical scheme code with high resolution shock capturing, adaptive mesh refinement and innovative treatment of vacuum exterior [34].

This strongly suggests that the cosmic censorship is violated within spherical collapse. This is because the GRLP solution describes naked singularity formation for $0 < k < 0.0105$. Hence, this example is one of the strongest counterexamples against the cosmic censorship. On the other hand, the GRLP solution would be unstable against nonspherical perturbation for $0 < k < 1/9$, as the direct consequence of the linear perturbation analysis [10]. There remains much to study in nonspherical collapse.

3.3. Convergence and critical phenomena

Suppose we have a generic one-parameter family of initial data sets parametrised by p. Then, there generally exists a threshold value p^* for black hole formation. A near-critical collapse first approaches a self-similar solution and deviates away eventually. This self-similar solution that sits at the threshold is called a critical solution. The scaling law for the formed black hole for supercritical collapse is given as

$$M_{\mathrm{BH}} \propto |p - p^*|^{\gamma}, \tag{18}$$

for $p \approx p^*$,where γ is called a critical exponent. This is observed only as a result of fine-tuning the parameter p to be $p \approx p^*$. The critical solution and critical exponent do not depend on the prepared one-parameter family of initial data sets, which is called universality. The above phenomena were first observed in numerical simulation by Choptuik [6] for the spherical collapse of a massless scalar field and are called critical phenomena. It is found that critical phenomena are seen in a variety of systems, such as a perfect fluid with $p = k\rho$ [3, 34, 27, 8]. See [11] for an extensive review on this subject.

The critical behaviour turns out to be well understood by renormalisation group approach [21]. We consider the space of functions of z. We can regard this as the space of initial data sets. There the critical solution H_{ss} is characterised as a fixed point in this space with a single unstable mode. This means that the fixed point has a stable manifold of codimension one. Let $\lambda(>0)$ and F_{rel} be the eigenvalue and eigenfunction of this unstable mode. The one-parameter family of initial data sets corresponds to a curve in the space of initial data sets and therefore generically has intersection with the stable manifold. The intersection is actually the initial data for the exact critical collapse, i.e. $p = p^*$, which we denote as H_{c}. The initial data set for near-critical collapse is then given by

$$h(0,z) = H_{\rm init}(z) = H_{\rm c}(z) + \epsilon F(z), \quad \epsilon = p - p^*, \tag{19}$$

where τ is chosen to be 0 for the initial time. For large τ, the deviation from the critical solution is dominated by the unique unstable mode of the fixed point. The near-critical solution is then approximated as

$$h(\tau, z) \approx H_{\rm ss}(x) + \epsilon e^{\lambda\tau} F_{\rm rel}(z). \tag{20}$$

Assuming that the deviation becomes of order unity at the black hole formation, we get the scaling law for the black hole mass as

$$M_{\rm BH} = O(r) = O(e^{-\tau}) \propto |p - p^*|^\gamma, \tag{21}$$

where $\gamma = 1/\lambda$. In fact, for the system of a perfect fluid with $p = k\rho$, the GRHA solution has a single unstable mode and acts as a critical solution. The observed critical exponent in the numerical experiment of gravitational collapse agrees very well with the above expected value obtained from the eigenvalue analysis. From this point of view, the GRLP solution, which has no unstable mode, corresponds to an attractor of codimension zero, i.e. a global attractor.

The critical behaviour itself has an important implication to the cosmic censorship. In the limit of $p \to p^*$ from the supercritical regime, we will have a black hole of arbitrarily small mass, which can be regarded as a naked singularity because the curvature strength near the black hole horizon scales as $1/M^2$. More directly, it is found that the Choptuik critical solution, which is discretely self-similar, actually has a naked singularity at the centre [26]. However, it should be noted that this naked singularity is realised as a consequence of exact fine-tuning and hence nongeneric. This is very different from the naked-singular GRLP solution, in which naked singularity generically appears because the solution acts as a global attractor.

Lastly, we briefly review the Newtonian collapse of isothermal gas in the present context. This system is much simpler than but still very similar to the general relativistic system of a perfect fluid. There are a discrete set of spherically symmetric self-similar solutions with analytic initial data, including a homogeneous collapse, Larson–Penston solution, Hunter (a) solution and so on [23, 33, 19, 37]. The kink mode was also studied in this system [29]. The numerical experiment of gravitational collapse and the normal mode analysis show that there exist both convergence and critical phenomena [24, 16]. The Larson–Penston solution acts as a global attractor solution, while the Hunter (a) solution acts as a critical solution. This example shows that general relativity is not very essential to the appearance of critical behaviour.

Figure 5 schematically shows the renormalisation group flow for this system. The global attractor solution (the Larson–Penston solution) has no unstable mode, while the critical solution (the Hunter (a) solution) has a single unstable mode. A generic initial data set approaches the global attractor solution. On the other hand, only if we tune the parameter near the critical value, the supercritical initial data set first approaches the critical solution, deviates away from it and then approaches the global attractor solution. This interplay of the critical and global attractor solutions was first numerically observed in the Newtonian collapse of an isothermal gas [16] and subsequently in the general relativistic system of a perfect fluid $p = k\rho$ with sufficiently small positive k [34].

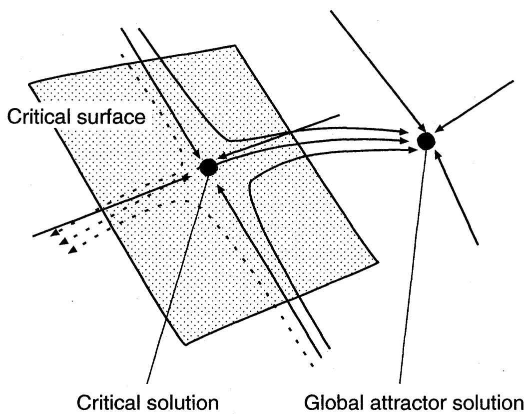

Fig. 5. The interplay of convergence and critical phenomena.

§4. Summary

The spacetime singularity has been a central issue in general relativity over several decades. Recently numerical relativity has been developed and can reveal the properties of generic spacetime singularities. We have focused on the role of self-similar solutions in singularity formation in gravitational collapse. The general theory of relativity as well as Newtonian gravity admits self-similar solutions. This is due to the scale-invariance of the theory. The self-similar solutions are important not only because they are dynamical and inhomogeneous solutions easier to obtain but also because they may play important roles in the asymptotic behaviour of more general solutions in certain circumstances. The numerical relativity experiments of gravitational collapse and the semi-analytical studies on self-similar solutions reveal that there is a self-similar solution which acts as a global attractor in the spherical collapse of a perfect fluid and then the cosmic censorship will be violated if it is formulated within spherical symmetry. We have also seen that the stability analysis of self-similar solutions gives a unified picture of the convergence to an attractor and the critical behaviour in gravitational collapse in terms of attractors of codimension zero and one, respectively. The interplay of these two behaviours has been recently observed in numerical simulation of gravitational collapse both in Newtonian gravity and general relativity. Since both the critical and convergence behaviours are not only in general relativity but also in Newtonian gravity, these two are considered to be common characteristics of gravitational physics and possibly of a wider class of scale-free nonlinear systems.

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Department of Physics
Rikkyo University
Tokyo
Japan

E-mail address: `harada@rikkyo.ac.jp`