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Good formal structure for meromorphic flat connections on smooth projective surfaces

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Abstract.

We prove the algebraic version of a conjecture of C. Sabbah on the existence of the good formal structure for meromorphic flat connections on surfaces after some blow up.

§1. Introduction

1.1. Main result

Let X be a smooth complex projective surface, and let D be a normal crossing divisor of X. Let $(\mathcal{E}, \nabla)$ be a flat meromorphic connection on (X, D), i.e., $\mathcal{E}$ is a locally free $\mathcal{O}_X(*D)$-module, and $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{X/C}$ is a flat connection. We prove a conjecture of C. Sabbah under the algebraicity assumption.

Theorem 1.1. *There exists a regular birational projective morphism $\pi : \widetilde{X} \longrightarrow X$ such that $\pi^{-1}(\mathcal{E}, \nabla)$ has the good formal structure.*

See Subsection 2.4 for good formal structure. For explanation of the meaning of the theorem, let us recall a very classical fact in the curve case. (See the introduction of [16] for more details, for example.) Let C be a smooth projective curve, and let $Z \subset C$ be a finite subset. Let $(\mathcal{E}, \nabla)$ be a meromorphic connection on (C, Z), i.e., $\mathcal{E}$ is a locally free $\mathcal{O}(*Z)$-module with a connection ∇. Let P be any point of Z, and let (U, t) be a holomorphic coordinate neighbourhood around P such that $t(P) = 0$. For understanding of the local structure of $\mathcal{E}$ around P, the first step is to look at the formal structure. Take a ramified covering $\varphi_P : (\widetilde{U}, t_d) \longrightarrow (U, t)$ given by $t = t_d^d$, where d is a large integer, for example

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divided by $(\mathrm{rank}(\mathcal{E})!)^3$. Let $\widetilde{P} \in \widetilde{U}$ be the inverse image of P. According to the Hukuhara–Turrittin theorem, the formal completion of $\varphi_P^*(\mathcal{E}, \nabla)$ at $\widetilde{P}$ is decomposed into the direct sum $\bigoplus_{\mathfrak{a} \in \boldsymbol{C}((t_d))/\boldsymbol{C}[\![t_d]\!]}(\mathcal{E}_{\mathfrak{a}}, \nabla_{\mathfrak{a}})$, where $\nabla_{\mathfrak{a}} - d\mathfrak{a}$ have regular singularities. (In the curve case, the algebraicity assumption is not necessary. In fact, the decomposition can be obtained for any connections on formal curves.) The second step is to take a lift of the formal decomposition to a flat decomposition on each small sector, which leads us to the Stokes structure. It is known that meromorphic flat bundles on curves are equivalent to flat bundles with Stokes structures on punctured curves.

It is a challenging and foundational problem to obtain a generalization in the higher dimensional case. A systematic study was initiated by H. Majima, who developed an asymptotic analysis in the higher dimensional case. (See [16].) Briefly speaking, his result gives a generalization of the second step, i.e., a lifting of the formal decomposition to decompositions on small sectors. Inspired by Majima's work, Sabbah developed an asymptotic analysis in another framework. (See [23].) Then, he proposed a conjecture concerning the first step, which says that the claim of Theorem 1.1 may hold without the algebraicity assumption, and he established it in the case $\mathrm{rank}(\mathcal{E}) \leq 5$. He also reduced the problem to the study of turning points contained in the smooth part of the divisor D, without any assumption on the rank.

Sabbah gave some interesting applications of the conjecture, one of which is a conjecture of B. Malgrange on the absence of the confluence phenomena for flat meromorphic connections. Recently, Y. André [1] proved Malgrange's conjecture motivated by Sabbah's conjecture. M. Hien [7] developed a theory of rapid decay homology for meromorphic flat connections on surfaces under Sabbah's conjecture.

In this paper, we will give a proof of the algebraic version of Sabbah's conjecture. In [19], the author study the correspondence of semisimple algebraic holonomic D-modules and polarizable wild pure twistor D-modules through wild harmonic bundles on smooth projective surfaces or the higher dimensional varieties, which is related with a conjecture of M. Kashiwara [9]. Theorem 1.1 has the foundational importance in this study.

Remark 1.2. *Very recently, K. Kedlaya* [13] *announced a proof of Sabbah's conjecture without the algebraicity assumption.*

1.2. Main ideas

Let k be an algebraically closed field, and let $(\mathcal{E}, \nabla)$ be a meromorphic flat connection on $k[\![s]\!]((t))$. If the characteristic number p of

k is positive, we always assume that p is much larger than $\operatorname{rank}\mathcal{E}$ and the Poincaré rank of $\mathcal{E}$ with respect to t. Let ∇_t denote the induced relative connection $\mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{k[\![s]\!]((t))/k[\![s]\!]}$. The induced connection $(\mathcal{E}, \nabla_t) \otimes k((s))((t))$ is denoted by $(\mathcal{E}_1, \nabla_1)$. The specialization of $(\mathcal{E}, \nabla_t)$ at $s = 0$ is denoted by $(\mathcal{E}_0, \nabla_0)$. We have the set of the irregular values $\operatorname{Irr}(\mathcal{E}_0, \nabla_0) \subset k((t_d))/k[\![t_d]\!]$ and $\operatorname{Irr}(\mathcal{E}_1, \nabla_1) \subset k((s_d))((t_d))/k((s_d))[\![t_d]\!]$, where t_d and s_d denote d-th roots of t and s, respectively. Briefly and imprecisely speaking, one of the main issues is how to compare $\operatorname{Irr}(\mathcal{E}_i, \nabla_i)$ $(i = 0, 1)$. Ideally, we hope that $\operatorname{Irr}(\mathcal{E}_1, \nabla_1)$ is contained in $k[\![s]\!]((t_d))/k[\![s, t_d]\!]$, and that the specialization at $s = 0$ gives $\operatorname{Irr}(\mathcal{E}_0, \nabla_0)$. However, they are not true, in general.

If $p > 0$, we have the p-curvature ψ (resp. ψ_i) of the connection ∇ (resp. ∇_i).

$$\psi \in \operatorname{End}(\mathcal{E}) \otimes F^*\Omega^1_{k[\![s]\!]((t))/k},$$

$$\psi_1 \in \operatorname{End}(\mathcal{E}_1) \otimes F^*\Omega^1_{k((s))((t))/k((s))},$$

$$\psi_0 \in \operatorname{End}(\mathcal{E}_0) \otimes F^*\Omega^1_{k((t))/k}$$

Here, F denotes the absolute Frobenius map. In the following, we use the symbol $\psi(t\partial_t)$ to denote $\psi\big(F^*(t\partial_t)\big)$, for simplicity. Let $\mathcal{S}p(\psi(t\partial_t))$ denote the set of the eigenvalues of $\psi(t\partial_t)$, which is contained in $\mathcal{A}_t$, where $\mathcal{A}$ denotes a finite extension of $k[\![s, t]\!]$ and $\mathcal{A}_t$ denotes a localization of $\mathcal{A}$ with respect to t. Similarly, let $\mathcal{S}p(\psi_i(t\partial_t))$ denote the set of the eigenvalues of $\psi_i(t\partial_t)$ for $i = 0, 1$, and then $\mathcal{S}p(\psi_0(t\partial_t)) \subset k((t_d))$ and $\mathcal{S}p(\psi_1(t\partial_t)) \subset k((s_d))((t_d))$ for some appropriate $d \in \mathbb{Z}_{>0}$. We may have the natural inclusion $\kappa_1 : \mathcal{A}_t \longrightarrow k((s_d))((t_d))$ and the specialization $\kappa_0 : \mathcal{A}_t \longrightarrow k((t_d))$ at $s = 0$. Clearly, $\psi_i(t\partial_t)$ $(i = 0, 1)$ are naturally obtained from $\psi(t\partial_t)$ by κ_i, and hence $\mathcal{S}p(\psi_i(t\partial_t))$ are obtained from $\mathcal{S}p(\psi(t\partial_t))$ by κ_i. The irregular values of ∇_i can be related with the negative part of the eigenvalues of $\psi_i(t\partial_t)$ (Lemma 2.7), where the negative part of $f = \sum f_j \cdot t_d^j \in R((t_d))$ is defined to be $f_- := \sum_{j<0} f_j \cdot t_d^j$. Thus, we obtain the following diagram:

$$\begin{array}{ccccc} \mathcal{S}p\big(\psi_1(t\partial_t)\big) & \xleftarrow{\kappa_1} & \mathcal{S}p\big(\psi(t\partial_t)\big) & \xrightarrow{\kappa_0} & \mathcal{S}p\big(\psi_0(t\partial_t)\big) \\ \downarrow & & & & \downarrow \\ \operatorname{Irr}(\mathcal{E}_1, \nabla_1) & & & & \operatorname{Irr}(\mathcal{E}_0, \nabla_0) \end{array}$$

But, we should remark that $\kappa_0(\alpha)_-$ and $\kappa_1(\alpha)_-$ cannot be directly related, in general.

If the ramification of $\mathcal{A}$ over $k[\![s, t]\!]$ may occur only along the divisor $\{t = 0\}$, then $\mathcal{S}p(\psi(t\partial_t))$ is contained in $k[\![s]\!]((t_d))$, and $\kappa_0(\alpha)_-$ is the specialization of $\kappa_1(\alpha)_-$ at $s=0$ for any $\alpha \in \mathcal{S}p(\psi(t\partial_t))$. Hence, we can compare the irregular values of ∇_i $(i=0,1)$ in this simplest case.

We have to consider what happens if the ramification of $\mathcal{A}$ may be non-trivial. As the second simplest case, we assume that the ramification may occur only at the normal crossing divisor $(t) \cup (s')$ of $\mathrm{Spec}^f\, k[\![s,t]\!]$, where $s' = s + t\cdot h(t)$. Then, $\mathcal{S}p(\psi(t\partial_t))$ is contained in $k[\![s'_d]\!]((t_d))$, where s'_d denotes a d-th root of s'. We assume, moreover, that $\mathcal{S}p(\psi(t\partial_t))$ are contained in $k[\![s]\!]((t_d)) + k[\![s'_d, t_d]\!]$. Then, the negative part of the eigenvalues behave well with respect to the specialization, and we can compare the irregular values of ∇_i $(i=0,1)$ in this mildly ramified case (See Lemma 3.2).

We would like to apply such consideration to our problem. Essentially, the problem is the following, although we will argue it in a different way. Let $(\mathcal{E},\nabla)$ be a meromorphic flat connection with a lattice E on (X,D) such that $\nabla(E) \subset E \otimes \Omega^1_X(ND)$. For simplicity, we assume that everything is defined over $\mathbb{Z}$. We have the mod p-reductions $(\mathcal{E}_p, \nabla_p) := (\mathcal{E},\nabla)\otimes \overline{\mathbb{F}}_p$ over $(X_p, D_p) := (X,D)\otimes\overline{\mathbb{F}}_p$ with the lattice $E_p = E\otimes \overline{\mathbb{F}}_p$, where $\overline{\mathbb{F}}_p$ denotes an algebraic closure of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let $\psi_p \in \mathrm{End}(E_p)\otimes F^*\Omega_{X_p}(ND_p)$ denote the p-curvature. We have the spectral manifold

$$\Sigma_p(\psi_p) := \bigl\{(x,\omega)\,\bigm|\,\omega \text{ eigenvalues of } \psi_{p|x}\bigr\} \subset F^*\bigl(\Omega^1_{X_p}\otimes\mathcal{O}(ND_p)\bigr).$$

For simplicity, we assume that ψ_p has distinct eigenvalues at the generic point. Then, we hope that the ramification of the projection π_p of $\Sigma_p(\psi_p)$ to X_p may happen along a normal crossing divisor, after some blow up, i.e., $R(\pi_p) := \bigl\{x \in X_p \,\bigm|\, \pi_p \text{ is not etale at } x\bigr\}$ is normal crossing. If we fix p, it is easy to obtain such a birational map. (Recall $\dim X = 2$.) However, for our problem, we would like to control the ramification for almost all p at once. So we need something more.

Here, we recall the important observation of J. Bost, Y. Laszlo and C. Pauly [14] which says that we have Σ'_p contained in $\Omega^1_{X_p}\otimes\mathcal{O}(ND)$, such that $\Sigma_p(\psi_p)$ is the pull back of Σ'_p. So, we have only to control the ramification curves $R(\pi'_p)$ of the projection π'_p of Σ'_p to X_p. Then, it is not difficult to see that the arithmetic genus of $R(\pi'_p)$ are dominated, independently of p. So, the complexity of the singularities of these ramification curves are bounded, and thus we can control them uniformly. (See Section 4.)

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It is an extremely great pleasure for the author to dedicate this paper to Masaki Kashiwara with admiration for his great works and his leading role in the development of contemporary mathematics.

§2. Preliminary

2.1. Notation

Let R be a ring, and let t be a formal variable. We use the symbol $R[\![t]\!]$ (resp. $R(\!(t)\!)$) to denote the ring of formal power series (resp. the ring of formal Laurent power series) over R. Let $R(\!(t)\!)_{<0}$ denote the subset $\bigl\{\sum_{j<0} a_j \cdot t^j \in R(\!(t)\!)\bigr\}$. For any $f = \sum a_j \cdot t^j \in R(\!(t)\!)$, we put $\mathrm{ord}_t(f) := \min\{j \mid a_j \neq 0\}$. If we are given two variables s and t, we use the symbol $R[\![s]\!](\!(t)\!)$ to denote the ring of formal Laurent power series over $R[\![s]\!]$. The symbol $R(\!(t)\!)[\![s]\!]$ is used to denote the ring of formal power series over $R(\!(t)\!)$. We have $R[\![s]\!](\!(t)\!) \subsetneq R(\!(t)\!)[\![s]\!]$.

For a given integer $d > 0$ and a formal variable t, we use the symbol t_d as a d-th root of t, i.e., $t_d^d = t$. For any $f = \sum f_j \cdot t_d^j \in R(\!(t_d)\!)$, we put $f_- := \sum_{j<0} f_j \cdot t_d^j$, which is called the negative part of f. If d' is a factor of d, we regard $R(\!(t_{d'})\!)$ as the subring of $R(\!(t_d)\!)$. For any $f \in R(\!(t_d)\!)$, we put $\mathrm{ord}_t(f) := d^{-1} \cdot \mathrm{ord}_{t_d}(f)$. The definition is consistent for the inclusions $R(\!(t)\!) \subset R(\!(t_{d'})\!) \subset R(\!(t_d)\!)$. Let us consider the case in which R is a ring over $\mathbb{Z}/p\mathbb{Z}$ for some prime p. If d is prime to p, the derivation $t\partial_t$ of $R(\!(t)\!)$ has the natural lift to $R(\!(t_d)\!)$, which is same as $d^{-1} \cdot t_d \partial_{t_d}$. We put $I_t(g) := \sum (d/j) \cdot g_j \cdot t_d^j$ for any $g = \sum_{j \not\equiv 0 \bmod p} g_j \cdot t^j \in R(\!(t)\!)$. We have $t\partial_t\bigl(I_t(g)\bigr) = g$ and $I_t\bigl(t\partial_t g\bigr) = g$.

When R is a subring of $\boldsymbol{C}$ finitely generated over $\mathbb{Z}$, let $S(R,p)$ denote the set of the generic points of the irreducible components of

$\mathrm{Spec}(R \otimes \mathbb{Z}/p\mathbb{Z})$ for each prime number p. We put $S(R) := \bigcup_p S(R,p)$. For each $\eta \in S(R)$, let $k(\eta)$ denote the corresponding field, and let $\overline{\eta} \longrightarrow \eta$ denote a morphism such that $k(\overline{\eta})$ is an algebraic closure of $k(\eta)$.

We use the symbol $M_r(R)$ to denote the set of the r-th square matrices over R, in general.

2.2. Irregular value

2.2.1. *Definition* Let k be a field, whose characteristic number is denoted by p. Let E be a locally free $k[\![t]\!]$-module of rank r. We use the symbol $E(\!(t)\!)$ to denote $E \otimes k(\!(t)\!)$. Let ∇ be a meromorphic connection of $E(\!(t)\!)$ such that $\nabla(\partial_t)(E) \subset E \cdot t^{-\mu}$ for some non-negative integer μ.

Assumption 2.1. *If $p > 0$, we assume that r and μ are sufficiently smaller than p, say $10 \cdot r! \cdot \mu < p$.*

Let $\overline{k}$ denote an algebraic closure of k. Then, it is known (see [1], for example) that we have the unique subset $\mathrm{Irr}\big(E(\!(t)\!), \nabla\big) \subset k(\!(t_d)\!)/k[\![t_d]\!]$ and the unique decomposition

$$(1) \qquad \big(E(\!(t)\!), \nabla\big) \otimes \overline{k}(\!(t_d)\!) \simeq \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(E(\!(t)\!),\nabla)} \big(E_{\mathfrak{a}}(\!(t_d)\!), \nabla_{\mathfrak{a}}\big)$$

for some appropriate factor d of $r!$, such that the following holds:

- For any element $\mathfrak{a} \in \mathrm{Irr}(E(\!(t)\!), \nabla)$, take a lift $\widetilde{\mathfrak{a}} \in k(\!(t_d)\!)$, and then $\nabla_{\mathfrak{a}} - d\widetilde{\mathfrak{a}} \cdot \mathrm{id}_{E_{\mathfrak{a}}}$ is a logarithmic connection of $E_{\mathfrak{a}} \neq 0$. The elements of $\mathrm{Irr}(E(\!(t)\!), \nabla)$ or their lifts are called the irregular values of (E, ∇).

The decomposition (1) is called the irregular decomposition in this paper. We usually use the natural lifts of $\mathfrak{a}$ in $k(\!(t_d)\!)_{<0}$, and denote them by the same letter $\mathfrak{a}$. We have $\mathrm{ord}_t(\mathfrak{a}) \geq -\mu + 1$, and d is a factor of $r!$, and hence $\mathrm{ord}_{t_d}(\mathfrak{a}) > -p$ under the assumption 2.1.

If the irregular decomposition exists on $k(\!(t)\!)$, we say that (E, ∇) is unramified. The following lemma easily follows from the uniqueness of the irregular decomposition.

Lemma 2.2. *Let k' be an algebraic extension of k, and let d' be a divisor of d. If all the irregular values are contained in $k'(\!(t_{d'})\!)$, then $(E, \nabla) \otimes k'(\!(t_{d'})\!)$ is unramified.* *Q.E.D.*

2.2.2. *Connection form of Deligne–Malgrange lattice* We have another characterization of the irregular values. For simplicity, we assume that $\big(E(\!(t)\!), \nabla\big)$ is unramified and that k is algebraically closed.

Definition 2.3. *We say that E is a Deligne–Malgrange lattice of $E((t))$, if the irregular decomposition* (1) *is given on $k[\![t]\!]$ not only on $k((t))$, i.e., $E = \bigoplus E_{\mathfrak{a}}$, and moreover $(\nabla(t\partial_t) - t\partial_t\mathfrak{a})E_{\mathfrak{a}} \subset E_{\mathfrak{a}}$.*

If E is Deligne–Malgrange, we have the logarithmic connection

$$\nabla^{\mathrm{reg}} = \bigoplus \nabla^{\mathrm{reg}}_{\mathfrak{a}},$$

where $\nabla^{\mathrm{reg}}_{\mathfrak{a}} := \nabla_{\mathfrak{a}} - d\mathfrak{a}\cdot \mathrm{id}_{E_{\mathfrak{a}}}$. We say E is a strict Deligne–Malgrange lattice, if $\alpha-\beta$ are not integers for any two distinct eigenvalues α,β of $\mathrm{Res}(\nabla^{\mathrm{reg}})$.

Remark 2.4. *In* [19], *we use the terminology "Deligne–Malgrange lattice" in a more restricted meaning. A lattice as above is called "unramifiedly good lattice" in* [19].

Let $\boldsymbol{v}$ be any frame of E. Let $A \in M_r\big(k((t))\big)$ be determined by $\nabla(t\partial_t)\boldsymbol{v} = \boldsymbol{v}\cdot A$. Let $\mathcal{S}p(A) \subset k((t_d))$ denote the set of the eigenvalues of A for some appropriate d. For any $\alpha \in \mathcal{S}p(A)$, we have the negative part $\alpha_- \in k((t_d))_{<0}$ and $I_t(\alpha_-) \in k((t_d))_{<0}$ as explained in Subsection 2.1.

Lemma 2.5. *If E is Deligne–Malgrange, we have* $\mathrm{Irr}\big(E((t)),\nabla\big) = \big\{I_t(\alpha_-)\,\big|\,\alpha\in\mathcal{S}p(A)\big\}$.

Proof We take a frame $\boldsymbol{v}_1$ of E compatible with the irregular decomposition, and A_1 is determined as above. Then, A_1 has the decomposition corresponding to the irregular decomposition, $A_1 = \bigoplus(t\partial_t\mathfrak{a}+R_{\mathfrak{a}})$, where $R_{\mathfrak{a}} \in M_r(k[\![t]\!])$. Hence the claim of the lemma clearly holds for the frame $\boldsymbol{v}_1$.

For any frame $\boldsymbol{v}$ of E, we have $G \in \mathrm{GL}\big(k[\![t]\!]\big)$ such that $\boldsymbol{v} = \boldsymbol{v}_1\cdot G$. We have the relation $A = G^{-1}\cdot A_1\cdot G + G^{-1}\cdot t\partial_t G$, i.e., $G\cdot A\cdot G^{-1} = A_1 + (t\partial_t G)\cdot G^{-1}$, where $t\partial_t G\cdot G^{-1} \in M_r(k[\![t]\!])$. Hence, the claim is reduced to the following general lemma.

Lemma 2.6. *Let $\Gamma \in M_r(k[\![t]\!])$ be a diagonal matrix whose (i,i)-entry is given by α_i. Let B be any element of $t^m\cdot M_r(k[\![t]\!])$ for a positive integer $m>0$. Then, any eigenvalue $\beta\in k[\![t_d]\!]$ of $\Gamma+B$ satisfies $\mathrm{ord}_t(\beta-\alpha_i)\geq m$ for some α_i.*

Proof Let $e_1,\ldots,e_r$ denote the canonical base of $k((t))^r$. Let $v = \sum f_i\cdot e_i$ be an eigenvector of $\Gamma+B$ corresponding to the eigenvalue β. We may assume $\mathrm{ord}_t(f_{i_0}) = 0$ for some i_0. We obtain $\mathrm{ord}_t\big((\alpha_i-\beta)\cdot f_i\big)\geq m$ for any i, and hence $\mathrm{ord}_t(\alpha_{i_0}-\beta)\geq m$. Thus, we obtain Lemma 2.6 and Lemma 2.5. Q.E.D.

2.2.3. *p-curvature* In the case $p > 0$, we have the other characterization of the irregular values. For simplicity, we assume $k = \overline{k}$. Let $\mathrm{Fr} : k((t)) \longrightarrow k((t))$ be the absolute Frobenius morphism, i.e., $\mathrm{Fr}(f) = f^p$. Applying Fr to the coefficients, we obtain the homomorphism $k((t))[T] \longrightarrow k((t))[T]$, which is also denoted by Fr. Let ψ be the p-curvature of ∇. (See [10] and [11], for example). Due to the observation of Bost–Laszlo–Pauly [14], there exists a polynomial $P_\nabla(T) \in k((t))[T]$ of degree r, such that $\det\bigl(T - \psi(t\partial_t)\bigr) = \mathrm{Fr}\bigl(P_\nabla\bigr)(T)$. Let $\mathrm{Sol}(P_\nabla)$ denote the set of the solutions of $P_\nabla(T) = 0$. Then $\mathrm{Sol}(P_\nabla) \subset k((t_d))$ for some appropriate factor d of $r!$. Because $\nabla(\partial_t)(E) \subset E \cdot t^{-\mu}$, we have $\psi(\partial_t)(E) \subset E \cdot t^{-\mu \cdot p}$. Hence we have $\mathrm{ord}_t(\alpha) \geq -\mu + 1$ for any solution $\alpha \in \mathrm{Sol}(P_\nabla)$. Under the assumption 2.1, we obtain $\mathrm{ord}_{t_d}(\alpha) > -p$ for any $\alpha \in \mathrm{Sol}(P_\nabla)$.

Lemma 2.7. *Under the assumption* 2.1*,*

$$\mathrm{Irr}(E((t)), \nabla) = \bigl\{ I_t(\alpha_-) \bigm| \alpha \in \mathrm{Sol}(P_\nabla) \bigr\}.$$

Proof We may assume that (E, ∇) is unramified and Deligne–Malgrange. Hence, we have only to consider the case in which (E, ∇) has the unique irregular value, i.e., $\nabla = d\mathfrak{a} \cdot \mathrm{id}_E + \nabla^{\mathrm{reg}}$, where $\mathfrak{a} \in k((t))_{<0}$, $\mathrm{ord}_t(\mathfrak{a}) > -p$, and ∇^{reg} is logarithmic. Let ψ_{reg} denote the p-curvature of ∇^{reg}. By a general formula ([22], [11], see also Lemma 3.4 of [25]), we have $\psi(t\partial_t) = \psi_{\mathrm{reg}}(t\partial_t) + (t\partial_t\mathfrak{a})^p$, where $\psi_{\mathrm{reg}}(t\partial_t) \in M_r(k[\![t]\!])$. Then the claim of the lemma follows from Lemma 2.6. Q.E.D.

2.3. Preliminary from elementary algebra

The following arguments are standard and well known. We would like to be careful about some finiteness, and we give just an outline. Let R be an integral domain whose quotient field is denoted by K. Let $P_t(T) \in R[\![t]\!][T]$ be a monic polynomial:

$$P_t(T) = T^r + \sum_{j=0}^{r-1} a_j(t) \cdot T^j = \sum_{j \geq 0} P_j(T) \cdot t^j$$

Lemma 2.8. *Assume that $P_0(T) = \overline{h}_1(T) \cdot \overline{h}_2(T)$ in $K[T]$ such that $\overline{h}_1$ and $\overline{h}_2$ are monic polynomials and coprime. Then, we have the decomposition $P(T) = h_1(T) \cdot h_2(T)$ in $R'[\![t]\!][T]$, where R' is the localization of R with respect to some $f_1, \ldots, f_m \in R$ depending on $P_0(T)$, and $h_i(T)$ are monics such that $h_i(T)_{|t=0} = \overline{h}_i(T)$.*

Proof There exist $F_i \in K[T]$ $(i = 1, 2)$ such that

$$1 = \overline{h}_1(T) \cdot F_1(T) + \overline{h}_2(T) \cdot F_2(T).$$

We may take a finite localization R' of R so that $\overline{h}_i, F_i \in R'[T]$. For any $Q(T) \in R'[T]$, we have $\overline{h}_1 \cdot (F_1 Q) + \overline{h}_2 \cdot (F_2 Q) = Q$. Take $H, G \in R'[T]$ such that $\deg_T(H) < \deg_T(\overline{h}_1)$ and $F_2 Q = \overline{h}_1 \cdot G + H$. We put $\alpha := F_1 \cdot Q + G \cdot \overline{h}_2$, and then we have $\overline{h}_1 \cdot \alpha + \overline{h}_2 \cdot H = Q$. Note $\deg_T(\overline{h}_1) + \deg_T(\overline{h}_2) = \deg_T P_0 = r$. If $\deg_T Q < r$, we have

$$\deg_T(\alpha) + \deg_T \overline{h}_1 \leq \max\bigl(\deg_T Q,\ \deg_T \overline{h}_2 + \deg_T H\bigr) < r.$$

Hence, $\deg_T(\alpha) < r - \deg_T(\overline{h}_1) = \deg_T(\overline{h}_2)$.

Assume we are given $h_{a,j}(T)$ ($a = 1, 2,\ j = 1, \ldots, L$) such that $\deg_T h_{a,j} < \deg_T(\overline{h}_a)$ and the following holds modulo t^{L+1}:

$$\left(\overline{h}_1(T) + \sum_{j=1}^{L} h_{1,j}(T) t^j\right) \cdot \left(\overline{h}_2(T) + \sum_{j=1}^{L} h_{2,j}(T) t^j\right) - \sum_{j=0}^{L} P_j(T) t^j \equiv 0$$

By using the above remark, it is easy to show that we can take $h_{a,L+1}$ ($a = 1, 2$) such that $\deg_T h_{a,L+1} < \deg_T(\overline{h}_a)$ and the following holds modulo t^{L+2}:

$$\left(\overline{h}_1(T) + \sum_{j=1}^{L+1} h_{1,j}(T) t^j\right) \cdot \left(\overline{h}_2(T) + \sum_{j=1}^{L+1} h_{2,j}(T) t^j\right) - \sum_{j=0}^{L+1} P_j(T) t^j \equiv 0$$

Thus, by an inductive argument, we can construct the desired h_1 and h_2. Q.E.D.

Lemma 2.9. *Let $P_t(T) \in R[\![t]\!][T]$ (resp. $R(\!(t)\!)[T]$) be a monic polynomial. There exists an appropriate number e, such that the roots of $P_t(T)$ are contained in $R'[\![t_e]\!]$ (resp. $R'(\!(t_e)\!)$) where R' is obtained from R by finite algebraic extensions and finite localizations.*

Proof Let $P_t(T) = \sum_{j=0}^{n} a_j(t) \cdot T^j$. We may assume that $n!$ is invertible in R. Let $\nu(P_t)$ denote the number $\min_j\{\mathrm{ord}_t(a_j)/(n-j)\}$. We use the induction on the numbers $\deg_T P_t$ and $\nu(P_t)$. For simplicity, we use the symbol ν instead of $\nu(P_t)$, and let d be the minimal positive integer such that $\nu \in d^{-1} \cdot \mathbb{Z}$. We formally use the symbol t^ν to denote $t_d^{d \cdot \nu}$. We have the following monic polynomial:

$$\begin{aligned} Q_t(T') := t^{-n\nu} P_t(t^\nu T') &= \sum_{j=0}^{n} a_j(t) t^{-(n-j)\nu} \cdot T'^j \\ &= \sum_{j=0}^{n} b_j(t) \cdot T'^j \in R[\![t_d]\!][T'] \end{aligned}$$

We have $d^{-1} \cdot \mathrm{ord}_{t_d}(b_j) = \mathrm{ord}_t(a_j) - (n-j) \cdot \nu \geq 0$, and we have $\mathrm{ord}_{t_d}(b_{j_0}) = 0$ for some j_0. We put $Q_0(T') = \sum_{j=0}^{n} b_j(0) T'^j \in R[T']$.

Case 1 Assume $Q_0(T')$ has at least two different roots. Then, there exists a finite algebraic extension K_1 of K such that we have the decomposition $Q_0(T') = \overline{h}_1(T') \cdot \overline{h}_2(T')$ in $K_1[T']$, and $\overline{h}_1$ and $\overline{h}_2$ are coprime. Due to Lemma 2.8, we have $Q_t(T') = h_1(T') \cdot h_2(T')$ in $R_1'[\![t_d]\!][T']$, where R_1' is obtained from R by finite algebraic extensions and finite localizations. By the hypothesis of the induction on the degree with respect to T', the roots of $h_i(T')$ $(i = 1, 2)$ are contained in $R_2[\![t_d]\!]$, where R_2 is obtained from R_1' by finite algebraic extensions and finite localizations. Any root of $P_t(T)$ is of the form $t^{\nu}\alpha$, where $\alpha \in R_2[\![t_d]\!]$ is a root of $h_1(T')$ or $h_2(T')$.

Case 2 In the case $Q_0(T') = (T' - \alpha)^n$, we have $n\alpha \in R$, and hence $\alpha \in R$. We have $\mathrm{ord}_t(a_n)/n = \mathrm{ord}_t(a_{n-1})/(n-1) = \nu$, and hence $\nu \in \mathbb{Z}$ and $d = 1$. We put $H_t(T) := P_t(T + t^{\nu}\alpha) = \sum_{j=0}^{n} c_j(t) T^j$. We have $\min\bigl(\mathrm{ord}(c_j)(n-j)^{-1}\bigr) > \nu$.

We continue the process. If we reach the case 1, we can reduce the degree with respect to T. If we do not reach the case 1, it is shown that $P_t(T) = (T - \mathfrak{a})^n$ for some $\mathfrak{a} \in R[\![t]\!]$ (resp. $\mathfrak{a} \in R(\!(t)\!)$). Thus we are done. Q.E.D.

Corollary 2.10. *Any $P(s,t)(T) \in R[\![s,t]\!][T]$ (resp. $R[\![s]\!](\!(t)\!)[T]$) has the roots in $R'_P(\!(s_d)\!)[\![t_d]\!]$ (resp. $R'_P(\!(s_d)\!)(\!(t_d)\!)$). Here R'_P is obtained from R, depending on P, by finite algebraic extensions and finite localizations, and d denotes an appropriate positive integer, depending on $\deg_T P$.* Q.E.D.

Let R be an integral domain such that $\mathbb{Z} \subset R$. Let K denote the quotient field of R. Let $(\mathcal{E}, \nabla)$ be a meromorphic connection on $R(\!(t)\!)$.

Lemma 2.11. *There exists an extension R' obtained from R by finite algebraic extensions and finite localizations, with the following property:*

- *The irregular values of $(\mathcal{E}, \nabla) \otimes K(\!(t)\!)$ are contained in $R'(\!(t_d)\!)$.*
- *The irregular decomposition and a Deligne–Malgrange lattice are defined on $R'(\!(t_d)\!)$.*

Proof We need only a minor modification for the argument given in [15], and hence we give just an outline. We put $D = \nabla(t\partial_t)$. Let $\mathcal{K}$ be the quotient field of $R(\!(t)\!)$. By applying the argument of Deligne [4] to $\mathcal{E} \otimes \mathcal{K}$ with the derivation D, we can take $e \in \mathcal{E}$ such that $e, D(e), \ldots, D^{r-1}(e)$ give a base of the $\mathcal{K}$-vector space $\mathcal{E} \otimes \mathcal{K}$. We have the relation $D^r e +$

$\sum_{j=0}^{r-1} a_j \cdot D^j e = 0$ where $a_j \in \mathcal{K}$. There exists a finite localization R_1 of R such that $a_j \in R_1((t))$.

We put $\nu := \min_j\{\mathrm{ord}_t(a_j)/(r-j)\}$. Note that $\nu \geq 0$ implies the regularity of the connection. Let d denote the minimal positive integer such that $d \cdot \nu \in \mathbb{Z}$. We put $f_{i+1} := t^{-\nu \cdot i} D^i e$ $(i = 0, \ldots, r-1)$, and $\boldsymbol{f} = (f_i \,|\, i = 1, \ldots, r)$. Let $A \in M_r\big(R_1((t_d))\big)$ be determined by $D\boldsymbol{f} = \boldsymbol{f}A$. Then, A is of the form $t^{\nu}\big(A_0 + t_d \cdot A_1(t_d)\big)$ such that (i) $A_1 \in M_r\big(R_1[\![t_d]\!]\big)$, (ii) $A_0 \in M_r(R_1)$ whose (i,j)-entries are as follows:

$$\big(A_0\big)_{i,j} = \begin{cases} 1 & (i = j+1) \\ -(t^{(-r+i-1)\nu} a_{i-1})_{|t_d=0} & (j = r) \\ 0 & (\text{otherwise}) \end{cases}$$

By the choice, one of $(A_0)_{i,r}$ is not 0.

Case 1 Let us consider the case in which A_0 has at least two distinct eigenvalues. There exists a finite extension R_2 such that (i) we have $G \in \mathrm{GL}_r(R_2)$ for which $G^{-1}A_0G$ is Jordan, (ii) the difference of any two distinct eigenvalues of A_0 are invertible in R_2. By a standard argument (see [15], or the proof of Lemma 2.17 below), we can show that there exists $G_1 \in \mathrm{GL}_r\big(R_2[\![t_d]\!]\big)$ such that (i) $G_{1\,|\,t_d=0} = G$, (ii) let $\boldsymbol{g} = \boldsymbol{f} \cdot G_1$ and $D\boldsymbol{g} = \boldsymbol{g} \cdot B$, then B is decomposed into a direct sum of matrices with smaller sizes. Hence, we obtain a decomposition into connections with lower ranks. Thus, we can reduce the problem to the lower rank case.

Case 2 If A_0 has the unique eigenvalues $\alpha \in R_1$, it can be shown that $d = 1$ and $\nu \in \mathbb{Z}_{<0}$, as in the proof of Lemma 2.9. We put $\nabla' = \nabla - t^{\nu}\alpha \cdot dt/t$ and $D' = \nabla'(t\partial_t)$. Let $\mathcal{K}_1$ be the quotient field of $R_1((t))$. It can be shown that $e, D'e, \ldots, (D')^{r-1}e$ give a base of $\mathcal{E} \otimes \mathcal{K}_1$. Let a'_j be determined by $D'^{\,r} e + \sum a'_j \cdot D'^{\,j} e = 0$. Then, we have $a'_j \in R_1((t))$ and $\nu(\nabla') = \min\{\mathrm{ord}_t(a'_j)/(r-j)\} \geq \nu + |\nu|/r$. We continue the process. After the finite steps, we will arrive at the case 1 or the case $\nu(\nabla') \geq 0$.

Q.E.D.

Corollary 2.12. *For a meromorphic connection $(\mathcal{E}, \nabla)$ on $R[\![s]\!]((t))$, there exists an extension R', which is obtained from R by finite algebraic extensions and finite localizations, with the following property:*

- *The irregular values of $(\mathcal{E}, \nabla) \otimes K((s))((t))$ are contained in $R'((s_d))((t_d))$.*
- *The irregular decomposition and a Deligne–Malgrange lattice are defined on $R'((s_d))((t_d))$.* *Q.E.D.*

2.4. Good formal structure

Let X be a complex algebraic surface, with a simple normal crossing divisor D. Let $(\mathcal{E},\nabla)$ be a meromorphic flat connection on (X,D). We recall the notion of good formal structure by following [23].

If P is a smooth point of D, we take a holomorphic coordinate (U,t,s) around P such that $t^{-1}(0)=U\cap D$. For a positive integer d, we take a ramified covering $\varphi_d : U_d \longrightarrow U$ given by $(t_d,s)\longmapsto (t_d^d,s)$. We put $D_d:=\{t_d=0\}\subset U_d$. Let $M(U_d,D_d)$ (resp. $H(U_d)$) denote the space of meromorphic (resp. holomorphic) functions on U_d whose poles are contained in D_d. For any element $\mathfrak{a}$ of $M(U_d,D_d)/H(U_d)$, we have the natural lift to $M(U_d,D_d)$ which is also denoted by $\mathfrak{a}$. Let $\widehat{D}_d$ denote the formal space obtained as the completion of U_d along D_d. (See [3], for example.)

Definition 2.13. *We say that $(\mathcal{E},\nabla)$ has the good formal structure at P, if the following holds for some (U,t,s) and some $d\in\mathbb{Z}_{>0}$:*

- *We have the finite subset $\mathrm{Irr}(\mathcal{E},\nabla)\subset M(U_d)/H(U_d)$ and the decomposition:*

$$\varphi_d^*(\mathcal{E},\nabla)_{|\widehat{D}_d}=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\mathcal{E},\nabla)}(\mathcal{E}_{\mathfrak{a}},\nabla_{\mathfrak{a}})$$

  Here $\nabla_{\mathfrak{a}}^{\mathrm{reg}}:=\nabla_{\mathfrak{a}}-d\mathfrak{a}\cdot\mathrm{id}_{\mathcal{E}_{\mathfrak{a}}}$ are regular.
- *For any non-zero $\mathfrak{a}\in\mathrm{Irr}(\mathcal{E},\nabla)$, the 0-divisor of $\mathfrak{a}$ has no intersection with D_d.*
- *For any two distinct $\mathfrak{a},\mathfrak{b}\in\mathrm{Irr}(\mathcal{E},\nabla)$, the 0-divisor of $\mathfrak{a}-\mathfrak{b}$ has no intersection with D_d.*

If P is a cross point of D, we take a holomorphic coordinate (U,t,s) such that $D\cap U=\{t\cdot s=0\}$. For each $d\in\mathbb{Z}_{>0}$, we take a ramified covering $\varphi_d : U_d\longrightarrow U$ given by $(t_d,s_d)\longmapsto(t_d^d,s_d^d)$. We put $D_d:=\{t_d\cdot s_d=0\}$ and $P_d:=(0,0)$. Let $\widehat{P}_d$ denote the formal space obtained as the completion of U_d at P_d.

Let $M(U_d,D_d)$ (resp. $H(U_d)$) denote the space of the meromorphic (holomorphic) functions on U_d whose poles are contained in D_d. For any element $\mathfrak{a}$ of $M(U_d,D_d)/H(U_d)$, we have the natural lift to $M(U_d,D_d)$, which is also denoted by $\mathfrak{a}$.

We use the partial order $\leq_{\mathbb{Z}^2}$ on $\mathbb{Z}^2$ given by $(a_1,a_2)\leq_{\mathbb{Z}^2}(a_1',a_2')\Longleftrightarrow a_i\leq a_i'$ $(i=1,2)$. For any element $f=\sum f_{i,j}\cdot s^i\cdot t^j\in M(U_d,D_d)$, let $\mathrm{ord}(f)$ denote the minimum of the set $\min\big\{(i,j)\,\big|\,f_{i,j}\neq 0\big\}$, if it exists.

Definition 2.14. *We say that $(\mathcal{E},\nabla)$ has the good formal structure if the following holds:*

- *We have the finite subset* $\mathrm{Irr}(\mathcal{E},\nabla) \subset M(U_d)/H(U_d)$ *and the decomposition for some* $d \in \mathbb{Z}_{>0}$*:*

$$\varphi_d^*(\mathcal{E},\nabla)_{|\widehat{P}_d} = \bigoplus_{\mathfrak{a}\in \mathrm{Irr}(\mathcal{E},\nabla)} (\mathcal{E}_{\mathfrak{a}},\nabla_{\mathfrak{a}})$$

Here $\nabla_{\mathfrak{a}}^{\mathrm{reg}} := \nabla_{\mathfrak{a}} - d\mathfrak{a}\cdot \mathrm{id}_{\mathcal{E}_{\mathfrak{a}}}$ *are regular.*
- $\mathrm{ord}(\mathfrak{a})$ *exists in* $\mathbb{Z}^2_{\leq 0} - \{(0,0)\}$ *for each non-zero* $\mathfrak{a} \in \mathrm{Irr}(\mathcal{E},\nabla)$.
- $\mathrm{ord}(\mathfrak{a}-\mathfrak{b})$ *exists in* $\mathbb{Z}^2_{\leq 0} - \{(0,0)\}$ *for any two distinct* $\mathfrak{a},\mathfrak{b} \in \mathrm{Irr}(\mathcal{E},\nabla)$. *And the set* $\{\mathrm{ord}(\mathfrak{a}-\mathfrak{b}) \mid \mathfrak{a},\mathfrak{b} \in \mathrm{Irr}(\mathcal{E},\nabla)\}$ *is totally ordered with respect to the above order* $\leq_{\mathbb{Z}^2}$.

Definition 2.15. *A point* P *is called turning with respect to* $(\mathcal{E},\nabla)$, *if* $(\mathcal{E},\nabla)$ *does not have a good formal structure at* P.

2.5. A sufficient condition for the existence of the good formal structure

2.5.1. *Preliminary* Let E be a free $\boldsymbol{C}[\![s,t]\!]$-module. Let $\nabla_t : E \longrightarrow E\otimes\Omega^1_{\boldsymbol{C}[\![s]\!]((t))/\boldsymbol{C}[\![s]\!]}[s^{-1}]$ be a connection such that the following holds for some $k \geq 1$ and $p \geq 0$:

$$\nabla_t(t^{k+1}s^p\partial_t)(E) \subset E$$

In that case, $\nabla_t\big(t^{k+1}s^p\partial_t\big)$ induces an endomorphism of $E_0 := E_{|t=0}$, which is denoted by F_0.

Lemma 2.16. *If* F_0 *is invertible, any meromorphic flat section* $f = \sum_{j\geq -N} f_j\cdot t^j$ *of* E *is* 0.

Proof Let f be a meromorphic flat section of E. Assume $f \neq 0$. We may assume that $-N = \min\{j \mid f_j \neq 0\}$. From $\nabla(t^{k+1}s^p\partial_t)f = 0$, we have $F_0(f_{-N}) = 0$. Because F_0 is invertible, we obtain $f_{-N} = 0$, which contradicts with the choice of N. Q.E.D.

Lemma 2.17. *Assume the following:*

- *We have a decomposition* $(E_0,F_0) = (E_0^{(1)},F_0^{(1)})\oplus(E_0^{(2)},F_0^{(2)})$.
- *The eigenvalues of* $F_0^{(i)}$ *are contained in* $\boldsymbol{C}[\![s]\!]$. *If* $\mathfrak{b}_i$ $(i=1,2)$ *are eigenvalues of* $F_0^{(i)}$, *we have* $(\mathfrak{b}_1-\mathfrak{b}_2)_{|s=0} \neq 0$.

Then, we have the unique ∇_t*-flat decomposition* $E = E^{(1)}\oplus E^{(2)}$ *such that the restriction to* $t=0$ *is the same as* $E_0 = E_0^{(1)}\oplus E_0^{(2)}$.

Proof We closely follow the argument in [15]. Let $\boldsymbol{v}$ be a frame of E such that $\boldsymbol{v}_{|t=0}$ is compatible with the decomposition $E_0 = E_0^{(1)}\oplus E_0^{(2)}$.

Then, $\boldsymbol{v}$ is divided as $(\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)})$, where $\boldsymbol{v}^{(i)}_{|t=0}$ are frames of $E_0^{(i)}$. Let $A = \sum_{j=0}^{\infty} A_j(s) \cdot t^j$ be determined by the following:

$$\nabla(t^{k+1}s^p\partial_t)\boldsymbol{v} = \boldsymbol{v} \cdot A.$$

We have the following decomposition corresponding to the decomposition of the frame $\boldsymbol{v} = (\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)})$:

$$A_j = \begin{pmatrix} A_j^{(11)} & A_j^{(12)} \\ A_j^{(21)} & A_j^{(22)} \end{pmatrix} \tag{2}$$

By the assumption, we have $A_0^{(12)} = 0$ and $A_0^{(21)} = 0$. For a change of frames from $\boldsymbol{v}$ to $\boldsymbol{v} \cdot G$, we have the following:

$$\nabla(t^{k+1}s^p\partial_t)\big(\boldsymbol{v}\cdot G\big) = \big(\boldsymbol{v}\cdot G\big)\cdot\widetilde{A}(G), \quad \widetilde{A}(G) := G^{-1}AG + t^{k+1}s^pG^{-1}\partial_t G$$

We consider a formal transform G of the following form:

$$G = I + \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}, \quad X = \sum_{j=1}^{\infty} X_j(s)\cdot t^j, \quad Y = \sum_{j=1}^{\infty} Y_j(s)\cdot t^j$$

Here the entries of $X_j(s)$ and $Y_j(s)$ are contained in $\boldsymbol{C}[\![s]\!]$. We want to determine X_j and Y_j by the following condition:

- The $(1,2)$-component and the $(2,1)$-component of $\widetilde{A}(G)$ are 0.
- The $(1,1)$-component of $\widetilde{A}(G)$ is of the form $A_0^{(11)} + B^{(11)}$, where the entries of $B^{(11)}$ are contained in $t\cdot\boldsymbol{C}[\![s,t]\!]$. Similarly, the $(2,2)$-component is of the form $A_0^{(22)} + B^{(22)}$, where the entries of $B^{(22)}$ are contained in $t \cdot \boldsymbol{C}[\![s,t]\!]$.

We obtain the following equations for Y and $B^{(11)}$:

$$A^{(11)} + A^{(12)}Y - A_0^{(11)} - B^{(11)} = 0,$$

$$A^{(21)} + A^{(22)}Y + t^{k+1}s^p\partial_t Y - Y(A_0^{(11)} + B^{(11)}) = 0$$

Then, we obtain the following equation for Y:

$$\begin{aligned} A_0^{(22)}Y - YA_0^{(11)} - Y(A^{(11)} - A_0^{(11)}) + (A^{(22)} - A_0^{22})Y \\ - YA^{(12)}Y + t^{k+1}s^p\partial_t Y + A^{(21)} = 0 \end{aligned}$$

For the expansion $Y = \sum_{j=1}^{\infty} Y_j(s) \cdot t^j$, we obtain the following equations:

$$(3)\quad A_0^{(22)} Y_j - Y_j A_0^{(11)} - \sum_{\substack{l+m=j\\ l,m\geq 1}} Y_l A_m^{(11)} + \sum_{\substack{l+m=j\\ l,m\geq 1}} A_l^{(22)} Y_m \\ - \sum_{\substack{l+m+n=j,\\ l,m,n\geq 1}} Y_l A_m^{(12)} Y_n + (j-k) s^p Y_{j-k} \cdot \chi_{j\geq k} + A_j^{(21)} = 0$$

Here $\chi_{j\geq k} = 0$ if $j < k$ and $\chi_{j\geq k} = 1$ if $j \geq k$. When we are given Y_m $(1 \leq m \leq j-1)$ whose entries are contained in $\boldsymbol{C}[\![s]\!]$, we have the unique solution Y_j of (3), whose entries are contained in $\boldsymbol{C}[\![s]\!]$. Hence, we have appropriate Y and $B^{(11)}$. Similarly, we have appropriate X and $B^{(22)}$. Thus, we can conclude the existence of the desired decomposition $E = E^{(1)} \oplus E^{(2)}$. The uniqueness follows from Lemma 2.16. Q.E.D.

Let us consider the case in which ∇_t comes from a flat meromorphic connection $\nabla : E \longrightarrow E \otimes \Omega^1_{\boldsymbol{C}[\![s]\!]((t))/\boldsymbol{C}}[s^{-1}]$.

Lemma 2.18. *Assume the hypothesis in Lemma* 2.17. *The decomposition $E = E^{(1)} \oplus E^{(2)}$ is ∇-flat.*

Proof We may assume $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$ is compatible with the decomposition $E = E^{(1)} \oplus E^{(2)}$. Let A and B be determined by the following:

$$\nabla(t^{k+1}\partial_t)\boldsymbol{v} = \boldsymbol{v} \cdot A, \quad A = \begin{pmatrix} A^{(11)} & 0 \\ 0 & A^{(22)} \end{pmatrix}$$

$$\nabla(\partial_s)\boldsymbol{v} = \boldsymbol{v} \cdot B, \quad B = \begin{pmatrix} B^{(11)} & B^{(12)} \\ B^{(21)} & B^{(22)} \end{pmatrix}$$

From the relation $\left[\nabla(\partial_s), \nabla(t^{k+1}\partial_t)\right]$, we have the following equation for $B^{(12)}$:

$$A^{(11)} B^{(12)} - B^{(12)} A^{(22)} + t^{k+1} \partial_t B^{(12)} = 0$$

Assume $B^{(12)} \neq 0$. We have the expression $B^{(12)} = \sum_{j\geq -N} B_j^{(12)} \cdot t^j$, and we may assume $B_{-N}^{(12)} \neq 0$. However, we have the relation $B_{-N}^{(12)} A_0^{(11)} - A_0^{(22)} B_{-N}^{(12)} = 0$, and hence $B_{-N}^{(12)} = 0$. Thus, we arrive at the contradiction, and we can conclude $B^{(12)} = 0$. Similarly, we obtain $B^{(21)} = 0$. Q.E.D.

2.5.2. *A condition* Let E be a free $\boldsymbol{C}[\![s,t]\!]$-module with a flat meromorphic connection $\nabla : E \longrightarrow E \otimes \Omega^1_{\boldsymbol{C}[\![s]\!]((t))/\boldsymbol{C}}$. We have the induced relative connection $\nabla_t : E \longrightarrow E \otimes \Omega^1_{\boldsymbol{C}[\![s]\!]((t))/\boldsymbol{C}[\![s]\!]}$.

We put $\mathfrak{K} := \boldsymbol{C}((s))((t))$. We put $(\mathcal{E}_{\mathfrak{K}}, \nabla_{\mathfrak{K}}) := (E, \nabla_t) \otimes \mathfrak{K}$ and $E_{\mathfrak{K}} := E \otimes \boldsymbol{C}((s))[\![t]\!]$. We assume that $E_{\mathfrak{K}}$ is a strict Deligne–Malgrange lattice. The intersection of $E_{\mathfrak{K}}$ and $E(*t)$ in $\mathcal{E}_{\mathfrak{K}}$ is the same as E, which gives a characterization of E.

Proposition 2.19. *Assume the following:*

- $\mathrm{Irr}(\mathcal{E}_{\mathfrak{K}}, \nabla_{\mathfrak{K}})$ *is contained in* $\boldsymbol{C}[\![s]\!]((t))/\boldsymbol{C}[\![s,t]\!]$.
- $\mathrm{Irr}(\mathcal{E}_{\mathfrak{K}}, \nabla_{\mathfrak{K}})$ *is good, in the following sense:*
  - *Let* $\mathfrak{a} \in \mathrm{Irr}(\mathcal{E}_{\mathfrak{K}}, \nabla_{\mathfrak{K}})$ *be non-zero. We have* $\mathfrak{a}_{\mathrm{ord}_t(\mathfrak{a})}(0) \neq 0$ *for the expression* $\mathfrak{a} = \sum_{j \geq \mathrm{ord}_t(\mathfrak{a})} \mathfrak{a}_j(s) \cdot t^j$.
  - *Similarly, we have* $(\mathfrak{a} - \mathfrak{b})_{\mathrm{ord}_t(\mathfrak{a}-\mathfrak{b})}(0) \neq 0$ *for any two distinct* $\mathfrak{a}, \mathfrak{b} \in \mathrm{Irr}(\mathcal{E}_{\mathfrak{K}}, \nabla_{\mathfrak{K}})$.

Then, $\big(E(*t), \nabla\big)$ *has the good formal structure.*

Proof We put $k(E) := -\min\big\{\mathrm{ord}_t(\mathfrak{a}) \,\big|\, \mathfrak{a} \in \mathrm{Irr}(\nabla_{\mathfrak{K}})\big\}$. (We define $\mathrm{ord}_t(0) := 0$.) Assume $k(E) \geq 1$. Because $E_{\mathfrak{K}}$ is a Deligne–Malgrange lattice of $\mathcal{E}_{\mathfrak{K}}$, we have $\nabla(t^{k(E)+1}\partial_t)E \subset E$. Let F_0 denote the endomorphism of $E_{|0}$ induced by $\nabla(t^{k(E)+1}\partial_t)$. The eigenvalues of F_0 are given by $(t^{k(E)+1}\partial_t\mathfrak{a})_{|t=0}$ ($\mathfrak{a} \in \mathrm{Irr}(\nabla_{\mathfrak{K}})$). By using Lemma 2.17 and Lemma 2.18, we obtain the decomposition:

$$(E, \nabla) = \bigoplus_{\mathfrak{b} \in S} (E_{\mathfrak{b}}, \nabla_{\mathfrak{b}}), \quad S := \Big\{\mathfrak{b} = \big(t^{k(E)}\mathfrak{a}\big)_{|t=0} \in \boldsymbol{C}[\![s]\!] \,\Big|\, \mathfrak{a} \in \mathrm{Irr}(\nabla_{\mathfrak{K}})\Big\}$$

Take any $\mathfrak{a} \in \mathrm{Irr}(\nabla_{\mathfrak{K}})$ such that $(t^{k(E)}\mathfrak{a})_{|t=0} = \mathfrak{b}$, and put $\nabla'_{\mathfrak{b}} := \nabla_{\mathfrak{b}} - d\mathfrak{a}$. Then, $(E_{\mathfrak{b}}, \nabla'_{\mathfrak{b}})$ also satisfy the assumption of this lemma, and we have $k(E_{\mathfrak{b}}) \leq k(E) - 1$. If $k(E_{\mathfrak{b}}) \geq 1$, we may apply the above argument to $(E_{\mathfrak{b}}, \nabla'_{\mathfrak{b}})$. By the inductive argument, we obtain the flat decomposition $(E, \nabla) = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla_{\mathfrak{K}})} (E_{\mathfrak{a}}, \nabla_{\mathfrak{a}})$ such that $\nabla^{\mathrm{reg}}_{\mathfrak{a}}(t\partial_t)(E_{\mathfrak{a}}) \subset E_{\mathfrak{a}}$ for $\nabla^{\mathrm{reg}}_{\mathfrak{a}} := \nabla_{\mathfrak{a}} - d\mathfrak{a}$.

Let us show $\nabla^{\mathrm{reg}}_{\mathfrak{a}}(\partial_s)E_{\mathfrak{a}} \subset E_{\mathfrak{a}}$. Let $\boldsymbol{v}_{\mathfrak{a}}$ be a frame of $E_{\mathfrak{a}}$. Let A and B be determined by the following:

$$\nabla^{\mathrm{reg}}_{\mathfrak{a}}(t\partial_t)\boldsymbol{v}_{\mathfrak{a}} = \boldsymbol{v}_{\mathfrak{a}} \cdot A, \quad A = \sum_{j=0}^{\infty} A_j \cdot t^j,$$

$$\nabla^{\mathrm{reg}}_{\mathfrak{a}}(\partial_s)\boldsymbol{v}_{\mathfrak{a}} = \boldsymbol{v}_{\mathfrak{a}} \cdot B, \quad B = \sum_{j=-N}^{\infty} B_j \cdot t^j$$

From the commutativity $\big[\nabla^{\mathrm{reg}}_{\mathfrak{a}}(t\partial_t),\, \nabla^{\mathrm{reg}}_{\mathfrak{a}}(\partial_s)\big] = 0$, we have the following equation:

$$AB + t\partial_t B - BA + \partial_s A = 0.$$

Assume $N > 0$. Then, we have the equation $A_0 B_{-N} - B_{-N} A_0 - N B_{-N} = 0$. Because $\alpha - \beta \notin \mathbb{Z} - \{0\}$ for two distinct eigenvalues of A_0, we obtain $B_{-N} = 0$. Hence, we have $N \leq 0$, i.e., the entries of B are contained in $\boldsymbol{C}[\![t, s]\!]$. Q.E.D.

2.6. Adjustment of the residue of a logarithmic connection

Let k be a field whose characteristic number is 0. Let E be a free $k[\![t]\!]$-module with a meromorphic connection ∇ such that $t\nabla(\partial_t)(E) \subset E$. Let E_0 denote the specialization of E at $t = 0$. We have the well defined endomorphism $\mathrm{Res}(\nabla)$ of E_0. To distinguish the dependence on E, we denote it by $\mathrm{Res}_E(\nabla)$. We recall the following standard lemma.

Lemma 2.20. *We can take a lattice E' of $E \otimes k(\!(t)\!)$ such that (i) ∇ is logarithmic with respect to E', (ii) $\alpha - \beta \notin \mathbb{Z}$ for any distinct eigenvalues of* $\mathrm{Res}_{E'}(\nabla)$.

Proof We give only an outline. Let S denote the set of the eigenvalues of $\mathrm{Res}_E(\nabla)$. We say $\alpha \leq \beta$ for $\alpha, \beta \in S$ if $\beta - \alpha \in \mathbb{Z}_{\geq 0}$. It determines the partial order on S. We put $\rho(E) := \max\bigl\{\beta - \alpha \bigm| \alpha \leq \beta,\ \alpha, \beta \in S\bigr\}$. If $\rho(E) = 0$, we have nothing to do. We will reduce the number $\rho(E)$ by replacing E.

Let S denote the maximal elements β of S such that there exists $\alpha \in S$ with $\alpha < \beta$. Let $\overline{k}$ denote the algebraic closure of k. We have the generalized eigen decomposition $E_0 \otimes \overline{k} = \bigoplus_{\alpha \in S} \mathbb{E}_\alpha$. Note that S is preserved by the action of the Galois group of $\overline{k}$ over k. It is easy to see that $\bigoplus_{\alpha \in S} \mathbb{E}_\alpha$ comes from the subspace V of E_0. Let $E^{(1)} := t^{-1} \cdot E$. The specialization $E_0^{(1)}$ of $E^{(1)}$ at $t = 0$ is naturally isomorphic to E_0 up to constant multiplication. Hence, V determines the subspace $V^{(1)} \subset E_0^{(1)}$. Let $E^{(2)}$ denote the kernel of the naturally defined morphism $E^{(1)} \longrightarrow E_0^{(1)}/V^{(1)}$. Then, it can be checked $\rho(E^{(2)}) \leq \rho(E) - 1$. Q.E.D.

§3. Mildly ramified connection

3.1. Positive characteristic case

Let k be an algebraically closed field whose characteristic number p is positive. Let C be a smooth divisor of $\mathrm{Spec}^f\, k[\![s, t]\!]$, which intersects with the divisor $\{t = 0\}$ transversally. We can take a morphism $\mathrm{Spec}^f\, k[\![u]\!] \simeq C \subset \mathrm{Spec}^f\, k[\![s, t]\!]$ given by $\bigl(s(u), t(u)\bigr)$. We may assume $t(u) = u$ and $s(u) = u \cdot h(u)$. We put $s' := s - h(t) \cdot t$. Then, C is given by the ideal generated by s'. We also have $k[\![s', t]\!] \simeq k[\![s, t]\!]$. For any positive integer d, let s'_d denote a d-th root of s'.

Let $\mathcal{E}$ be a free $k[\![s]\!]((t))$-module. Let $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{k[\![s]\!]((t))/k}$ be a flat meromorphic connection. Let ψ denote the p-curvature of ∇. Let Fr denote the absolute Frobenius map $k[\![s]\!]((t)) \longrightarrow k[\![s]\!]((t))$. It induces the ring homomorphism $\mathrm{Fr} : k[\![s]\!]((t))[T] \longrightarrow k[\![s]\!]((t))[T]$ by

$$\mathrm{Fr}\Bigl(\sum a_j \cdot T^j\Bigr) = \sum \mathrm{Fr}(a_j) \cdot T^j.$$

Due to an observation of Bost–Laszlo–Pauly ([14], see also Lemma 4.4 below), we have $P_s(T), P_t(T) \in k[\![s]\!]((t))[T]$ such that

$$\det\bigl(T - \psi(\partial_s)\bigr) = \mathrm{Fr}\bigl(P_s\bigr)(T), \quad \det\bigl(T - \psi(t\partial_t)\bigr) = \mathrm{Fr}\bigl(P_t\bigr)(T).$$

In general, the roots of the polynomials $P_s(T)$ and $P_t(T)$ are contained in $k((s_d))((t_d))$ for some appropriate integer d.

Definition 3.1. *Let C be a smooth divisor transversal with the divisor $\{t = 0\}$. We say that $(\mathcal{E}, \nabla)$ is mildly ramified along $\{t = 0\} \cup C$, if the following conditions are satisfied:*

(1) *The roots of the polynomials $P_s(T) = 0$ and $P_t(T) = 0$ are contained in $k[\![s'_d]\!]((t_d))$ for some $d \in \mathbb{Z}_{>0}$, where s'_d is taken for C as above.*
(2) *The roots are of the form $\alpha + \beta$, where $\alpha \in k[\![s']\!]((t_d))$ and $\beta \in k[\![s'_d, t_d]\!]$.*

We say that $(\mathcal{E}, \nabla)$ is mildly ramified, if it is mildly ramified along $\{t = 0\} \cup C$ for some C.

The connection ∇ induces the relative connection $\nabla_t : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_{k[\![s]\!]((t))/k[\![s]\!]}$. We put $\mathfrak{K} := k((s))((t))$ and $\mathfrak{k} := k((t))$. Both of them are equipped with the differential ∂_t. We have the natural inclusion $k[\![s]\!]((t)) \subset \mathfrak{K}$, and the specialization $k[\![s]\!]((t)) \longrightarrow \mathfrak{k}$ at $s = 0$. The morphisms are equivariant with respect to ∂_t. Therefore, we have the induced connections of $\mathcal{E} \otimes \mathfrak{K}$ and $\mathcal{E} \otimes \mathfrak{k}$, which are also denoted by ∇_t.

Lemma 3.2. *Assume that $(\mathcal{E}, \nabla)$ is mildly ramified at $\{t = 0\} \cup C$. Then, the irregular values of $\bigl(\mathcal{E} \otimes \mathfrak{K}, \nabla_t\bigr)$ are contained in $k[\![s]\!]((t_d))_{<0}$, and their specialization at $s = 0$ give the irregular values for $\bigl(\mathcal{E} \otimes \mathfrak{k}, \nabla_t\bigr)$. The induced map $\mathrm{Irr}(\mathcal{E} \otimes \mathfrak{K}, \nabla_t) \longrightarrow \mathrm{Irr}(\mathcal{E} \otimes \mathfrak{k}, \nabla_t)$ is surjective.*

Proof Let $\mathrm{Sol}(P_t)$ denote the set of the solutions of $P_t(T) = 0$. By assumption, any element of $\mathrm{Sol}(P_t)$ is of the form $\alpha + \beta$ as above. We have the natural map $\kappa_1 : k[\![s'_d]\!]((t_d)) \longrightarrow k((s'_d))((t_d)) \simeq k((s_d))((t_d))$. The image of $\mathrm{Sol}(P_t)$ via κ_1 gives the set of the solutions of $P_t(T) = 0$ in $k((s_d))((t_d))$. We remark that the image of $k[\![s'_d, t_d]\!]$ via κ_1 is contained in $k((s_d))[\![t_d]\!]$. Hence, we have $\kappa_1(\alpha + \beta)_- = \kappa_1(\alpha_-) \in k[\![s]\!]((t_d))_{<0}$ for any

$\alpha+\beta \in \mathrm{Sol}(P_t)$. Then, the first claim follows from the characterization of the irregular value given in Lemma 2.7.

On the other hand, let us take the specialization of $P_t(T)$ to $s=0$, which are denoted by $P_{t,0}(T) \in \mathfrak{k}[T]$. Let $\mathrm{Sol}(P_{t,0})$ denote the solution of the equation $P_{t,0}(T)=0$, which is contained in $k(\!(t_d)\!)$ for some appropriate d. Then, $\mathrm{Sol}(P_{t,0})$ is the image of $\mathrm{Sol}(P_t)$ by the composite κ_2 of the following morphisms:

$$k[\![s'_d]\!](\!(t_d)\!) \simeq k[\![s]\!](\!(t_d)\!)[U]/\bigl(U^d - s'(s,t)\bigr) \\ \longrightarrow k(\!(t_d)\!)[U]/\bigl(U^d - s'(0,t)\bigr) \longrightarrow k(\!(t_d)\!)$$

The last map is given by the substitution $U = s'(0,t)^{1/d} \in k(\!(t_d)\!)$ for some choice of $s'(0,t)^{1/d}$. Any element of $k[\![s'_d, t_d]\!]$ is mapped into $k[\![t_d]\!]$ via κ_2, and the image of any element of $k[\![s]\!](\!(t_d)\!) = k[\![s']\!](\!(t_d)\!)$ via κ_2 is given by the natural specialization at $s=0$. Hence, for any $\kappa_2(\alpha+\beta) \in \mathrm{Sol}\bigl(P_{t,0}(T)\bigr)$, we have $\kappa_2(\alpha+\beta)_- = \kappa_2(\alpha_-)$. Then, the second and third claims follow from the characterization of the irregular values in Lemma 2.7. Q.E.D.

Let $\varphi : \mathrm{Spec}^f\, k[\![v]\!] \longrightarrow \mathrm{Spec}^f\, k[\![s,t]\!]$ be a morphism given by $\varphi^*(s) = v\cdot \overline{h}_0(v)$ and $\varphi^*(t) = v^a$ for some $a>0$. We assume a is sufficiently smaller than p. We consider the morphism $\Phi : \mathrm{Spec}^f\, k[\![u,v]\!] \longrightarrow \mathrm{Spec}^f\, k[\![s,t]\!]$ given by $\Phi^* s = u + v\cdot \overline{h}_0(v)$ and $\Phi^* t = v^a$. Then, we have

$$\Phi^* s' = \Phi^*\bigl(s - h(t)\cdot t\bigr) = u + v\cdot \overline{h}_0(v) - h(v^a)\cdot v^a = u + v\cdot h_1(v).$$

In particular, the divisor $Y = \{\Phi^*(s') = 0\}$ is smooth and transversal to the divisor $\{v=0\}$.

Lemma 3.3. *$\Phi^*(E,\nabla)$ is mildly ramified along $\{v=0\}\cup Y$.*

Proof We have $\Phi^*(ds) = a_{1,1}\cdot du + a_{1,2}\cdot dv/v$ and $\Phi^*(dt/t) = a_{2,1}\cdot du + a_{2,2}\cdot dv/v$, where $a_{i,j}$ are contained in $k[\![u,v]\!]$. Let $\psi_{\Phi^*\nabla}$ be the p-curvature of $\Phi^*\nabla$. Due to a formula of O. Gabber (Appendix of [12]) we have the following:

$$\psi_{\Phi^*\nabla}(\partial_u) = a_{1,1}^p \cdot \Phi^*\bigl(\psi(\partial_s)\bigr) + a_{2,1}^p\cdot \Phi^*\bigl(\psi(t\partial_t)\bigr),$$

$$\psi_{\Phi^*\nabla}(v\partial_v) = a_{1,2}^p \cdot \Phi^*\bigl(\psi(\partial_s)\bigr) + a_{2,2}^p\cdot \Phi^*\bigl(\psi(t\partial_t)\bigr)$$

Then, it is easy to check the claim of the lemma by using the commutativity of $\psi(\partial_s)$ and $\psi(t\partial_t)$. Q.E.D.

3.2. Mixed characteristic case

Let R be a subring of $\boldsymbol{C}$ finitely generated over $\mathbb{Z}$. Let $\mathcal{E}_R$ be a free $R[\![s]\!]((t))$-module, and let $\nabla : \mathcal{E}_R \longrightarrow \mathcal{E}_R \otimes \Omega^1_{R[\![s]\!]((t))/R}$ be a meromorphic flat connection. For each $\eta \in S(R)$, we put $\mathcal{E}_{\overline{\eta}} := \mathcal{E}_R \otimes_{R[\![s]\!]((t))} k(\overline{\eta})[\![s]\!]((t))$, and we have the induced meromorphic flat connection ∇ of $\mathcal{E}_{\overline{\eta}}$.

Definition 3.4. *We say that $(\mathcal{E}_R, \nabla)$ is mildly ramified, if $(\mathcal{E}_{\overline{\eta}}, \nabla)$ is mildly ramified for any $\eta \in S(R)$. Note that the ramification curves may depend on η.*

If $(\mathcal{E}_R, \nabla)$ is mildly ramified, it is easy to show that $(\mathcal{E}_R, \nabla) \otimes_{R[\![s]\!]((t))} R'[\![s]\!]((t))$ is also mildly ramified for any $R' \subset \boldsymbol{C}$ finitely generated over R.

3.3. Complex number field case

Let $\mathcal{E}_{\boldsymbol{C}}$ be a free $\boldsymbol{C}[\![s]\!]((t))$-module with a meromorphic connection $\nabla : \mathcal{E}_{\boldsymbol{C}} \longrightarrow \mathcal{E}_{\boldsymbol{C}} \otimes \Omega^1_{\boldsymbol{C}[\![s]\!]((t))/\boldsymbol{C}}$.

Definition 3.5. *We say that $(\mathcal{E}_{\boldsymbol{C}}, \nabla)$ is algebraic, if there exists a subring $R \subset \boldsymbol{C}$ finitely generated over $\mathbb{Z}$, a free $R[\![s]\!]((t))$-module $\mathcal{E}_R$ with a meromorphic connection $\nabla : \mathcal{E}_R \longrightarrow \mathcal{E}_R \otimes \Omega^1_{R[\![s]\!]((t))/R}$ such that $(\mathcal{E}_R, \nabla) \otimes_{R[\![s]\!]((t))} \boldsymbol{C}[\![s]\!]((t)) \simeq (\mathcal{E}_{\boldsymbol{C}}, \nabla)$. Such $(\mathcal{E}_R, \nabla)$ is called an R-model of $(\mathcal{E}_{\boldsymbol{C}}, \nabla)$.*

Definition 3.6. *Let $(\mathcal{E}_{\boldsymbol{C}}, \nabla)$ be algebraic. We say $(\mathcal{E}_{\boldsymbol{C}}, \nabla)$ is mildly ramified, if an R-model of $(\mathcal{E}_{\boldsymbol{C}}, \nabla)$ is mildly ramified for some R.*

We put $\mathfrak{K}_{\boldsymbol{C}} := \boldsymbol{C}((s))((t))$ and $\mathfrak{k}_{\boldsymbol{C}} := \boldsymbol{C}((t))$. We have the induced relative connection $\nabla_t : \mathcal{E}_{\boldsymbol{C}} \longrightarrow \mathcal{E}_{\boldsymbol{C}} \otimes \Omega^1_{\boldsymbol{C}[\![s]\!]((t))/\boldsymbol{C}[\![s]\!]}$. We put $\big(\mathcal{E}_{\mathfrak{K}_{\boldsymbol{C}}}, \nabla_t\big) := \big(\mathcal{E}_{\boldsymbol{C}}, \nabla_t\big) \otimes \mathfrak{K}_{\boldsymbol{C}}$ and $\big(\mathcal{E}_{\mathfrak{k}_{\boldsymbol{C}}}, \nabla_t\big) := \big(\mathcal{E}_{\boldsymbol{C}}, \nabla_t\big) \otimes \mathfrak{k}_{\boldsymbol{C}}$.

Proposition 3.7. *Assume that $(\mathcal{E}_{\boldsymbol{C}}, \nabla)$ is algebraic and mildly ramified. Then the irregular values of $(\mathcal{E}_{\mathfrak{K}_{\boldsymbol{C}}}, \nabla_t)$ are contained in $\boldsymbol{C}[\![s]\!]((t_d))_{<0}$ for some $d \in \mathbb{Z}_{>0}$, and their specializations at $s = 0$ give the irregular values of $(\mathcal{E}_{\mathfrak{k}_{\boldsymbol{C}}}, \nabla_t)$. The induced map $\mathrm{Irr}(\mathcal{E}_{\mathfrak{K}_{\boldsymbol{C}}}, \nabla_t) \longrightarrow \mathrm{Irr}(\mathcal{E}_{\mathfrak{k}_{\boldsymbol{C}}}, \nabla_t)$ is surjective.*

Proof We take a subring $R \subset \boldsymbol{C}$ finitely generated over $\mathbb{Z}$, and an R-model $(\mathcal{E}_R, \nabla)$ of $(\mathcal{E}_{\boldsymbol{C}}, \nabla)$. We may assume that the irregular decomposition of $(\mathcal{E}_{\mathfrak{K}_{\boldsymbol{C}}}, \nabla_t)$ is defined on $R((s_d))((t_d))$ (Corollary 2.12):

$$(4) \qquad (\mathcal{E}_R, \nabla_t) \otimes R((s_d))((t_d)) = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathcal{E}_{\mathfrak{K}_{\boldsymbol{C}}}, \nabla_t)} \big(\mathcal{E}_{\mathfrak{a}}, \nabla_{\mathfrak{a},t}\big)$$

We may also have a Deligne–Malgrange lattice $\bigoplus E_{\mathfrak{a}} \subset \bigoplus \mathcal{E}_{\mathfrak{a}}$.

Let p be a sufficiently large prime, and let η be any point of $S(R,p)$. We put $\mathfrak{K}_{\overline{\eta}} := k(\overline{\eta})((s_d))((t_d))$ and $\mathfrak{k}_{\overline{\eta}} := k(\overline{\eta})((t_d))$. We have the decomposition of $(\mathcal{E}_{\mathfrak{K}_{\overline{\eta}}}, \nabla_t) := (\mathcal{E}_R, \nabla_t) \otimes \mathfrak{K}_{\overline{\eta}}$ induced by (4):

$$(\mathcal{E}_{\mathfrak{K}_{\overline{\eta}}}, \nabla_t) = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathcal{E}_{\mathfrak{K}_C}, \nabla_t)} \big(\mathcal{E}_{\mathfrak{a},\overline{\eta}}, \nabla_{\mathfrak{a},t}\big)$$

Let $\mathcal{F}_{\overline{\eta}}$ denote the naturally induced morphism $R((s_d))((t_d)) \longrightarrow \mathfrak{K}_{\overline{\eta}}$ and $R((t_d)) \longrightarrow \mathfrak{k}_{\overline{\eta}}$. Since $\nabla_{\mathfrak{a},t} - d\mathfrak{a} \cdot \mathrm{id}_{\mathcal{E}_{\mathfrak{a},\overline{\eta}}}$ are logarithmic with respect to the lattice $E_{\mathfrak{a},\overline{\eta}}$, we can conclude that $\mathrm{Irr}\big(\mathcal{E}_{\mathfrak{K}_{\overline{\eta}}}, \nabla_t\big)$ is the image of $\mathrm{Irr}\big(\mathcal{E}_{\mathfrak{K}_C}, \nabla_t\big)$ via the map $\mathcal{F}_{\overline{\eta}}$. Due to Lemma 3.2, $\mathcal{F}_{\overline{\eta}}(\mathfrak{a})$ are contained in $k(\overline{\eta})[\![s]\!]((t_d))_{<0}$ for any $\mathfrak{a} \in \mathrm{Irr}(\mathcal{E}_{\mathfrak{k}_C}, \nabla_t)$. Then, it follows that $\mathfrak{a}$ are contained in $R[\![s]\!]((t_d))_{<0}$. Moreover, $\mathcal{F}_{\overline{\eta}}\big(\mathfrak{a}_{|s=0}\big) = \mathcal{F}_{\overline{\eta}}\big(\mathfrak{a}\big)_{|s=0}$ give the irregular values of $(\mathcal{E}_{\mathfrak{k}_{\overline{\eta}}}, \nabla_t)$ for any $\mathfrak{a} \in \mathrm{Irr}(\mathcal{E}_{\mathfrak{K}_C}, \nabla_t)$, due to Lemma 3.2. To conclude that $\mathfrak{a}_{|s=0}$ gives the irregular values of $(\mathcal{E}_{\mathfrak{k}_C}, \nabla_t)$, we use the following lemma.

Lemma 3.8. *Let $(\mathcal{E}, \nabla)$ be a meromorphic connection on $R((t))$. Let $\mathfrak{a} \in R((t_d))_{<0}$. If $\mathcal{F}_{\overline{\eta}}(\mathfrak{a})$ are the irregular values for $(\mathcal{E}, \nabla)_{\overline{\eta}}$ on $k(\overline{\eta})((t))$ for any $\overline{\eta}$, then $\mathfrak{a}$ is an irregular value for $(\mathcal{E}, \nabla) \otimes \boldsymbol{C}((t))$.*

Proof Due to Corollary 2.12, we may assume to have the irregular decomposition $(\mathcal{E}, \nabla) = \bigoplus_i (\mathcal{E}_i, d\mathfrak{a}_i + \nabla_i^{\mathrm{reg}})$ on $R((t_d))$. Then, for some i, there are infinitely many $\eta \in S(R)$ such that $\mathcal{F}_{\overline{\eta}}(\mathfrak{a}) - \mathcal{F}_{\overline{\eta}}(\mathfrak{a}_i) = 0$ in $k(\overline{\eta})((t_d))_{<0}$. It implies $\mathfrak{a} = \mathfrak{a}_i$. Thus, we obtain Lemma 3.8. Q.E.D.

Let us return to the proof of Proposition 3.7. Let $\mathfrak{b} \in \mathrm{Irr}(\mathcal{E}_{\mathfrak{k}_C}, \nabla_t)$. Due to the surjectivity in Lemma 3.2, there exists $\mathfrak{a} \in \mathrm{Irr}(\mathcal{E}_{\mathfrak{K}_C}, \nabla_t)$ such that $\mathcal{F}_{\overline{\eta}}(\mathfrak{a}_{|s=0}) = \mathcal{F}_{\overline{\eta}}(\mathfrak{b})$ in $k((\overline{\eta}))((t_d))_{<0}$ for infinitely many $\eta \in S(R)$. It implies $\mathfrak{a}_{|s=0} = \mathfrak{b}$. Hence, we obtain the surjectivity of the induced map $\mathrm{Irr}(\mathcal{E}_{\mathfrak{K}_C}, \nabla_t) \longrightarrow \mathrm{Irr}(\mathcal{E}_{\mathfrak{k}_C}, \nabla_t)$. Thus the proof of Proposition 3.7 is finished. Q.E.D.

Let $\varphi_C : \mathrm{Spec}^f \boldsymbol{C}[\![v]\!] \longrightarrow \mathrm{Spec}^f \boldsymbol{C}[\![s,t]\!]$ be an algebraic morphism, i.e., there exist a morphism $\mathrm{Spec}\, A_1 \longrightarrow \mathrm{Spec}\, A_2$ for some regular rings A_i $(i = 1,2)$ finitely generated over $\boldsymbol{C}$, such that the completion at some closed points is isomorphic to φ_C. We assume $\varphi_C^*(t) \neq 0$. We have the induced map $\varphi_{<0}^* : \boldsymbol{C}[\![s]\!]((t_d))/\boldsymbol{C}[\![s,t_d]\!] \longrightarrow \boldsymbol{C}((v_d))/\boldsymbol{C}[\![v_d]\!]$ for any d.

Proposition 3.9. *Assume that $(\mathcal{E}_C, \nabla)$ is algebraic and mildly ramified. Then, the set of the irregular values of $\varphi_C^*(\mathcal{E}_C, \nabla)$ is given by the image of $\mathrm{Irr}\big(\mathcal{E}_C, \nabla_t\big)$ via $\varphi_{<0}^*$.*

Proof By extending R, we may assume that φ is induced from $\varphi_R : \mathrm{Spec}^f R[\![v]\!] \longrightarrow \mathrm{Spec}^f R[\![s,t]\!]$ given by $\varphi_R^*(t) = v^a$ and $\varphi_R^*(s) =$

$v \cdot h(v)$. We have the induced map $\varphi_R^* : R(\!(s_d)\!)(\!(t_d)\!) \longrightarrow R(\!(v_d)\!)$. Let $\Phi : \mathrm{Spec}^f\, R[\![u,v]\!] \longrightarrow \mathrm{Spec}^f\, R[\![s,t]\!]$ be given by $t = v^a$ and $s = u+v\cdot h(v)$. Then, $\Phi^*(\mathcal{E},\nabla)$ is mildly ramified due to Lemma 3.3.

We put $\mathfrak{K}(u,v) := \boldsymbol{C}(\!(u_d)\!)(\!(v_d)\!)$ and $\mathcal{E}_{\mathfrak{K}(u,v)} := \mathcal{E}_{\boldsymbol{C}} \otimes \mathfrak{K}(u,v)$ on which the relative connection ∇_v is induced. We have the induced map $R[\![s]\!](\!(t_d)\!)/R[\![s,t_d]\!] \longrightarrow R[\![u]\!](\!(v_d)\!)/R[\![u,v_d]\!]$, which is denoted by $\Phi^*_{<0}$. Then, we have only to show that $\mathrm{Irr}(\mathcal{E}_{\mathfrak{K}(u,v)},\nabla_v)$ is the same as the image of $\mathrm{Irr}\bigl(\mathcal{E}_{\mathfrak{K}_{\boldsymbol{C}}},\nabla_t\bigr)$ via the map $\Phi^*_{<0}$ due to Proposition 3.7. Since both of them are contained in $\boldsymbol{C}[\![u]\!](\!(v_d)\!)/\boldsymbol{C}[\![u,v_d]\!]$, we have only to compare them in $\boldsymbol{C}(\!(u)\!)(\!(v_d)\!)/\boldsymbol{C}(\!(u)\!)[\![v_d]\!]$.

The meromorphic connection $\bigl(\mathcal{E}_{\mathfrak{K}_{\boldsymbol{C}}} \otimes \boldsymbol{C}(\!(s)\!)(\!(t_d)\!),\nabla_t\bigr)$ is unramified, because the irregular values are contained in $\boldsymbol{C}[\![s]\!](\!(t_d)\!)/\boldsymbol{C}[\![s,t_d]\!]$. By Lemma 2.20, we have a strict Deligne–Malgrange lattice $E_{\boldsymbol{C}}$ which is the free $\boldsymbol{C}(\!(s)\!)[\![t_d]\!]$-module, and the irregular decomposition with respect to the relative connection ∇_t:

$$\bigl(E_{\boldsymbol{C}},\nabla_t\bigr) = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\mathcal{E}_{\mathfrak{K}_{\boldsymbol{C}}},\nabla_t)} \bigl(E_{\mathfrak{a}},\nabla_{\mathfrak{a},t}\bigr)$$

Due to the uniqueness of the irregular decomposition and the commutativity of $\nabla(\partial_s)$ and $\nabla(t\partial_t)$, it is standard to show that $\nabla(\partial_s)\bigl(E_{\mathfrak{a}}(\!(t_d)\!)\bigr) \subset E_{\mathfrak{a}}(\!(t_d)\!)$. (See the proof of Lemma 2.18, for example.) Hence, it is the decomposition of the meromorphic flat connection:

$$\bigl(E_{\boldsymbol{C}},\nabla\bigr) = \bigoplus \bigl(E_{\mathfrak{a}},\nabla_{\mathfrak{a}}\bigr).$$

We put $\nabla'_{\mathfrak{a}} := \nabla_{\mathfrak{a}} - d\mathfrak{a}$. By construction, we have $\nabla'_{\mathfrak{a}}(t_d\partial_{t_d})(E_{\mathfrak{a}}) \subset E_{\mathfrak{a}}$. Since $E_{\boldsymbol{C}}$ is assumed to be strict Deligne–Malgrange, it can be shown that $\nabla'_{\mathfrak{a}}(\partial_s)(E_{\mathfrak{a}}) \subset E_{\mathfrak{a}}$ by a standard argument. (See the last part of the proof of Proposition 2.19, for example.) We put $\nabla' = \bigoplus \nabla'_{\mathfrak{a}}$.

Let $\boldsymbol{v}$ be a frame of $E_{\boldsymbol{C}}$ compatible with the irregular decomposition. Let A and B be determined by $\nabla'\boldsymbol{v} = \boldsymbol{v}\cdot(A\cdot dt_d/t_d + B\cdot ds)$. Then, $A,B \in M_r\bigl(\boldsymbol{C}(\!(s)\!)[\![t_d]\!]\bigr)$. We remark $\Phi^*(s)^{-k} \in \boldsymbol{C}(\!(u)\!)[\![v]\!]$ for any integer k. Then, it is easy to see that $E_{\boldsymbol{C}}\otimes\boldsymbol{C}(\!(u)\!)[\![v_d]\!]$ gives a Deligne–Malgrange lattice of $\mathcal{E}\otimes\boldsymbol{C}(\!(u)\!)(\!(v_d)\!)$ with respect to $\nabla(v_d\partial_{v_d})$, and the irregular decomposition of $\Phi^*(\mathcal{E},\nabla)$ is given as follows:

$$\mathcal{E}\otimes\boldsymbol{C}(\!(u)\!)(\!(v_d)\!) \simeq \bigoplus_{\mathfrak{b}\in\boldsymbol{C}(\!(u)\!)(\!(v_d)\!)/\boldsymbol{C}(\!(u)\!)[\![v_d]\!]} \left(\bigoplus_{\Phi^*_{<0}(\mathfrak{a})=\mathfrak{b}} E_{\mathfrak{a}}\otimes\boldsymbol{C}(\!(u)\!)(\!(v_d)\!)\right).$$

Thus, we are done. Q.E.D.

§4. Resolution of turning points

4.1. Resolution of the discriminants of polynomials

Let R be a regular subring of $\boldsymbol{C}$ which is finitely generated over $\mathbb{Z}$. Let X_R be a smooth projective surface over R. Let D_R be a simply effective normal crossing divisor of X_R. We assume that $X_R \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is smooth or empty for each p. Let N and r be positive integers.

Take $\eta \in S(R,p)$. We put $X_{\overline{\eta}} := X_R \otimes_R k(\overline{\eta})$. We denote the function field of $X_{\overline{\eta}}$ by $K(X_{\overline{\eta}})$. Let

$$\mathcal{P}^{(a)}(T) \in \bigoplus_{j=0}^{r} H^0\big(X_{\overline{\eta}}, \mathcal{O}_X(jND_{\overline{\eta}})\big) \cdot T^{r-j}$$

be monic polynomials ($a = 1, \ldots, L$). The tuple $\big(\mathcal{P}^{(a)} \,\big|\, a = 1, \ldots, L\big)$ is denoted by $\boldsymbol{\mathcal{P}}$. We regard them as elements of $K(X_{\overline{\eta}})[T]$. Let $\mathcal{P}^{(a)} = \prod_{i=1}^{m(a)} (\mathcal{P}_i^{(a)})^{e(i,a)}$ be the irreducible decompositions. The monic polynomials $\mathcal{P}_i^{(a)}$ are contained in $\bigoplus_{j=0}^{r_i(a)} H^0\big(X_{\overline{\eta}}, \mathcal{O}_X(jND_{\overline{\eta}})\big) \cdot T^{r_i(a)-j}$, where $r_i(a) := \deg_T \mathcal{P}_i^{(a)}$. We regard the discriminants $\mathrm{disc}(\mathcal{P}_i^{(a)})$ as the elements of the function field $K(X_{\overline{\eta}})$. There exists a constant $M_1 > 0$, which is independent of the choice of $\overline{\eta}$ and p, such that $\mathrm{disc}(\mathcal{P}_i^{(a)})$ are contained in $H^0\big(X_{\overline{\eta}}, \mathcal{O}_X(M_1 \cdot D_{\overline{\eta}})\big)$. We put as follows:

$$\mathrm{disc}(\boldsymbol{\mathcal{P}}) := \prod_{a=1}^{L} \prod_{j=1}^{m(a)} \mathrm{disc}(\mathcal{P}_j^{(a)}) \in K(X_{\overline{\eta}})$$

There exists a constant $M_2 > 0$, which is independent of the choice of $\overline{\eta}$ and p, such that $\mathrm{disc}(\boldsymbol{\mathcal{P}})$ is contained in $H^0\big(X_{\overline{\eta}}, \mathcal{O}_X(M_2 \cdot D_{\overline{\eta}})\big)$. Let $Z(\boldsymbol{\mathcal{P}})$ denote the 0-set of $\mathrm{disc}(\boldsymbol{\mathcal{P}})$, when we regard $\mathrm{disc}(\boldsymbol{\mathcal{P}})$ as a section of the line bundle $\mathcal{O}_X(M_2 \cdot D_{\overline{\eta}})$. We may assume $D_{\overline{\eta}} \subset Z(\boldsymbol{\mathcal{P}})$, by making M_2 larger. Since $Z(\boldsymbol{\mathcal{P}})$ is a member of some bounded family, it is easy to show the following lemma by using the flattening stratifications (see Lecture 8 in [20]) and the semi-continuity theorem (see Chapter III. 12 in [6]) for any flat family.

Lemma 4.1. *There exists a constant M_3, which is independent of $\overline{\eta}$ and p, such that the arithmetic genus of $Z(\boldsymbol{\mathcal{P}})$ is smaller than M_3.* *Q.E.D.*

Let P be any closed point of $X_{\overline{\eta}}$. We put $(X_{\overline{\eta}}^{(0)}, P^{(0)}) := (X_{\overline{\eta}}, P)$. Inductively, let $\pi^{(i)} : X_{\overline{\eta}}^{(i)} \longrightarrow X_{\overline{\eta}}^{(i-1)}$ be the blow up at $P^{(i-1)}$, and let us take a point $P^{(i)} \in \pi^{(i)}(P^{(i-1)})$. Let π_i denote the naturally induced map $X^{(i)} \longrightarrow X$. By the classical arguments (see Section V.3 in [6], for example), we can show the following lemma.

Lemma 4.2. *There exists some i_0, independent of the choice of p, $\overline{\eta}$ and the points $P^{(i)}$, such that the divisor $(\pi^{(i)})^{-1}Z(\boldsymbol{\mathcal{P}})$ is normal crossing around the exceptional divisor $(\pi^{(i)})^{-1}(P^{(i-1)})$ for any $i \geq i_0$.*

Proof We give only an outline. We use the notation p_a to denote the arithmetic genus. Let Y denote the reduced scheme associated to $Z(\boldsymbol{\mathcal{P}})$. Let $Y = \bigcup Y_j$ denote the irreducible decomposition. We have $p_a(Y) \leq M_3$ and $p_a(Y_j) \leq M_3$. Let $\widetilde{Y}_i$ denote the inverse image of Y via π_i with the reduced structure. Let $\widetilde{Y}_{i,j}$ denote the strict transform of Y_j via π_i. Let $C_{i,q}$ denote the strict transform of $(\pi^{(q)})^{-1}(P^{(q-1)})$ via the natural map $X^{(i)}_{\overline{\eta}} \longrightarrow X^{(q)}_{\overline{\eta}}$. We have $\widetilde{Y}_i = \bigcup_j \widetilde{Y}_{i,j} \cup \bigcup_q C_{i,q}$. Let $r_{P^{(q)}}(\widetilde{Y}_q)$ denote the multiplicity of $P^{(q)}$ in $\widetilde{Y}_q$. We use the symbol $r_{P^{(q)}}(\widetilde{Y}_{q,j})$ in a similar meaning. We have the following equality:

$$p_a(\widetilde{Y}_{i,j}) = p_a(Y_j) - \sum_{q \leq i-1} \frac{1}{2} r_{P^{(q)}}(\widetilde{Y}_{q,j}) \cdot \bigl(r_{P^{(q)}}(\widetilde{Y}_{q,j}) - 1\bigr)$$

By our choice, $P^{(i)} \notin \widetilde{Y}_{i,j}$, or $P^{(i)}$ is a smooth point of $\widetilde{Y}_{i,j}$ for any $i \geq i(1)$ if $P^{(i(1))}$ is a smooth point of $\widetilde{Y}_{i(1),j}$. Hence, we obtain $r_{P^{(i)}}(\widetilde{Y}_{i,j}) \leq 1$ if i is sufficiently large. We also have the following equality:

$$p_a(\widetilde{Y}_i) = p_a(Y) - \sum_{q \leq i-1} \frac{1}{2} \bigl(r_{P^{(q)}}(\widetilde{Y}_q) - 1\bigr) \cdot \bigl(r_{P^{(q)}}(\widetilde{Y}_q) - 2\bigr)$$

Assume $r_{P^{(q)}}(\widetilde{Y}_q) = 2$. Then, as explained in the proof of Theorem 3.9 in Section V of [6], there are three possibility:

- $\widetilde{Y}_q$ is normal crossing around $P^{(q)}$.
- Let $\widetilde{Y}'_{q+1}$ denote the strict transform of $\widetilde{Y}_q$ via $\pi^{(q+1)}$. Then, it is nonsingular in a neighbourhood of $(\pi^{(q+1)})^{-1}(P^{(q)})$, and $\widetilde{Y}'_{q+1}$ and $(\pi^{(q+1)})^{-1}(P^{(q)})$ intersect at one point with multiplicity 2. If $P^{(q+1)}$ and $P^{(q+2)}$ are also singular points of $\widetilde{Y}_{q+1}$ and $\widetilde{Y}_{q+2}$ respectively, we have $r_{P^{(q+2)}}(\widetilde{Y}_{q+2}) = 3$.
- $\widetilde{Y}'_{q+1}$ and $(\pi^{(q+1)})^{-1}(P^{(q)})$ intersects at one point, whose multiplicity in $\widetilde{Y}'_{q+1}$ is 2. If $P^{(q+1)}$ is singular point of a $\widetilde{Y}_{q+1}$, we have $r_{P^{(q+1)}}(\widetilde{Y}_{q+1}) = 3$.

Hence, we obtain that $\widetilde{Y}_i$ are normal crossing around $P^{(i)}$ for sufficiently large i. Q.E.D.

Let $i \geq i_0$. Let $C_i(\boldsymbol{\mathcal{P}})$ denote the closure of $Z(\boldsymbol{\mathcal{P}}) \cap (X_{\overline{\eta}} - D_{\overline{\eta}})$ in $X_{\overline{\eta}}^{(i)}$. We take a local coordinate neighbourhood $(U^{(i)}, s^{(i)}, t^{(i)})$ around $P^{(i)}$ such that (i) $(t^{(i)})^{-1}(0)$ is $U^{(i)} \cap \left(\pi^{(i)}\right)^{-1}(P^{(i-1)})$, (ii) if $P^{(i)}$ is contained in $C_i(\boldsymbol{\mathcal{P}})$, then $(s^{(i)})^{-1}(0) = U^{(i)} \cap C_i(\boldsymbol{\mathcal{P}})$, (ii)' if $P^{(i)}$ is not contained in $C_i(\boldsymbol{\mathcal{P}})$, then $s^{(i)}$ may be anything. According to generalized Abhyankar's lemma (see Expose XIII Section 5 of [28]), any solution of the equations $\pi_i^* \mathcal{P}_j^{(a)}(T) = 0$ $(a = 1, \ldots, L,\ j = 1, \ldots, m(a))$ is contained in $k(\overline{\eta})[\![s_d^{(i)}]\!]((t_d^{(i)}))$ for some appropriate d, which is a factor of $r!$.

Lemma 4.3. *There exists an i_1, which is independent of the choice of $\overline{\eta}$, p and the points $P^{(i)}$, such that the following holds for any $i \geq i_1$:*

- *Any roots of $\pi_i^* \mathcal{P}_j^{(a)}(T)$ $(a = 1, \ldots, L, j = 1, \ldots, m(a))$ are contained in $k(\overline{\eta})[\![s_d^{(i)}, t_d^{(i)}]\!] + k(\overline{\eta})[\![s^{(i)}]\!]((t_d^{(i)}))$.*

Proof If $P^{(i_0)}$ is not contained in $C_{i_0}(\boldsymbol{\mathcal{P}})$, then $P^{(i)} \not\in C_i(\boldsymbol{\mathcal{P}})$ for any $i \geq i_0$, and the claim is obvious in this case. Assume $P^{(i_0)}$ is contained in $C_{i_0}(\boldsymbol{\mathcal{P}})$. Let $\alpha_l^{(i_0)} \in k(\overline{\eta})[\![s_d^{(i_0)}]\!]((t_d^{(i_0)}))$ be any solution of $\pi_{i_0}^* \mathcal{P}_j^{(a)}(T) = 0$ for some (a, j). Note that there exists a constant M_4, which is independent of the choice of $\overline{\eta}$, p, and the sequence of the points $P^{(i)}$, with the following property:

- The orders of the poles of the coefficients of $\pi_{i_0}^* \mathcal{P}_j^{(a)}(T)$ with respect to $t^{(i_0)}$ are dominated by M_4,

Hence, there exists a constant M_5, which is independent of the choice of $\overline{\eta}$, p, the sequence of the points $P^{(i)}$, (a, j) and $\alpha_l^{(i_0)}$, with the following property:

- The order of the pole of $\alpha_l^{(i_0)}$ with respect to $t_d^{(i_0)}$ are dominated by M_5.

If $P^{(i)}$ are contained in $C_i(\boldsymbol{\mathcal{P}})$ for $i \geq i_0$, we may assume $(\pi^{(i)})^*(s^{(i-1)}) = s^{(i)} \cdot t^{(i)}$ and $(\pi^{(i)})^*(t^{(i-1)}) = t^{(i)}$. Hence, the pull back of $\alpha_l^{(i_0)}$ via $X_{\overline{\eta}}^{(i)} \longrightarrow X_{\overline{\eta}}^{(i_0)}$ are contained in $k(\overline{\eta})[\![s_d^{(i)}, t_d^{(i)}]\!] + k(\overline{\eta})((t_d^{(i)}))$ for any sufficiently large i. Q.E.D.

4.2. Proof of Theorem 1.1

If we take a sufficiently large R, then $\mathcal{E}$, ∇, X and D come from $\mathcal{E}_R$, ∇_R, X_R and D_R which are defined over R. We may also assume that we have the canonical lattice $E_R \subset \mathcal{E}_R$ defined over R. (See [17]. Although it is obtained in an analytic way, it is algebraic according to GAGA [24] as remarked in [17].) By applying a theorem of Sabbah (Proposition

4.3.1 in [23], see also Theorem 5.4.1 in [1]), we may assume that any cross points of D are not turning. Let P be a turning point contained in a smooth part of D. Let U be a Zariski neighbourhood of P with an étale morphism $(x, y) : U \longrightarrow A^2$ such that $x^{-1}(0) = D \cap U$. For simplicity, U does not contain any other turning points than P. We may assume that P and (U, x, y) are also defined over R. We have only to take a proper birational map $\pi : U' \longrightarrow U$ such that $\pi^{-1}(\mathcal{E}, \nabla)$ has no turning points. On U, we have the vector field $x\partial_x$ and ∂_y. By taking blow up of X outside of U, and by extending D, we may assume that $x\partial_x$ and ∂_y are sections of $\Theta_X(M_0 D)$. We have a positive number M_0' such that $\nabla(E_R) \subset E_R(M_0' D_R)$. Hence, we have the constant M_1 such that $\nabla(x\partial_x)(E_R)$ and $\nabla(\partial_y)(E_R)$ are contained in $E_R(M_1 D_R)$.

Take a large prime p. For any $\eta \in S(R, p)$, let $\mathcal{E}_{\overline{\eta}}, E_{\overline{\eta}}, \nabla_{\overline{\eta}}, X_{\overline{\eta}}, D_{\overline{\eta}}, P_{\overline{\eta}}$ and $U_{\overline{\eta}}$ denote the induced objects over $k(\overline{\eta})$. Let ψ be the p-curvature of $(\mathcal{E}_{\overline{\eta}}, \nabla_{\overline{\eta}})$. We put $\psi_x := \psi(x\partial_x)$ and $\psi_y := \psi(\partial_y)$. Because $\nabla(\partial_x)\big(E_{\overline{\eta}}\big) \subset E_{\overline{\eta}}\big(M_1 D_{\overline{\eta}}\big)$ and $\nabla(y\partial_y)\big(E_{\overline{\eta}}\big) \subset E_{\overline{\eta}}(M_1 D_{\overline{\eta}})$, we have

$$\psi_x, \psi_y \in \operatorname{End}(E_{\overline{\eta}}) \otimes \mathcal{O}(pM_1 D).$$

Hence, the characteristic polynomials $\det(T - \psi_x)$ and $\det(T - \psi_y)$ are contained in $\bigoplus_{j=0}^{n} H^0\big(X_{\overline{\eta}}, \mathcal{O}(pjM_1 D)\big) \cdot T^{n-j}$. Due to the excellent observation of Bost, Laszlo and Pauly [14], we have the following lemma.

Lemma 4.4. *We have the polynomials $\mathcal{P}_{x,\overline{\eta}}(T)$ and $\mathcal{P}_{y,\overline{\eta}}(T)$ in $\bigoplus_{j=0}^{n} H^0\big(X_{\overline{\eta}}, \mathcal{O}(jM_1 D)\big) \cdot T^{n-j}$ satisfying*

$$\det(T - \psi_x) = \operatorname{Fr}^* \mathcal{P}_{x,\overline{\eta}}(T), \quad \det(T - \psi_y) = \operatorname{Fr}^* \mathcal{P}_{y,\overline{\eta}}(T),$$

where $\operatorname{Fr} : X_{\overline{\eta}} \longrightarrow X_{\overline{\eta}}$ *denotes the absolute Frobenius morphism.*

Proof We reproduce the argument in [14] for the convenience of readers. Let T' be a formal variable. According to the Cartier descent, we have only to show $\partial_y \det(1 - T'\psi_\kappa) = 0$ and $x\partial_x \det(1 - T'\psi_\kappa) = 0$ for $\kappa = x, y$. Let $\boldsymbol{v}$ be a local frame of $E_{\overline{\eta}}$. Let A, B, and Ψ_κ ($\kappa = x, y$) be determined by $\nabla \boldsymbol{v} = \boldsymbol{v}(Ady + Bdx/x)$, $\psi_\kappa \boldsymbol{v} = \boldsymbol{v} \cdot \Psi_\kappa$. Because $(\partial_y)^p = 0$ and $(x\partial_x)^p = x\partial_x$, we have $\psi_x = \nabla(x\partial_x)^p - \nabla(x\partial_x)$ and $\psi_y = \nabla(\partial_y)^p$. Since ∇ is flat, we have the commutativity $[\nabla(\partial_y), \nabla(x\partial_x)] = [\nabla(\partial_y), \nabla(\partial_y)] = [\nabla(x\partial_x), \nabla(x\partial_x)] = 0$. Hence, we have $[\nabla(\partial_y), \psi_\kappa] = [\nabla(x\partial_x), \psi_\kappa] = 0$ for $\kappa = x, y$. Therefore, $\partial_y \Psi_\kappa + [A, \Psi_\kappa] = x\partial_x \Psi_\kappa + [B, \Psi_\kappa] = 0$, and thus $\operatorname{Tr}(\Psi_\kappa^n \partial_y \Psi_\kappa) = \operatorname{Tr}(\Psi_\kappa^n x\partial_x \Psi_\kappa) = 0$.

Recall $\partial_y \det(M) = \det(M) \cdot \operatorname{Tr}(M^{-1}\partial_y M)$ for an invertible matrix M. For $M = \mathrm{id} - T'\Psi_\kappa$, we have $M^{-1} = \sum T'^n \Psi_\kappa^n$ and

$$\partial_y \det(\mathrm{id} - T'\Psi_\kappa) = -T' \det(\mathrm{id} - T'\Psi_\kappa) \cdot \sum_{n=0} T'^n \operatorname{Tr}(\Psi_\kappa^n \partial_y \Psi_\kappa) = 0$$

Similarly, we have $x\partial_x \det(\mathrm{id} - T'\Psi_s) = 0$. Q.E.D.

Inductively, we construct the blow up $\pi^{(i)} : X^{(i)} \longrightarrow X^{(i-1)}$ as follows. First, let $\pi^{(1)} : X^{(1)} \longrightarrow X$ be the blow up at P, and we put $\pi_1 := \pi^{(1)}$. Let $\pi^{(2)} : X^{(2)} \longrightarrow X^{(1)}$ denote the blow up at the turning points of $\pi_1^*(\mathcal{E}, \nabla)$ contained in $\pi_1^{-1}(U)$, and we put $\pi_2 := \pi^{(1)} \circ \pi^{(2)}$. When $\pi^{(i)} : X^{(i)} \longrightarrow X^{(i-1)}$ is given, let $\pi_i : X^{(i)} \longrightarrow X$ denote the naturally induced morphism, and let $\pi^{(i+1)} : X^{(i+1)} \longrightarrow X^{(i)}$ be the blow up at the turning points of $\pi_i^*(\mathcal{E}, \nabla)$ contained in $\pi_i^{-1}(U)$.

We take subrings $R^{(i)} \subset \boldsymbol{C}$ such that (i) $R^{(i-1)} \subset R^{(i)}$, and $R^{(i)}$ is smooth and finitely generated over $R^{(i-1)}$, (ii) $X^{(i)}$ and the turning points contained in $\pi_i^{-1}(U)$ are defined over $R^{(i)}$. Let $\eta \in S(R, p)$. We take geometric points $\overline{\eta}(i)$ of $S(R^{(i)}, p)$ for any i with the morphisms $\overline{\eta}(i) \longrightarrow \overline{\eta}(i-1) \longrightarrow \eta$ compatible with $\operatorname{Spec} R^{(i)} \longrightarrow \operatorname{Spec} R^{(i-1)} \longrightarrow \operatorname{Spec} R$. For $j \leq i$, $X^{(j)}$ are defined over $R^{(i)}$, and we have $X^{(j)}_{R^{(i)}} \otimes_{R^{(i)}} k(\overline{\eta}(i)) \simeq X^{(j)}_{\overline{\eta}(j)} \otimes_{\overline{\eta}(j)} k(\overline{\eta}(i))$. And the objects over them are naturally related by the pull backs.

Let $\mathcal{P}_{\kappa,\overline{\eta}}(T) = \prod \mathcal{P}_{\kappa,\overline{\eta},j}(T)^{e(\kappa,\overline{\eta},j)}$ be the irreducible decompositions for $\kappa = x, y$. Applying Lemma 4.2 and Lemma 4.3, we can show that there exist i_1 and p_1 such that the following claims hold if $i \geq i_1$, $p \geq p_1$:

- Let C be any exceptional divisor with respect to $\pi^{(i)}_{\overline{\eta}(i)}$. Then, $\bigcup_{j,\kappa} \pi^{-1}_{i,\overline{\eta}(i)}\big(\operatorname{disc}(\mathcal{P}_{\kappa,\overline{\eta},j}) \cup D_{\overline{\eta}(i)}\big)$ are normal crossing around C.
- Let $Z^{(i)}_{\kappa,\overline{\eta}(i),j}$ denote the closure of $\operatorname{disc}(\mathcal{P}_{\kappa,\overline{\eta},j}) \cap (X_{\overline{\eta}(i)} - D_{\overline{\eta}(i)})$ in $X^{(i)}_{\overline{\eta}(i)}$. If C intersects at Q with $Z^{(i)}_{\kappa,\overline{\eta}(i),j}$ for some (κ, j), we take a coordinate neighbourhood (U_Q, z, w) such that $w^{-1}(0) = C \cap U_Q$ and $z^{-1}(0)$ is $Z^{(i)}_{\kappa,\overline{\eta}(i),j} \cap U_Q$. Then, any solutions of $\mathcal{P}_{\kappa,\overline{\eta},j}(T) = 0$ are contained in $k(\overline{\eta}(i))[\![z_d, w_d]\!] + k(\overline{\eta}(i))[\![z]\!]((w_d))$.

We remark that the completion of $\pi^*_{i,\overline{\eta}(i)}(\mathcal{E}, \nabla)$ at such Q is mildly ramified, which can be shown by the argument used in the proof of Lemma 3.3.

Due to a theorem of Sabbah (Proposition 4.3.1 in [23]), we can take a regular birational projective map $F : \overline{X} \longrightarrow X^{(i_1)}$ as follows:

- F is the blow up along an ideal supported at the cross points of the divisor $\pi_{i_1}^{-1}(P)$.
- Any cross points of the divisor $G^{-1}(P)$ are not turning points for $(\overline{\mathcal{E}}, \overline{\nabla}) := G^*(\mathcal{E}, \nabla)$, where $G := \pi_{i_1} \circ F$.

Let Q be a point of the smooth part of $G^{-1}(P) \subset \overline{X}$ which is a turning point for $(\overline{\mathcal{E}}, \overline{\nabla})$. We remark that $F(Q) \in X^{(i_1)}$ is contained in some exceptional divisor with respect to $\pi^{(i_1)}$. We take a subring $R_0 \subset \boldsymbol{C}$ finitely generated over $R^{(i_1)}$, on which Q is defined. We may also have a neighbourhood U_Q with an étale map $(u,v) : U_Q \longrightarrow A^2$ around Q such that $v^{-1}(0) = G^{-1}(P) \cap U_Q$. By considering the completion at Q, we obtain the free $R_0[\![u]\!]((v))$-module $\widehat{\mathcal{E}}_{R_0}$ with a meromorphic connection $\widehat{\nabla}_{R_0}$.

Lemma 4.5. $(\widehat{\mathcal{E}}_{R_0}, \widehat{\nabla}_{R_0})$ *is mildly ramified.*

Proof Let $\overline{\eta}_0$ be a geometric point of $\operatorname{Spec} R_0$ over some $\overline{\eta}(i_1) \in S(R^{(i_1)})$. We have only to show that $(\widehat{\mathcal{E}}_{\overline{\eta}_0}, \widehat{\nabla}_{\overline{\eta}_0})$ is mildly ramified. Assume $F(Q)$ is a cross point of the divisor $\pi_{i_1}^{-1}(P)$. Then, $F_{\overline{\eta}_0}(Q_{\overline{\eta}_0})$ is not contained in any $Z^{(i_1)}_{\kappa,\overline{\eta}_0,j}$, and hence the ramification around $Q_{\overline{\eta}_0}$ may occur only along $G_{\overline{\eta}_0}^{-1}(P_{\overline{\eta}_0})$. If $F(Q)$ is contained in the smooth part of $\pi_{i_1}^{-1}(P)$, the claim follows from our choice of i_1. Q.E.D.

Then, we can control the irregular values for $(\overline{\mathcal{E}}, \overline{\nabla})$.

Lemma 4.6. *Let S be the set of the irregular values of $(\overline{\mathcal{E}}, \overline{\nabla}_v) \otimes \boldsymbol{C}((u))((v))$.*

- *S is contained in $\boldsymbol{C}[\![u]\!]((v_d))/\boldsymbol{C}[\![u, v_d]\!]$ for some appropriate d.*
- *For any curve $\varphi : C \longrightarrow \widetilde{X}$ such that $\varphi(C) \cap D_Q = \{Q\}$, where D_Q denotes the exceptional divisor containing Q, the irregular values of $\varphi^*(\overline{\mathcal{E}}, \overline{\nabla})$ are given by the negative parts of $\varphi^*\mathfrak{a}$ $(\mathfrak{a} \in S)$.*

Proof It follows from Proposition 3.7, Proposition 3.9 and Lemma 4.5. Q.E.D.

In the following, we use classical topology instead of Zariski topology, although we can also argue in Zariski topology. Let $\mathcal{U}$ be a neighbourhood of Q in $\overline{X}$. We will shrink $\mathcal{U}$ without mention, if it is necessary. Let $\varphi : \widetilde{\mathcal{U}} \longrightarrow \mathcal{U}$ be the ramified covering given by $(u, v_d) \longmapsto (u, v_d^d)$ for some appropriate d. We put $\mathcal{G} := \mathbb{Z}/d\mathbb{Z}$ which naturally acts on $\widetilde{\mathcal{U}}$. We put $\mathcal{D}_d := \{v_d = 0\}$. Let $M(\widetilde{\mathcal{U}})$ (resp. $H(\widetilde{\mathcal{U}})$) denote the space of meromorphic (resp. holomorphic) functions whose poles are contained in $\mathcal{D}_d$. For each $\mathfrak{a} \in M(\widetilde{\mathcal{U}})/H(\widetilde{\mathcal{U}})$, we use the same symbol to denote the natural lift to $M(\widetilde{\mathcal{U}})_{<0}$. Due to Lemma 4.6, there exists the finite subset $S \subset M(\mathcal{U}_d)/H(\mathcal{U}_d)$ which gives the irregular values of $(\overline{\mathcal{E}}, \overline{\nabla}_v) \otimes \boldsymbol{C}((u))((v))$. (The meromorphic property of the irregular values is shown in Theorem 2.3.1 of [23], for example.) Let S_1 denote the set of pairs $(\mathfrak{a}, \mathfrak{b}) \in S^2$ such that $\mathfrak{a} \neq \mathfrak{b}$.

We put $\varphi(\mathfrak{a}) := \prod_{\sigma\in\mathcal{G}} \sigma^*\mathfrak{a}$ for any $\mathfrak{a} \in S$ which give the meromorphic functions $\varphi(\mathfrak{a})$ on $\mathcal{U}$. For any $(\mathfrak{a}, \mathfrak{b}) \in S_1$, we have the meromorphic functions $\varphi(\mathfrak{a} - \mathfrak{b})$ on $\mathcal{U}$, similarly. The union of the zero and the pole of $\varphi(\mathfrak{a})$ is denoted by $|\varphi(\mathfrak{a})|$. We use the symbol $|\varphi(\mathfrak{a} - \mathfrak{b})|$ in a similar meaning.

We can take a blow up $\kappa : \mathcal{U}_1 \longrightarrow \mathcal{U}$ along an ideal supported in $\{Q\}$, such that the following holds:

- $\kappa^{-1}\big(|\varphi(\mathfrak{a})| \cup D_Q\big)$ and $\kappa^{-1}\big(|\varphi(\mathfrak{a}-\mathfrak{b})| \cup D_Q\big)$ are normal crossing for any $\mathfrak{a} \in S$ and $(\mathfrak{a}, \mathfrak{b}) \in S_1$. Here D_Q denotes the component of $G^{-1}(P)$ such that $Q \in D_Q$.
- The zero and the pole of $\kappa^{-1}(\varphi(\mathfrak{a}))$ have no intersections for any $\mathfrak{a} \in S$. The zero and the pole of $\kappa^{-1}\big(\varphi(\mathfrak{a} - \mathfrak{b})\big)$ have no intersections for any $(\mathfrak{a}, \mathfrak{b}) \in S_1$.
- For any $(\mathfrak{a}, \mathfrak{b}), (\mathfrak{a}', \mathfrak{b}') \in S_1$, the ideals generated by $\kappa^{-1}(\varphi(\mathfrak{a} - \mathfrak{b}))$ and $\kappa^{-1}(\varphi(\mathfrak{a}' - \mathfrak{b}'))$ are principal.

Applying Sabbah's theorem, we can take $\nu : \mathcal{U}'' \longrightarrow \mathcal{U}'$ such that any cross points of the divisor $(\kappa \circ \nu)^{-1}(Q)$ are not turning. We put $\widetilde{\kappa} := \kappa \circ \nu$, for which the above three conditions are satisfied. For any point Q' of the smooth part of $\widetilde{\kappa}^{-1}(Q)$, the irregular values of $\widetilde{\kappa}^{-1}(\overline{\mathcal{E}}, \overline{\nabla})$ around Q' are given by the negative parts of $\widetilde{\kappa}^{-1}(\mathfrak{a})$ due to Lemma 4.6. By using Proposition 2.19, we can conclude that Q' is not a turning point. Therefore, we have no turning points in $\widetilde{\kappa}^{-1}(Q)$. Applying the procedure to any turning points for $(\overline{\mathcal{E}}, \overline{\nabla})$ contained in $G^{-1}(U)$, we can resolve them. Thus the proof of Theorem 1.1 is finished. Q.E.D.

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