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Principal $\widehat{\mathfrak{sl}_3}$ subspaces and quantum Toda Hamiltonian

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Abstract.

We study a class of representations of the Lie algebra $\mathfrak{n}\otimes\mathbb{C}[t,t^{-1}]$, where $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{sl}_3$. We derive Weyl-type (bosonic) character formulas for these representations. We establish a connection between the bosonic formulas and the Whittaker vector in the Verma module for the quantum group $U_v(\mathfrak{sl}_3)$. We also obtain a fermionic formula for an eigenfunction of the $\mathfrak{sl}_3$ quantum Toda Hamiltonian.

§1. Introduction

Let $\mathfrak{n} = \mathbb{C}e_{21} \oplus \mathbb{C}e_{32} \oplus \mathbb{C}e_{31}$ be the nilpotent subalgebra of the complex Lie algebra $\mathfrak{sl}_3$, and let $\widehat{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t,t^{-1}]$ be the corresponding current algebra. In this paper we study a class of $\widehat{\mathfrak{n}}$-modules. The simplest example of the modules in question is the principal subspace V^k of the level k vacuum representation of $\widehat{\mathfrak{sl}_3}$ (see [FS]). Namely, let M^k be the level k vacuum representation of the affine Lie algebra $\widehat{\mathfrak{sl}_3}$. Fix a highest weight vector $v^k \in M^k$. Then

$$V^k = U(\widehat{\mathfrak{n}}) \cdot v^k.$$

The principal subspaces are studied in [AKS], [C], [CLM], [FS], [G], [LP], [P]. In particular, the following fermionic formula is available for

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the character of V^k:
(1.1)
$$\mathrm{ch}V^k = \sum_{\substack{n_1,\dots,n_k\geq 0\\ m_1,\dots,m_k\geq 0}} \frac{z_1^{\sum_{i=1}^k in_i} z_2^{\sum_{i=1}^k im_i} q^{\sum_{i,j=1}^k \min(i,j)(n_in_j-m_in_j+m_im_j)}}{(q)_{n_1}\cdots(q)_{n_k}(q)_{m_1}\cdots(q)_{m_k}},$$

where $(a)_n = \prod_{i=0}^{n-1}(1-aq^i)$. One of our results is the following new formula for $\mathrm{ch}V^k$. For non-negative integers d_1, d_2, set

$$I_{d_1,d_2}(z_1,z_2) = \frac{(qz_1^{-1}z_2^{-1})_{d_1+d_2}}{(q)_{d_1}(q)_{d_2}(qz_1^{-1})_{d_1}(qz_2^{-1})_{d_2}(qz_1^{-1}z_2^{-1})_{d_1}(qz_1^{-1}z_2^{-1})_{d_2}},$$
$$J_{d_1,d_2}(z_1,z_2) = \frac{I_{d_1,d_2}(z_1,z_2)}{(qz_1)_\infty(qz_2)_\infty(qz_1z_2)_\infty}.$$

We show that
(1.2)
$$\mathrm{ch}V^k = \sum_{d_1,d_2\geq 0} z_1^{kd_1}z_2^{kd_2}q^{k(d_1^2+d_2^2-d_1d_2)}J_{d_1,d_2}(z_1q^{2d_1-d_2}, z_2q^{2d_2-d_1}).$$

In the right hand side, each summand is understood as a power series expansion in z_1, z_2. Formula (1.2) was conjectured in [FS1].

The functions $I_{d_1,d_2}(z_1,z_2)$ are known to be the coefficients of the expansion of an eigenfunction of the $\mathfrak{sl}_3$ quantum Toda Hamiltonian (see [GL]). Namely, define the generating function

$$I(Q_1,Q_2,z_1,z_2) = \sum_{d_1,d_2\geq 0} Q_1^{d_1}Q_2^{d_2}I_{d_1,d_2}(z_1,z_2).$$

The $\mathfrak{sl}_3$ quantum Toda Hamiltonian $\widehat{H}$ is an operator of the form

$$\widehat{H} = q^{\partial/\partial t_0} + q^{\partial/\partial t_1}(1-Q_1) + q^{\partial/\partial t_2}(1-Q_2)$$

acting on the space of functions in variables Q_1, Q_2. The variables t_i are introduced by $Q_i = e^{t_{i-1}-t_i}$ and

$$q^{\partial/\partial t_j} : t_i \mapsto t_i + \delta_{ij}\ln q.$$

Let the variables p_1, p_2 be such that $z_1 = p_1^{-2}p_2$, $z_2 = p_1p_2^{-2}$. Then
(1.3)
$$\widehat{H}\left(p_1^{\frac{t_0-t_1}{\ln q}}p_2^{\frac{t_1-t_2}{\ln q}}I(Q_1,Q_2,z_1,z_2)\right) = (p_1+p_1^{-1}p_2+p_2^{-1})I(Q_1,Q_2,z_1,z_2)$$

(see [GL], [E], [BF]). Equation (1.3) can be rewritten as a set of recurrent relations

$$(1.4)\quad \begin{aligned}&(p_1(q^{d_1}-1)+p_1^{-1}p_2(q^{d_2-d_1}-1)+p_2^{-1}(q^{-d_2}-1))I_{d_1,d_2}(z_1,z_2)\\ &\quad = p_2p_1^{-1}q^{d_2-d_1}I_{d_1-1,d_2}(z_1,z_2)+p_2^{-1}q^{-d_2}I_{d_1,d_2-1}(z_1,z_2).\end{aligned}$$

In this paper we call (1.4) the Toda recursion.

One of the consequences of the formulas (1.1) and (1.2) is the following recurrence relations for the rational functions $I_{d_1,d_2}(z_1,z_2)$:

$$(1.5)\qquad I_{d_1,d_2}(z_1,z_2)=\sum_{n_1=0}^{d_1}\sum_{n_2=0}^{d_2}\frac{z_1^{-n_1}z_2^{-n_2}q^{n_1^2+n_2^2-n_1n_2}}{(q)_{d_1-n_1}(q)_{d_2-n_2}}I_{n_1,n_2}(z_1,z_2),$$

which leads to the fermionic formula

(1.6)

$$I_{d_1,d_2}(z_1,z_2)=\sum_{\{n_i\}_{i>0},\{m_i\}_{i>0}}\frac{z_1^{-\sum_{i>0}n_i}z_2^{-\sum_{i>0}m_i}q^{\sum_{i>0}(n_i^2+m_i^2-n_im_i)}}{(q)_{d_1-n_1}(q)_{n_1-n_2}\cdots(q)_{d_2-m_1}(q)_{m_1-m_2}\cdots},$$

where the sum is over all sequences $\{n_i\}$, $\{m_i\}$ such that

$$n_i,m_i\in\mathbb{Z}_{\geq 0},\quad d_1\geq n_1\geq n_2\geq\ldots,\quad d_2\geq m_1\geq m_2\geq\ldots,$$

and n_i,m_i vanish for almost all i. We conjecture that the obvious generalization of (1.6) to the case of $\mathfrak{sl}_n$ gives the coefficients of an eigenfunction for the corresponding quantum Toda Hamiltonian.

Let us briefly explain our approach to the computation of the character formulas. Recall (see [K1]) the Weyl–Kac formula for the character of an integrable irreducible representation M_λ of a Kac–Moody Lie algebra. It is written as a sum over the set of extremal vectors in M_λ. We call the summands the contributions of the extremal vectors. There are two different ways to compute these contributions. The first one is algebraic (see [K1], [Kum]) and uses the BGG resolution. The second uses the realization of M_λ as a dual space of sections of a certain line bundle on the generalized flag manifold and the Lefschetz fixed point formula (see [Kum]). We want to obtain a formula of the same structure by a combinatorial method. Let us explain our method on the example of M_λ. In this case the extremal vectors are labeled by elements of the affine Weyl group W, and the character formula can be written as

$$(1.7)\qquad \mathrm{ch}M_\lambda=\sum_{w\in W}\exp(w\lambda)\lim_{n\to\infty}(\exp(-w(n\lambda))\mathrm{ch}M_{n\lambda}).$$

Roughly speaking, we compute the character of $M_{n\lambda}$ in the "vicinity" of the extremal vectors and sum up the results. We apply the same approach to the characters of $\widehat{\mathfrak{n}}$-modules. We use combinatorial tools to compute the terms corresponding to the limit $\lim_{n\to\infty}(\exp(-w(n\lambda))\mathrm{ch}\, M_{n\lambda})$ in (1.7). Thus to obtain a *bosonic* formula for the character of an $\widehat{\mathfrak{n}}$-module M we follow the three steps:

(i) find (guess) the set of extremal vectors of M;
(ii) find (guess) the contribution of each vector;
(iii) prove that the sum of all contributions equals to the character of M.

Step (i) is more or less easy, while steps (ii) and (iii) are subtler. For example, for the principal subspace V^k, the extremal vectors are labeled by $\mathbb{Z}^2_{\geq 0}$ and the corresponding bosonic formula is given by (1.2). In particular, the contributions of the extremal vectors are given by $J_{d_1,d_2}(z_1q^{2d_1-d_2}, z_2q^{2d_2-d_1})$. In order to complete step (iii) for V^k, we introduce a set of $\widehat{\mathfrak{n}}$-modules which contains, in particular, all principal subspaces in integrable $\widehat{\mathfrak{sl}}_3$-modules. We describe these $\widehat{\mathfrak{n}}$-modules below.

Let

$$e_{ij}[n] = e_{ij} \otimes t^n \in \widehat{\mathfrak{n}}, \quad e_{ij}(z) = \sum_{n\in\mathbb{Z}} e_{ij}[n] \otimes z^{-n-1}.$$

The module V^k can be described as a cyclic $\widehat{\mathfrak{n}}$-module with a cyclic vector v such that the relations

$$e_{21}(z)^{k+1} = 0, \quad e_{32}(z)^{k+1} = 0$$

hold on V^k, and that the cyclic vector satisfies

$$e_{ij}[n]v = 0 \quad (n \geq 0).$$

Let k_1, k_2, l_1, l_2, l_3 be non-negative integers satisfying $k_1 \leq k_2$. The module $U^{k_1,k_2}_{l_1,l_2,l_3}$ is a cyclic $\widehat{\mathfrak{n}}$-module with a cyclic vector v such that the relations

$$e_{21}(z)^{k_1+1} = 0, \ e_{32}(z)^{k_2+1} = 0$$

hold on $U^{k_1,k_2}_{l_1,l_2,l_3}$, and that the cyclic vector satisfies

$$\begin{aligned} &e_{21}[n]v = 0, \quad e_{32}[n]v = 0 \quad (n > 0), \\ &e_{21}[0]^{l_1+1}v = 0, \quad e_{31}[1]^{l_3+1}v = 0, \quad e_{32}[0]^{l_2+1}v = 0. \end{aligned}$$

Similarly, the module $V^{k_1,k_2}_{l_1,l_2,l_3}$ is a cyclic $\widehat{\mathfrak{n}}$-module with the cyclic vector v such that the relations

$$e_{21}(z)^{k_1+1} = 0, \ e_{32}(z)^{k_2+1} = 0$$

hold on $V^{k_1,k_2}_{l_1,l_2,l_3}$, and that the cyclic vector satisfies

$$\begin{aligned}
&e_{21}[n]v=0,\quad e_{32}[n]v=0,\quad e_{31}[n]v=0\quad (n>0),\\
&e_{21}[0]^{l_1+1}v=0,\quad e_{32}[0]^{l_2+1}v=0,\\
&e_{21}[0]^{\alpha}e_{31}[0]^{l_3+1-\alpha}v=0\quad (0\le\alpha\le l_3+1).
\end{aligned}$$

The structure of the set of extremal vectors is more complicated for the modules $U^{k_1,k_2}_{l_1,l_2,l_3}$ and $V^{k_1,k_2}_{l_1,l_2,l_3}$ than for V^k. The extremal vectors of the modules $U^{k_1,k_2}_{l_1,l_2,l_3}$ and $V^{k_1,k_2}_{l_1,l_2,l_3}$ are labeled by $(\mathbf{d},\sigma)$, where $\mathbf{d}\in\mathbb{Z}^3_{\ge 0}$ and σ is an element of the Weyl group of $\mathfrak{sl}_3$. On the other hand, the computation of the contributions of extremal vectors for $U^{k_1,k_2}_{l_1,l_2,l_3}$ and $V^{k_1,k_2}_{l_1,l_2,l_3}$ is simpler than for V^k. We write these contributions explicitly and prove that they sum up to the characters of the modules $U^{k_1,k_2}_{l_1,l_2,l_3}$ and $V^{k_1,k_2}_{l_1,l_2,l_3}$ when the parameters l_1,l_2,l_3 belong to a certain region. To do that, we show that the characters and the sums of the contributions satisfy the same set of recurrent relations. We also show that the solution of these recursion relations is unique.

The principal space V^k is isomorphic to $U^{k,k}_{0,0,0}$. Equating the bosonic formula and (1.2), we arrive at the identity

$$I_{d_1,d_2}(z_1,z_2)=\sum_{n=0}^{\min(d_1,d_2)} I_{d_1,d_2,n}(z_1,z_2), \tag{1.8}$$

where

$$I_{d_1,d_2,n}(z_1,z_2)=\frac{1}{(q)_{d_1-n}(q)_{d_2-n}(q)_n(qz_1^{-1})_{d_1-n}}\times$$
$$\frac{(qz_2)_\infty}{(qz_1^{-1}z_2^{-1})_n(q^{d_1-2n+1}z_2)_\infty(q^{-d_1+2n+1}z_2^{-1})_{d_2-n}(qz_2)_{d_1-n}(qz_2^{-1})_n}.$$

The functions $I_{d_1,d_2}(z_1,z_2)$ in the limit $q\to 1$ appear in the study of the Whittaker functions (see [IS]). For generic q, they are also closely related to the Whittaker vectors in Verma modules of quantum groups (see [Kos], [E], [S]). We interpret (1.8) in terms of the representation theory. Namely, let ω and $\bar\omega$ be the Whittaker vectors in the Verma modules of the quantum groups $U_{q^{1/2}}(\mathfrak{sl}_3)$ and $U_{q^{-1/2}}(\mathfrak{sl}_3)$. We fix the decompositions of ω and $\bar\omega$ in the Gelfand–Tsetlin bases:

$$\omega=\sum_{d_1,d_2\ge 0}\sum_{n=0}^{\min(d_1,d_2)}\omega_{d_1,d_2,n},\quad \bar\omega=\sum_{d_1,d_2\ge 0}\sum_{n=0}^{\min(d_1,d_2)}\bar\omega_{d_1,d_2,n}.$$

We prove that

$$I_{d_1,d_2,n} = (\omega_{d_1,d_2,n}, \bar{\omega}_{d_1,d_2,n}),$$
$$I_{d_1,d_2} = \Big(\sum_{n=0}^{\min(d_1,d_2)} \omega_{d_1,d_2,n}, \sum_{n=0}^{\min(d_1,d_2)} \bar{\omega}_{d_1,d_2,n} \Big),$$

where (,) denotes the dual pairing. For the connection of the Whittaker vectors and Toda equations see also [GL], [BF], [GKLO].

Our paper is organized as follows.

In Section 2, we recall known bosonic and fermionic formulas for the simplest case of $\mathfrak{sl}_2$. This section is meant to be an illustration of the discussions which follow. In Section 3, we introduce the family of $\widehat{\mathfrak{n}}$-modules

(1.9)
$$\{U^{k_1,k_2}_{l_1,l_2,l_3} \mid (l_1,l_2,l_3) \in P^{k_1,k_2}_U\}, \quad \{V^{k_1,k_2}_{l_1,l_2,l_3} \mid (l_1,l_2,l_3) \in P^{k_1,k_2}_V\},$$

where the index sets P^{k_1,k_2}_U, P^{k_1,k_2}_V are defined in the text (see (3.23), (3.24)). Studying the structure of some of their subquotients, we derive a recurrent upper estimate for their characters. In Section 4, we introduce another family of $\widehat{\mathfrak{n}}$-modules

(1.10)
$$\{(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3} \mid (l_1,l_2,l_3) \in \bar{R}^{k_1,k_2}_U\}, \{(V_{VO})^{k_1,k_2}_{l_1,l_2,l_3} \mid (l_1,l_2,l_3) \in R^{k_1,k_2}_V\}$$

using the vertex operator construction. They are parametrized by subsets $\bar{R}^{k_1,k_2}_U \subset P^{k_1,k_2}_U$, $R^{k_1,k_2}_V \subset P^{k_1,k_2}_V$ (see (3.39),(3.38)), and are quotients of the corresponding modules (1.9). For these modules we derive a recurrent lower estimate for the characters. In Section 5, we show the uniqueness of solutions for the recurrent estimates (see Proposition 5.1 for the precise statement), and prove that in the parameter regions $\bar{R}^{k_1,k_2}_U$ and R^{k_1,k_2}_V, the modules (1.9) and (1.10) are isomorphic (Theorem 5.3). In Section 6, we proceed to write bosonic formulas for these modules utilizing an inductive structure with respect to the rank of the algebra. We start by recalling previous results on bosonic formulas for modules over the abelian subalgebra $\widehat{\mathfrak{a}} \subset \widehat{\mathfrak{n}}$ spanned by $e_{21}[n]$, $e_{31}[n]$ ($n \in \mathbb{Z}$). To make distinction we call the latter modules $\mathfrak{sl}_3$ *small principal subspaces.* Combining the characters for the $\mathfrak{sl}_2$ principal subspaces and those of the $\mathfrak{sl}_3$ small principal subspaces, we present a family of formal series. Then we prove that these formal series coincide with the characters of (1.9) in appropriate regions (Theorem 6.2; the region of validity for $V^{k_1,k_2}_{l_1,l_2,l_3}$ is R^{k_1,k_2}_V, while for $U^{k_1,k_2}_{l_1,l_2,l_3}$ it is a subset $\tilde{R}^{k_1,k_2}_U \subset \bar{R}^{k_1,k_2}_U$, see (5.5)). In Section 7, we consider the special case $k_1 = k_2$. In this case some of the terms in the bosonic formula can be combined to give a simpler result of the type discussed in [GL]. We obtain such a formula (Theorem 7.3), which includes as a particular case the character formula for the principal subspace V^k (Corollary 7.8). In the final Section 8, we discuss the connection to Whittaker vectors in the Verma modules of quantum groups.

Throughout the text, e_{ab} denotes the matrix unit with 1 at the (a, b)-th place and 0 elsewhere. For a graded vector space $M = \oplus_{m_1,\cdots,m_l,d\in\mathbb{Z}} M_{m_1,\cdots,m_l,d}$ with finite dimensional homogeneous components $M_{m_1,\cdots,m_l,d}$, we call the formal Laurent series

$$\mathrm{ch}_{z_1,\cdots,z_l,q} M = \sum_{m_1,\cdots,m_l,d\in\mathbb{Z}} (\dim M_{m_1,\cdots,m_l,d}) z_1^{m_1} \cdots z_l^{m_l} q^d$$

the *character* of M. In the text we deal with the case $l = 1$ or 2. We often suppress q from the notation as it does not change throughout the paper. For two formal series with integer coefficients

$$f^{(i)} = \sum_{m_1,\cdots,m_l,d} f^{(i)}_{m_1,\cdots,m_l,d} z_1^{m_1} \cdots z_l^{m_l} q^d \quad (i = 1, 2),$$

we write $f^{(1)} \le f^{(2)}$ to mean $f^{(1)}_{m_1,\cdots,m_l,d} \le f^{(2)}_{m_1,\cdots,m_l,d}$ for all $m_1, \cdots, m_l, d$.

§2. Bosonic formula for the case of $\widehat{\mathfrak{sl}}_2$.

In this section, we study the characters of the principal subspaces of integrable modules for $\widehat{\mathfrak{sl}}_2$. We present fermionic and bosonic formulas for these characters. The contributions of the extremal vectors are calculated in two different ways, one from the combinatorial set which labels a monomial basis, and another from the fermionic formula.

2.1. Principal spaces for $\widehat{\mathfrak{sl}}_2$

Consider the Lie algebra $\mathfrak{sl}_2 = \mathbb{C}e_{12} \oplus \mathbb{C}e_{21} \oplus \mathbb{C}(e_{11} - e_{22})$ where e_{ab} are the 2×2 matrix units. We also consider the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ spanned by the central element c and

$$e[n] = e_{12} \otimes t^n, \ f[n] = e_{21} \otimes t^n, \ h[n] = (e_{11} - e_{22}) \otimes t^n \ (n \in \mathbb{Z}).$$

Let M_l^k be the level k irreducible highest weight module of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ with the highest weight l $(0 \le l \le k)$. On M_l^k, c acts as a scalar k. The module M_l^k has a highest weight vector v_l^k characterized up to scalar multiple by

$$x[n]v_l^k = 0 \quad (x = e, f, h; n > 0),$$
$$e[0]v_l^k = 0, \quad f[0]^{l+1}v_l^k = 0, \quad h[0]v_l^k = lv_l^k.$$

Let $\hat{\mathfrak{n}}$ be the subalgebra of $\widehat{\mathfrak{sl}}_2$ generated by $f[n]$ $(n \in \mathbb{Z})$. Let V_l^k be the subspace of M_l^k generated by $f[-n]$ $(n \geq 0)$ from v_l^k. The space V_l^k is called the principal subspace of M_l^k. It is known [FS] that V_l^k is isomorphic to the cyclic module $U(\hat{\mathfrak{n}})v_l^k$ with the defining relations

$$f(z)^{k+1} = 0, \quad f[0]^{l+1}v_l^k = 0, \quad f[n]v_l^k = 0 \ (n > 0),$$

where $f(z) = \sum_{n\in\mathbb{Z}} f[n]z^{-n-1}$. It is graded by weight m and degree d. Namely, we have the decomposition $V_l^k = \bigoplus_{m,d=0}^{\infty}\left(V_l^k\right)_{m,d}$, where $\left(V_l^k\right)_{m,d}$ is spanned by the monomial vectors

$$f[0]^{a_0}f[-1]^{a_1}f[-2]^{a_2}\cdots v_l^k \tag{2.1}$$

with

$$\sum_{j=0}^{\infty} a_j = m, \quad \sum_{j=0}^{\infty} ja_j = d. \tag{2.2}$$

A basis of the principal subspace V_l^k is described by the set of integer points $\mathcal{P}_l^k = {}^{\mathbb{R}}\mathcal{P}_l^k \cap \mathbb{Z}^\infty$ where

$$\begin{aligned}{}^{\mathbb{R}}\mathcal{P}_l^k = \{(a_n)_{n\geq 0} \mid a_n \in \mathbb{R}_{\geq 0},\ a_n = 0 \\ \text{for almost all } n,\ a_j + a_{j+1} \leq k\ (j \geq 0),\ a_0 \leq l\}.\end{aligned}$$

Proposition 2.1. [FS] *The set of monomial vectors* (2.1) *with the exponents* $(a_n)_{n\geq 0}$ *taken from the set* $\mathcal{P}_l^k$ *constitutes a basis of* V_l^k.

The condition $a_j + a_{j+1} \leq k$ comes from the integrability condition $f(z)^{k+1} = 0$.

The character of V_l^k is the formal power series

$$\chi_l^k(z) = \sum_{m,d=0}^{\infty} \dim\left(V_l^k\right)_{m,d} z^m q^d.$$

Two different formulas are known for this quantity.

Proposition 2.2. [FS] *The character $\chi_l^k(z)$ is given by*

$$\chi_l^k(z) = \sum_{m_1,\dots,m_k=0}^{\infty} \frac{q^{\sum_{i,j=1}^k m_i m_j \min(i,j) - \sum_{i=1}^k \min(i,l) m_i} z^{\sum_{i=1}^k i m_i}}{(q)_{m_1} \cdots (q)_{m_k}}. \tag{2.3}$$

This formula is called fermionic.

Proposition 2.3. [FL] *The character $\chi_l^k(z)$ is given by*

$$\begin{aligned}\chi_l^k(z) = \sum_{n=0}^{\infty} &\frac{z^{nk} q^{n^2k-nl}}{(q^{2n}z)_\infty (q)_n (q^{-2n+1}z^{-1})_n} \\ &+ \sum_{n=0}^{\infty} \frac{z^{nk+l} q^{n^2k+nl}}{(q^{2n+1}z)_\infty (q)_n (q^{-2n}z^{-1})_{n+1}}.\end{aligned} \tag{2.4}$$

This formula is called bosonic.

2.2. Recursion relation

The recursion relation for the characters

$$\chi_l^k(z) = \sum_{a=0}^{l} z^a \chi_{k-a}^k(qz)$$

follows immediately from the definition of $\mathcal{P}_l^k$. It can be rewritten as

$$\chi_l^k(z) = \chi_{l-1}^k(z) + z^l \chi_{k-l}^k(qz). \tag{2.5}$$

In this form, the recursion can be explained by the representation theory as follows. Let $V_l^k[i]$ be the $U(\mathfrak{n})$-module identical with V_l^k as a vector space, where $f[n]$ acts as $f[n+i]$. We denote the vector v_l^k considered as the cyclic vector of the module $V_l^k[i]$ by $v_l^k[i]$. The identity (2.5) for the characters corresponds to a short exact sequence of $U(\hat{\mathfrak{n}})$-modules.

Proposition 2.4. *There is an exact sequence of $U(\hat{\mathfrak{n}})$ modules*

$$0 \to V_{k-l}^k[-1] \xrightarrow{\iota} V_l^k[0] \to V_{l-1}^k[0] \to 0,$$

where the homomorphism ι is defined by

$$\iota(v_{k-l}^k[-1]) = f[0]^l v_l^k[0].$$

Proof. Let $\langle f[0]^l \rangle$ be the submodule of $V_l^k[0]$ generated by the vector $f[0]^l v_l^k[0]$. By the definition, there is an isomorphism

$$V_l^k[0]/\langle f[0]^l \rangle \simeq V_{l-1}^k[0].$$

On the other hand, the integrability condition $f(z)^{k+1}=0$ implies

$$f[-1]^{k-l+1}f[0]^l v_l^k[0]=0.$$

Therefore, the mapping ι is well-defined. The character identity implies that it is an inclusion. Q.E.D.

2.3. Contributions of extremal vectors

Let $\mathcal{P}$ be a convex subset of a real vector space. A point in $\mathcal{P}$ is called extremal if and only if it is not a linear combination $\theta P_1+(1-\theta)P_2$ $(0<\theta<1)$ of two distinct points $P_1,P_2\in\mathcal{P}$. A general principle in "counting" the number of integer points in convex polygons is to write it as the sum of contributions from the extremal points. In this subsection, we show how we can guess the bosonic formula (2.4) by using this principle.

Weyl's character formula is of this kind: the simplest case is

$$\frac{1-z^{l+1}}{1-z}=\frac{1}{1-z}+\frac{z^l}{1-z^{-1}}. \tag{2.6}$$

The polygon in this example is the interval $[0,l]$. For the character, instead of counting the number of integer points, we count z^n for each integer point n in $[0,l]$. The extremal points are 0 and l. The contribution from an extremal point is counted as the sum of z^n over the integer points near that point in the limit $l\to\infty$. For 0 this is $1+z+z^2+\cdots=1/(1-z)$, and for l this is $z^l+z^{l-1}+\cdots=z^l/(1-z^{-1})$. These are the two terms in the right hand side of (2.6). To obtain the left hand side of the formula, we write the second term as $-z^{l+1}/(1-z)$. Because of rewriting $z^l+z^{l-1}+\cdots$ to $-z^{l+1}-z^{l+2}-\cdots$, the obtained formula contains both positive and negative coefficient terms. The formula (2.4) should be understood as an equality of non-negative power series in z. The n-th summand of the first sum (respectively, of the second sum) in the right hand side contains negative coefficient terms if and only if n is odd (respectively, even). This is the difference between the bosonic formula (2.4) and the fermionic one (2.3). In the latter, each term corresponds to an integer point, and therefore, the formula consists of positive coefficient terms only.

There is an important point in counting contributions of extremal points. For the characters of the principal subspaces, we count monomials z^mq^d using two linear functions (2.2). It means that we count not the integer points in the infinite dimensional polygon ${}^{\mathbb{R}}\mathcal{P}_l^k$ but rather the integer points with multiplicities in a polygon in $\mathbb{R}^2$, which is the image of ${}^{\mathbb{R}}\mathcal{P}_l^k$ by the mapping (m,d). Not all extremal points remain to be extremal when they are projected to $\mathbb{R}^2$. We guess that the contributions from such extremal points in ${}^{\mathbb{R}}\mathcal{P}_l^k$ that are projected to a non-extremal point in the image cancel out. Thus we count only the contributions from the extremal points in the image.

In the case of the Weyl–Kac character formula for integrable modules, the relevant extremal points in the weight space are given by the Weyl group orbit of the highest weight. In the case of the principal subspaces, we also take the extremal points in ${}^{\mathbb{R}}\mathcal{P}_l^k$ whose weights belong to the orbit of the highest weight. They are

(2.7)

$$w_{2n}:$$
$$a_0 = l, a_1 = k - l, \ldots, a_{2n-2} = l, a_{2n-1} = k - l, a_j = 0 \ (j \geq 2n),$$

(2.8)

$$w_{2n+1}:$$
$$a_0 = l, a_1 = k - l, \ldots, a_{2n-1} = k - l, a_{2n} = l, a_j = 0 \ (j \geq 2n+1),$$

where $n \geq 0$. The monomials $z^{m(w)}q^{d(w)}$ at these points are given by

$$z^{m(w)}q^{d(w)} = \begin{cases} z^{nk}q^{n^2k-nl} & \text{for } w = w_{2n}; \\ z^{nk+l}q^{n^2k+nl} & \text{for } w = w_{2n+1}. \end{cases}$$

The contribution from an extremal point is defined to be the formal series obtained in the limit

$$z^{m(w)}q^{d(w)} \times \lim_{l,k-l\to\infty} z^{-m(w)}q^{-d(w)}\chi_l^k(z)$$

The contribution can be calculated in several different ways. Here we present two such calculations.

A direct calculation using $\mathcal{P}_l^k$ is possible for the present case of $\widehat{\mathfrak{sl}}_2$. In the limit the shape of ${}^{\mathbb{R}}\mathcal{P}_l^k$ in the vicinity of each extremal point w becomes a cone of the form $w + {}^{\mathbb{R}}\mathcal{C}$. For the extremal point

$$w = w_{2n} = (l, k-l, l, k-l, \ldots, l, k-l, 0, 0, 0, \ldots),$$

the cone ${}^{\mathbb{R}}\mathcal{C}$ is generated by the following vectors:

$$
\begin{array}{llllllllllll}
(\ 0, & 0, & 0, & 0,\dots, & -1, & 1, & 0, & 0, & 0,\dots), & q, \\
 & & & & \dots & & & & & \\
(\ 0, & 0, & -1, & 1,\dots, & -1, & 1, & 0, & 0, & 0,\dots), & q^{n-1}, \\
(-1, & 1, & -1, & 1,\dots, & -1, & 1, & 0, & 0, & 0,\dots), & q^{n}, \\
(\ 0, & 0, & 0, & 0,\dots, & 0, & -1, & 0, & 0, & 0,\dots), & q^{-2n+1}z^{-1}, \\
 & & & & \dots & & & & & \\
(\ 0, & 0, & 0, & -1,\dots, & 1, & -1, & 0, & 0, & 0,\dots), & q^{-n-1}z^{-1}, \\
(\ 0, & -1, & 1, & -1,\dots, & 1, & -1, & 0, & 0, & 0,\dots), & q^{-n}z^{-1}, \\
(\ 0, & 0, & 0, & 0,\dots, & 0, & 0, & 1, & 0, & 0,\dots), & q^{2n}z, \\
(\ 0, & 0, & 0, & 0,\dots, & 0, & 0, & 0, & 1, & 0,\dots), & q^{2n+1}z, \\
 & & & & \dots, & & & & &
\end{array}
$$

where in the first line -1 follows $2(n-1)$ zeros. After each vector we wrote the corresponding monomial. Note that all these vectors are linearly independent and the integer points in the cone, $\mathcal{C} = {}^{\mathbb{R}}\mathcal{C} \cap \mathbb{Z}^{\infty}$, are linear combinations of the generating vectors with non-negative integer coefficients. Therefore, the contribution of the extremal point w_{2n} is the sum of products of powers of these monomials. Thus we get the summands in the first term in the right hand side of (2.4). The case of $w = w_{2n+1}$ is similar.

The second calculation of the contribution uses the fermionic formula. In the fermionic formula, terms are labeled by $(m_1, \dots, m_k)$. In this case, the term corresponding to each point is not a monomial but a formal power series. It is written explicitly in (2.3). It is known that there is a mapping $\Phi^k : \mathcal{P}^k_k \to \mathbb{Z}^k_{\geq 0}$ such that the sum of monomials over $(\Phi^k)^{-1}(m_1, m_2, \dots, m_k) \cap \mathcal{P}^k_l$ is the series corresponding to $(m_1, m_2, \dots, m_k)$. The mapping is defined inductively as follows. Let $a = (a_n)_{n\geq 0} \in \mathcal{P}^k_k$. If $a_i + a_{i+1} < k$ for all i we have $a \in \mathcal{P}^{k-1}_{k-1}$, and we define $\Phi^k(a) = (\Phi^{k-1}(a), 0)$. If $a_i + a_{i+1} = k$ for some i, define $b \in \mathcal{P}^k_k$ by

$$
b_j = \begin{cases} a_j & (j < i); \\ a_{j+2} & (j \geq i). \end{cases}
$$

Then, we define $\Phi^k(a) = \Phi^k(b) + (0, 0, \cdots, 0, 1)$. For example, consider the extremal points $w_{2n+\epsilon} \in {}^{\mathbb{R}}\mathcal{P}^k_l$ given by (2.7), (2.8). We have

$$
\Phi^k(w_{2n+\epsilon}) = (0, 0, \dots, 0, \overset{l-\mathrm{th}}{\epsilon}, 0, \dots, \overset{k-\mathrm{th}}{n}).
$$

The points of the cone ${}^{\mathbb{R}}\mathcal{C}$ are linear combinations of the generating vectors. For $M \in \mathbb{Z} > 0$, let $w_{2n+\epsilon} + \mathcal{C}(M)$ be the subset of $w_{2n+\epsilon} + \mathcal{C}$ such that the sum of the coefficients in front of the generating vectors is bounded by M. Set

$$\mathcal{M}^n_\epsilon(N) = \{(m_1, m_2, \cdots, m_{l-1}, m_l, m_{l+1}, \cdots, m_{k-1}, m_k) \\ \mid \sum_{0\le i\le N} m_{k-i} = n, \sum_{|i|\le N} m_{l+i} = \epsilon\}.$$

For any M, if $k-l-N, l-N, N$ are large enough, the points in $w_{2n+\epsilon} + \mathcal{C}(M)$ are mapped to $\mathcal{M}^n_\epsilon(N)$. Thus, the contribution of the extremal point $w_{2n+\epsilon}$ is equal to the sum of series corresponding to the points in $\mathcal{M}^n_\epsilon(N)$ in the limit $k-l-N, l-N, N \to \infty$.

Let us check this statement by direct calculation. We first prepare two lemmas.

Lemma 2.5.

$$\sum_{m_1,m_2,\ldots=0}^{\infty} \frac{q^{\sum_{i,j=1}^\infty m_i m_j \min(i,j)} z^{\sum_{i=1}^\infty i m_i}}{\prod_{i=1}^\infty (q)_{m_i}} = \frac{1}{(zq)_\infty}. \tag{2.9}$$

Proof. According to [FS], Theorem 2.7.1, a restriction of the summation in (2.9) to a region $m_{k+1} = m_{k+2} = \cdots = 0$ gives a formula for the character of the space $\mathbb{C}[e_1, e_2 \ldots]/I_k$, where I_k is an ideal generated by elements

$$\sum_{i_1+\cdots+i_{k+1}=n} e_{i_1} \ldots e_{i_{k+1}}, \quad n \ge k+1.$$

Therefore the $k \to \infty$ limit of the characters of $\mathbb{C}[e_1, e_2 \ldots]/I_k$ is equal to the character of the polynomial algebra itself, which is given by the right hand side of (2.9). Q.E.D.

Lemma 2.6. *Set*

$$g_{n,k}(z) = \sum_{(n_0,n_1,\ldots),\ \sum_{j=0}^\infty n_j = n} \frac{q^{\sum_{i,j=0}^\infty n_i n_j \min(k-i,k-j)} z^{\sum_{i=0}^\infty (k-i) n_i}}{\prod_{i=0}^\infty (q)_{n_i}}.$$

Then we have the equality

$$g_{n,k}(z) = \frac{q^{n^2 k} z^{nk}}{(q)_n (q^{-2n+1} z^{-1})_n}. \tag{2.10}$$

Proof. Splitting the sum in $g_{n,k}(z)$ into $n+1$ subsums corresponding to $n_0 = 0, 1, \ldots, n$, we obtain

$$g_{n,k}(z) = \sum_{i=0}^{n} \frac{q^{i^2 k} z^{ik}}{(q)_i} g_{n-i,k-1}(q^{2i} z).$$

This recursion has a unique solution of the form $g_{n,k}(z) \in 1 + z\mathbb{C}[[z]]$. One can check the recursion is satisfied by (2.10). The check reduces to the equality

$$\sum_{i=0}^{n} (-1)^i \frac{(q)_n}{(q)_i (q)_{n-i}} (q^n z)_i q^{i(i+1)/2 - in} = q^{n^2} z^n.$$

Q.E.D.

Let $F^n_\epsilon(N)$ be a sum

$$\sum \frac{q^{\sum_{i,j=1}^{k} m_i m_j \min(i,j) - \sum_{i=1}^{k} \min(i,l) m_i} z^{\sum_{i=1}^{k} i m_i}}{(q)_{m_1} \cdots (q)_{m_k}},$$

where the terms are labeled by $(m_1, \ldots, m_k) \in \mathcal{M}^n_\epsilon(N)$.

Proposition 2.7. *The $N, l-N, k-l-N \to \infty$ limit of $F^n_\epsilon(N)$ is equal to*

$$\frac{z^{nk} q^{n^2 k - nl}}{(q^{2n} z)_\infty (q)_n (q^{-2n+1} z^{-1})_n} \tag{2.11}$$

if $\epsilon = 0$ and is equal to

$$\frac{z^{nk+l} q^{n^2 k + nl}}{(q^{2n+1} z)_\infty (q)_n (q^{-2n} z^{-1})_{n+1}} \tag{2.12}$$

if $\epsilon = 1$.

Proof. Since we want to pass to the limit $N, l-N, k-l-N \to \infty$ it is natural to split any k-tuple $(m_1, \ldots, m_k) \in \mathcal{M}^n_\epsilon(N)$ into three groups: m_i with i "around" 1 (not too big), m_i with i "around" l and m_i with i "around" k. An important point is that in the limit $N, l-N, k-l-N \to \infty$ the following statement holds: for any m_i from the first group and m_j from the second group (or m_i from the second group and m_j from the third group) one has $i < j$. In what follows we leave the notation m_i for the entries from the first group and introduce new notations

$$p_s = m_{l+s}, \ n_j = m_{k-j}$$

for the second and third groups. Here $s \in \mathbb{Z}$, $j \in \mathbb{Z}_{\geq 0}$ and for $(m_1, \ldots, m_k) \in \mathcal{M}_\epsilon^n(N)$ we have $\sum_{s\in\mathbb{Z}} p_s = \epsilon$, $\sum_{j\geq 0} n_j = n$.

Suppose $\epsilon = 1$. We note that the limit of $F_\epsilon^n(N)$ is equal to a sum of terms labeled by three infinite sequences of non-negative integers: $(m_i)_{i\geq 1}$, $(p_{l+s})_{s\in\mathbb{Z}}$ and $(n_j)_{j\geq 0}$ with the properties $\sum_{s\in\mathbb{Z}} p_{l+s} = 1$, $\sum_{j\geq 0} n_j = n$. A term is given by the formula

$$\frac{q^{\sum_{i,j\geq 1} m_i m_j \min(i,j) + 2\sum_{i\geq 1} i m_i + 2\sum_{i\geq 1, j\geq 0} i m_i n_j - \sum_{i\geq 1} i m_i} z^{\sum_{i\geq 1} i m_i}}{\prod_{i\geq 1} (q)_{m_i}} \times \frac{q^{l+s-\min(l+s,l)+2\sum_{j\geq 0}(l+s)n_j} z^{l+s}}{1-q} \times \frac{q^{\sum_{i,j\geq 0} n_i n_j \min(k-i,k-j) - \sum_{j\geq 0} l n_j} z^{\sum (k-i) n_i}}{\prod_{j\geq 0} (q)_{n_j}}$$

Using Lemmas 2.5 and 2.6 and the equality $\sum_{j\geq 0} n_j = n$ we obtain that the sum over all $(m_i)_{i\geq 1}$, $(n_j)_{j\geq 0}$ and $s \in \mathbb{Z}$ of the terms above is equal to

$$\frac{1}{(zq^{2n+2})_\infty} \frac{q^{2ln} z^l}{1-q} \left(\sum_{s\geq 0} z^s q^{(2n+1)s} + \sum_{s<0} z^s q^{2ns} \right) \frac{z^{nk} q^{nk^2 - nl}}{(q)_n (q^{-2n+1} z^{-1})_n},$$

which is equal to (2.12). The case $\epsilon = 0$ is similar (even simpler since all $p_s = 0$). Proposition is proved. Q.E.D.

§3. Highest weight $\widehat{\mathfrak{n}}$-modules

In this section, we introduce a family of modules which generalize the principal subspaces of integrable $\widehat{\mathfrak{sl}}_3$-modules.

We fix the notation as follows. Let $\widehat{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}]$ denote the current algebra over the nilpotent subalgebra $\mathfrak{n} = \mathbb{C}e_{21} \oplus \mathbb{C}e_{31} \oplus \mathbb{C}e_{32}$ of $\mathfrak{sl}_3$. The basis elements $e_{ab}[n] = e_{ab} \otimes t^n$ $(1 \leq b < a \leq 3,\ n \in \mathbb{Z})$ of $\widehat{\mathfrak{n}}$ satisfy the relations

$$[e_{32}[m], e_{21}[n]] = e_{31}[m+n], [e_{21}[m], e_{31}[n]] = 0, [e_{32}[m], e_{31}[n]] = 0.$$

We set

$$e_{ab}(z) = \sum_{n\in\mathbb{Z}} e_{ab}[n] z^{-n-1}.$$

With the degree assignment

$$\deg e_{21}[n] = (1, 0, -n), \quad \deg e_{32}[n] = (0, 1, -n),$$

$\widehat{\mathfrak{n}}$ is a $\mathbb{Z}^2_{\geq 0} \times \mathbb{Z}$-graded Lie algebra. All $\widehat{\mathfrak{n}}$-modules considered in this paper are graded $\widehat{\mathfrak{n}}$-modules.

3.1. Principal subspaces for $\widehat{\mathfrak{sl}}_3$

For non-negative integers k, l_1, l_2 satisfying $l_1 + l_2 \leq k$, let $M^k_{l_1,l_2}$ be the level k integrable highest weight $\widehat{\mathfrak{sl}}_3$-module of highest weight (l_1, l_2). The highest weight vector $v^k_{l_1,l_2}$ is characterized up to scalar multiple by

$$\begin{aligned}
&x[n]v^k_{l_1,l_2} = 0 \quad (x \in \mathfrak{sl}_3; n > 0),\\
&e_{ab}[0]v^k_{l_1,l_2} = 0 \quad (a < b),\\
&(e_{11}[0] - e_{22}[0])v^k_{l_1,l_2} = l_1 v^k_{l_1,l_2}, \quad (e_{22}[0] - e_{33}[0])v^k_{l_1,l_2} = l_2 v^k_{l_1,l_2},\\
&e_{21}[0]^{l_1+1}v^k_{l_1,l_2} = 0, \quad e_{32}[0]^{l_2+1}v^k_{l_1,l_2} = 0.
\end{aligned}$$

We call the $U(\widehat{\mathfrak{n}})$-submodule

$$V^k_{l_1,l_2} = U(\widehat{\mathfrak{n}}) \cdot v^k_{l_1,l_2} \subset M^k_{l_1,l_2}$$

the *principal subspace* of $M^k_{l_1,l_2}$. The following relations for $v = v^k_{l_1,l_2}$ take place in $V^k_{l_1,l_2}$:

$$e_{21}[n]v = 0 \quad (n > 0), \tag{3.1}$$

$$e_{31}[n]v = 0 \quad (n > 0), \tag{3.2}$$

$$e_{32}[n]v = 0 \quad (n > 0), \tag{3.3}$$

$$e_{21}[0]^{l_1+1}v = 0, \tag{3.4}$$

$$e_{32}[0]^{l_2+1}v = 0, \tag{3.5}$$

$$e_{21}(z)^{k+1} = 0, \quad e_{32}(z)^{k+1} = 0\,. \tag{3.6}$$

We remark that also

$$e_{21}[0]^{\alpha}e_{31}[0]^{l_1+l_2-\alpha+1}v = 0 \qquad (0 \leq \alpha \leq l_1 + l_2 + 1) \tag{3.7}$$

holds, due to the following lemma.

Lemma 3.1. *Let w be a vector in an $\mathfrak{n}$-module such that $e_{21}^{l_1+1}w = 0$ and $e_{32}^{l_2+1}w = 0$ for some non-negative integers l_1, l_2. Then $e_{21}^{\alpha}e_{31}^{\beta}w = 0$ holds for all $\alpha, \beta \geq 0$ with $\alpha + \beta = l_1 + l_2 + 1$.*

Proof. Let W be the irreducible $\mathfrak{sl}_3$-module with highest weight (l_1, l_2). It is simple to check that the lemma holds for the highest weight vector of W. On the other hand, W is isomorphic to the quotient of the free left $U(\mathfrak{n})$-module $U(\mathfrak{n})$ by the submodule generated by $e_{21}^{l_1+1}$, $e_{32}^{l_2+1}$. Hence $U(\mathfrak{n}) \cdot w$ is a quotient of W, and the assertion follows. Q.E.D.

Our goal is to find a formula for the character of $V^k_{l_1,l_2}$.

3.2. Modules $U^{k_1,k_2}_{l_1,l_2,l_3}$, $V^{k_1,k_2}_{l_1,l_2,l_3}$

We shall introduce a family of cyclic $\widehat{\mathfrak{n}}$-modules $U^{k_1,k_2}_{l_1,l_2,l_3}$, $V^{k_1,k_2}_{l_1,l_2,l_3}$ parametrized by non-negative integers k_1, k_2 and l_1, l_2, l_3. The parameters k_1, k_2 play a role similar to that of the level of representations, while l_1, l_2, l_3 correspond to the highest weight. Throughout the paper we assume that

$$k_1 \le k_2.$$

Definition 3.2. We define $V^{k_1,k_2}_{l_1,l_2,l_3}$ to be the $\widehat{\mathfrak{n}}$-module generated by a non-zero vector v under the following defining relations:

$$e_{21}[n]v = 0 \quad (n > 0), \tag{3.8}$$
$$e_{31}[n]v = 0 \quad (n > 0), \tag{3.9}$$
$$e_{32}[n]v = 0 \quad (n > 0), \tag{3.10}$$
$$e_{21}[0]^{l_1+1}v = 0, \tag{3.11}$$
$$e_{21}[0]^{\alpha}e_{31}[0]^{l_3+1-\alpha}v = 0 \quad (0 \le \alpha \le l_3 + 1), \tag{3.12}$$
$$e_{32}[0]^{l_2+1}v = 0, \tag{3.13}$$
$$e_{21}(z)^{k_1+1} = 0, \quad e_{32}(z)^{k_2+1} = 0. \tag{3.14}$$

Definition 3.3. We define $U^{k_1,k_2}_{l_1,l_2,l_3}$ to be the $\widehat{\mathfrak{n}}$-module generated by a non-zero vector v with the following defining relations:

$$e_{21}[n]v = 0 \quad (n > 0), \tag{3.15}$$
$$e_{31}[n]v = 0 \quad (n > 1), \tag{3.16}$$
$$e_{32}[n]v = 0 \quad (n > 0), \tag{3.17}$$
$$e_{31}[1]^{l_3+1}v = 0, \tag{3.18}$$
$$e_{21}[0]^{\alpha}e_{31}[1]^{l_1+1-\alpha}v = 0 \quad (0 \le \alpha \le l_1 + 1), \tag{3.19}$$
$$e_{32}[0]^{l_2+1}v = 0, \tag{3.20}$$
$$e_{21}(z)^{k_1+1} = 0, \quad e_{32}(z)^{k_2+1} = 0. \tag{3.21}$$

Taking commutators of $e_{32}[0]$ and (3.14) or (3.21), we obtain also the relations

$$e_{21}(z)^{\alpha}e_{31}(z)^{k_1-\alpha+1} = 0 \qquad (0 \le \alpha \le k_1 + 1). \tag{3.22}$$

We use the following notation for the characters of these modules:

$$\psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1, z_2) = \mathrm{ch}_{z_1,z_2,q} V^{k_1,k_2}_{l_1,l_2,l_3},$$
$$\varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1, z_2) = \mathrm{ch}_{z_1,z_2,q} U^{k_1,k_2}_{l_1,l_2,l_3}.$$

Note that our characters are normalized in such a way that the degree of the cyclic vectors is $(0,0,0)$.

Remark 3.4. From the definition it readily follows that $V^{k,k}_{l_1,l_2,l_1+l_2} \simeq U^{k,k}_{l_1,l_2,0}$. The principal subspace $V^{k}_{l_1,l_2}$ is its quotient. Indeed, comparing (3.8)–(3.13) with (3.1)–(3.6) and the remark after that, we see that there is a surjection of $\widehat{\mathfrak{n}}$-modules

$$V^{k,k}_{l_1,l_2,l_1+l_2} \to V^{k}_{l_1,l_2} \to 0.$$

Later it will turn out to be an isomorphism (see Corollary 5.4).

Some of the modules are the same which follows immediately from the definition.

Lemma 3.5. *We have*

$$\begin{aligned}
V^{k_1,k_2}_{l_1,l_2,l_3} &= V^{k_1,k_2}_{k_1,l_2,l_3} \quad (l_1 > k_1),\\
V^{k_1,k_2}_{l_1,l_2,l_3} &= V^{k_1,k_2}_{l_1,k_2,l_3} \quad (l_2 > k_2),\\
V^{k_1,k_2}_{l_1,l_2,l_3} &= V^{k_1,k_2}_{l_1,l_2,k_1} \quad (l_3 > k_1),\\
V^{k_1,k_2}_{l_1,l_2,l_3} &= V^{k_1,k_2}_{l_3,l_2,l_3} \quad (l_1 > l_3),\\
V^{k_1,k_2}_{l_1,l_2,l_3} &= V^{k_1,k_2}_{l_1,l_2,l_1+l_2} \quad (l_3 > l_1 + l_2).
\end{aligned}$$

Proof. For example, (3.11), (3.13) and Lemma 3.1 imply that $e_{21}[0]^\alpha e_{31}[0]^{l_1+l_2-\alpha+1}v = 0$ for all $0 \le \alpha \le l_1 + l_2 + 1$. This proves the last relation. Other cases can be verified similarly using either the definition, (3.14) or (3.22). Q.E.D.

Lemma 3.6. *We have*

$$\begin{aligned}
U^{k_1,k_2}_{l_1,l_2,l_3} &= U^{k_1,k_2}_{k_1,l_2,l_3} \quad (l_1 > k_1),\\
U^{k_1,k_2}_{l_1,l_2,l_3} &= U^{k_1,k_2}_{l_1,k_2,l_3} \quad (l_2 > k_2),\\
U^{k_1,k_2}_{l_1,l_2,l_3} &= U^{k_1,k_2}_{l_1,l_2,\min(l_1,l_2)} \quad (l_3 > \min(l_1,l_2)).
\end{aligned}$$

Proof. By (3.19) we have $e_{31}[1]^{l_1+1}v = 0$. Taking the commutator of $e_{21}[1]$ with $e_{32}[0]^{l_2+1}$ and using $e_{21}[1]v = 0$ and (3.20), we obtain $e_{31}[1]^{l_2+1}v = 0$. This proves the third relation. The other relations are proved similarly. Q.E.D.

In view of Lemmas 3.5–3.6, we may restrict our attention to the modules

$$\{U^{k_1,k_2}_{l_1,l_2,l_3} \mid (l_1,l_2,l_3) \in P_U^{k_1,k_2}\}, \quad \{V^{k_1,k_2}_{l_1,l_2,l_3} \mid (l_1,l_2,l_3) \in P_V^{k_1,k_2}\},$$

where the parameter regions are the following subsets of $\mathbb{Z}^3_{\geq 0}$:

(3.23)
$$P_U^{k_1,k_2} = \{(l_1,l_2,l_3) \mid 0 \leq l_1 \leq k_1,\ 0 \leq l_2 \leq k_2,\ 0 \leq l_3 \leq \min(l_1,l_2)\},$$
(3.24)
$$P_V^{k_1,k_2} = \{(l_1,l_2,l_3) \mid 0 \leq l_1 \leq k_1,\ 0 \leq l_2 \leq k_2,\ l_1 \leq l_3 \leq \min(l_1+l_2,k_1)\}.$$

3.3. Subquotient modules

In this subsection we study a recurrent structure for some subquotients of $U^{k_1,k_2}_{l_1,l_2,l_3}$, $V^{k_1,k_2}_{l_1,l_2,l_3}$. Denote by $T_{m,n}$ the automorphism of $\widehat{\mathfrak{n}}$ given by

$$T_{m,n}e_{21}[i] = e_{21}[i-m], \quad T_{m,n}e_{31}[i] = e_{31}[i-m-n], \quad T_{m,n}e_{32}[i] = e_{32}[i-n].$$

For an $\widehat{\mathfrak{n}}$-module M, we denote by $M[m,n]$ the module with the same underlying vector space on which $x \in \widehat{\mathfrak{n}}$ acts as $T_{m,n}x$. For a cyclic $\widehat{\mathfrak{n}}$-module M with a cyclic vector v and $f \in U(\widehat{\mathfrak{n}})$, we use the notation

$$\langle f \rangle = U(\widehat{\mathfrak{n}}) \cdot fv.$$

In what follows, we set $U^{k_1,k_2}_{l_1,l_2,l_3} = 0$, $V^{k_1,k_2}_{l_1,l_2,l_3} = 0$ if one of l_i's is negative.

Lemma 3.7. *We have an exact sequence of $\widehat{\mathfrak{n}}$-modules*

$$V^{k_1,k_2}_{l_3,k_2-l_2,l_1}[1,-1] \to U^{k_1,k_2}_{l_1,l_2,l_3} \to U^{k_1,k_2}_{l_1,l_2-1,l_3} \to 0.$$

Proof. Consider the submodule $\langle e_{32}[0]^{l_2} \rangle$ of $U^{k_1,k_2}_{l_1,l_2,l_3}$. By the definition we have an exact sequence

$$0 \to \langle e_{32}[0]^{l_2} \rangle \to U^{k_1,k_2}_{l_1,l_2,l_3} \to U^{k_1,k_2}_{l_1,l_2-1,l_3} \to 0.$$

We show that there is a surjection

$$V^{k_1,k_2}_{l_3,k_2-l_2,l_1}[1,-1] \to \langle e_{32}[0]^{l_2} \rangle \to 0.$$

It suffices to verify the following relations for $v_1 = e_{32}[0]^{l_2}v$ where v denotes the generator of $U^{k_1,k_2}_{l_1,l_2,l_3}$:

$$e_{21}[n]v_1 = 0 \quad (n > 1), \tag{3.25}$$
$$e_{31}[n]v_1 = 0 \quad (n > 0), \tag{3.26}$$
$$e_{32}[n]v_1 = 0 \quad (n > -1), \tag{3.27}$$
$$e_{21}[1]^{l_3+1}v_1 = 0, \tag{3.28}$$
$$e_{21}[1]^{\alpha}e_{31}[0]^{l_1+1-\alpha}v_1 = 0, \tag{3.29}$$
$$e_{32}[-1]^{k_2-l_2+1}v_1 = 0. \tag{3.30}$$

Equation (3.25) follows from $e_{21}[n]v = 0$, $[e_{32}[0], e_{21}[n]] = e_{31}[n]$, $e_{31}[n]v = 0$ and $[e_{31}[n], e_{32}[0]] = 0$ with $n > 1$. Equation (3.26) follows from (3.16) for $n \geq 2$. For $n = 1$ it follows from $e_{21}[1]v = 0$ and (3.20) by using $[e_{32}[0], e_{21}[1]] = e_{31}[1]$. Equation (3.27) follows from (3.17) and (3.20). Equation (3.28) follows from (3.18), $e_{21}[1]v = 0$ and $[e_{32}[0], e_{21}[1]] = e_{31}[1]$.

Let us prove (3.29). We first assume $\alpha > l_2$. Then

$$e_{21}[1]^{\alpha}e_{31}[0]^{l_1+1-\alpha}e_{32}[0]^{l_2}v = e_{31}[0]^{l_1+1-\alpha}(e_{21}[1]^{\alpha}e_{32}[0]^{l_2}v) = 0,$$

because $e_{21}[1]v = 0$ and thus

$$e_{21}[1]^{\alpha}e_{32}[0]^{l_2}v = -l_2^2 e_{21}[1]^{\alpha-l_2}e_{31}[1]^{l_2}v = 0.$$

Now suppose $\alpha \leq l_2$. From (3.19) we have

$$e_{21}[0]^{l_1+1-\alpha}e_{31}[1]^{\alpha}v = 0.$$

From (3.20), using $e_{21}[1]v = 0$, we obtain

$$e_{32}[0]^{l_2+1-\alpha}e_{31}[1]^{\alpha}v = 0.$$

A variant of Lemma 3.1 and the last two equations above lead to

$$e_{31}[0]^{l_1+1-\alpha}e_{32}[0]^{l_2-\alpha}e_{31}[1]^{\alpha}v = 0.$$

It follows that

$$e_{21}[1]^{\alpha}e_{31}[0]^{l_1+1-\alpha}e_{32}[0]^{l_2}v = -l_2\alpha e_{31}[0]^{l_1+1-\alpha}e_{31}[1]^{\alpha}e_{32}[0]^{l_2-\alpha}v = 0.$$

Finally, (3.30) is a consequence of $e_{32}(z)^{k_2+1} = 0$. Q.E.D.

Lemma 3.8. *We have an exact sequence*

$$U^{k_1,k_2}_{l_3,k_2-l_2,l_1}[0,-1] \to V^{k_1,k_2}_{l_1,l_2,l_3} \to V^{k_1,k_2}_{l_1,l_2-1,l_3} \to 0.$$

Proof. We consider the submodule $\langle e_{32}[0]^{l_2}\rangle$ of $V^{k_1,k_2}_{l_1,l_2,l_3}$, so that

$$0 \to \langle e_{32}[0]^{l_2}\rangle \to V^{k_1,k_2}_{l_1,l_2,l_3} \to V^{k_1,k_2}_{l_1,l_2-1,l_3} \to 0$$

is exact. We then show that there is a surjection

$$U^{k_1,k_2}_{l_3,k_2-l_2,l_1}[0,-1] \to \langle e_{32}[0]^{l_2}\rangle \to 0$$

by checking the defining relations. The rest of the proof is similar to that of Lemma 3.7, we omit the details.

Q.E.D.

Lemma 3.9. *We have exact sequences*

$$V^{k_1,k_2}_{l_1-l_3,l_2-l_3,k_1-l_3} \to U^{k_1,k_2}_{l_1,l_2,l_3} \to U^{k_1,k_2}_{l_1,l_2,l_3-1} \to 0\,, \tag{3.31}$$

$$U^{k_1,k_2}_{k_1-l_1,l_1+l_2,l_3-l_1}[-1,0] \to V^{k_1,k_2}_{l_1,l_2,l_3} \to V^{k_1,k_2}_{l_1-1,l_2,l_3} \to 0\,. \tag{3.32}$$

Proof. We repeat the argument of Lemmas 3.7 and 3.8.

To show (3.31), we take the submodule generated by $e_{31}[1]^{l_3}v \in U^{k_1,k_2}_{l_1,l_2,l_3}$ and check the exact sequence

$$0 \to \langle e_{31}[1]^{l_3}\rangle \to U^{k_1,k_2}_{l_1,l_2,l_3} \to U^{k_1,k_2}_{l_1,l_2,l_3-1} \to 0,$$
$$V^{k_1,k_2}_{l_1-l_3,l_2-l_3,l_1+l_2-2l_3} \to \langle e_{31}[1]^{l_3}\rangle \to 0.$$

For the proof of (3.32), we take $e_{21}[0]^{l_1}v \in V^{k_1,k_2}_{l_1,l_2,l_3}$ and verify

$$0 \to \langle e_{21}[0]^{l_1}\rangle \to V^{k_1,k_2}_{l_1,l_2,l_3} \to V^{k_1,k_2}_{l_1-1,l_2,l_3} \to 0,$$
$$U^{k_1,k_2}_{k_1-l_1,l_1+l_2,l_3-l_1}[-1,0] \to \langle e_{21}[0]^{l_1}\rangle \to 0.$$

Q.E.D.

From Lemmas 3.7–3.9, we obtain the following upper estimate for the characters of all modules in the parameter regions (3.23),(3.24) :

(3.33)
$$\varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \le \varphi^{k_1,k_2}_{l_1,l_2-1,l_3}(z_1,z_2) + z_2^{l_2}\psi^{k_1,k_2}_{l_3,k_2-l_2,l_1}(q^{-1}z_1,qz_2),$$
(3.34)
$$\psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \le \psi^{k_1,k_2}_{l_1,l_2-1,l_3}(z_1,z_2) + z_2^{l_2}\varphi^{k_1,k_2}_{l_3,k_2-l_2,l_1}(z_1,qz_2),$$
(3.35)
$$\varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \le \varphi^{k_1,k_2}_{l_1,l_2,l_3-1}(z_1,z_2) + (q^{-1}z_1z_2)^{l_3}\psi^{k_1,k_2}_{l_1-l_3,l_2-l_3,k_1-l_3}(z_1,z_2),$$
(3.36)
$$\psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \le \psi^{k_1,k_2}_{l_1-1,l_2,l_3}(z_1,z_2) + z_1^{l_1}\varphi^{k_1,k_2}_{k_1-l_1,l_1+l_2,l_3-l_1}(qz_1,z_2).$$

In general, the equality does not hold, however, we will see that it does hold in a restricted range of the parameters. Define the following sets:

$$R_U^{k_1,k_2} = \{(l_1,l_2,l_3) \mid k_1 \le l_1+l_2-l_3 \le k_2\} \cap P_U^{k_1,k_2}, \tag{3.37}$$

$$R_V^{k_1,k_2} = \{(l_1,l_2,l_3) \mid 0 \le l_1+l_2-l_3 \le k_2-k_1\} \cap P_V^{k_1,k_2}, \tag{3.38}$$

$$\bar{R}_U^{k_1,k_2} = \{(l_1,l_2,l_3) \mid 0 \le l_1+l_2-l_3 \le k_2\} \cap P_U^{k_1,k_2}. \tag{3.39}$$

Theorem 3.10. *The following recurrence relations hold.*
If $(l_1,l_2,l_3) \in R_U^{k_1,k_2}$, then

(3.40)
$$\varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) = \varphi^{k_1,k_2}_{l_1,l_2-1,\min(l_3,l_2-1)}(z_1,z_2) + z_2^{l_2}\psi^{k_1,k_2}_{l_3,k_2-l_2,l_1}(q^{-1}z_1,qz_2)\,.$$

If $(l_1,l_2,l_3) \in R_V^{k_1,k_2}$, then

(3.41)
$$\psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) = \psi^{k_1,k_2}_{l_1,l_2-1,\min(l_3,l_1+l_2-1)}(z_1,z_2) + z_2^{l_2}\varphi^{k_1,k_2}_{l_3,k_2-l_2,l_1}(z_1,qz_2)\,.$$

If $(l_1,l_2,l_3) \in \bar{R}_U^{k_1,k_2}$ and either $l_1+l_2-l_3 \neq k_2$ or $l_3=0$, then

$$\begin{aligned}\varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) &= \varphi^{k_1,k_2}_{l_1,l_2,l_3-1}(z_1,z_2)\\ &+(q^{-1}z_1z_2)^{l_3}\psi^{k_1,k_2}_{l_1-l_3,l_2-l_3,\min(k_1-l_3,l_1+l_2-2l_3)}(z_1,z_2),\end{aligned} \tag{3.42}$$

If $(l_1,l_2,l_3) \in R_V^{k_1,k_2}$, then

$$\begin{aligned}\psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) &= \psi^{k_1,k_2}_{l_1-1,l_2,\min(l_3,l_1+l_2-1)}(z_1,z_2)\\ &+ z_1^{l_1}\varphi^{k_1,k_2}_{k_1-l_1,l_1+l_2,l_3-l_1}(qz_1,z_2).\end{aligned} \tag{3.43}$$

The proof of Theorem 3.10 will be completed in Section 5.

We remark that, in (3.42), the condition $l_1+l_2-l_3 \neq k_2$ is imposed so that the parameters of the first term in the right hand side stay within the region $\bar{R}_U^{k_1,k_2}$. In all other cases, the parameters appearing in the right hand side belong to the proper region ($\bar{R}_U^{k_1,k_2}$ for $\varphi^{k_1,k_2}_{l_1,l_2,l_3}$ and $R_V^{k_1,k_2}$ for $\psi^{k_1,k_2}_{l_1,l_2,l_3}$). In what follows we refer to (3.40)–(3.43) as *short exact sequence (SES) recursion.*

§4. Vertex operators

In this section we construct another family of $\widehat{\mathfrak{n}}$-modules

$$\{(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3} \mid (l_1,l_2,l_3) \in \bar{R}_U^{k_1,k_2}\}, \{(V_{VO})^{k_1,k_2}_{l_1,l_2,l_3} \mid (l_1,l_2,l_3) \in R_V^{k_1,k_2}\},$$

as subspaces of tensor products of certain Fock modules. These modules have the following properties.

(i) $(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3}$ is a quotient of $U^{k_1,k_2}_{l_1,l_2,l_3}$, and $(V_{VO})^{k_1,k_2}_{l_1,l_2,l_3}$ is a quotient of $V^{k_1,k_2}_{l_1,l_2,l_3}$.

(ii) The characters

$$(\varphi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) = \mathrm{ch}_{z_1,z_2,q}(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3},$$
$$(\psi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) = \mathrm{ch}_{z_1,z_2,q}(V_{VO})^{k_1,k_2}_{l_1,l_2,l_3}$$

satisfy the following inequalities.

If $(l_1,l_2,l_3) \in R^{k_1,k_2}_U$, then

$$(4.1)\quad (\varphi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \geq (\varphi_{VO})^{k_1,k_2}_{l_1,l_2-1,\min(l_3,l_2-1)}(z_1,z_2) + z_2^{l_2}(\psi_{VO})^{k_1,k_2}_{l_3,k_2-l_2,l_1}(q^{-1}z_1,qz_2).$$

If $(l_1,l_2,l_3) \in R^{k_1,k_2}_V$, then

$$(4.2)\quad (\psi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \geq (\psi_{VO})^{k_1,k_2}_{l_1,l_2-1,\min(l_3,l_1+l_2-1)}(z_1,z_2) + z_2^{l_2}(\varphi_{VO})^{k_1,k_2}_{l_3,k_2-l_2,l_1}(z_1,qz_2).$$

If $(l_1,l_2,l_3) \in \bar{R}^{k_1,k_2}_U$ and $l_1+l_2-l_3 \neq 0$ or $l_3 = 0$, then

$$(4.3)\quad (\varphi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \geq (\varphi_{VO})^{k_1,k_2}_{l_1,l_2,l_3-1}(z_1,z_2) + (q^{-1}z_1z_2)^{l_3}(\psi_{VO})^{k_1,k_2}_{l_1-l_3,l_2-l_3,\min(k_1-l_3,l_1+l_2-2l_3)}(z_1,z_2).$$

If $(l_1,l_2,l_3) \in R^{k_1,k_2}_V$, then

$$(4.4)\quad (\psi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \geq (\psi_{VO})^{k_1,k_2}_{l_1-1,l_2,\min(l_3,l_1+l_2-1)}(z_1,z_2) + z_1^{l_1}(\varphi_{VO})^{k_1,k_2}_{k_1-l_1,l_1+l_2,l_3-l_1}(qz_1,z_2).$$

These inequalities differ from (3.40)–(3.43) by the change of the sign $=$ to the sign $\geq$.

Let us recall some constructions from the theory of lattice vertex operator algebras (see [D], [K2]). Let $\mathfrak{h}$ be a two-dimensional complex vector space with a basis a,b and an inner product defined by the $\mathfrak{sl}_3$ Cartan matrix:

$$(a,a) = 2,\ (b,b) = 2,\ (a,b) = -1.$$

Let

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}1$$

be the corresponding Heisenberg Lie algebra with the bracket

$$[\alpha[i], \beta[j]] = i\delta_{i,-j}(\alpha, \beta) \quad (\alpha, \beta \in \mathfrak{h}),$$

where $\alpha[i] = \alpha \otimes t^i$. For $\alpha \in \mathfrak{h}$ define the Fock representation $\mathcal{F}_\alpha$ generated by a vector $|\alpha\rangle$ such that

$$\beta[n]|\alpha\rangle = 0, \ n > 0, \qquad \beta[0]|\alpha\rangle = (\beta, \alpha)|\alpha\rangle.$$

Set $L = \mathbb{Z}(2a + b)/3 \oplus \mathbb{Z}(a + 2b)/3$. For $\alpha \in L$ we consider the corresponding vertex operators acting on the direct sum of Fock spaces $\bigoplus_{\alpha \in L} \mathcal{F}_\alpha$:

$$\Gamma_\alpha(z) = S_\alpha z^{\alpha[0]} \exp(-\sum_{n<0} \frac{\alpha[n]}{n} z^{-n}) \exp(-\sum_{n>0} \frac{\alpha[n]}{n} z^{-n}),$$

where $z^{\alpha[0]}$ acts on $\mathcal{F}_\beta$ by $z^{(\alpha,\beta)}$ and the operator S_α is defined by

$$S_\alpha|\beta\rangle = |\alpha + \beta\rangle, \quad [S_\alpha, \beta[n]] = 0 \quad (n \neq 0, \alpha, \beta \in \mathfrak{h}).$$

The Fourier decomposition is given by

$$\Gamma_\alpha(z) = \sum_{n \in \mathbb{Z}} \Gamma_\alpha[n] z^{-n-(\alpha,\alpha)/2}.$$

In particular,

$$\Gamma_\alpha[-(\alpha, \alpha)/2 - (\alpha, \beta)]|\beta\rangle = |\alpha + \beta\rangle.$$

We need three vertex operators corresponding to the vectors a, b and $c = a + b$. The Frenkel–Kac construction for level 1 modules (see [FK]) defines the action of $\widehat{\mathfrak{n}}$ on $\oplus_{\alpha \in L} \mathcal{F}_\alpha$ via the homomorphism

$$e_{21}[n] \mapsto \Gamma_a[n], \quad e_{32}[n] \mapsto \Gamma_b[n], \quad e_{31}[n] \mapsto \Gamma_c[n].$$

Let $v_{m,n}$ be a vacuum vector of $\mathcal{F}_{-(\frac{2m+n}{3}+1)a-(\frac{2n+m}{3}+1)b}$.

Lemma 4.1. *We have*

$$\begin{aligned}
e_{21}[i]v_{m,n} &= 0 \quad (i > m), \\
e_{32}[i]v_{m,n} &= 0 \quad (i > n), \\
e_{31}[i]v_{m,n} &= 0 \quad (i > m + n + 1), \\
e_{21}[m]v_{m,n} &= v_{m-2,n+1}, \\
e_{32}[n]v_{m,n} &= v_{m+1,n-2}, \\
e_{31}[m + n + 1]v_{m,n} &= v_{m-1,n-1}.
\end{aligned}$$

We denote by $W_{m,n} \hookrightarrow \oplus_{\alpha\in L}\mathcal{F}_\alpha$ the cyclic $\widehat{\mathfrak{n}}$-module with the cyclic vector $v_{m,n}$. The shift automorphism $T_{m,n}$ induces an isomorphism between $W_{0,0}[m,n]$ and $W_{m,n}$.

We also need one-parameter analogues of the modules $W_{m,n}$. Fix a one-dimensional space with a basis vector $\bar{b}$ and an inner product defined by $(\bar{b},\bar{b}) = 2$. The vertex operator $\Gamma_{\bar{b}}(z)$ acts in the direct sum of Fock modules over the Heisenberg algebra $\mathbb{C}\bar{b}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\cdot 1$. We set $W_n = \mathbb{C}[\{\Gamma_{\bar{b}}[i]\}_{i\in\mathbb{Z}}]\cdot v_n$ where $v_n = |-(n+1)\bar{b}/2\rangle$, and make it an $\widehat{\mathfrak{n}}$-module by letting $e_{32}[i]$ act by $\Gamma_{\bar{b}}[i]$ and $e_{21}[i], e_{31}[i]$ by 0. Then we have

$$e_{32}[i]v_n = 0 \quad (i > n),\ e_{32}[n]v_n = v_{n-2}.$$

Now we define the VO (vertex operator) versions of the spaces $U^{k_1,k_2}_{l_1,l_2,l_3}$ and $V^{k_1,k_2}_{l_1,l_2,l_3}$, utilizing the modules

$$W_{0,0},\ W_{0,-1},\ W_{-1,0},\ W_{-1,-1},\ W_0,\ W_{-1}$$

as building blocks.

Definition 4.2. Let $(l_1,l_2,l_3)\in\bar{R}^{k_1,k_2}_U$.
If $(l_1,l_2,l_3)\in R^{k_1,k_2}_U$, then we define

$$(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3} \subset W_{0,0}^{\otimes l_3}\otimes W_{0,-1}^{\otimes(l_1-l_3)}\otimes W_{-1,0}^{\otimes(k_1-l_1)}\otimes W_0^{\otimes(l_1+l_2-l_3-k_1)} \otimes W_{-1}^{\otimes(k_2-l_1-l_2+l_3)}$$

to be the cyclic $\widehat{\mathfrak{n}}$-module with the cyclic vector

$$w_1(l_1,l_2,l_3) = v_{0,0}^{\otimes l_3}\otimes v_{0,-1}^{\otimes(l_1-l_3)}\otimes v_{-1,0}^{\otimes(k_1-l_1)}\otimes v_0^{\otimes(l_1+l_2-l_3-k_1)} \otimes v_{-1}^{\otimes(k_2-l_1-l_2+l_3)}.$$

If $l_1+l_2-l_3<k_1$, then we define

$$(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3} \subset W_{0,0}^{\otimes l_3}\otimes W_{0,-1}^{\otimes(l_1-l_3)}\otimes W_{-1,0}^{\otimes(l_2-l_3)}\otimes W_{-1,-1}^{\otimes(k_1-l_1-l_2+l_3)} \otimes W_{-1}^{\otimes(k_2-k_1)}$$

to be the cyclic $\widehat{\mathfrak{n}}$-module with the cyclic vector

$$w_2(l_1,l_2,l_3) = v_{0,0}^{\otimes l_3}\otimes v_{0,-1}^{\otimes(l_1-l_3)}\otimes v_{-1,0}^{\otimes(l_2-l_3)}\otimes v_{-1,-1}^{\otimes(k_1-l_1-l_2+l_3)} \otimes v_{-1}^{\otimes(k_2-k_1)}.$$

Lemma 4.3. *Let $(l_1,l_2,l_3)\in \bar{R}_U^{k_1,k_2}$. Then there exists a surjective homomorphism of $\widehat{\mathfrak{n}}$-modules*

$$U^{k_1,k_2}_{l_1,l_2,l_3}\to (U_{VO})^{k_1,k_2}_{l_1,l_2,l_3}.$$

Proof. It is sufficient to check that the defining relations of $U^{k_1,k_2}_{l_1,l_2,l_3}$ are satisfied in $(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3}$. That follows from Lemma 4.1. Q.E.D.

Definition 4.4. If $(l_1,l_2,l_3)\in R_V^{k_1,k_2}$, then we define

$$(V_{VO})^{k_1,k_2}_{l_1,l_2,l_3}\subset W_{0,-1}^{\otimes l_1}\otimes W_{-1,0}^{\otimes(l_3-l_1)}\otimes W_{-1,-1}^{\otimes(k_1-l_3)}\otimes W_0^{\otimes(l_1+l_2-l_3)}\otimes W_{-1}^{\otimes(k_2-k_1-l_1-l_2+l_3)}$$

to be the cyclic $\widehat{\mathfrak{n}}$-module with the cyclic vector

$$w_3(l_1,l_2,l_3)=v_{0,-1}^{\otimes l_1}\otimes v_{-1,0}^{\otimes(l_3-l_1)}\otimes v_{-1,-1}^{\otimes(k_1-l_3)}\otimes v_0^{\otimes(l_1+l_2-l_3)}\otimes v_{-1}^{\otimes(k_2-k_1-l_1-l_2+l_3)}.$$

Lemma 4.5. *Let $(l_1,l_2,l_3)\in R_V^{k_1,k_2}$. Then there exists a surjective homomorphism of $\widehat{\mathfrak{n}}$-modules*

$$V^{k_1,k_2}_{l_1,l_2,l_3}\to (V_{VO})^{k_1,k_2}_{l_1,l_2,l_3}.$$

Proof. The lemma follows from Lemma 4.1. Q.E.D.

Theorem 4.6. *There exist the following complexes of $\widehat{\mathfrak{n}}$-modules which are exact in the first and third terms.*

For $(l_1,l_2,l_3)\in R_U^{k_1,k_2}$, we have

$$0\to (V_{VO})^{k_1,k_2}_{l_3,k_2-l_2,l_1}[1,-1]\xrightarrow{\iota_1}(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3}\to(U_{VO})^{k_1,k_2}_{l_1,l_2-1,\min(l_3,l_2-1)}\to 0, \tag{4.5}$$

such that $\iota_1(T_{1,-1}(w_3(l_3,k_2-l_2,l_1)))=e_{32}^{l_2}[0]w_1(l_1,l_2,l_3)$.

For $(l_1,l_2,l_3)\in R_V^{k_1,k_2}$, we have

$$0\to (U_{VO})^{k_1,k_2}_{l_3,k_2-l_2,l_1}[0,-1]\xrightarrow{\iota_2}(V_{VO})^{k_1,k_2}_{l_1,l_2,l_3}\to(V_{VO})^{k_1,k_2}_{l_1,l_2-1,\min(l_3,l_1+l_2-1)}\to 0, \tag{4.6}$$

such that $\iota_2(T_{0,-1}(w_1(l_3,k_2-l_2,l_1)))=e_{32}^{l_2}[0]w_3(l_1,l_2,l_3)$.

For $(l_1,l_2,l_3)\in \overline{R}_U^{k_1,k_2}$ *with* $l_1+l_2-l_3\neq k_2$ *or* $l_3=0$,

$$(4.7)\quad 0\to (V_{VO})^{k_1,k_2}_{l_1-l_3,l_2-l_3,\min(k_1-l_3,l_1+l_2-2l_3)} \xrightarrow{\iota_3} (U_{VO})^{k_1,k_2}_{l_1,l_2,l_3} \to (U_{VO})^{k_1,k_2}_{l_1,l_2,l_3-1}\to 0,$$

such that $\iota_3(w_3(l_1-l_3,l_2-l_3,k_1-l_3))=e_{31}^{l_3}[1]w_1(l_1,l_2,l_3)$ *in the case of* $(l_1,l_2,l_3)\in R_U^{k_1,k_2}$ *and* $\iota_3(w_3(l_1-l_3,l_2-l_3,l_1+l_2-2l_3))=e_{31}^{l_3}[1]w_2(l_1,l_2,l_3)$ *otherwise.*

For $(l_1,l_2,l_3)\in R_V^{k_1,k_2}$,

$$(4.8)\qquad 0\to (U_{VO})^{k_1,k_2}_{k_1-l_1,l_1+l_2,l_3-l_1}[-1,0] \xrightarrow{\iota_4} (V_{VO})^{k_1,k_2}_{l_1,l_2,l_3} \to (V_{VO})^{k_1,k_2}_{l_1-1,l_2,\min(l_3,l_1+l_2-1)}\to 0,$$

such that $\iota_4(T_{-1,0}(w_1(k_1-l_1,l_1+l_2,l_3-l_1)))=e_{21}[0]^{l_1}w_3(l_1,l_2,l_3)$.

In these formulas, if one of the indices is negative, then the corresponding term is understood as zero.

Corollary 4.7. *Inequalities* (4.1), (4.2), (4.3) *and* (4.4) *are satisfied.*

The rest of the section is devoted to the proof of Theorem 4.6. We start with the proof of the existence of the embeddings.

Proposition 4.8. *Let* $(l_1,l_2,l_3)\in R_U^{k_1,k_2}$. *Then we have embeddings of* $\widehat{\mathfrak{n}}$*-modules*

$$(V_{VO})^{k_1,k_2}_{l_3,k_2-l_2,l_1}[1,-1] \xrightarrow{\iota_1} (U_{VO})^{k_1,k_2}_{l_1,l_2,l_3},$$
$$(V_{VO})^{k_1,k_2}_{l_1-l_3,l_2-l_3,k_1-l_3} \xrightarrow{\iota_3} (U_{VO})^{k_1,k_2}_{l_1,l_2,l_3},$$

such that $\iota_1(T_{1,-1}(w_3(l_3,k_2-l_2,l_1)))=e_{32}^{l_2}[0]w_1(l_1,l_2,l_3)$ *and* $\iota_3(w_3(l_1-l_3,l_2-l_3,k_1-l_3))=e_{31}^{l_3}[1]w_1(l_1,l_2,l_3)$.

Let $(l_1,l_2,l_3)\in \bar{R}_U^{k_1,k_2}$ *and* $l_1+l_2-l_3<k_1$. *Then we have an embedding of* $\widehat{\mathfrak{n}}$*-modules*

$$(V_{VO})^{k_1,k_2}_{l_1-l_3,l_2-l_3,l_1+l_2-2l_3} \xrightarrow{\iota_3} (U_{VO})^{k_1,k_2}_{l_1,l_2,l_3},$$

such that $\iota_3(w_3(l_1-l_3,l_2-l_3,l_1+l_2-2l_3))=e_{31}^{l_3}[1]w_2(l_1,l_2,l_3)$.

Let $(l_1,l_2,l_3)\in R_V^{k_1,k_2}$. *Then we have embeddings of* $\widehat{\mathfrak{n}}$*-modules*

$$(U_{VO})^{k_1,k_2}_{l_3,k_2-l_2,l_1}[0,-1] \xrightarrow{\iota_2} (V_{VO})^{k_1,k_2}_{l_1,l_2,l_3},$$
$$(U_{VO})^{k_1,k_2}_{k_1-l_1,l_1+l_2,l_3-l_1}[-1,0] \xrightarrow{\iota_4} (V_{VO})^{k_1,k_2}_{l_1,l_2,l_3},$$

such that $\iota_2(T_{0,-1}(w_1(l_3,k_2-l_2,l_1)))=e_{32}^{l_2}[0]w_3(l_1,l_2,l_3)$ *and* $\iota_4(T_{-1,0}(w_1(k_1-l_1,l_1+l_2,l_3-l_1)))=e_{21}[0]^{l_1}w_3(l_1,l_2,l_3)$.

Proof. We prove the first embedding. The proof of the other embeddings is similar. By Lemma 4.1 we have

$$
\begin{aligned}
&e_{32}[0]^{l_2}(v_{0,0}^{\otimes l_3} \otimes v_{0,-1}^{\otimes(l_1-l_3)} \otimes v_{-1,0}^{\otimes(k_1-l_1)} \otimes v_0^{\otimes(l_1+l_2-l_3-k_1)} \otimes v_{-1}^{\otimes(k_2-l_1-l_2+l_3)})\\
=&v_{1,-2}^{\otimes l_3} \otimes v_{0,-1}^{\otimes(l_1-l_3)} \otimes v_{0,-2}^{\otimes(k_1-l_1)} \otimes v_{-2}^{\otimes(l_1+l_2-l_3-k_1)} \otimes v_{-1}^{\otimes(k_2-l_1-l_2+l_3)}.
\end{aligned}
$$

The vector in the second line coincides with the cyclic vector of $(V_{VO})_{l_3,k_2-l_2,l_1}^{k_1,k_2}[1,-1]$. Q.E.D.

We prepare some facts about the isomorphisms between the modules $U_{l_1,l_2,l_3}^{k_1,k_2}$ and $(U_{VO})_{l_1,l_2,l_3}^{k_1,k_2}$.

Proposition 4.9. *Let $l_3 = \min(l_1, l_2)$. Then we have the isomorphism*

$$U_{l_1,l_2,l_3}^{k_1,k_2} \simeq (U_{VO})_{l_1,l_2,l_3}^{k_1,k_2},$$

and the corresponding character is given by the fermionic formula

(4.9)

$$
\begin{aligned}
\phi_{l_1,l_2,l_3}^{k_1,k_2}(z_1,z_2) =&(\phi_{VO})_{l_1,l_2,l_3}^{k_1,k_2}(z_1,z_2)\\
=&\sum_{\substack{m_1,\dots,m_{k_1}\geq 0\\ n_1,\dots,n_{k_2}\geq 0}} \frac{q^{Q(m,n)-\sum_{i=1}^{k_1}\min(l_1,i)m_i-\sum_{i=1}^{k_2}\min(l_2,i)n_i}}{(q)_{m_1}\cdots(q)_{m_{k_1}}(q)_{n_1}\cdots(q)_{n_{k_2}}},
\end{aligned}
$$

where

$$
\begin{aligned}
Q(m,n) = \sum_{i,j=1}^{k_1} \min(i,j)m_im_j +& \sum_{i,j=1}^{k_2} \min(i,j)n_in_j\\
&-\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\min(i,j)m_in_j.
\end{aligned}
$$

Proof. Because of the existence of the surjection of $\widehat{\mathfrak{n}}$-modules

$$U_{l_1,l_2,l_3}^{k_1,k_2} \to (U_{VO})_{l_1,l_2,l_3}^{k_1,k_2}, \tag{4.10}$$

it is sufficient to prove (4.9).

For the rest of the proof, we assume $l_3 = \min(l_1, l_2)$. In this case the defining relations of $U_{l_1,l_2,l_3}^{k_1,k_2}$ reduce to (3.15), (3.16), (3.17) and

$$e_{21}(z)^{k_1+1} = 0, \quad e_{32}(z)^{k_2+1} = 0, \quad e_{21}[0]^{l_1+1}v = 0, \quad e_{32}[0]^{l_2+1}v = 0,$$

where v is a cyclic vector of $U^{k_1,k_2}_{l_1,l_2,l_3}$. Denote the fermionic formula in the right hand side of (4.9) by $F^{k_1,k_2}_{l_1,l_2}(z_1,z_2)$. By using the Gordon filtration technique and repeating the proof in [AKS] in the case $k_1 < k_2$, we obtain an upper estimate

$$F^{k_1,k_2}_{l_1,l_2}(z_1,z_2) \geq \varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2).$$

From surjection (4.10) we know that

$$\varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \geq (\varphi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2).$$

To finish the proof it remains to show the inequality

$$(\varphi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \geq F^{k_1,k_2}_{l_1,l_2}(z_1,z_2). \tag{4.11}$$

The space $(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3}$ is generated by the modes of

$$e_{21}(z) = \sum_{i=1}^{k_1} \Gamma_{a_i}(z) \quad \text{and} \quad e_{32}(z) = \sum_{j=1}^{k_2} \Gamma_{b_j}(z)$$

from the vector $w_1(l_1,l_2,l_3)$ or $w_2(l_1,l_2,l_3)$ (see Definition 4.2). The vertex operators $\Gamma_{a_i}(z)$, $\Gamma_{b_j}(z)$ correspond to the vectors a_i, $1 \leq i \leq k_1$, and b_j, $1 \leq j \leq k_2$, respectively, with the scalar products given by

$$(a_{i_1},a_{i_2}) = 2\delta_{i_1,i_2}, \quad (b_{j_1},b_{j_2}) = 2\delta_{j_1,j_2}, \quad (a_i,b_j) = -\delta_{i,j}.$$

For a non-zero complex number ε, set

$$e^{\varepsilon}_{21}(z) = \sum_{i=1}^{k_1} \varepsilon^i \Gamma_{a_i}(z), \quad e^{\varepsilon}_{32}(z) = \sum_{j=1}^{k_2} \varepsilon^j \Gamma_{b_j}(z).$$

The currents $\Gamma_{a_i}(z)$ are mutually commutative and $\Gamma_{a_i}(z)^2 = 0$. The same holds for the currents $\Gamma_{b_j}(z)$. Hence we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-i(i+1)/2}(e^{\varepsilon}_{21}(z))^i = i!\Gamma_{a_1}(z)\dots\Gamma_{a_i}(z),$$
$$\lim_{\varepsilon\to 0} \varepsilon^{-i(i+1)/2}(e^{\varepsilon}_{32}(z))^i = i!\Gamma_{b_1}(z)\dots\Gamma_{b_i}(z).$$

Consider the subspace generated from the vector $w_1(l_1,l_2,l_3)$ or $w_2(l_1,l_2,l_3)$ by the modes of the operators

$$\mathbf{a}_i(z) = \Gamma_{a_1}(z)\dots\Gamma_{a_i}(z) \quad (1 \leq i \leq k_1),$$
$$\mathbf{b}_j(z) = \Gamma_{b_1}(z)\dots\Gamma_{b_j}(z) \quad (1 \leq j \leq k_2),$$

and let $(\overline{\varphi}_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ be its character. Clearly, we have

$$(\varphi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \geq (\overline{\varphi}_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2).$$

The operator $\mathbf{a}_i(z)$ is a vertex operator corresponding to the vector $\mathbf{a}_i = a_1 + \cdots + a_i$, and $\mathbf{b}_j(z)$ is a vertex operator corresponding to the vector $\mathbf{b}_j = b_1 + \cdots + b_j$. The scalar products of these vectors are given by

$$(\mathbf{a}_{i_1}, \mathbf{a}_{i_2}) = 2\min(i_1,i_2),\ (\mathbf{b}_{j_1}, \mathbf{b}_{j_2}) = 2\min(j_1,j_2),$$
$$(\mathbf{a}_i, \mathbf{b}_j) = -\min(i,j).$$

Using the standard technique (see for example [FJMMT]), we obtain

$$(\overline{\varphi}_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) = F^{k_1,k_2}_{l_1,l_2}(z_1,z_2).$$

Inequality (4.11) follows.

The proposition is proved. Q.E.D.

Corollary 4.10. *We have the isomorphisms*

$$W_0 \simeq U^{0,1}_{0,1,0},\ W_{-1} \simeq U^{0,1}_{0,0,0},$$
$$W_{0,0} \simeq U^{1,1}_{1,1,1},\ W_{0,-1} \simeq U^{1,1}_{1,0,0},\ W_{-1,0} \simeq U^{1,1}_{0,1,0},\ W_{-1,-1} \simeq U^{1,1}_{0,0,0}.$$

Next, we prove the surjections in Theorem 4.6 and show that all sequences are in fact complexes.

Proposition 4.11. *Let $(l_1,l_2,l_3) \in R^{k_1,k_2}_U$ and $l_2 > 0$. Then there exists a surjective homomorphism of $\widehat{\mathfrak{n}}$-modules*

$$(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3}/\langle e_{32}[0]^{l_2}\rangle \to (U_{VO})^{k_1,k_2}_{l_1,l_2-1,\min(l_3,l_2-1)}$$

such that, if $l_1+l_2-1-\min(l_3,l_2-1) \geq k_1$ then

$$w_1(l_1,l_2,l_3) \mapsto w_1(l_1,l_2-1,\min(l_3,l_2-1)),$$

and if $l_1+l_2-1-\min(l_3,l_2-1) = k_1-1$ then

$$w_1(l_1,l_2,l_3) \mapsto w_2(l_1,l_2-1,\min(l_3,l_2-1)).$$

Proof. We consider the case $l_2 > l_3$ and $l_1+l_2-1-l_3 < k_1$. The other cases are similar. Replacing a factor $v_{-1,0}$ in $w_1(l_1,l_2,l_3)$ by the factor $v_{-1,-1}$, we obtain $w_2(l_1,l_2-1,l_3)$. We have an obvious surjective homomorphism of $\widehat{\mathfrak{n}}$-modules (see Corollary 4.10)

$$W_{-1,0} \to W_{-1,-1}, \quad v_{-1,0} \mapsto v_{-1,-1}.$$

Therefore, we obtain a surjective homomorphism

$$(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3} \to (U_{VO})^{k_1,k_2}_{l_1,l_2-1,l_3}, \quad w_1(l_1,l_2,l_3) \mapsto w_2(l_1,l_2-1,l_3).$$

In addition, the vector $e_{32}[0]^{l_2} w_1(l_1,l_2,l_3)$ maps to zero, because

$$e_{32}[0]^{l_2} w_2(l_1,l_2-1,l_3) = 0.$$

The proposition is proved. Q.E.D.

Corollary 4.12. *Sequence (4.5) is exact in the first and third terms.*

Proposition 4.13. *Suppose* $(l_1,l_2,l_3) \in R_U^{k_1,k_2}$, $l_1 + l_2 - l_3 < k_2$ *and* $l_3 > 0$. *Then we have a surjective homomorphism of* $\widehat{\mathfrak{n}}$*-modules*

$$(4.12) \qquad \begin{gathered}(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3} / \langle e_{31}[1]^{l_3} \rangle \to (U_{VO})^{k_1,k_2}_{l_1,l_2,l_3-1}, \\ w_1(l_1,l_2,l_3) \mapsto w_1(l_1,l_2,l_3-1).\end{gathered}$$

Proof. Replacing a factor $v_{0,0} \otimes v_{-1}$ in $w_1(l_1,l_2,l_3)$ by the factor $v_{0,-1} \otimes v_0$, we obtain $w_1(l_1,l_2,l_3-1)$. By Definition 4.2,

$$U(\widehat{\mathfrak{n}}) \cdot (v_{0,0} \otimes v_{-1}) = (U_{VO})^{1,2}_{1,1,1}, \quad U(\widehat{\mathfrak{n}}) \cdot (v_{0,-1} \otimes v_0) = (U_{VO})^{1,2}_{1,1,0}.$$

Therefore to construct (4.12) it is sufficient to construct a homomorphism

$$(U_{VO})^{1,2}_{1,1,1} \to (U_{VO})^{1,2}_{1,1,0}, \quad v_{0,0} \otimes v_{-1} \mapsto v_{0,-1} \otimes v_0.$$

Using Proposition 4.9, we have

$$(U_{VO})^{1,2}_{1,1,1} \simeq U^{1,2}_{1,1,1} \to U^{1,2}_{1,1,0} \to (U_{VO})^{1,2}_{1,1,0}.$$

The proposition follows from

$$e_{31}[1]^{l_3} w_1(l_1,l_2,l_3-1) = 0.$$

Q.E.D.

Proposition 4.14. *Suppose* $(l_1,l_2,l_3) \in \bar{R}_U^{k_1,k_2}$, $l_1 + l_2 - l_3 < k_1$ *and* $l_3 > 0$. *Then we have a surjection of* $\widehat{\mathfrak{n}}$*-modules*

$$\begin{gathered}(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3} / \langle e_{31}[1]^{l_3} \rangle \to (U_{VO})^{k_1,k_2}_{l_1,l_2,l_3-1}, \\ w_2(l_1,l_2,l_3) \mapsto w_2(l_1,l_2,l_3-1).\end{gathered}$$

Proof. Replacing a factor $v_{0,0} \otimes v_{-1,-1}$ in $w_2(l_1, l_2, l_3)$ by the factor $v_{0,-1} \otimes v_{-1,0}$, we obtain $w_2(l_1, l_2, l_3 - 1)$. We have

$$U(\widehat{\mathfrak{n}}) \cdot v_{0,0} \otimes v_{-1,-1} = (U_{VO})^{2,2}_{1,1,1}, \quad U(\widehat{\mathfrak{n}}) \cdot v_{0,-1} \otimes v_{-1,0} = (U_{VO})^{2,2}_{1,1,0}.$$

We have a surjective homomorphism $(U_{VO})^{2,2}_{1,1,1} \to (U_{VO})^{2,2}_{1,1,0}$, which is the composition

$$(U_{VO})^{2,2}_{1,1,1} \simeq U^{2,2}_{1,1,1} \to U^{2,2}_{1,1,0} \to (U_{VO})^{2,2}_{1,1,0}.$$

Therefore we obtain a surjection

$$(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3} \to (U_{VO})^{k_1,k_2}_{l_1,l_2,l_3-1}.$$

The proposition follows from

$$e_{31}[1]^{l_3} w(l_1, l_2, l_3 - 1) = 0.$$

Q.E.D.

Corollary 4.15. *Sequence (4.7) is exact in the first and third terms.*

Proposition 4.16. *Let $(l_1, l_2, l_3) \in R^{k_1,k_2}_V$ and $l_2 > 0$. Then there exists a surjective homomorphism of $\widehat{\mathfrak{n}}$-modules*

$$\begin{gathered}(V_{VO})^{k_1,k_2}_{l_1,l_2,l_3} / \langle e_{32}[0]^{l_2} \rangle \to (V_{VO})^{k_1,k_2}_{l_1,l_2-1,\min(l_3,l_1+l_2-1)}, \\ w_3(l_1, l_2, l_3) \mapsto w_3(l_1, l_2 - 1, \min(l_3, l_1 + l_2 - 1)).\end{gathered}$$

Proof. The proof is similar to the proof of Proposition 4.11. Q.E.D.

Corollary 4.17. *Sequence (4.6) is exact in the first and third terms.*

Proposition 4.18. *Let $(l_1, l_2, l_3) \in R^{k_1,k_2}_V$ and $l_1 > 0$. Then there exists a surjective homomorphism of $\widehat{\mathfrak{n}}$-modules*

$$\begin{gathered}(V_{VO})^{k_1,k_2}_{l_1,l_2,l_3} / \langle e_{21}[0]^{l_1} \rangle \to (V_{VO})^{k_1,k_2}_{l_1-1,l_2,\min(l_3,l_1+l_2-1)}, \\ w_3(l_1, l_2, l_3) \mapsto w_3(l_1 - 1, l_2, \min(l_3, l_1 + l_2 - 1)).\end{gathered}$$

Proof. The proof is done similarly to the proof of other cases with the help of the surjective homomorphism

$$(U_{VO})^{1,2}_{1,1,0} \to (U_{VO})^{1,2}_{0,1,0}, \; v_{0,-1} \otimes v_0 \mapsto v_{-1,0} \otimes v_{-1}, \tag{4.13}$$

which we construct below.

First, we show that

$$(U_{VO})^{1,2}_{1,1,0} \simeq U^{1,2}_{1,1,0}. \tag{4.14}$$

By (3.33), we have the inequality

$$\varphi^{1,2}_{1,1,0}(z_1,z_2) \le \varphi^{1,2}_{1,0,0}(z_1,z_2) + z_2\psi^{1,2}_{0,1,1}(q^{-1}z_1,qz_2).$$

By the definition we have an isomorphism

$$V^{1,2}_{0,1,1} \simeq U^{1,2}_{0,1,0},$$

Therefore

$$\varphi^{1,2}_{1,1,0}(z_1,z_2) \le \varphi^{1,2}_{1,0,0}(z_1,z_2) + z_2\varphi^{1,2}_{0,1,0}(q^{-1}z_1,qz_2).$$

By Proposition 4.11 and $(V_{VO})^{1,2}_{0,1,1} \simeq (U_{VO})^{1,2}_{0,1,0} \subset W_{-1,0}\otimes W_{-1}$, we have the inequality

$$(\varphi_{VO})^{1,2}_{1,1,0}(z_1,z_2) \ge (\varphi_{VO})^{1,2}_{1,0,0}(z_1,z_2) + z_2(\varphi_{VO})^{1,2}_{0,1,0}(q^{-1}z_1,qz_2).$$

By Proposition 4.9, we obtain the following diagram:

$$\begin{array}{ccccc}
\varphi^{1,2}_{1,1,0}(z_1,z_2) & \le & z_2\,\varphi^{1,2}_{0,1,0}(q^{-1}z_1,qz_2) & + & \varphi^{1,2}_{1,0,0}(z_1,z_2) \\
\vee| & & \| & & \| \\
(\varphi_{VO})^{1,2}_{1,1,0}(z_1,z_2) & \ge & z_2\,(\varphi_{VO})^{1,2}_{0,1,0}(q^{-1}z_1,qz_2) & + & (\varphi_{VO})^{1,2}_{1,0,0}(z_1,z_2)
\end{array}$$

From here we obtain isomorphism (4.14).

Map (4.13) is a composition of the following mappings:

$$(U_{VO})^{1,2}_{1,1,0} \simeq U^{1,2}_{1,1,0} \to U^{1,2}_{0,1,0} \to (U_{VO})^{1,2}_{0,1,0}.$$

Q.E.D.

Corollary 4.19. *Sequence (4.8) is exact in the first and third terms.*

Theorem 4.6 is proved.

§5. Uniqueness of the solution to the SES-recursion

5.1. The main case of the recursion

Recall the definition of the regions of the parameters R^{k_1,k_2}_U, R^{k_1,k_2}_V, $\overline{R}^{k_1,k_2}_U$ (see (3.37)–(3.39)).

Let $\bar\varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ $((l_1,l_2,l_3)\in \bar R^{k_1,k_2}_U)$ and $\bar\psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ $((l_1,l_2,l_3)\in R^{k_1,k_2}_V)$ be formal power series in variables z_1,z_2 whose coefficients are Laurent power series in q.

We use the following convention. A series with a negative index is understood to be zero. If $l_3>\min(l_1,l_2)$, then $\bar\varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$

$:= \bar{\varphi}^{k_1,k_2}_{l_1,l_2,\min(l_1,l_2)}(z_1,z_2)$. If $l_3 > l_1+l_2$, then $\bar{\psi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ $:= \bar{\psi}^{k_1,k_2}_{l_1,l_2,l_1+l_2}(z_1,z_2)$.

Assume that the following inequalities hold:

If $(l_1,l_2,l_3) \in R^{k_1,k_2}_U$, then

$$(5.1)\quad \bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \le \bar{\varphi}^{k_1,k_2}_{l_1,l_2-1,l_3}(z_1,z_2) + z_2^{l_2}\bar{\psi}^{k_1,k_2}_{l_3,k_2-l_2,l_1}(q^{-1}z_1,qz_2).$$

If $(l_1,l_2,l_3) \in R^{k_1,k_2}_V$, then

$$(5.2)\quad \bar{\psi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \le \bar{\psi}^{k_1,k_2}_{l_1,l_2-1,l_3}(z_1,z_2) + z_2^{l_2}\bar{\varphi}^{k_1,k_2}_{l_3,k_2-l_2,l_1}(z_1,qz_2).$$

If $(l_1,l_2,l_3) \in \bar{R}^{k_1,k_2}_U$ and either $l_1+l_2-l_3 \ne k_2$ or $l_3 = 0$, then

(5.3)
$$\bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \le \bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3-1}(z_1,z_2) + (q^{-1}z_1z_2)^{l_3}\bar{\psi}^{k_1,k_2}_{l_1-l_3,l_2-l_3,k_1-l_3}(z_1,z_2).$$

If $(l_1,l_2,l_3) \in R^{k_1,k_2}_V$, then

$$(5.4)\quad \bar{\psi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) \le \bar{\psi}^{k_1,k_2}_{l_1-1,l_2,l_3}(z_1,z_2) + z_1^{l_1}\bar{\varphi}^{k_1,k_2}_{k_1-l_1,l_1+l_2,l_3-l_1}(qz_1,z_2).$$

We call formal power series of the forms

$$p(z_1,z_2)\bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3}(q^a z_1,q^b z_2) \quad ((l_1,l_2,l_3) \in \bar{R}^{k_1,k_2}_U),$$
$$p(z_1,z_2)\bar{\psi}^{k_1,k_2}_{l_1,l_2,l_3}(q^a z_1,q^b z_2) \quad ((l_1,l_2,l_3) \in R^{k_1,k_2}_V)$$

higher degree series if $a,b \in \mathbb{Z}_{\ge 0}$, $p(z_1,z_2)$ is a polynomial in z_1,z_2 whose coefficients are Laurent polynomials in q and if $p(0,0)=0$.

Let $F(z_1,z_2)$ and $G(z_1,z_2)$ be formal power series in variables z_1,z_2. We write $F(z_1,z_2) \le_* G(z_1,z_2)$ if there exist higher degree series $H_1(z_1,z_2)$, $\dots, H_s(z_1,z_2)$ such that $F(z_1,z_2) \le G(z_1,z_2) + \sum_{i=1}^s H_i(z_1,z_2)$.

Lemma 5.1. *Under the assumptions above, let* $F(z_1,z_2)$ *be either* $\bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$, *for some* $(l_1,l_2,l_3) \in \bar{R}^{k_1,k_2}_U$, *or* $\bar{\psi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$, *for some* $(l_1,l_2,l_3) \in R^{k_1,k_2}_V$. *Then there exist* $m_1,m_2 \in \mathbb{Z}_{\ge 1}$ *such that* $F(z_1,z_2) \le_* \bar{\psi}^{k_1,k_2}_{0,0,0}(q^{m_1}z_1,q^{m_2}z_2)$.

Proof. First, consider the case $F(z_1,z_2) = \bar{\psi}^{k_1,k_2}_{0,0,0}(z_1,z_2)$. We use the given inequalities as follows:

$$\bar{\psi}^{k_1,k_2}_{0,0,0}(z_1,z_2) \le \bar{\varphi}^{k_1,k_2}_{0,k_2,0}(z_1,qz_2) \le \bar{\psi}^{k_1,k_2}_{0,k_2,k_1}(z_1,qz_2) \le \bar{\varphi}^{k_1,k_2}_{k_1,k_2,k_1}(qz_1,qz_2).$$

Here we used (5.2) then (5.3) then (5.4). Then we use inequality (5.1) k_2 times to obtain:

$$\bar{\varphi}^{k_1,k_2}_{k_1,k_2,k_1}(qz_1,qz_2)\leq_*\bar{\varphi}^{k_1,k_2}_{k_1,0,0}(qz_1,qz_2).$$

Finally, using (5.3) followed by the k_1 applications of (5.4), we obtain:

$$\bar{\varphi}^{k_1,k_2}_{k_1,0,0}(qz_1,qz_2)\leq\bar{\psi}^{k_1,k_2}_{k_1,0,k_1}(qz_1,qz_2)\leq_*\bar{\psi}^{k_1,k_2}_{0,0,0}(qz_1,qz_2).$$

Combining, we obtain: $\bar{\psi}^{k_1,k_2}_{0,0,0}(z_1,z_2)\leq_*\bar{\psi}^{k_1,k_2}_{0,0,0}(qz_1,qz_2)$.

Next, consider the case $F(z_1,z_2) = \bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$, $(l_1,l_2,l_3) \in R^{k_1,k_2}_U$.

We start using inequality (5.1) followed by (5.3). We repeat this step l_3 times. Then we apply (5.3) one more time. We get

$$\bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)\leq_*\bar{\varphi}^{k_1,k_2}_{l_1,l_2-l_3,0}(z_1,z_2)\leq\bar{\psi}^{k_1,k_2}_{l_1,l_2-l_3,k_1}(z_1,z_2).$$

Then we use the l_1 applications of (5.4) and after that the l_2-l_3 applications of (5.2). We obtain

$$\bar{\psi}^{k_1,k_2}_{l_1,l_2-l_3,k_1}(z_1,z_2)\leq_*\bar{\psi}^{k_1,k_2}_{0,l_2-l_3,k_1}(z_1,z_2)\leq_*\bar{\psi}^{k_1,k_2}_{0,0,0}(z_1,z_2).$$

Combining, we obtain

$$\bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)\leq_*\bar{\psi}^{k_1,k_2}_{0,0,0}(z_1,z_2)\leq_*\bar{\psi}^{k_1,k_2}_{0,0,0}(qz_1,qz_2).$$

Next, we consider the case $F(z_1,z_2) = \bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$, $(l_1,l_2,l_3) \in \bar{R}^{k_1,k_2}_U$ and $(l_1,l_2,l_3) \notin R^{k_1,k_2}_U$.

Let $l = \min(l_3, k_1-l_1-l_2+l_3)$. We use l applications of (5.3) and obtain

$$\bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)\leq_*\bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3-l}(z_1,z_2).$$

We have either $(l_1,l_2,l_3-l) \in R^{k_1,k_2}_U$, or $l_3-l=0$. In the first case we are reduced to the situation treated above. In the second case, we use (5.3) to obtain

$$\bar{\varphi}^{k_1,k_2}_{l_1,l_2,0}(z_1,z_2)\leq\bar{\psi}^{k_1,k_2}_{l_1,l_2,k_1}(z_1,z_2) = \bar{\psi}^{k_1,k_2}_{l_1,l_2,l_1+l_2}(z_1,z_2),$$

where the last equality holds since $l_1+l_2\leq k_1$. Using l_2 times (5.2) and then l_1 times (5.4) we obtain:

$$\bar{\psi}^{k_1,k_2}_{l_1,l_2,l_1+l_2}(z_1,z_2)\leq_*\bar{\psi}^{k_1,k_2}_{l_1,0,l_1}(z_1,z_2)\leq_*\bar{\psi}^{k_1,k_2}_{0,0,0}(z_1,z_2).$$

Last, consider the case $F(z_1,z_2)=\bar\psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$, $(l_1,l_2,l_3)\in R^{k_1,k_2}_V$. We use (5.2) l_2+1 times and obtain:

$$\bar\psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)\le_*\bar\psi^{k_1,k_2}_{l_1,0,l_1}(z_1,z_2)\le\bar\varphi^{k_1,k_2}_{l_1,k_2,l_1}(z_1,qz_2).$$

Therefore we are again reduced to the previous cases and the lemma is proved. Q.E.D.

Corollary 5.2. *Let $\bar\varphi^{k_1,k_2}_{l_1,l_2,l_3}$, $(l_1,l_2,l_3)\in\bar R^{k_1,k_2}_U$, and $\bar\psi^{k_1,k_2}_{l_1,l_2,l_3}$, $(l_1,l_2,l_3)\in R^{k_1,k_2}_V$, be formal power series in variables z_1,z_2, such that inequalities (5.1)–(5.4) are satisfied.*

Assume that all coefficients of the power series are Laurent power series in q with non-negative integer coefficients. Assume also that the formal power series $\bar\psi^{k_1,k_2}_{0,0,0}$ has no constant term.

Then all these formal power series are identically zero.

Proof. Suppose the contrary, and let n_1,n_2 be non-negative integers such that one of the series has a non-trivial coefficient of $z_1^{n_1}z_2^{n_2}$ and all coefficients of $z_1^{\alpha_1}z_2^{\alpha_2}$ of all power series are zero if either $\alpha_1<n_1$ or $\alpha_2<n_2$. By the assumption and Lemma 5.1, we have $n_1+n_2>0$.

Let n_3 be an integer such that the coefficient of $z_1^{n_1}z_2^{n_2}q^{n_3}$ is non-zero in one of the series $F(z_1,z_2)$ and coefficients of $z_1^{n_1}z_2^{n_2}q^n$ of $\bar\psi^{k_1,k_2}_{0,0,0}$ vanish for $n<n_3$.

By Lemma 5.1, we have

$$F(z_1,z_2)\le\sum_{i=1}^s H_i(z_1,z_2)+\bar\psi^{k_1,k_2}_{0,0,0}(z_1q^{m_1},z_2q^{m_2}),$$

where H_i are higher degree series and $m_1,m_2\ge1$. Clearly, the coefficient of $z_1^{n_1}z_2^{n_2}q^{n_3}$ is zero on the right hand side of this inequality and does not vanish on the left hand side, which is a contradiction. Q.E.D.

5.2. All inequalities are equalities

Recall that we have a set of $\hat{\mathfrak{n}}$-modules $(U_{VO})^{k_1,k_2}_{l_1,l_2,l_3}$, $U^{k_1,k_2}_{l_1,l_2,l_3}$ for $(l_1,l_2,l_3)\in\bar R^{k_1,k_2}_U$ and $(V_{VO})^{k_1,k_2}_{l_1,l_2,l_3}$, $V^{k_1,k_2}_{l_1,l_2,l_3}$ for $(l_1,l_2,l_3)\in R^{k_1,k_2}_V$, whose characters we denote by $(\varphi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$, $\varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ and $(\psi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$, $\psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$, respectively.

Theorem 5.3. *We have*

$$\begin{aligned}(\varphi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)&=\varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)\qquad((l_1,l_2,l_3)\in\bar R^{k_1,k_2}_U),\\(\psi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)&=\psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)\qquad((l_1,l_2,l_3)\in R^{k_1,k_2}_V),\end{aligned}$$

and therefore surjections in Lemmas 4.3 and 4.5 are isomorphisms of $\hat{\mathfrak{n}}$*-modules.*

Moreover, Theorem 3.10 holds.

Proof. The theorem immediately follows from Corollary 5.2 applied to the series

$$\bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) = \varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) - (\varphi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2),$$
$$\bar{\psi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) = \psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) - (\psi_{VO})^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2).$$

Q.E.D.

Corollary 5.4. *The principal subspace* $V^k_{l_1,l_2} \subset M^k_{l_1,l_2}$ *is isomorphic to* $V^{k,k}_{l_1,l_2,l_1+l_2}$.

Proof. As we noted in Remark 3.4, there is a surjective homomorphism of $\widehat{\mathfrak{n}}$-modules

$$V^{k,k}_{l_1,l_2,l_1+l_2} \to V^k_{l_1,l_2} \to 0.$$

On the other hand, taking tensor products of the Frenkel–Kac construction we obtain a surjection

$$V^k_{l_1,l_2} \to (V_{VO})^{k,k}_{l_1,l_2,l_1+l_2} \to 0.$$

Hence the assertion follows from Theorem 5.3. Q.E.D.

5.3. Other cases

In this section we describe two more versions of Corollary 5.2 which we use later to establish the bosonic formulas for our characters.

Set

$$\widetilde{R}^{k_1,k_2}_U = \{(l_1,l_2,l_3) \mid k_1 - 1 \le l_1 + l_2 - l_3 \le k_2\} \cap P^{k_1,k_2}_U. \tag{5.5}$$

Proposition 5.5. *Let* $\bar{\varphi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ *(*$(l_1,l_2,l_3) \in \widetilde{R}^{k_1,k_2}_U$*) and* $\bar{\psi}^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ *(*$(l_1,l_2,l_3) \in R^{k_1,k_2}_V$*) be formal power series in variables* z_1, z_2 *such that equations* (3.40)–(3.43) *are satisfied (where* (3.42) *is assumed for* $(l_1,l_2,l_3) \in \widetilde{R}^{k_1,k_2}_U$ *and either* $l_1 + l_2 - l_3 \neq k_2$ *or* $l_3 = 0$*).*

Assume that all coefficients of the power series are Laurent power series in q*. Assume also that the formal power series* $\bar{\psi}^{k_1,k_2}_{0,0,0}(z_1,z_2)$ *has no constant term.*

Then all these formal power series are identically zero.

Proof. The proof of the proposition is similar to the proof of Corollary 5.2. Q.E.D.

Consider the case $k_1 = k_2 = k$. Then if $(l_1, l_2, l_3) \in R_U^{k,k}$, then $l_3 = l_1 + l_2 - k$. And if $(l_1, l_2, l_3) \in R_V^{k,k}$, then $l_3 = l_1 + l_2$.

Let $\bar{\varphi}^{k,k}_{l_1,l_2,l_3}(z_1, z_2)$ $((l_1, l_2, l_3) \in R_U^{k,k})$ and $\bar{\psi}^{k,k}_{l_1,l_2,l_3}(z_1, z_2)$ $((l_1, l_2, l_3) \in R_V^{k,k})$ be formal power series in variables z_1, z_2 such that the equations

$$\bar{\psi}^{k,k}_{l_1,l_2,l_1+l_2}(z_1, z_2) = \bar{\psi}^{k,k}_{l_1,l_2-1,l_1+l_2-1}(z_1, z_2) + z_2^{l_2} \bar{\varphi}^{k,k}_{l_1+l_2,k-l_2,l_1}(z_1, qz_2),$$
$$\bar{\psi}^{k,k}_{l_1,l_2,l_1+l_2}(z_1, z_2) = \bar{\psi}^{k,k}_{l_1-1,l_2,l_1+l_2-1}(z_1, z_2) + z_1^{l_1} \bar{\varphi}^{k,k}_{k-l_1,l_1+l_2,l_2}(qz_1, z_2)$$

and

$$\begin{aligned}\bar{\varphi}^{k,k}_{l_1,l_2,l_1+l_2-k}(z_1, z_2) = {} & \bar{\varphi}^{k,k}_{l_1,l_2-1,l_1+l_2-k-1}(z_1, z_2) \\ & + z_2^{l_2} \bar{\psi}^{k,k}_{l_1+l_2-k,k-l_2,l_1}(q^{-1}z_1, qz_2) \\ & + (q^{-1}z_1z_2)^{l_1+l_2-k} \bar{\psi}^{k,k}_{k-l_2,k-l_1-1,2k-l_1-l_2-1}(z_1, z_2)\end{aligned}$$

are satisfied.

Note that the last equation is obtained from equations (3.40) and (3.42) via eliminating the term $\bar{\varphi}^{k,k}_{l_1,l_2,l_1+l_2-k-1}(z_1, z_2)$.

Proposition 5.6. *Assume that all coefficients of all power series $\bar{\varphi}^{k,k}_{l_1,l_2,l_3}(z_1, z_2)$ $((l_1, l_2, l_3) \in R_U^{k,k})$ and $\bar{\psi}^{k,k}_{l_1,l_2,l_3}(z_1, z_2)$ $((l_1, l_2, l_3) \in R_V^{k,k})$ are Laurent power series in q. Assume also that the formal power series $\bar{\psi}^{k,k}_{0,0,0}(z_1, z_2)$ has no constant term.*

Then all these formal power series are identically zero.

Proof. The proof of the proposition is similar to the proof of Corollary 5.2. Q.E.D.

§6. Bosonic formulas for the characters of $\widehat{\mathfrak{n}}$-modules

In this section we write explicit solutions of recursion relations (3.40)–(3.43) in the regions $\tilde{R}_U^{k_1,k_2}$ and $R_V^{k_1,k_2}$ in the bosonic form. First, we prepare notation and recall basic facts about the small principal $\widehat{\mathfrak{sl}_3}$ subspaces (see [FJLMM1]).

6.1. The small principal subspaces.

Let $\widehat{\mathfrak{a}}$ denote the abelian Lie algebra spanned by $e_{21}[n], e_{31}[n]$, $n \in \mathbb{Z}$. For non-negative integers k, l_1, l_2 satisfying $l_1 + l_2 \le k$, define $X^k_{l_1,l_2}$ to be the cyclic $\widehat{\mathfrak{a}}$-module with a cyclic vector v and the defining relations

$$\begin{aligned}& e_{21}[n]v = e_{31}[n]v = 0 \quad (n > 0), \\ & e_{21}[0]^{l_1+1}v = 0, \\ & e_{21}[0]^{\alpha} e_{31}[0]^{\beta} v = 0 \quad (\alpha + \beta = l_1 + l_2 + 1), \\ & e_{21}(z)^{\alpha} e_{31}(z)^{\beta} = 0 \quad (\alpha + \beta = k + 1).\end{aligned}$$

The space $X^k_{l_1,l_2}$ has a monomial basis of the form

$$\dots e_{31}[-1]^{a_3}e_{21}[-1]^{a_2}e_{31}[0]^{a_1}e_{21}[0]^{a_0}v,$$

where $\{a_i\}_{i\geq 0}$ run over sequences of non-negative integers such that $a_i=0$ for almost all i and that satisfy the conditions

$$a_0\leq l_1,\ a_0+a_1\leq l_1+l_2,$$
$$a_i+a_{i+1}+a_{i+2}\leq k.$$

Let $\chi^k_{l_1,l_2}(z_1,z_2)$ denote the character of $X^k_{l_1,l_2}$ (normalized in such a way that the degree of the cyclic vector v is $(0,0,0)$). The description of the monomial basis of $X^k_{l_1,l_2}$ leads to the following recursion relations

$$\chi^k_{l_1,l_2}(z_1,z_2)=\chi^k_{l_1-1,l_2}(z_1,z_2)+z_1^{l_1}\chi^k_{l_2,k-l_1-l_2}(z_1z_2,qz_2^{-1}). \tag{6.1}$$

We now write a formula for $\chi^k_{l_1,l_2}(z_1,z_2)$. Let the quantities $p(m,n,s,z_1,z_2)$ $(m,n\in\mathbb{Z}_{\geq 0},\ 0\leq s\leq 5)$ be given by

$$\begin{aligned}
p^k_{l_1,l_2}(m,n,0,z_1,z_2)&=z_1^{km}z_2^{kn},\\
p^k_{l_1,l_2}(m,n,1,z_1,z_2)&=z_1^{km+l_1}z_2^{kn},\\
p^k_{l_1,l_2}(m,n,2,z_1,z_2)&=z_1^{km+l_1+l_2}z_2^{kn+l_2},\\
p^k_{l_1,l_2}(m,n,3,z_1,z_2)&=z_1^{km+l_1+l_2}z_2^{kn+l_1+l_2},\\
p^k_{l_1,l_2}(m,n,4,z_1,z_2)&=z_1^{km+l_1}z_2^{kn+l_1+l_2},\\
p^k_{l_1,l_2}(m,n,5,z_1,z_2)&=z_1^{km}z_2^{kn+l_2}.
\end{aligned}$$

Let the quantities $a(m,n,s)$ $(m,n\in\mathbb{Z}_{\geq 0},\ 0\leq s\leq 5)$ be given by

$$\begin{aligned}
a^k_{l_1,l_2}(m,n,0)&=kQ_2(m,n)-ml_1-nl_2,\\
a^k_{l_1,l_2}(m,n,1)&=kQ_2(m,n)+(m-n)l_1-nl_2,\\
a^k_{l_1,l_2}(m,n,2)&=kQ_2(m,n)+(m-n)l_1+ml_2,\\
a^k_{l_1,l_2}(m,n,3)&=kQ_2(m,n)+nl_1+ml_2,\\
a^k_{l_1,l_2}(m,n,4)&=kQ_2(m,n)+nl_1+(-m+n)l_2,\\
a^k_{l_1,l_2}(m,n,5)&=kQ_2(m,n)-ml_1+(-m+n)l_2,
\end{aligned}$$

where $Q_2(m,n)=m^2+n^2-mn$.

Let the quantities $d(m,n,s,z_1,z_2)$ $(m,n\in\mathbb{Z}_{\geq 0},\ 0\leq s\leq 5)$ be defined by

$$d(m,n,0,z_1,z_2)=(q)_n(q)_{m-n}(z_1q^{2m-n})_\infty(z_1^{-1}q^{-2m+n+1})_{m-n}$$
$$\times(z_1z_2q^{m+n})_\infty(z_1^{-1}z_2^{-1}q^{-m-n+1})_n(z_2q^{2n-m})_{m-n}(z_2^{-1}q^{-2n+m+1})_n,$$
$$d(m,n,1,z_1,z_2)=-z_1^{-1}q^{-2m+n}d(m,n,0,z_1,z_2),$$
$$d(m,n,2,z_1,z_2)=-z_1^{-1}z_2^{-1}q^{-m-n}\frac{1-z_2q^n}{1-z_1^{-1}q^{-m}}d(m,n,1,z_1,z_2),$$
$$d(m,n,3,z_1,z_2)=-z_2^{-1}q^{-2n+m}d(m,n,2,z_1,z_2),$$
$$d(m,n,4,z_1,z_2)=-z_1q^{2m-n}\frac{1-z_1^{-1}q^{-m}}{1-q^{m-n}}d(m,n,3,z_1,z_2),$$
$$d(m,n,5,z_1,z_2)=-z_1z_2q^{n+m}d(m,n,4,z_1,z_2).$$

We note that for all s the quantities $d(m,n,s,z_1,z_2)$ factor in a similar way as $d(m,n,0,z_1,z_2)$ does. For instance

$$d(m,n,5,z_1,z_2)=(q)_n(q)_{m-n-1}(z_1q^{2m-n})_\infty(z_1^{-1}q^{-2m+n+1})_{m-n}$$
$$\times(z_1z_2q^{m+n})_\infty(z_1^{-1}z_2^{-1}q^{-m-n+1})_n(z_2q^{2n-m+1})_{m-n}(z_2^{-1}q^{-2n+m})_{n+1}.$$

We also note the following simple relations:

$$\begin{aligned}
d(m,m-n,5,z_1z_2,z_2^{-1}) &= \frac{1-z_2q^n}{1-q^n}d(m,n,0,z_1,z_2),\\
d(m,m-n,4,z_1z_2,z_2^{-1}) &= \frac{1-z_2q^n}{1-q^n}d(m,n,1,z_1,z_2),\\
d(m,m-n,3,z_1z_2,z_2^{-1}) &= \frac{1-z_1^{-1}q^{-m}}{1-z_1^{-1}z_2^{-1}q^{-m}}d(m,n,2,z_1,z_2).
\end{aligned}$$

For all integers k,l_1,l_2, we define

(6.2)
$$(\chi_B)^k_{l_1,l_1+l_2}(z_1,z_2)=\sum_{m\geq n\geq 0,s=0,\dots,5}\frac{p^k_{l_1,l_2}(m,n,s,z_1,z_2)q^{a^k_{l_1,l_2}(m,n,s)}}{d(m,n,s,z_1,z_2)}.$$

Remark 6.1. We deal with expressions $(\chi_B)^k_{l_1,l_1+l_2}(q^\alpha z_1,q^\beta z_2)$, $(\chi_B)^k_{l_1,l_1+l_2}(q^\alpha z_1z_2,q^\beta z_2^{-1})$ etc. All these expressions are sums where each term is a ratio of a monomial in z_1,z_2 and of a product of factors of the form $(1-z_1^iz_2^jq^k)$, where either $i\geq 0,j\geq 0$ or $i\leq 0$, $j\leq 0$, $i+j<0$. In the first case we expand

$$\frac{1}{1-z_1^iz_2^jq^k}=\sum_{\alpha=0}^\infty(z_1^iz_2^jq^k)^\alpha$$

and in the second case

$$\frac{1}{1-z_1^i z_2^j q^k} = (-z_1^{-i} z_2^{-j} q^{-k}) \sum_{\alpha=0}^{\infty} (z_1^i z_2^j q^k)^{-\alpha}.$$

Using these expansions we can always rewrite our expressions as formal power series in the variables z_1, z_2 whose coefficients are Laurent power series in q.

The following result can be extracted from [FJLMM1]. For completeness we give a proof.

Theorem 6.2. *For non-negative integers k, l_1, l_2 such that $l_1 + l_2 \le k$, we have*

$$\chi^k_{l_1,l_2}(z_1, z_2) = (\chi_B)^k_{l_1,l_2}(z_1, z_2).$$

Proof. The formal power series $\chi^k_{l_1,l_2}(z_1, z_2)$ $(0 \le l_1, l_2, l_1 + l_2 \le k)$ are uniquely determined by

(i) relations (6.1),
(ii) the normalization $\chi^k_{l_1,l_2}(0,0) = 1$,
(iii) the initial condition $\chi^k_{-1,l_2}(z_1, z_2) = 0$.

This follows from the [FJLMM1], Proposition 2.5 and also can be proved directly using the notion of the higher degree series (see Section 5). All the conditions above can be verified for $(\chi_B)^k_{l_1,l_2}(z_1, z_2)$ by a direct computation. Q.E.D.

6.2. The bosonic formula for the $\widehat{\mathfrak{n}}$-modules.

For all integers k_1, k_2, l_1, l_2, l_3 we introduce the formal power series in z_1, z_2 whose coefficients are Laurent series in q:

(6.3)

$$(\varphi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1, z_2) = \sum_{i\ge 0} \frac{z_2^{ik_2} q^{i^2 k_2 - il_2} (\chi_B)^{k_1}_{l_3,l_1}(q^{i-1} z_1 z_2, q^{-2i+1} z_2^{-1})}{(q)_i (q^{2i} z_2)_\infty (q^{-2i+1} z_2^{-1})_i} + \sum_{i\ge 0} \frac{z_2^{ik_2+l_2} q^{i^2 k_2 + il_2} (\chi_B)^{k_1}_{l_3,l_1}(q^{-i-1} z_1, q^{2i+1} z_2)}{(q)_i (q^{2i+1} z_2)_\infty (q^{-2i} z_2^{-1})_{i+1}},$$

(6.4)

$$(\psi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1, z_2) = \sum_{i\ge 0} \frac{z_2^{ik_2} q^{i^2 k_2 - il_2} (\chi_B)^{k_1}_{l_1,l_3}(q^{-i} z_1, q^{2i} z_2)}{(q)_i (q^{2i} z_2)_\infty (q^{-2i+1} z_2^{-1})_i} + \sum_{i\ge 0} \frac{z_2^{ik_2+l_2} q^{i^2 k_2 + il_2} (\chi_B)^{k_1}_{l_1,l_3}(q^{i} z_1 z_2, q^{-2i} z_2^{-1})}{(q)_i (q^{2i+1} z_2)_\infty (q^{-2i} z_2^{-1})_{i+1}}.$$

Proposition 6.3. *For all integers k_1, k_2, l_1, l_2, l_3 we have*

(6.5) $(\varphi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$
$$= (\varphi_B)^{k_1,k_2}_{l_1,l_2-1,l_3}(z_1,z_2) + z_2^{l_2}(\psi_B)^{k_1,k_2}_{l_3,k_2-l_2,l_1}(q^{-1}z_1, qz_2),$$

(6.6) $(\psi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$
$$= (\psi_B)^{k_1,k_2}_{l_1,l_2-1,l_3}(z_1,z_2) + z_2^{l_2}(\varphi_B)^{k_1,k_2}_{l_3,k_2-l_2,l_1}(z_1, qz_2),$$

(6.7) $(\varphi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$
$$= (\varphi_B)^{k_1,k_2}_{l_1,l_2,l_3-1}(z_1,z_2) + (q^{-1}z_1z_2)^{l_3}(\psi_B)^{k_1,k_2}_{l_1-l_3,l_2-l_3,k_1-l_3}(z_1,z_2),$$

(6.8) $(\psi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$
$$= (\psi_B)^{k_1,k_2}_{l_1-1,l_2,l_3}(z_1,z_2) + z_1^{l_1}(\varphi_B)^{k_1,k_2}_{k_1-l_1,l_1+l_2,l_3-l_1}(qz_1, z_2).$$

Proof. We prove (6.5). The proof of other formulas is similar. We have

$$(\varphi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) - z_2^{l_2}(\psi_B)^{k_1,k_2}_{l_3,k_2-l_2,l_1}(q^{-1}z_1, qz_2)$$
$$= \sum_{i\ge 0} \frac{z_2^{ik_2} q^{i^2k_2-il_2}(\chi_B)^{k_1}_{l_3,l_1}(q^{i-1}z_1z_2, q^{-2i+1}z_2^{-1})[1-(1-q^i)]}{(q)_i(q^{2i}z_2)_\infty(q^{-2i+1}z_2^{-1})_i}$$
$$+ \sum_{i\ge 0} \frac{z_2^{ik_2+l_2} q^{i^2k_2+il_2}(\chi_B)^{k_1}_{l_3,l_1}(q^{-i-1}z_1, q^{2i+1}z_2)[1-(1-q^{-i}z_2^{-1})]}{(q)_i(q^{2i+1}z_2)_\infty(q^{-2i}z_2^{-1})_{i+1}}$$
$$= (\varphi_B)^{k_1,k_2}_{l_1,l_2-1,l_3}(z_1,z_2).$$

Q.E.D.

Theorem 6.4. *If $(l_1,l_2,l_3) \in \tilde{R}_U^{k_1,k_2}$, then*

$$(\varphi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) = \varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2).$$

If $(l_1,l_2,l_3) \in R_V^{k_1,k_2}$, then

$$(\psi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2) = \psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2).$$

Proof. Consider $\varphi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ ($(l_1,l_2,l_3) \in \tilde{R}_U^{k_1,k_2}$) and $\psi^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ ($(l_1,l_2,l_3) \in R_V^{k_1,k_2}$). They satisfy recursion relations (3.40)–(3.43).
The series $(\varphi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ ($(l_1,l_2,l_3) \in \tilde{R}_U^{k_1,k_2}$) and $(\psi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ ($(l_1,l_2,l_3) \in R_V^{k_1,k_2}$) satisfy relations (6.5)–(6.8).

We want to use the uniqueness of the solution of (3.40)–(3.43) (see Proposition 5.5). So we compare two sets of relations: (3.40)–(3.43) and (6.5)–(6.8). The difference is two-fold. Firstly, the expressions of the form $\min(a,b)$ do not enter the right hand sides of (6.5)–(6.8). Secondly, it is not assumed that the series $(\varphi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ and $(\psi_B)^{k_1,k_2}_{l_1,l_2,l_3}(z_1,z_2)$ vanish unless all l_i are non-negative. Therefore our theorem follows from the Proposition 5.5 and the following relations:

1) $(\psi_B)^{k_1,k_2}_{-1,l_2,l_3}(z_1,z_2)=0$,
2) $(\psi_B)^{k_1,k_2}_{l_1,-1,l_3}(z_1,z_2)=0$,
3) $(\varphi_B)^{k_1,k_2}_{l_1,-1,l_3}(z_1,z_2)=0$,
4) $(\varphi_B)^{k_1,k_2}_{l_1,l_2,-1}(z_1,z_2)=0$,
5) $(\varphi_B)^{k_1,k_2}_{l_1,l_2,\min(l_1,l_2)}(z_1,z_2)=(\varphi_B)^{k_1,k_2}_{l_1,l_2,\min(l_1,l_2)+1}(z_1,z_2)$,
6) $(\psi_B)^{k_1,k_2}_{l_1,l_2,l_1+l_2}(z_1,z_2)=(\psi_B)^{k_1,k_2}_{l_1,l_2,l_1+l_2+1}(z_1,z_2)$.

We prove the last formula. The proof of the rest is similar. We need to show that

$$\begin{aligned}
&\sum_{i\geq 0}\frac{z_2^{ik_2}q^{i^2k_2-il_2}}{(q)_i(q^{2i}z_2)_\infty(q^{-2i+1}z_2^{-1})_i}(\chi_B)^{k_1}_{l_1,l_1+l_2}(q^{-i}z_1,q^{2i}z_2)\\
&\quad+\sum_{i\geq 0}\frac{z_2^{ik_2+l_2}q^{i^2k_2+il_2}}{(q)_i(q^{2i+1}z_2)_\infty(q^{-2i}z_2^{-1})_{i+1}}(\chi_B)^{k_1}_{l_1,l_1+l_2}(q^{i}z_1z_2,q^{-2i}z_2^{-1})\\
&=\sum_{i\geq 0}\frac{z_2^{ik_2}q^{i^2k_2-il_2}}{(q)_i(q^{2i}z_2)_\infty(q^{-2i+1}z_2^{-1})_i}(\chi_B)^{k_1}_{l_1,l_1+l_2+1}(q^{-i}z_1,q^{2i}z_2)\\
&\quad+\sum_{i\geq 0}\frac{z_2^{ik_2+l_2}q^{i^2k_2+il_2}}{(q)_i(q^{2i+1}z_2)_\infty(q^{-2i}z_2^{-1})_{i+1}}(\chi_B)^{k_1}_{l_1,l_1+l_2+1}(q^{i}z_1z_2,q^{-2i}z_2^{-1}).
\end{aligned}$$

It is sufficient to show that the coefficients of $z_2^{ik_2}q^{i^2k_2}$ for each i are equal:

$$\begin{aligned}
-q^{-i(l_2+2)}z_2^{-1}&(\chi_B)^{k_1}_{l_1,l_1+l_2}(q^{-i}z_1,q^{2i}z_2)\\
&+q^{il_2}z_2^{l_2}(\chi_B)^{k_1}_{l_1,l_1+l_2}(q^{i}z_1z_2,q^{-2i}z_2^{-1})\\
=-q^{-i(l_2+2)}z_2^{-1}&(\chi_B)^{k_1}_{l_1,l_1+l_2+1}(q^{-i}z_1,q^{2i}z_2)\\
&+q^{il_2}z_2^{l_2}(\chi_B)^{k_1}_{l_1,l_1+l_2+1}(q^{i}z_1z_2,q^{-2i}z_2^{-1}).
\end{aligned}\tag{6.9}$$

Each term in (6.9) has the form

$$\sum_{m\geq n\geq 0}\sum_{s=0}^{5}p^{k_1}_{l_1,l_2}(m,n,s,z_1,z_2)q^{a^{k_1}_{l_1,l_2}(m,n,s)}g_s(z_1,z_2,q)$$

where g_s are independent of k_1, l_1 and l_2.

Equating coefficients of $p^{k_1}_{l_1,l_2}(m,n,0,z_1,z_2)$ we are led to show

$$\begin{aligned}(6.10)\quad &\frac{-q^{-2i}q^{a^{k_1}_{l_1,l_2}(m,n,0)}}{d(m,n,0,q^{-i}z_1,q^{2i}z_2)}+\frac{z_2q^{a^{k_1}_{l_1,l_2}(m,m-n,5)}}{d(m,m-n,5,q^iz_1z_2,q^{-2i}z_2^{-1})}\\ &=\frac{-q^{-2i}q^{a^{k_1}_{l_1,l_2+1}(m,n,0)}}{d(m,n,0,q^{-i}z_1,q^{2i}z_2)}+\frac{q^{-2i}q^{a^{k_1}_{l_1,l_2+1}(m,m-n,5)}}{d(m,m-n,5,q^iz_1z_2,q^{-2i}z_2^{-1})}.\end{aligned}$$

Equation (6.10) is equivalent to the equation

$$\frac{d(m,n,0,q^{-i}z_1,q^{2i}z_2)}{d(m,m-n,5,q^iz_1z_2,q^{-2i}z_2^{-1})}=\frac{1-q^n}{1-z_2q^{n+2i}},$$

which can be checked by a direct calculation. In a similar way we check that the coefficients of all monomials $p^k_{l_1,l_2}(m,n,s,z_1,z_2)$ coincide. The theorem is proved. Q.E.D.

§7. Case of $k_1 = k_2$ and Toda recursion

In this section we restrict to the case of $k_1 = k_2 = k$. If $(l_1,l_2,l_3) \in R^{k,k}_V$, then $l_3 = l_1 + l_2$, and if $(l_1,l_2,l_3) \in R^{k,k}_U$, then $l_3 = l_1 + l_2 - k$. As a result, in (6.3) and (6.4) several terms have the same dependence on k, l_1, l_2. In principle, these terms can be summed up. However, a direct summation is not completely obvious. We use our recursion to obtain the result of the summation, which turns out to have a factorized form.

Set

(7.1)

$$I_{d_1,d_2}(z_1,z_2)=\frac{(qz_1^{-1}z_2^{-1})_{d_1+d_2}}{(q)_{d_1}(q)_{d_2}(qz_1^{-1})_{d_1}(qz_2^{-1})_{d_2}(qz_1^{-1}z_2^{-1})_{d_1}(qz_1^{-1}z_2^{-1})_{d_2}}.$$

Proposition 7.1. *The functions $I_{d_1,d_2}(z_1,z_2)$ satisfy the following recurrence relation*

$$\begin{aligned}&(z_1^{-1}(q^{d_1}-1)+(q^{d_2-d_1}-1)+z_2(q^{-d_2}-1))I_{d_1,d_2}(z_1,z_2)\\ &=q^{d_2-d_1}I_{d_1-1,d_2}(z_1,z_2)+z_2q^{-d_2}I_{d_1,d_2-1}(z_1,z_2).\end{aligned}$$

We call this relation the *Toda recursion.*

Set further

$$\bar{J}_{d_1,d_2}(z_1,z_2)=\frac{1}{(z_1)_\infty(z_2)_\infty(z_1z_2)_\infty}I_{d_1,d_2}(z_1,z_2).$$

Definition 7.2. We define the series $A^s_{d_1,d_2}(z_1,z_2)$, $B^s_{d_1,d_2}(z_1,z_2)$ as follows:

$$\begin{aligned}
&A^s_{d_1,d_2}(z_1,z_2)=\bar{J}_{d_1,d_2}(w_1,w_2)f^s(w_1,w_2),\\
&B^0_{d_1,d_2}(z_1,qz_2)=(1-z_2^{-1}q^{d_1-d_2})(-w_2)\bar{J}_{d_1,d_2}(w_1,w_2),\\
&B^1_{d_1,d_2}(z_1,qz_2)=(1-z_1^{-1}z_2^{-1}q^{-d_1})w_1^2w_2\bar{J}_{d_1,d_2}(w_1,w_2),\\
&B^2_{d_1-1,d_2-1}(z_1,qz_2)=(1-q^{d_2})(-w_1)\bar{J}_{d_1,d_2}(w_1,w_2),\\
&B^3_{d_1-1,d_2-1}(z_1,qz_2)=(1-q^{d_1-d_2}z_2^{-1})w_1w_2^2\bar{J}_{d_1,d_2}(w_1,w_2),\\
&B^4_{d_1,d_2-1}(z_1,qz_2)=(1-q^{-d_1}z_1^{-1}z_2^{-1})(-w_1^2w_2^2)\bar{J}_{d_1,d_2}(w_1,w_2),\\
&B^5_{d_1,d_2-1}(z_1,qz_2)=(1-q^{d_2})\bar{J}_{d_1,d_2}(w_1,w_2),
\end{aligned}$$

where $w_1=z_1q^{2d_1-d_2}$, $w_2=z_2q^{2d_2-d_1}$, and

$$\begin{aligned}
&f^0(w_1,w_2)=1, f^1(w_1,w_2)=-w_1, f^2(w_1,w_2)=w_1^2w_2,\\
&f^3(w_1,w_2)=-w_1^2w_2^2, f^4(w_1,w_2)=w_1w_2^2, f^5(w_1,w_2)=-w_2.
\end{aligned}$$

In this section we prove the following theorem.

Theorem 7.3. *We have*

$$\begin{aligned}
(7.2)\qquad \psi^{k,k}_{l_1,l_2,l_1+l_2}(z_1,z_2)&=\sum_{d_1,d_2\geq 0} z_1^{kd_1}z_2^{kd_2}q^{k(d_1^2+d_2^2-d_1d_2)}\\
&\times\Big(q^{-l_1d_1-l_2d_2}A^0_{d_1,d_2}(z_1,z_2)+z_1^{l_1}q^{l_1(d_1-d_2)-l_2d_2}A^1_{d_1,d_2}(z_1,z_2)\\
&+z_1^{l_1+l_2}z_2^{l_2}q^{l_1(d_1-d_2)+l_2d_1}A^2_{d_1,d_2}(z_1,z_2)\\
&+(z_1z_2)^{l_1+l_2}q^{l_1d_2+l_2d_1}A^3_{d_1,d_2}(z_1,z_2)\\
&+z_1^{l_1}z_2^{l_1+l_2}q^{l_2(d_2-d_1)+l_1d_2}A^4_{d_1,d_2}(z_1,z_2)\\
&+z_2^{l_2}q^{-l_1d_1-l_2(d_1-d_2)}A^5_{d_1,d_2}(z_1,z_2)\Big)
\end{aligned}$$

and

$$\begin{aligned}
(7.3)\qquad \varphi^{k,k}_{l_1,l_2,l_1+l_2-k}(z_1,z_2)&=\sum_{d_1,d_2\geq 0} z_1^{kd_1}z_2^{kd_2}q^{k(d_1^2+d_2^2-d_1d_2)}\\
&\times\Big(q^{-l_1d_1-l_2d_2}B^0_{d_1,d_2}(z_1,z_2)+z_1^{l_1}q^{l_1(d_1-d_2)-l_2d_2}B^1_{d_1,d_2}(z_1,z_2)\\
&+z_1^{l_1+l_2}z_2^{l_2}q^{l_1(d_1-d_2)+l_2d_1}B^2_{d_1,d_2}(z_1,z_2)\\
&+(z_1z_2)^{l_1+l_2}q^{l_1d_2+l_2d_1}B^3_{d_1,d_2}(z_1,z_2)\\
&+z_1^{l_1}z_2^{l_1+l_2}q^{l_2(d_2-d_1)+l_1d_2}B^4_{d_1,d_2}(z_1,z_2)\\
&+z_2^{l_2}q^{-l_1d_1-l_2(d_1-d_2)}B^5_{d_1,d_2}(z_1,z_2)\Big).
\end{aligned}$$

The strategy of the proof is as follows. We show that the right hand sides of (7.2) and (7.3) satisfy the recursion relations provided certain relations for $A^s_{d_1,d_2}$ and $B^s_{d_1,d_2}$ are satisfied. We prove that these relations hold for $A^s_{d_1,d_2}$, $B^s_{d_1,d_2}$ of the form given in Definition 7.2. This proves Theorem 7.3 because of the uniqueness of the solution of the recursion relations (Proposition 5.6).

Substituting (7.2) and (7.3) into the recursion relations, one can easily verify the following three Lemmas.

Lemma 7.4. *The relations*

$$\begin{aligned}
&(1-q^{d_2})A^0_{d_1,d_2}(z_1,z_2)=B^5_{d_1,d_2-1}(z_1,qz_2),\\
&(1-q^{d_2})A^1_{d_1,d_2}(z_1,z_2)=B^2_{d_1-1,d_2-1}(z_1,qz_2),\\
&(1-q^{-d_1}z_1^{-1}z_2^{-1})A^2_{d_1,d_2}(z_1,z_2)=B^1_{d_1,d_2}(z_1,qz_2),\\
&(1-q^{-d_1}z_1^{-1}z_2^{-1})A^3_{d_1,d_2}(z_1,z_2)=B^4_{d_1,d_2-1}(z_1,qz_2),\\
&(1-q^{d_1-d_2}z_2^{-1})A^4_{d_1,d_2}(z_1,z_2)=B^3_{d_1-1,d_2-1}(z_1,qz_2),\\
&(1-q^{d_1-d_2}z_2^{-1})A^5_{d_1,d_2}(z_1,z_2)=B^0_{d_1,d_2}(z_1,qz_2)
\end{aligned}$$

imply the recursion

(7.4)
$$\psi^{k,k}_{l_1,l_2,l_1+l_2}(z_1,z_2)=\psi^{k,k}_{l_1,l_2-1,l_1+l_2-1}(z_1,z_2)+z_2^{l_2}\varphi^{k,k}_{l_1+l_2,k-l_2,l_1}(z_1,qz_2).$$

Lemma 7.5. *The relations*

$$\begin{aligned}
&(1-q^{d_1})A^0_{d_1,d_2}(z_1,z_2)=B^1_{d_1-1,d_2}(qz_1,z_2),\\
&(1-q^{d_2-d_1}z_1^{-1})A^1_{d_1,d_2}(z_1,z_2)=B^0_{d_1,d_2}(qz_1,z_2),\\
&(1-q^{d_2-d_1}z_1^{-1})A^2_{d_1,d_2}(z_1,z_2)=B^3_{d_1-1,d_2-1}(qz_1,z_2),\\
&(1-q^{-d_2}z_1^{-1}z_2^{-1})A^3_{d_1,d_2}(z_1,z_2)=B^2_{d_1-1,d_2}(qz_1,z_2),\\
&(1-q^{-d_2}z_1^{-1}z_2^{-1})A^4_{d_1,d_2}(z_1,z_2)=B^5_{d_1,d_2}(qz_1,z_2),\\
&(1-q^{d_1})A^5_{d_1,d_2}(z_1,z_2)=B^4_{d_1-1,d_2-1}(qz_1,z_2)
\end{aligned}$$

imply the recursion

(7.5)
$$\psi^{k,k}_{l_1,l_2,l_1+l_2}(z_1,z_2)=\psi^{k,k}_{l_1-1,l_2,l_1+l_2-1}(z_1,z_2)+z_1^{l_1}\varphi^{k,k}_{k-l_1,l_1+l_2,l_2}(qz_1,z_2).$$

Lemma 7.6. *The relations*

$$\begin{aligned}
(1-q^{d_2})B^0_{d_1,d_2}(z_1,z_2) &= z_1^{-1}z_2^{-1}q^{-d_1+1}A^3_{d_1-1,d_2-1}(z_1,z_2)\\
&\quad + A^5_{d_1,d_2-1}(q^{-1}z_1,qz_2),\\
(1-q^{d_2})B^1_{d_1,d_2}(z_1,z_2) &= z_2^{-1}q^{d_1-d_2+1}A^4_{d_1,d_2-1}(z_1,z_2)\\
&\quad + A^2_{d_1,d_2-1}(q^{-1}z_1,qz_2),\\
(1-q^{-d_1}z_1^{-1}z_2^{-1})B^2_{d_1,d_2}(z_1,z_2) &= z_2^{-1}q^{d_1-d_2+1}A^5_{d_1+1,d_2}(z_1,z_2)\\
&\quad + A^1_{d_1+1,d_2}(q^{-1}z_1,qz_2),\\
(1-q^{-d_1}z_1^{-1}z_2^{-1})B^3_{d_1,d_2}(z_1,z_2) &= q^{d_2+1}A^0_{d_1+1,d_2+1}(z_1,z_2)\\
&\quad + A^4_{d_1+1,d_2}(q^{-1}z_1,qz_2),\\
(1-q^{d_1-d_2}z_2^{-1})B^4_{d_1,d_2}(z_1,z_2) &= q^{d_2+1}A^1_{d_1,d_2+1}(z_1,z_2)\\
&\quad + A^3_{d_1,d_2}(q^{-1}z_1,qz_2),\\
(1-q^{d_1-d_2}z_2^{-1})B^5_{d_1,d_2}(z_1,z_2) &= z_1^{-1}z_2^{-1}q^{-d_1+1}A^2_{d_1-1,d_2}(z_1,z_2)\\
&\quad + A^0_{d_1,d_2}(q^{-1}z_1,qz_2)
\end{aligned}$$

imply the 3-term relation

$$\begin{aligned}
(7.6)\quad \varphi^{k,k}_{l_1,l_2,l_1+l_2-k}(z_1,z_2) &= \varphi^{k,k}_{l_1,l_2-1,l_1+l_2-k-1}(z_1,z_2)+\\
&\quad z_2^{l_2}\psi^{k,k}_{l_1+l_2-k,k-l_2,l_1}(q^{-1}z_1,qz_2)+\\
&\quad (q^{-1}z_1z_2)^{l_1+l_2-k}\psi^{k,k}_{k-l_2,k-l_1-1,2k-l_1-l_2-1}(z_1,z_2).
\end{aligned}$$

Proposition 7.7. *The series $A^s_{d_1,d_2}$, $B^s_{d_1,d_2}$ satisfy all relations from Lemmas* 7.4, 7.5, 7.6.

Proof. The proposition is proved by a direct calculation. Q.E.D.

Proof of Theorem 7.3. Theorem 7.3 follows from Lemmas 7.4–7.6, Proposition 7.7 and Proposition 5.6. Q.E.D.

Recall that $\psi^{k,k}_{0,0,0}(z_1,z_2)$ is equal to the character of the principal subspace V^k of the level k vacuum $\widehat{\mathfrak{sl}_3}$ module. In this case, one can further sum the six terms in (7.3) to obtain the following corollary:

Corollary 7.8.

(7.7)
$$\mathrm{ch}V^k = \sum_{d_1,d_2\ge 0} z_1^{kd_1}z_2^{kd_2}q^{k(d_1^2+d_2^2-d_1d_2)}J_{d_1,d_2}(z_1q^{2d_1-d_2},z_2q^{2d_2-d_1}),$$

where $J_{d_1,d_2}(z_1,z_2)=\bar{J}_{d_1,d_2}(z_1,z_2)(1-z_1)(1-z_2)(1-z_1z_2)$.

Corollary 7.9. *The functions $I_{d_1,d_2}(z_1,z_2)$ satisfy the following relations:*

$$I_{d_1,d_2}(z_1,z_2)=\sum_{n_1=0}^{d_1}\sum_{n_2=0}^{d_2}\frac{z_1^{-n_1}z_2^{-n_2}q^{n_1^2+n_2^2-n_1n_2}}{(q)_{d_1-n_1}(q)_{d_2-n_2}}I_{n_1,n_2}(z_1,z_2). \tag{7.8}$$

Proof. We recall the fermionic formula for the character of V^k:

$$\mathrm{ch}V^k=\sum_{\substack{n_1,\dots,n_k\ge0\\ m_1,\dots,m_k\ge0}}\frac{z_1^{\sum in_i}z_2^{\sum im_i}q^{\sum_{i,j=1}^k\min(i,j)(n_in_j-m_in_j+m_im_j)}}{(q)_{n_1}\cdots(q)_{n_k}(q)_{m_1}\cdots(q)_{m_k}}.$$

Summing up all terms with the fixed values of n_k and m_k we obtain the relation

$$\mathrm{ch}V^k=\sum_{n,m\ge0}\frac{z_1^{kn}z_2^{km}q^{k(n^2+m^2-mn)}}{(q)_n(q)_m}\mathrm{ch}V^{k-1}(q^{2n-m}z_1,q^{2m-n}z_2). \tag{7.9}$$

Substituting (7.7) into (7.9) we obtain (7.8). Q.E.D.

Corollary 7.10. *The function $I_{d_1,d_2}(z_1,z_2)$ is given by the fermionic formula*

$$I_{d_1,d_2}(z_1,z_2)=\sum_{\{m_i\}_{i>0},\{n_i\}_{i>0}}\frac{z_1^{-\sum_{i>0}n_i}z_2^{-\sum_{i>0}m_i}q^{\sum_{i>0}(n_i^2+m_i^2-n_im_i)}}{(q)_{d_1-n_1}(q)_{n_1-n_2}\cdots(q)_{d_2-m_1}(q)_{m_1-m_2}\cdots},$$

where the sum is over all sequences $\{m_i\}_{i>0},\{n_i\}_{i>0}$ satisfying $m_i,n_i\in\mathbb{Z}_{\ge0}$ and $d_1\ge n_1\ge n_2\ge\cdots$, $d_2\ge m_1\ge m_2\ge\cdots$, and $m_i=n_i=0$ for almost all i.

Proposition 7.11. *We have*

$$I_{d_1,d_2}(z_1,z_2)=\sum_{n=0}^{\min(d_1,d_2)}I_{d_1,d_2,n}(z_1,z_2),$$

where

$$I_{d_1,d_2,n}(z_1,z_2)=\frac{1}{(q)_{d_1-n}(q)_{d_2-n}(q)_n(qz_1^{-1})_{d_1-n}}\times\frac{(qz_2)_\infty}{(qz_1^{-1}z_2^{-1})_n(q^{d_1-2n+1}z_2)_\infty(q^{-d_1+2n+1}z_2^{-1})_{d_2-n}(qz_2)_{d_1-n}(qz_2^{-1})_n}. \tag{7.10}$$

Proof. Using the equality $\psi^{k,k}_{0,0,0}(z_1,z_2) = \mathrm{ch}V^k$ and formula(6.4) we obtain

$$(7.11)\quad \mathrm{ch}V^k = \sum_{i\geq 0} \frac{z_2^{ik} q^{i^2 k}}{(q)_i (q^{2i} z_2)_\infty (q^{-2i+1} z_2^{-1})_i} (\chi_B)^k_{0,0}(q^{-i} z_1, q^{2i} z_2)$$
$$+ \sum_{i\geq 0} \frac{z_2^{ik} q^{i^2 k}}{(q)_i (q^{2i+1} z_2)_\infty (q^{-2i} z_2^{-1})_{i+1}} (\chi_B)^k_{0,0}(q^{i} z_1 z_2, q^{-2i} z_2^{-1}).$$

Recall formula (6.2). Because of the equalities

$$p^k_{0,0}(m,n,s,z_1,z_2) = z_1^{km} z_2^{kn}, \quad a^k_{0,0}(m,n,s) = k(m^2+n^2-mn),$$

the right hand side of (7.11) can be rewritten as

$$\sum_{d_1,d_2\geq 0} z_1^{kd_1} z_2^{kd_2} q^{k(d_1^2+d_2^2-d_1 d_2)} \sum_{n=0}^{\min(d_1,d_2)} J_{d_1,d_2,n}(z_1 q^{2d_1-d_2}, z_2 q^{2d_2-d_1}),$$

where $J_{d_1,d_2,n}(z_1,z_2) = \frac{1}{(qz_1)_\infty (qz_2)_\infty (qz_1 z_2)_\infty} I_{d_1,d_2,n}(z_1,z_2)$. Now the proposition follows from Corollary 7.8. Q.E.D.

§8. Whittaker vector and the character of the vacuum module

8.1. The quantum group $U_v(\mathfrak{sl}_3)$

Let $U_v(\mathfrak{sl}_3)$ be the quantum group associated to $\mathfrak{sl}_3$. The quantum group $U_v(\mathfrak{sl}_3)$ is an associative algebra over the field of rational functions $\mathbb{C}(v)$ in formal variable v with generators $K^{\pm 1}$, E_i, F_i, $i=1,2$, satisfying the standard commutation relations:

$$K_i K_i^{-1} = 1, K_i K_j = K_j K_i, K_i E_i = v^2 E_i K_i,$$
$$K_i F_i = v^{-2} F_i K_i, K_i E_j = v^{-1} E_j K_i, K_i F_j = v F_j K_i,$$
$$E_i F_i - F_i E_i = \frac{K_i - K_i^{-1}}{v - v^{-1}}, E_i F_j = F_j E_i,$$
$$E_i^2 E_j - (v+v^{-1}) E_i E_j E_i + E_j E_i^2 = 0,$$
$$F_i^2 F_j - (v+v^{-1}) F_i F_j F_i + F_j F_i^2 = 0$$

for all $i,j = 1,2$, $i \neq j$.

We extend the algebra $U_v(\mathfrak{sl}_3)$ by the operators $K_i^{\pm 1/3}$ in the obvious way and use the same notation $U_v(\mathfrak{sl}_3)$ for the resulting algebra.

Let $Z \in U_v(\mathfrak{sl}_3)$ be the quadratic Casimir operator given by:

$$Z = v^{-2}K_1^{-4/3}K_2^{-2/3} + K_1^{2/3}K_2^{-2/3} + v^2K_1^{2/3}K_2^{4/3}$$
$$+ (v - v^{-1})^2(v^{-1}F_1E_1K_1^{-1/3}K_2^{-2/3}$$
$$+ vF_2E_2K_1^{2/3}K_2^{1/3} + v^{-1}F_{13}E_{13}K_1^{-1/3}K_2^{1/3}),$$

where $F_{13} = F_2F_1 - vF_1F_2$ and $E_{13} = E_1E_2 - vE_2E_1$.

Lemma 8.1. *The element Z is in the center of $U_v(\mathfrak{sl}_3)$.*

Proof. The lemma is proved by a direct calculation. Q.E.D.

We have an anti-isomorphism of $\mathbb{C}(v)$-algebras τ given by:

$$\tau: \ U_v(\mathfrak{sl}_3) \to U_v(\mathfrak{sl}_3), \quad E_i \mapsto F_i, \quad F_i \mapsto E_i, \quad K_i \mapsto K_i.$$

We have an isomorphism of $\mathbb{C}(v)$-algebras ν given by:

$$\nu: \ U_v(\mathfrak{sl}_3) \to U_v(\mathfrak{sl}_3), \quad E_i \mapsto E_{3-i}, \quad F_i \mapsto F_{3-i}, \quad K_i \mapsto K_{3-i}.$$

We also consider the quantum group $U_{v^{-1}}(\mathfrak{sl}_3)$ with parameter v^{-1}. Denote the generators of $U_{v^{-1}}(\mathfrak{sl}_3)$ by $\bar{E}_i, \bar{F}_i, \bar{K}_i$.

We have an isomorphism of $\mathbb{C}(v)$-algebras σ given by:

$$\sigma: \ U_v(\mathfrak{sl}_3) \to U_{v^{-1}}(\mathfrak{sl}_3), \quad E_i \mapsto \bar{E}_i, \quad F_i \mapsto \bar{F}_i, \quad K_i \mapsto \bar{K}_i^{-1}.$$

Clearly, all these maps commute: $\tau \circ \sigma = \sigma \circ \tau$, $\tau \circ \nu = \nu \circ \tau$, $\nu \circ \sigma = \sigma \circ \nu$.

8.2. Verma modules

Let λ_1, λ_2 be formal variables. Set $\lambda_3 = -\lambda_1 - \lambda_2$.

Let $\mathbb{C}(v, v^{\lambda_1}, v^{\lambda_2})$ be the field of rational functions in formal variables $v, v^{\lambda_1}, v^{\lambda_2}$. Let $\mathcal{R}$ be the ring spanned by all elements of the form $\sqrt{g}$, where $g \in \mathbb{C}(v, v^{\lambda_1}, v^{\lambda_2})$, with the obvious operations of addition and multiplication. In what follows, we consider the quantum groups with the extended coefficient ring: $U_v(\mathfrak{sl}_3) \otimes_{\mathbb{C}(v)} \mathcal{R}$, $U_{v^{-1}}(\mathfrak{sl}_3) \otimes_{\mathbb{C}(v)} \mathcal{R}$. We use the same notation $U_v(\mathfrak{sl}_3)$, $U_{v^{-1}}(\mathfrak{sl}_3)$ for the extended algebras.

Let $\mathcal{V}_v$ be the $U_v(\mathfrak{sl}_3)$-module generated by the highest weight vector w with the defining relations:

$$K_1 w = v^{\lambda_1 - \lambda_2} w, \qquad K_2 w = v^{\lambda_2 - \lambda_3} w, \qquad E_1 w = 0, \qquad E_2 w = 0.$$

Similarly, let $\mathcal{V}_{v^{-1}}$ be the $U_{v^{-1}}(\mathfrak{sl}_3)$-module generated by the highest weight vector $\bar{w}$ with the defining relations:

$$\bar{K}_1 \bar{w} = v^{\lambda_2 - \lambda_1} \bar{w}, \qquad \bar{K}_2 \bar{w} = v^{\lambda_3 - \lambda_2} \bar{w}, \qquad \bar{E}_1 w = 0, \qquad \bar{E}_2 w = 0.$$

We call $\mathcal{V}_v$ and $\mathcal{V}_{v^{-1}}$ the *Verma modules* over $U_v(\mathfrak{sl}_3)$ and $U_{v^{-1}}(\mathfrak{sl}_3)$, respectively.

We denote by $\mathcal{V}_v(d_1, d_2) \subset \mathcal{V}_v$ and by $\mathcal{V}_{v^{-1}}(d_1, d_2) \subset \mathcal{V}_{v^{-1}}$ the weight subspaces:

$$\begin{aligned}\mathcal{V}_v(d_1, d_2) = \{w_1 \in \mathcal{V}_v \mid K_1 w_1 &= v^{\lambda_1-\lambda_2-2d_1+d_2} w_1,\\ K_2 w_1 &= v^{\lambda_2-\lambda_3-2d_2+d_1} w_1\},\\ V_{v^{-1}}(d_1, d_2) = \{\bar{w}_2 \in \mathcal{V}_{v^{-1}} \mid \bar{K}_1 \bar{w}_2 &= v^{\lambda_2-\lambda_1+2d_1-d_2} \bar{w}_2,\\ \bar{K}_2 \bar{w}_2 &= v^{\lambda_3-\lambda_2+2d_2-d_1} \bar{w}_2\}.\end{aligned}$$

We have $\mathcal{V}_v = \oplus_{d_1,d_2=0}^{\infty} \mathcal{V}_v(d_1, d_2)$, $\mathcal{V}_{v^{-1}} = \oplus_{d_1,d_2=0}^{\infty} \mathcal{V}_{v^{-1}}(d_1, d_2)$.

Lemma 8.2. *There exists a unique non-degenerate $\mathcal{R}$-bilinear pairing $(\ ,\ ):\ \mathcal{V}_v \otimes \mathcal{V}_{v^{-1}} \to \mathcal{R}$ such that $(w, \bar{w}) = 1$ and*

$$(g w_1, \bar{w}_2) = (w_1, (\sigma \circ \tau)(g) \bar{w}_2),$$

for any $w_1 \in \mathcal{V}_v$, $\bar{w}_2 \in \mathcal{V}_{v^{-1}}$, $g \in U_v(\mathfrak{sl}_3)$.

Proof. The proof is standard. Q.E.D.

In what follows, we use the following notation:

$$[a] = \frac{v^a - v^{-a}}{v - v^{-1}}, \qquad [a]! = \prod_{i=1}^{a} [i], \qquad [a]_b = \prod_{i=0}^{b-1} [a+i].$$

The Verma module $\mathcal{V}_v$ has the Gelfand–Tsetlin basis

$$\{m_{d_1,d_2,n} \mid d_1, d_2, n \in \mathbb{Z}_{\geq 0}, n \leq \min(d_1, d_2)\}$$

(see [J]). In this basis the action of the $U_v(\mathfrak{sl}_3)$ is given by:

$$\begin{aligned}K_1 m_{d_1,d_2,n} &= v^{\lambda_1-\lambda_2-2d_1+d_2} m_{d_1,d_2,n},\\ K_2 m_{d_1,d_2,n} &= v^{\lambda_2-\lambda_3-2d_2+d_1} m_{d_1,d_2,n},\\ E_1 m_{d_1,d_2,n} &= \sqrt{b_1(d_1,d_2,n)}\, m_{d_1-1,d_2,n-1}\\ &\quad + \sqrt{b_2(d_1,d_2,n)}\, m_{d_1-1,d_2,n},\\ F_1 m_{d_1,d_2,n} &= \sqrt{b_1(d_1+1,d_2,n+1)}\, m_{d_1+1,d_2,n+1}\\ &\quad + \sqrt{b_2(d_1+1,d_2,n)}\, m_{d_1+1,d_2,n},\\ E_2 m_{d_1,d_2,n} &= \sqrt{a(d_1,d_2,n)}\, m_{d_1,d_2-1,n},\\ F_2 m_{d_1,d_2,n} &= \sqrt{a(d_1,d_2+1,n)}\, m_{d_1,d_2+1,n},\end{aligned}$$

where

$$
\begin{aligned}
a(d_1,d_2,n) &= [d_2-n][\lambda_2-\lambda_3+d_1-d_2-n+1],\\
b_1(d_1,d_2,n) &= \frac{[d_2-n+1][n][\lambda_2-\lambda_3-n+1][\lambda_1-\lambda_3-n+2]}{[\lambda_2-\lambda_3+d_1-2n+1][\lambda_2-\lambda_3+d_1-2n+2]},\\
b_2(d_1,d_2,n) &= \frac{[d_1-n][\lambda_2-\lambda_3+d_1-d_2-n][\lambda_2-\lambda_3+d_1-n+1]}{[\lambda_2-\lambda_3+d_1-2n][\lambda_2-\lambda_3+d_1-2n+1]}\\
&\quad\times\frac{1}{[\lambda_1-\lambda_2-d_1+n+1]}.
\end{aligned}
$$

Remark 8.3. Our formulas are identified with the formulas in [J] as follows. Let the vector $m(d_1,d_2,n)$ correspond to the Gelfand–Tsetlin pattern in [J] given by:

$$
\begin{pmatrix}
-\lambda_3 & & -\lambda_2 & & -\lambda_1\\
& -\lambda_3-n & & -\lambda_2-d_1+n & \\
& & -\lambda_3-d_2 & &
\end{pmatrix}
$$

Then the action of $g\in U_v(\mathfrak{sl}_3)$ in our paper is given by formulas for the action of $\nu(g)$ in [J].

Similarly, we have a Gelfand–Tsetlin basis of the Verma module $\mathcal{V}_{v^{-1}}$, $\{\bar{m}_{d_1,d_2,n}\mid d_1,d_2,n\in\mathbb{Z}_{\geq 0}, n\leq\min(d_1,d_2)\}$.

Lemma 8.4. *We have*

$$(m_{d_1,d_2,n},\bar{m}_{d_1',d_2',n'})=\delta_{d_1,d_1'}\delta_{d_2,d_2'}\delta_{n,n'}.$$

Proof. The Shapovalov form on $\mathcal{V}_v$ is the unique non-degenerate symmetric bilinear form such that the length of the highest weight vector w is 1 and every $g\in U_v(\mathfrak{sl}_3)$ is dual to $\tau(g)$. According to [J], the Gelfand–Tsetlin basis is orthonormal with respect to the Shapovalov form in $\mathcal{V}_v$. The lemma follows. Q.E.D.

8.3. Whittaker vectors

We call a series $\omega=\sum_{d_1,d_2=0}^{\infty}\omega_{d_1,d_2}$, $\omega_{d_1,d_2}\in\mathcal{V}_v(d_1,d_2)$, the *Whittaker vector* if $\omega_{0,0}$ is a highest weight vector and

$$E_1K_1\omega_{d_1,d_2}=\frac{1}{1-v^2}\omega_{d_1-1,d_2},\qquad E_2\omega_{d_1,d_2}=\frac{1}{1-v^2}\omega_{d_1,d_2-1}.$$

We call a series $\bar{\omega}=\sum_{d_1,d_2=0}^{\infty}\bar{\omega}_{d_1,d_2}$, $\bar{\omega}_{d_1,d_2}\in\mathcal{V}_{v^{-1}}(d_1,d_2)$, the *dual Whittaker vector* if $\bar{\omega}_{0,0}$ is a highest weight vector and

$$\bar{E}_1\bar{K}_1^2\bar{\omega}_{d_1,d_2}=\frac{v}{1-v^{-2}}\bar{\omega}_{d_1-1,d_2},\qquad \bar{E}_2\bar{K}_2\bar{\omega}_{d_1,d_2}=\frac{v}{1-v^{-2}}\bar{\omega}_{d_1,d_2-1}.$$

Remark 8.5. Because of the quantum Serre relations, there are no non-zero vectors invariant under the action of E_1 and E_2 (except for the multiples of the highest weight vector w). The operators $e_1 = E_1K_1$ and $e_2 = E_2$ satisfy

$$\begin{aligned} e_1^2e_2 - (1+v^{-2})e_1e_2e_1 + v^{-2}e_2e_1^2 = 0, \\ e_2^2e_1 - (1+v^2)e_2e_1e_2 + v^2e_1e_2^2 = 0, \end{aligned}$$

which does not prohibit the existence of non-trivial Whittaker vectors.

Let $r(d_1,d_2,n)$ and $s(d_1,d_2)$ be given by the formulas:

$$\begin{aligned} r(d_1,d_2,n) &= (\lambda_3-\lambda_2-d_1-1)n + n^2 + (\lambda_2-\lambda_1+1)d_1 + d_1^2, \\ s(d_1,d_2) &= -d_1^2 - d_2^2 + d_1d_2 + (\lambda_1-\lambda_2)d_1 + (\lambda_2-\lambda_3)d_2. \end{aligned}$$

Let $c(d_1,d_2,n)$ be given by the formula:

$$\begin{aligned} c(d_1,d_2,n) &= \frac{1}{[d_1-n]![d_2-n]![n]!} \\ &\times \frac{[\lambda_2-\lambda_3+2]_\infty}{[\lambda_1-\lambda_2-d_1+n+1]_{d_1-n}[\lambda_1-\lambda_3-n+2]_n[\lambda_2-\lambda_3+2]_{d_1-n}} \\ &\times \frac{1}{[\lambda_2-\lambda_3+d_1-d_2-n+1]_{d_2-n}[\lambda_2-\lambda_3+d_1-2n+2]_\infty} \\ &\frac{1}{[\lambda_2-\lambda_3-n+1]_n}. \end{aligned}$$

Note that $c(d_1,d_2,n)$ is a rational function in $v, v^{\lambda_1}, v^{\lambda_2}$, which is invariant under the change $v \mapsto v^{-1}$, $v^{\lambda_1} \mapsto v^{-\lambda_1}$, $v^{\lambda_2} \mapsto v^{-\lambda_2}$.

Set

$$\begin{aligned} \omega_{d_1,d_2,n} &= \frac{1}{(1-v^2)^{d_1+d_2}} v^{r(d_1,d_2,n)} \sqrt{c(d_1,d_2,n)}\; m_{d_1,d_2,n}, \\ \bar{\omega}_{d_1,d_2,n} &= \frac{1}{(1-v^{-2})^{d_1+d_2}} v^{-r(d_1,d_2,n)+s(d_1,d_2)} \sqrt{c(d_1,d_2,n)}\; \bar{m}_{d_1,d_2,n}. \end{aligned}$$

Theorem 8.6. *The Whittaker vector ω and the dual Whittaker vector $\bar{\omega}$ exist, are unique and are given by the formula:*

$$\omega_{d_1,d_2} = \sum_{n=0}^{\min(d_1,d_2)} \omega_{d_1,d_2,n}, \qquad \bar{\omega}_{d_1,d_2} = \sum_{n=0}^{\min(d_1,d_2)} \bar{\omega}_{d_1,d_2,n}.$$

Proof. The theorem is proved by a direct calculation. Q.E.D.

8.4. Toda recursion

In this section we describe a relation between the Whittaker vectors, the Toda recursion and the characters of $\hat{\mathfrak{n}}$-modules. Such relations hold with the following identification:

$$q = v^2, \qquad z_1 = q^{\lambda_1 - \lambda_2 + 1}, \qquad z_2 = q^{\lambda_2 - \lambda_3 + 1},$$

which we always assume in this section.

The following lemma can be extracted from [E].

Lemma 8.7. *The functions* $(\omega_{d_1,d_2}, \bar{\omega}_{d_1,d_2})$ *satisfy the Toda recursion given in Proposition 7.1.*

Proof. The Casimir operator $Z \in U_v(\mathfrak{sl}_3)$ acts in the cyclic module $\mathcal{V}_v$ by a constant which is readily computed on the highest weight vector. Therefore, we have

$$(Z\omega_{d_1,d_2}, \bar{\omega}_{d_1,d_2}) = (q^{-\lambda_1-1} + q^{-\lambda_2} + q^{-\lambda_3+1})(\omega_{d_1,d_2}, \bar{\omega}_{d_1,d_2}).$$

On the other hand,

$$\begin{aligned}
(Z\omega_{d_1,d_2}, \bar{\omega}_{d_1,d_2}) = ((&v^{-2}K_1^{-4/3}K_2^{-2/3} + K_1^{2/3}K_2^{-2/3} \\
&+ v^2 K_1^{2/3}K_2^{4/3})\omega_{d_1,d_2}, \bar{\omega}_{d_1,d_2}) \\
&+(v - v^{-1})^2\Big(v^{-1}(E_1K_1^{-1/3}K_2^{-2/3}\omega_{d_1,d_2}, \bar{E}_1\bar{\omega}_{d_1,d_2}) \\
&+ v(E_2K_1^{2/3}K_2^{1/3}\omega_{d_1,d_2}, \bar{E}_2\bar{\omega}_{d_1,d_2})\Big) \\
=&(q^{-\lambda_1+d_1-1} + q^{-\lambda_2-d_1+d_2} + q^{-\lambda_3-d_2+1})(\omega_{d_1,d_2}, \bar{\omega}_{d_1,d_2}) \\
-q^{-\lambda_2-d_1+d_2}&(\omega_{d_1-1,d_2}, \bar{\omega}_{d_1-1,d_2}) - q^{-\lambda_3-d_2+1}(\omega_{d_1,d_2-1}, \bar{\omega}_{d_1,d_2-1}).
\end{aligned}$$

In this computation we used $\bar{E}_{13}\bar{\omega}_{d_1,d_2} = 0$.

The lemma follows. Q.E.D.

Corollary 8.8. *We have* $I_{d_1,d_2}(z_1, z_2) = (\omega_{d_1,d_2}, \bar{\omega}_{d_1,d_2})$, *where* $I_{d_1,d_2}(z_1, z_2)$ *is given by* (7.1).

Proof. The corollary follows from the uniqueness of the solution of the Toda recursion, Lemma 8.7 and (7.1). Q.E.D.

Recall the functions $I_{d_1,d_2,n}(z_1, z_2)$ given by (7.10). The following theorem explains the meaning of these functions from the point of view of Whittaker vectors.

Theorem 8.9. *We have*

$$I_{d_1,d_2,n}(z_1, z_2) = (\omega_{d_1,d_2,n}, \bar{\omega}_{d_1,d_2,n}).$$

Proof. It is easy to see from the explicit formulas that there exist integers $r(d_1, d_2, n)$ such that

$$v^{r(d_1,d_2,n)}\sqrt{I_{d_1,d_2,n}(z_1,z_2)}\; m_{d_1,d_2,n} = \omega_{d_1,d_2,n}.$$

Let $\bar{I}_{d_1,d_2,n}(z_1,z_2)$ be obtained from $I_{d_1,d_2,n}(z_1,z_2)$ by the change $v \mapsto v^{-1}$, $v^{\lambda_1} \mapsto v^{-\lambda_1}$, $v^{\lambda_2} \mapsto v^{-\lambda_2}$. Then we have

$$v^{-r(d_1,d_2,n)+s(d_1,d_2)}\sqrt{\bar{I}_{d_1,d_2,n}(z_1,z_2)}\; \bar{m}_{d_1,d_2,n} = \bar{\omega}_{d_1,d_2,n}.$$

It is also easy to check explicitly that

$$v^{s(d_1,d_2)}\sqrt{\bar{I}_{d_1,d_2,n}(z_1,z_2)} = \sqrt{I_{d_1,d_2,n}(z_1,z_2)}.$$

The theorem follows. Q.E.D.

Remark 8.10. In fact, we guessed the formulas for the Whittaker vectors in Theorem 8.6 expecting that Theorem 8.9 is true.

We recover the result of Proposition 7.11.

Corollary 8.11. *We have* $I_{d_1,d_2}(z_1,z_2) = \sum_{n=0}^{\min(d_1,d_2)} I_{d_1,d_2,n}(z_1,z_2)$.

Remark 8.12. In [IS] the classical limit (corresponding to $v \to 1$) of the function $I_{d_1,d_2}(z_1,z_2)$ is written as a sum of $\min(d_1,d_2)$ factorized terms. The classical limits of our terms differ from the terms in [IS].

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References

[AKS] E. Ardonne, R. Kedem and M. Stone, Fermionic characters and arbitrary highest-weight integrable $\widehat{\mathfrak{sl}}_{r+1}$-modules, Comm. Math. Phys., **264** (2006), 427–464.

[BF] A. Braverman and M. Finkelberg, Finite-difference quantum Toda lattice via equivariant K-theory, Transform. Groups, **10** (2005), 363–386.

[C] C. Calinescu, Principal subspaces of higher-level deformed $\widehat{\mathfrak{sl}_2}$-modules, math.0611534.

[CLM] C. Calinescu, J. Lepowsky and A. Millas, Vertex-algebraic structure of the principal subspaces of certain $A_1^{(1)}$-modules, I: level one case, math.07041759.

[D] C. Dong, Vertex algebras associated with even lattices, J. Algebra, **161** (1993), 245–265.

[E] P. Etingof, Whittaker functions on quantum groups and q-deformed Toda operators, In: Differential Topology, Infinite-dimensional Lie Algebras and Applications, Amer. Math. Soc. Transl. Ser. 2, **194**, Amer. Math. Soc., Providence, RI, 1999, pp. 9–25.

[FJLMM1] B. Feigin, M. Jimbo, S. Loktev, T. Miwa and E. Mukhin, Bosonic formulas for (k, l)-admissible partitions, Ramanujan J., **7** (2003), 485–517.

[FJLMM2] B. Feigin, M. Jimbo, S. Loktev, T. Miwa and E. Mukhin, Addendum to "Bosonic formulas for (k, l)-admissible partitions", Ramanujan J., **7** (2003), 519–530.

[FJMMT] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin and Y. Takeyama, Fermionic formulas for $(k, 3)$-admissible configurations, Publ. Res. Inst. Math. Sci., **40** (2004), 125–162.

[FKLMM] B. Feigin, R. Kedem, S. Loktev, T. Miwa and E. Mukhin, Combinatorics of $\widehat{\mathfrak{sl}_2}$ spaces of coinvariants: dual functional realization and recursion, Compositio Math., **134** (2002), 193–241.

[FL] B. Feigin and S. Loktev, On finitization of the Gordon identities, math.QA/0006221, Funct. Anal. Appl.

[FS] B. Feigin and A. Stoyanovsky, Quasi-particle models for the representations of Lie algebras and geometry of flag manifold, hep-th/9308079, RIMS preprint, **942**; Functional models for the representations of current algebras and the semi-infinite Schubert cells, Funct. Anal. Appl., **28** (1994), 55–72.

[FS1] B. Feigin and A. Stoyanovsky, unpublished.

[FK] I. Frenkel and V. Kac, Basic representations of affine Lie algebras and dual resonance models, Invent. Math., **62** (1980), 23–66.

[G] G. Georgiev, *Combinatorial construction of modules for infinite-dimensional Lie algebras. I. Principal subspace*, J. Pure Appl. Algebra, **112** (1996), 247–286.

[GKLO] A. Gerasimov, S. Kharchev, D. Lebedev and S. Oblezin, On a Gauss–Givental representation of quantum Toda chain wave function, math/0608152.

[GL] A. Givental and Y.-P. Lee, Quantum K-theory on flag manifolds, finite-difference Toda lattice and quantum groups, Invent. Math., **151** (2003), 193–219.

[IS] T. Ishii and E. Stade, New formulas for Whittaker functions on $GL_n(\mathbb{R})$, J. Funct. Anal., **244** (2007), 289–314.

[J] M. Jimbo, Quantum R matrix related to the generalized Toda system :án algebraic approach, Lecture Notes in Phys., **246** (1986), 335–361.

[K1] V. Kac, Infinite dimensional Lie algebras, 3rd ed., Cambridge Univ. Press, Cambridge, 1990.

[K2] V. Kac, Vertex algebras for beginners, 2nd ed., Univ. Lecture Ser., **10**, Amer. Math. Soc., Providence, RI, 1998.

[Kos] B. Kostant, On Whittaker vectors and representation theory, Invent. Math., **48** (1978), 101–184.

[Kum] S. Kumar, Kac–Moody groups, their flag varieties and representation theory, Progr. Math., **204**, Birkhäuser Boston, Boston, MA, 2002.

[LP] J. Lepowsky and M. Primc, Structure of the standard modules for the affine Lie algebra $A_1^{(1)}$, Contemp. Math., **46**, Amer. Math. Soc., Providence, RI, 1985.

[P] M. Primc, Vertex operator construction of standard modules for $A_n^{(1)}$, Pacific J. Math., **162** (1994), 143–187.

[S] A. Sevostyanov, Quantum deformation of Whittaker modules and Toda lattice, Duke Math. J., **105** (2000), 211–238.

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