

Subharmonic bifurcation from relative equilibria in reversible systems with rotation symmetry

André Vanderbauwhede

Abstract.

In this paper we study the bifurcation of subharmonic tori from branches of symmetric relative equilibria in reversible systems with an additional S^1 -symmetry. The analysis is based on a detailed study of the Poincaré map, which appears to be generated by a reduced vectorfield on the section. The results are applied to a simple example system, but the bifurcation behaviour described by our theoretical results can also be observed in, for example, the spherical pendulum and the Furuta pendulum.

§1. Introduction

Consider a system

$$(1) \quad \dot{x} = f(x),$$

with $x \in \mathbb{R}^{2n}$ and $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ a C^∞ -smooth vectorfield. Such system is *reversible* if there exists a linear involution $R \in \mathcal{L}(\mathbb{R}^{2n})$ such that $f(Rx) = -Rf(x)$ for all $x \in \mathbb{R}^{2n}$; we will assume that $\dim(\text{Fix}(R)) = \dim(\text{Fix}(-R)) = n$. Denoting by $\tilde{x}(t, x)$ the flow of (1) we have then that $\tilde{x}(t, Rx) = R\tilde{x}(-t, x)$, and if $\tilde{x}(t_0, x_0) \in \text{Fix}(R)$ for some $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^{2n}$ then the orbit $\{\tilde{x}(t, x_0) \mid t \in \mathbb{R}\}$ is invariant under R ; we call such orbits *symmetric*. If for some $t_0 < t_1$ we have $\tilde{x}(t_0, x_0) \in \text{Fix}(R)$, $\tilde{x}(t_1, x_0) \in \text{Fix}(R)$ and $\tilde{x}(t, x_0) \notin \text{Fix}(R)$ for $t_0 < t < t_1$, then $\tilde{x}(t, x_0)$ is both symmetric and periodic, with minimal period $T = 2(t_1 - t_0)$. Such symmetric periodic orbits appear typically in one-parameter families, since they are determined by the intersection of the n -dimensional subspace $\text{Fix}(R)$ with the $(n + 1)$ -dimensional manifold $\{\tilde{x}(t, x) \mid t \in \mathbb{R}, x \in \text{Fix}(R)\}$. When along such one-parameter family one finds an orbit which has a (non-trivial) pair of multipliers which are q -th roots of unity ($q \geq 3$) then one expects at this orbit the bifurcation of one or

more branches of q -subharmonics. The precise conditions for such subharmonic bifurcation and the detailed form of the bifurcating branches have been studied in [8]; see also [1] and [2].

Typically, two branches of q -subharmonic orbits will bifurcate from the primary family; the two branches are close to each other, like the boundaries of an Arnold tongue (i.e. at a distance ρ from the bifurcation point the separation between the two branches is of the order $(q-2)/2$). Along one of the two branches there is a pair of near-critical multipliers (i.e. close to 1) on the unit circle, on the other branch the near-critical multipliers are on the real axis. To see this typical subharmonic branching behaviour three conditions must be satisfied: (1) the critical periodic orbit along the primary family must have 1 as a multiplier with multiplicity 2, together with exactly one pair of simple multipliers which are q -th roots of unity; (2) this pair of multipliers must move with non-zero speed through the root of unity as one moves along the primary family (using an appropriate intrinsic parametrization); (3) some higher order coefficient (of order $q-1$) in the normal form of the Poincaré-map must be different from zero. In [3] it is shown what happens when the critical multipliers are not simple (condition (1) is not satisfied), and in a forthcoming paper we will study the case where the transversality conditions (2) fails. In this note we describe a particular class of systems where due to additional symmetries the condition (3) is not satisfied.

We will assume that the system (1) has an S^1 -symmetry which is compatible with the reversibility: there exists some $J_0 \in \mathcal{L}(\mathbb{R}^{2n})$ such that $J_0^2 = -I$, $J_0 R = -R J_0$ and

$$(2) \quad f(e^{J_0\theta} x) = e^{J_0\theta} f(x), \quad \forall \theta \in \mathbb{R}, \forall x \in \mathbb{R}^{2n}.$$

Observe that $(\theta, x) \in \mathbb{R} \times \mathbb{R}^{2n} \mapsto e^{J_0\theta} x \in \mathbb{R}^{2n}$ defines an S^1 -action on \mathbb{R}^{2n} , since $e^{2\pi J_0} = I$. Without loss of generality we can assume that both R and J_0 are orthogonal (this implies that R is symmetric and J_0 anti-symmetric). A symmetric *relative equilibrium* is a symmetric orbit of (1) (generated by some $x_0 \in \text{Fix}(R)$) which is at the same time an orbit under the S^1 -action: $\{\tilde{x}(t, x_0) \mid t \in \mathbb{R}\} = \{e^{J_0\theta} x_0 \mid \theta \in \mathbb{R}\}$. A necessary and sufficient condition for $x_0 \in \text{Fix}(R)$ to generate such symmetric relative equilibrium is that there exists some $\Omega_0 \in \mathbb{R}$ such that

$$(3) \quad f(x_0) = \Omega_0 J_0 x_0.$$

By imposing the condition $\Omega_0 \neq 0$ we exclude actual equilibria; replacing J_0 by $-J_0$ if necessary we can w.l.o.g. assume that $\Omega_0 > 0$. The corresponding orbit is then periodic, with minimal period $T_0 := 2\pi/\Omega_0$,

and $\tilde{x}(t, x_0) = e^{J_0 \Omega_0 t} x_0$. In the subsequent sections we will study subharmonic bifurcation from such symmetric relative equilibrium.

Observe that since both f and J_0 map $\text{Fix}(R)$ into $\text{Fix}(-R)$, we can rewrite the condition (3) for symmetric relative equilibria as $F(x, \Omega) = 0$, with $F : \text{Fix}(R) \times \mathbb{R} \rightarrow \text{Fix}(-R)$ defined by $F(x, \Omega) := f(x) - \Omega J_0 x$. Since $\text{Fix}(R)$ and $\text{Fix}(-R)$ have the same dimension we expect to see one-parameter families of symmetric relative equilibria.

Here is an example of a system satisfying our hypotheses. Take $n = 2$, identify \mathbb{R}^4 with \mathbb{C}^2 writing $x = (a, b)$, and consider the system

$$(4) \quad \begin{cases} \dot{a} &= ib, \\ \dot{b} &= ia - i|b|^2 b. \end{cases}$$

The reversibility and S^1 -equivariance are generated by respectively

$$(5) \quad R(a, b) := (\bar{a}, \bar{b}) \quad \text{and} \quad J_0(a, b) := i(a, b), \quad \forall (a, b) \in \mathbb{C}^2.$$

An extensive study of (4) can be found in [6]. There mainly the Hamiltonian structure of (4) was used; here we will mostly ignore this additional structure. The condition for a non-trivial symmetric relative equilibrium takes the form

$$(6) \quad b = \Omega a \quad \text{and} \quad a - b^3 = \Omega b, \quad (a, b, \Omega) \in \mathbb{R}^3, (a, b) \neq (0, 0).$$

It follows that $0 < b^2 = g(\Omega) := \Omega^{-1} - \Omega$ and hence either $\Omega < -1$ or $0 < \Omega < 1$. This gives us two branches of symmetric relative equilibria, both originating at the origin, and generated by the points

$$(7) \quad x_\Omega = \sqrt{g(\Omega)}(\Omega^{-1}, 1), \quad \text{with } \Omega \in I_- :=]-\infty, -1[\text{ or } \Omega \in I_+ :=]0, 1[.$$

§2. The Poincaré map

In order to study the bifurcation of subharmonics at a symmetric relative equilibrium $\gamma_0 := \{e^{J_0 \Omega_0 t} x_0 \mid t \in \mathbb{R}\}$ of (1) we will use the Poincaré map at x_0 defined by the R -invariant section $x_0 + \Sigma$, where $\Sigma := (J_0 x_0)^\perp$. First we introduce an appropriate coordinate system in a neighborhood of γ_0 .

Lemma 1. *There exists some $r_0 > 0$ such that the mapping*

$$(8) \quad \Psi : S^1 \times U_0 := (\mathbb{R}/2\pi\mathbb{Z}) \times \{y \in \Sigma \mid \|y\|^2 < r_0\} \longrightarrow \mathbb{R}^{2n}, \\ (\theta, y) \longmapsto e^{J_0 \theta}(x_0 + y)$$

is a diffeomorphism onto the open S^1 -invariant neighborhood $U := \Psi(S^1 \times U_0)$ of γ_0 .

Proof. We have $\Psi(\theta, 0) = e^{J_0\theta}x_0$ and $D\Psi(\theta, 0) \cdot (\tilde{\theta}, \tilde{y}) = e^{J_0\theta}(\tilde{\theta}J_0x_0 + \tilde{y})$ for all $\theta \in S^1$, $\tilde{\theta} \in \mathbb{R}$ and $\tilde{y} \in \Sigma$. Since Σ is the orthogonal complement of J_0x_0 this shows that $D\Psi(\theta, 0)$ is invertible (for each $\theta \in S^1$), and hence we can choose $r_0 > 0$ sufficiently small such that the restriction of Ψ to (a small neighborhood of θ in S^1) $\times U_0$ is a diffeomorphism onto an open neighborhood of $e^{J_0\theta}x_0$ in \mathbb{R}^{2n} . It follows that $U = \Psi(S^1 \times U_0)$ is an open neighborhood of γ_0 (we call this a *tubular neighborhood*); U is obviously S^1 -invariant. It remains to show that Ψ is injective on its full domain $S^1 \times U_0$ if we choose r_0 small enough; this injectivity is equivalent to the statement that for all $\theta \in \mathbb{R}$ and all $y \in U_0$ we have

$$(9) \quad e^{J_0\theta}(x_0 + y) \in x_0 + U_0 \implies \theta \in 2\pi\mathbb{Z}.$$

Suppose we can not choose $r_0 > 0$ small enough such that (9) holds. Then there exist sequences $(\theta_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in respectively \mathbb{R} and Σ such that

$$\theta_n \notin 2\pi\mathbb{Z}, \quad y'_n := e^{J_0\theta_n}(x_0 + y_n) - x_0 \in \Sigma,$$

$$\lim_{n \rightarrow \infty} y_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} y'_n = 0.$$

Adding multiples of 2π we can assume that the sequence (θ_n) is bounded and hence (taking subsequences) convergent: $\lim_{n \rightarrow \infty} \theta_n = \bar{\theta}$. Then $e^{J_0\bar{\theta}}x_0 - x_0 = \lim_{n \rightarrow \infty} y'_n = 0$, and therefore $\bar{\theta} \in 2\pi\mathbb{Z}$. But then the equality $e^{J_0\theta_n}(x_0 + y_n) = e^{J_0\bar{\theta}}(x_0 + y'_n)$ together with the fact that Ψ is a local diffeomorphism near $(\theta, 0)$ implies that $\theta_n = \bar{\theta}$ and $y_n = y'_n$ for n sufficiently large, giving us a contradiction since $\theta_n \notin 2\pi\mathbb{Z}$. Q.E.D.

Clearly $\mathbb{R}^{2n} = \text{span}_{\mathbb{R}}\{J_0x_0\} \oplus \Sigma$; by continuity and by choosing r_0 sufficiently small we have then also $\mathbb{R}^{2n} = \text{span}_{\mathbb{R}}\{J_0(x_0 + y)\} \oplus \Sigma$ for all $y \in U_0$. So there exist unique mappings $\Omega : U_0 \rightarrow \mathbb{R}$ and $h : U_0 \rightarrow \Sigma$ such that

$$(10) \quad f(x_0 + y) = \Omega(y)J_0(x_0 + y) + h(y), \quad \forall y \in U_0;$$

these mappings are smooth, with $\Omega(0) = \Omega_0$ and $h(0) = 0$. Moreover, the uniqueness of Ω and h combined with the reversibility of f imply that

$$(11) \quad \Omega(Ry) = \Omega(y) \quad \text{and} \quad h(Ry) = -Rh(y), \quad \forall y \in U_0.$$

In the tubular neighborhood U of γ_0 the vectorfield f is given by

$$(12) \quad f(e^{J_0\theta}(x_0 + y)) = e^{J_0\theta}(\Omega(y)J_0(x_0 + y) + h(y)), \quad \forall \theta \in \mathbb{R}, \forall y \in U_0.$$

Next we consider the solution $\tilde{x}(t, x_0 + y)$ starting at a point $x_0 + y$ in the section $x_0 + \Sigma$ and close to x_0 (i.e. $y \in U_0$). By Lemma 1 we can write

$$(13) \quad \tilde{x}(t, x_0 + y) = e^{J_0 \tilde{\theta}(t, y)}(x_0 + \tilde{y}(t, y))$$

for all $t \in \mathbb{R}$ such that $\tilde{x}(t, x_0 + y) \in U$, and with $\tilde{\theta}(t, y) \in \mathbb{R}$ and $\tilde{y}(t, y) \in U_0$. Expressing that this is a solution of (1) leads to

$$\begin{aligned} & e^{J_0 \tilde{\theta}(t, y)}(\dot{\tilde{\theta}}(t, y)J_0(x_0 + \tilde{y}(t, y)) + \dot{\tilde{y}}(t, y)) \\ &= e^{J_0 \tilde{\theta}(t, y)}(\Omega(\tilde{y}(t, y))(J_0(x_0 + \tilde{y}(t, y)) + h(\tilde{y}(t, y)))) \end{aligned}$$

and hence

$$\dot{\tilde{\theta}}(t, y) = \Omega(\tilde{y}(t, y)) \quad \text{and} \quad \dot{\tilde{y}}(t, y) = h(\tilde{y}(t, y)).$$

Combined with the initial conditions $\tilde{y}(0, y) = y$ and $\tilde{\theta}(0, y) = 0$ we obtain the following.

Theorem 2. *In the tubular neighborhood U given by Lemma 1 the flow of (1) is given by*

$$(14) \quad \tilde{x}(t, x_0 + y) = e^{J_0 \Omega(y)t}(x_0 + \tilde{y}(t, y)),$$

where $\tilde{y}(t, y)$ is the flow of the reversible system

$$(15) \quad \dot{y} = h(y)$$

on the $(2n - 1)$ -dimensional space Σ .

Proof. It follows from the foregoing that $\tilde{y}(t, y)$ as appearing in (13) must indeed be the flow of (15). The S^1 -equivariance of (1) implies that $\tilde{x}(t, e^{J_0 \theta} x) = e^{J_0 \theta} \tilde{x}(t, x)$ for all $\theta \in \mathbb{R}$; we also have $\tilde{y}(t, \tilde{y}(\tau, y)) = \tilde{y}(t + \tau, y)$ and $\tilde{x}(t, \tilde{x}(\tau, x_0 + y)) = \tilde{x}(t + \tau, x_0 + y)$ for all $t, \tau \in \mathbb{R}$. Bringing (13) in this last equality and working out leads to

$$\tilde{\theta}(t + \tau, y) = \tilde{\theta}(t, y) + \tilde{\theta}(\tau, y), \quad \forall t, \tau \in \mathbb{R}, \forall y \in U_0.$$

Differentiating in τ at $\tau = 0$ gives

$$\dot{\tilde{\theta}}(t, y) = \dot{\tilde{\theta}}(0, y) = \Omega(y) \implies \tilde{\theta}(t, y) = \Omega(y)t, \quad \forall (t, y) \in \mathbb{R} \times U_0.$$

Bringing this in (13) proves the theorem.

Q.E.D.

The proof above shows that

$$(16) \quad \Omega(\tilde{y}(\tau, y)) = \Omega(y), \quad \forall \tau \in \mathbb{R}, \forall y \in U_0.$$

Also, for $t, \tau \in \mathbb{R}$ and $y \in U_0$ we have

$$\begin{aligned} \tilde{x}(t, x_0 + \tilde{y}(\tau, y)) &= e^{J_0 \Omega(\tilde{y}(\tau, y))t} (x_0 + \tilde{y}(t, \tilde{y}(\tau, y))) \\ &= e^{J_0 \Omega(y)t} (x_0 + \tilde{y}(t + \tau, y)) \\ &= e^{-J_0 \Omega(y)\tau} \tilde{x}(t + \tau, x_0 + y); \end{aligned}$$

this proves that the orbits $\{\tilde{x}(t, x_0 + \tilde{y}(\tau, y)) \mid t \in \mathbb{R}\}$ and $\{\tilde{x}(t, x_0 + y) \mid t \in \mathbb{R}\}$ generated by respectively $x_0 + \tilde{y}(\tau, y)$ and $x_0 + y$ can be obtained one from the other by using the S^1 -symmetry.

It follows from Theorem 2 that for each sufficiently small $y \in \Sigma$ the *first return time* — defined as the time $T = T(y) > 0$ close to $T_0 = 2\pi/\Omega_0$ such that $\tilde{x}(T, x_0 + y) \in x_0 + \Sigma$ — is given by $T(y) = 2\pi/\Omega(y)$. The corresponding Poincaré map $P : U_0 \rightarrow \Sigma$, defined by $P(y) := \tilde{x}(T(y), x_0 + y) - x_0$, is then given by

$$(17) \quad P(y) = \tilde{y}(T(y), y) = \tilde{y}(2\pi/\Omega(y), y), \quad \forall y \in U_0.$$

We have $P(0) = 0$ and (using $\tilde{y}(t, 0) = 0$)

$$DP(0) = D_y \tilde{y}(T_0, 0) = e^{A_0 T_0}, \quad \text{with } A_0 := Dh(0) \in \mathcal{L}(\Sigma).$$

Moreover, it follows from (11) that $T(Ry) = T(y)$, $\tilde{y}(t, Ry) = R\tilde{y}(-t, y)$ and $P(Ry) = \tilde{y}(T(Ry), Ry) = R\tilde{y}(-T(y), y) = P^{-1}(y)$, i.e. the Poincaré map P is a *reversible map*, in the sense that

$$(18) \quad R \circ P \circ R^{-1} = P^{-1}.$$

We finish this section by finding the relation between the eigenvalues of A_0 and $DP(0)$, and the multipliers of the relative periodic orbit γ_0 . These multipliers are the eigenvalues of the monodromy matrix $M_0 := X(T_0)$, where $X(t)$ is the solution of the initial value problem

$$(19) \quad \dot{X}(t) = Df(e^{J_0 \Omega_0 t} x_0) \cdot X(t), \quad X(0) = I, \quad (X(t) \in \mathcal{L}(\mathbb{R}^{2n})).$$

It follows from the S^1 -equivariance of f that $Df(e^{J_0 \theta} x) = e^{J_0 \theta} Df(x) e^{-J_0 \theta}$ for all $(\theta, x) \in \mathbb{R} \times \mathbb{R}^{2n}$; bringing this in (19) and setting $X(t) = e^{J_0 \Omega_0 t} Y(t)$ shows that $Y(t) = e^{\tilde{A}_0 t}$, with $\tilde{A}_0 := Df(x_0) - \Omega_0 J_0$. Hence $X(t) = e^{J_0 \Omega_0 t} e^{\tilde{A}_0 t}$, and $M_0 = e^{\tilde{A}_0 T_0}$. Remember that x_0 is an equilibrium of the vectorfield $F(x) := f(x) - \Omega_0 J_0 x$, and observe that $\tilde{A}_0 = DF(x_0)$. Since F is reversible and $x_0 \in \text{Fix}(R)$ also \tilde{A}_0 is reversible: $R\tilde{A}_0 = -\tilde{A}_0 R$. Consequently, if $\lambda \in \mathbb{C}$ is an eigenvalue of

\tilde{A}_0 , then so is $-\lambda$, and both have the same algebraic and geometric multiplicities. In the same way M_0 is a reversible linear mapping, i.e. $R \circ M_0 \circ R = M_0^{-1}$, and if $\mu \in \mathbb{C}$ is a multiplier, then so is μ^{-1} . From this it is easy to see that 1 (which is always a multilier) must have an even (algebraic) multiplicity.

To see the relation between \tilde{A}_0 and A_0 we differentiate (12) in θ and y at the point $(\theta, y) = (0, 0)$; rearranging terms and using the definitions of \tilde{A}_0 and A_0 we find that

$$\begin{cases} \tilde{A}_0 \cdot J_0 x_0 &= 0, \\ \tilde{A}_0 \cdot y &= (D\Omega(0) \cdot y) J_0 x_0 + A_0 y, \end{cases} \quad \forall y \in \Sigma.$$

This shows that \tilde{A}_0 is triangular with respect to the splitting $\mathbb{R}^{2n} = \text{span}_{\mathbb{R}}\{J_0 x_0\} \oplus \Sigma$, and that A_0 and \tilde{A}_0 have the same eigenvalues with the same multiplicities, except for the eigenvalue 0 for which the multiplicity as an eigenvalue of A_0 is one lower than the multiplicity as an eigenvalue of \tilde{A}_0 . Hence the eigenvalues of $DP(0)$ are exactly the multipliers of the periodic orbit γ_0 , except that one has to lower the multiplicity of the trivial multiplier 1 by one.

§3. Subharmonic bifurcation

In order to study subharmonic bifurcation from a symmetric relative equilibrium γ_0 as described in the preceding sections we fix some $q \geq 1$ and make the following assumption:

(H1) The linear operator $\tilde{A}_0 = Df(x_0) - \Omega_0 J_0$ has:

- 0 as an eigenvalue with multiplicity 2;
- a pair of simple eigenvalues $\pm i\omega_0$, with $\omega_0 = \Omega_0 k q^{-1}$, $k > 0$ and $\text{gcd}(k, q) = 1$;
- no other purely imaginary eigenvalues $\pm i\mu$ such that $q\mu \in \Omega_0 \mathbb{Z}$.

In the case $q \geq 3$ this means that next to the multiplier 1, which should have the lowest possible multiplicity (namely 2), the periodic orbit γ_0 has just one pair of simple multipliers which are q -th roots of unity. We should mention that the cases $q = 1$ (plain bifurcation of periodic orbits) and $q = 2$ (period-doubling) are allowed. Under the hypothesis **(H1)** we will study the existence of q -periodic points of the Poincaré map P near the fixpoint $y = 0$; such a q -periodic point $y \in U_0 \subset \Sigma$ generates a q -subharmonic solution $\tilde{x}(t, x_0 + y)$ of (1).

The small q -periodic points $y \in U_0$ of the Poincaré map P are determined by the equation

$$P^{(q)}(y) = y, \quad \text{with } P^{(q)} := P \circ P \circ \dots \circ P \text{ (} q \text{ times);}$$

since $P(y) = \tilde{y}(T(y), y)$ and $T(\tilde{y}(\tau, y)) = T(y)$ this takes the form

$$(20) \quad \tilde{y}(qT(y), y) = y, \quad (y \in U_0).$$

From this we obtain immediately the following result.

Theorem 3. *A point $y \in U_0$ is a q -periodic point of P if and only if either*

- (i) *y is an equilibrium of the vectorfield h , i.e. $h(y) = 0$, $\tilde{y}(t, y) = y$ for all $t \in \mathbb{R}$, and $\{\tilde{x}(t, x_0 + y) = e^{J_0 \Omega(y)t}(x_0 + y) \mid t \in \mathbb{R}\}$ is a relative periodic solution of (1),*

or

- (ii) *$\tilde{y}(t, y)$ is a non-trivial periodic solution of (15), with some minimal period $\tilde{T}(y) > 0$, and*

$$(21) \quad \frac{qT(y)}{\tilde{T}(y)} \in \mathbb{N}.$$

In case (ii) each point of the (periodic) orbit $\{\tilde{y}(\tau, y) \mid \tau \in \mathbb{R}\}$ will also be a q -periodic point of P ; the corresponding subharmonic orbits $\{\tilde{x}(t, x_0 + \tilde{y}(\tau, y)) \mid t \in \mathbb{R}\}$ ($\tau \in \mathbb{R}$) of (1) are obtained one from the other by applying the S^1 -symmetry and all lie on a 2-torus $\{\tilde{x}(t, x_0 + \tilde{y}(\tau, y)) \mid t \in \mathbb{R} \tau \in \mathbb{R}\}$ which is invariant both for the flow of (1) and for the S^1 -action. We call such invariant torus filled with subharmonics a subharmonic torus.

It follows that we have to determine all sufficiently small equilibria and periodic orbits of (15); for the periodic orbits also the condition (21) must be satisfied. We know from Hopf bifurcation theory that for y sufficiently small the possible minimal periods $\tilde{T}(y)$ which can appear will be close to $2\pi/\omega$, with $\pm i\omega$ a pair of purely imaginary eigenvalues of $Dh(0) = A_0$. Taking the “limit for $y \rightarrow 0$ ” in (21) and using the fact that $T(0) = 2\pi/\Omega_0$ gives us the condition $q\omega \in \mathbb{N}\Omega_0$ for the eigenvalues $\pm i\omega$ which we have to consider. It follows from our hypothesis **(H1)** that there is only one pair of eigenvalues which satisfies this condition, namely the pair $\pm i\omega_0$. Hence we can restrict our search for small periodic orbits to those which have a minimal period near $\tilde{T}_0 := 2\pi/\omega_0$; the resonance condition (21) then takes the form

$$(22) \quad qT(y) = k\tilde{T}(y).$$

For the further analysis we rely on the general reduction theory for Hopf bifurcation as worked out in [5]; for our particular problem this theory tells us the following. There is a smooth one-to-one relation

between the small periodic orbits of (15) with a period near \tilde{T}_0 (this includes all small equilibria) and the small periodic orbits of a *reduced system* which lives on the subspace $V := \text{Ker}(e^{A_0\tilde{T}_0} - I)$. This reduced system is reversible but also equivariant with respect to the S^1 -action on V defined by $(\theta, v) \in S^1 \times V \mapsto e^{A_0\omega_0^{-1}\theta}v \in V$; also, its linearization at the origin is given by the restriction of A_0 to V . It follows from **(H1)** that 0 is a simple eigenvalue of A_0 and that the corresponding eigenvector belongs to $\text{Fix}(R)$ (the eigenvector J_0x_0 of \hat{A}_0 belongs to $\text{Fix}(-R)$ and hence $\dim(\text{Fix}(R) \cap \Sigma) = n$ and $\dim(\text{Fix}(-R) \cap \Sigma) = n - 1$). Hence $\dim(V) = 3$; one can easily show that by choosing an appropriate basis one can identify V with $\mathbb{R} \times \mathbb{C}$ in such a way that $A_0v = A_0(\alpha, z) = (\alpha, i\omega_0z)$ and $Rv = R(\alpha, z) = (\alpha, \bar{z})$ for all $v = (\alpha, z) \in V \cong \mathbb{R} \times \mathbb{C}$. The reduced system must then have the form

$$(23) \quad \begin{cases} \dot{\alpha} &= 0, \\ \dot{z} &= i\omega(\alpha, |z|^2)z; \end{cases}$$

here $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a smooth function with $\omega(0, 0) = \omega_0$. Since *all* solutions of (23) are periodic (or equilibria) the one-to-one relation given by the general reduction theory (which in principle must be restricted to periodic orbits) becomes a smooth conjugacy between the system (23) and the system (15) restricted to an appropriate invariant manifold. This allows us to formulate the following theorem.

Theorem 4. *Assume **(H1)**. Then the set of equilibria and periodic orbits of (15) with period near $\tilde{T}_0 = 2\pi\omega_0^{-1}$ forms locally near the origin a smooth 3-dimensional invariant submanifold of Σ on which the flow is smoothly conjugate to the flow on $V \cong \mathbb{R} \times \mathbb{C}$ given by (23).*

We will denote the (locally defined) conjugacy by $\Phi : V \rightarrow \Phi(V) \subset \Sigma$; the theory in [5] shows that Φ commutes with R . The flow of (23) can be written out explicitly:

$$\tilde{v}(t, v) = \tilde{v}(t, \alpha, z) = (\alpha, e^{i\omega(\alpha, |z|^2)t}z), \quad \forall v = (\alpha, z) \in V \cong \mathbb{R} \times \mathbb{C}.$$

For $z = 0$ we find a one-dimensional line $\mathbb{R} \times \{0\}$ of equilibria corresponding (locally near γ_0) to a one-parameter family $\{\gamma_\alpha \mid |\alpha| < \alpha_0\}$ of symmetric relative equilibria, given by

$$\begin{aligned} \gamma_\alpha &= \{\tilde{x}(t, x_\alpha) = e^{J_0\Omega_\alpha t}x_\alpha \mid t \in \mathbb{R}\} \\ &\text{with } x_\alpha := x_0 + \Phi(\alpha, 0) \text{ and } \Omega_\alpha := \Omega(\Phi(\alpha, 0)). \end{aligned}$$

The periodic orbit γ_α has period $T_\alpha := 2\pi\Omega_\alpha^{-1}$; next to the trivial multiplier 1 (which has multiplicity 2) there is also a pair of simple multipliers

of the form $e^{\pm 2\pi i \lambda(\alpha)}$, with $\lambda(\alpha) := \omega_\alpha \Omega_\alpha^{-1}$, $\omega_\alpha := \omega(\alpha, 0)$. All other solutions $\tilde{v}(t, \alpha, z)$ (with $z \neq 0$) of (23) are periodic, corresponding to (non-equilibrium) periodic solutions $\tilde{y}(t, \Phi(\alpha, z)) = \Phi(\tilde{v}(t, \alpha, z))$ of (15) with minimal period $\tilde{T}(\Phi(\alpha, z)) = 2\pi\omega(\alpha, |z|^2)^{-1}$. The set of q -periodic points of the Poincaré map P is then given by those points $\Phi(\alpha, z) \in \Sigma$ for which (22) is satisfied. Setting $z = \rho e^{i\vartheta}$ and using $\Omega(\tilde{y}(t, y)) = \Omega(y)$ (see (16)) in combination with the fact that $\Phi(\alpha, \rho e^{i\vartheta}) \in \{\Phi(\alpha, \tilde{v}(t, \rho)) = \tilde{y}(t, \Phi(\alpha, \rho)) \mid t \in \mathbb{R}\}$ shows that $T(\Phi(\alpha, \rho e^{i\vartheta})) = T(\Phi(\alpha, \rho))$ for all $\vartheta \in \mathbb{R}$; hence we have $T(\Phi(\alpha, \rho e^{i\vartheta})) = 2\pi\Omega(\alpha, \rho^2)^{-1}$ for some smooth $\Omega : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $\Omega(0, 0) = \Omega_0$. Bringing this in (22) gives the equation

$$(24) \quad \frac{\omega(\alpha, \rho^2)}{\Omega(\alpha, \rho^2)} = \frac{k}{q}.$$

For $\rho = 0$ this reduces to $\lambda(\alpha) = k/q$, which is satisfied for $\alpha = 0$ by (H1). Now we make a last hypothesis, namely we assume that the transversality condition

(H2) $\lambda'(0) \neq 0$

holds; this means that as we move along the family $\{\gamma_\alpha\}$ of symmetric relative equilibria (using α as a parameter) the multipliers $e^{\pm 2\pi i \lambda(\alpha)}$ move with non-zero speed through the q -th roots of unity $e^{\pm 2\pi i k/q}$. Under this transversality condition we can apply the implicit function theorem to solve (24) for $\alpha = \tilde{\alpha}(\rho^2)$, with $\tilde{\alpha} : \mathbb{R} \rightarrow \mathbb{R}$ a smooth function such that $\tilde{\alpha}(0) = 0$. Each of the points $\Phi(\tilde{\alpha}(\rho^2), \rho) \in \Sigma$ ($0 < \rho < \rho_0$) satisfies the condition (ii) of Theorem 3 and hence generates a subharmonic 2-torus $\mathcal{T}_\rho := \{\tilde{x}(t, x_0 + \tilde{y}(\tau, \Phi(\tilde{\alpha}(\rho^2), \rho))) \mid t \in \mathbb{R}, \tau \in \mathbb{R}\}$. Since $R(\tilde{\alpha}(\rho^2), \rho) = (\tilde{\alpha}(\rho^2), \rho)$ (on V) we have that $\Phi(\tilde{\alpha}(\rho^2), \rho) \in \text{Fix}(R)$ and hence also \mathcal{T}_ρ is invariant under R , i.e. \mathcal{T}_ρ is a symmetric subharmonic torus. Hence we have reached the following conclusion.

Theorem 5. *Let $\gamma_0 = \{e^{J_0 \Omega_0 t} x_0 \mid t \in \mathbb{R}\}$ be a symmetric relative equilibrium of the reversible and S^1 -equivariant system (1). Fix some $q \geq 1$ and assume (H1). Then γ_0 belongs to a one-parameter family $\{\gamma_\alpha \mid |\alpha| < \alpha_0\}$ of symmetric relative equilibria. If also (H2) holds then we have at γ_0 the bifurcation of a single branch $\{\mathcal{T}_\rho \mid 0 < \rho < \rho_0\}$ of R -invariant q -subharmonic 2-tori. Locally near γ_0 there are no other relative equilibria or q -subharmonic tori.*

It should be noted that the first part of the conclusion (on the existence of the family $\{\gamma_\alpha\}$) already holds if only the first item of (H1) is satisfied (this follows by redoing the analysis above for $q = 1$ and with $\dim(V) = 2m + 1$, where m depends on the number of pairs of eigenvalues of \tilde{A}_0 of the form $\pm i\Omega_0 k$, with $k \in \mathbb{N}$, $k \geq 1$).

We finish this section with a remark on the transversality condition **(H2)**. As such this hypothesis requires the use of the special parameter α when calculating the pair of critical multipliers $e^{\pm 2\pi i \lambda(\alpha)}$ of the branch $\{\gamma_\alpha\}$ of symmetric relative equilibria. However, it is possible to verify this hypothesis more directly, as we show now. Suppose that from direct calculation we have obtained a (smooth) local curve of points $\{\hat{x}(\sigma) \mid \sigma \in \mathbb{R}\} \subset \text{Fix}(R)$ which generate symmetric relative equilibria, i.e.

$$f(\hat{x}(\sigma)) = \hat{\Omega}(\sigma)J_0\hat{x}(\sigma), \quad \forall \sigma \in \mathbb{R}.$$

Suppose also that $\hat{x}(0) = x_0$ (and hence $\hat{\Omega}(0) = \Omega_0$), and that for small σ the linear operator $\hat{A}(\sigma) := Df(\hat{x}(\sigma)) - \hat{\Omega}(\sigma)J_0$ has

- 0 as an eigenvalue with multiplicity 2;
- a pair of simple eigenvalues $\pm i\hat{\omega}(\sigma)$, with $\hat{\omega}(0) = \hat{\Omega}(0)kq^{-1}$, $k > 0$ and $\text{gcd}(k, q) = 1$;
- no other purely imaginary eigenvalues $\pm i\mu$ such that $q\mu \in \Omega_0\mathbb{Z}$.

Then the symmetric relative equilibrium $\hat{\gamma}(\sigma) := \{e^{J_0\hat{\Omega}(\sigma)t}\hat{x}(\sigma) \mid t \in \mathbb{R}\}$ has next to the trivial multiplier 1 with multiplicity 2 also a pair of multipliers $e^{\pm 2\pi i \hat{\lambda}(\sigma)}$ with $\hat{\lambda}(\sigma) := \hat{\omega}(\sigma)\hat{\Omega}(\sigma)^{-1}$. Clearly our assumptions imply **(H1)**, and hence also the first part of Theorem 3 holds. It follows that there exists a smooth function $\hat{\alpha}(\sigma)$ with $\hat{\alpha}(0) = 0$ and such that $\hat{x}(\sigma) = x_{\hat{\alpha}(\sigma)}$ and $\hat{\lambda}(\sigma) = \lambda(\hat{\alpha}(\sigma))$. Differentiating at $\sigma = 0$ gives

$$\hat{x}'(0) = \hat{\alpha}'(0)D_\alpha\Phi(0, 0) \quad \text{and} \quad \hat{\lambda}'(0) = \lambda'(0)\hat{\alpha}'(0);$$

so, if $\hat{x}'(0) \neq 0$ then $\hat{\alpha}'(0) \neq 0$ and $\lambda'(0) \neq 0$ if and only if $\hat{\lambda}'(0) \neq 0$. Therefore, under the foregoing assumptions **(H2)** is satisfied if **(H2*)** $\hat{x}'(0) \neq 0$ and $\hat{\lambda}'(0) \neq 0$.

§4. Applications

Consider the system (4) already mentionned in the Introduction. Direct calculation shows that there are two branches of symmetric relative equilibria, generated by the points x_Ω (see (7)), with $\Omega \in I_- =]-\infty, 0[$ along one branch, and $\Omega \in I_+ =]0, 1[$ along the other; this is in agreement with the first part of Theorem 3. To find the bifurcation points of subharmonic tori we follow the approach described at the end of the last section, i.e. we calculate the eigenvalues of $Df(x_\Omega) - \Omega J_0$ (here $\sigma = \Omega$).

Explicitly this matrix is given by

$$\begin{pmatrix} 0 & \Omega & 0 & -1 \\ -\Omega & 0 & 1 & 0 \\ 0 & -1 & 0 & \Omega^{-1} \\ 1 & 0 & 2\Omega - 3\Omega^{-1} & 0 \end{pmatrix};$$

its eigenvalues are 0, with multiplicity 2, and $\pm i\hat{\omega}(\Omega)$, with $\hat{\omega}(\Omega) := \Omega\sqrt{1 + 3\Omega^{-4}}$. This last pair of eigenvalues is simple. It follows that $\hat{\lambda}(\Omega) = \Omega^{-1}\hat{\omega}(\Omega) = \sqrt{1 + 3\Omega^{-4}}$, and it is then straightforward to verify that

$$\frac{dx_\Omega}{d\Omega} \neq 0 \quad \text{and} \quad \hat{\lambda}'(\Omega) \neq 0, \quad \forall \Omega \in I_- \cup I_+.$$

It follows that for each $\Omega \in I_- \cup I_+$ for which $\hat{\lambda}(\Omega)$ is rational we will have the bifurcation of a subharmonic 2-torus at the relative equilibrium $\gamma_\Omega = \{e^{i\Omega t}x_\Omega \mid t \in \mathbb{R}\}$. More precisely, using the fact that

$$\{\hat{\lambda}(\Omega) \mid \Omega \in I_-\} =]1, 2[\quad \text{and} \quad \{\hat{\lambda}(\Omega) \mid \Omega \in I_+\} =]2, \infty[,$$

we can say the following:

- for each pair of integers (k, q) with $0 < q < k < 2q$ and $\text{gcd}(k, q) = 1$ there is a unique $\Omega = \Omega(k, q) \in I_-$ for which $\hat{\lambda}(\Omega) = k/q$; at $\gamma_{\Omega(k, q)}$ a branch of q -subharmonic tori bifurcates from the branch $\Gamma_- := \{\gamma_\Omega \mid \Omega \in I_-\}$ of relative equilibria;
- for each pair of integers (k, q) with $0 < q < 2q < k$ and $\text{gcd}(k, q) = 1$ there is a unique $\Omega = \Omega(k, q) \in I_+$ for which $\hat{\lambda}(\Omega) = k/q$; at $\gamma_{\Omega(k, q)}$ a branch of q -subharmonic tori bifurcates from the branch $\Gamma_+ := \{\gamma_\Omega \mid \Omega \in I_+\}$ of relative equilibria.

The paper [6] contains the result of a numerical study of these bifurcating branches of subharmonic tori. In that paper it is shown that each of the relative equilibria and each of the subharmonic tori correspond to a unique point in the *energy-momentum diagram* for (4); this diagram depicts the image of the *energy-momentum map* defined by $(a, b) \in \mathbb{C}^2 \mapsto (H(a, b), F(a, b))$, where $H(a, b) := a\bar{b} + \bar{a}b - \frac{1}{2}(b\bar{b})^2$ is the Hamiltonian for the system (4), and where $F(a, b) := a\bar{a} + b\bar{b}$ is the Hamiltonian which generates the S^1 -action on \mathbb{C}^2 . In Figure 1 (taken from [6]) one can see the two branches of relative equilibria and some of the bifurcating branches of subharmonic tori. It appears that the branches bifurcating from Γ_- connect to branches bifurcating from Γ_+ , forming “bridges” between the two families of relative equilibria; we refer to [6] for more details on the connection rules.

The example (4) is somewhat special, in the sense that it is not only an S^1 -equivariant reversible system, but at the same time it is also an

integrable Hamiltonian system with a well understood geometry — again see [6]. The same can be said about the spherical pendulum (see [4]) and the Furuta pendulum (see [7]), which are both more physical examples showing the bifurcation behaviour of subharmonic tori as described in this paper.

From the other side, our specific example (4) is just a special case of a larger class of systems which satisfy our basic hypotheses, but which are in general not Hamiltonian. These systems can briefly be described as follows. Identify \mathbb{R}^{2n} with \mathbb{C}^n , writing $x = (z_1, z_2, \dots, z_n)$. Let $A : \mathbb{C}^n \rightarrow \mathcal{L}(\mathbb{R}^n)$ be a smooth mapping (smooth when using $\mathbb{C}^n \cong \mathbb{R}^{2n}$) such that $A(e^{i\theta}x) = A(x)$ for all $\theta \in \mathbb{R}$ and all $x \in \mathbb{C}^n$. Then the system

$$(25) \quad \dot{x} = iA(x)x, \quad (x \in \mathbb{C}^n)$$

is both reversible and S^1 -equivariant, with $Rx = R(z_1, z_2, \dots, z_n) := (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$ and $J_0x = ix$. Symmetric relative equilibria are determined by the equation

$$(26) \quad A(x)x = \Omega x, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n = \text{Fix}(R);$$

this means that Ω must be a (real) eigenvalue of $L(x)$ such that $x \in \mathbb{R}^n$ is a corresponding eigenvector. From the preceding theory one expects such symmetric relative equilibria to appear in one-parameter families from which there bifurcate branches of subharmonic tori. Of course the

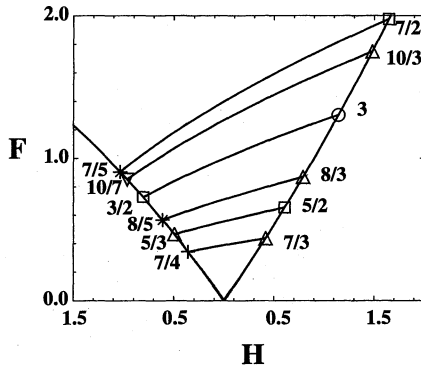


Fig. 1. The energy-momentum diagram for (4), with the two branches of relative equilibria, some subharmonic bifurcations and some connecting bridges. The numbers indicate the value of $\hat{\lambda}(\Omega)$. The figure is taken from [6].

explicit calculation of such branches and the verification of the hypotheses for subharmonic bifurcation will in general not be as easy as for our example (4).

§5. Final comment

The approach described in this note can be applied in much more general situations. For example, in order to avoid technical complications in the formulations, we have restricted to an S^1 -action without non-trivial isotropies; it is straightforward to adapt our results to general S^1 -actions. For more general (continuous) symmetries relative equilibria are not necessarily periodic, but some of our arguments can still be used to study for example the bifurcation of relative periodic orbits from branches of relative equilibria.

Acknowledgments. The author wants to acknowledge many years of collaboration with Emilio Freire, Jorge Galán and Francisco Javier Muñoz-Almaraz from which the work presented in this paper originated. He also acknowledges the support of The Fund for Scientific Research—Flanders (Belgium) and from the Spanish Ministry of Education through the grant BFM2003-00336.

References

- [1] M.-C. Ciocci and A. Vanderbauwhede, On the bifurcation and stability of periodic points in reversible and symplectic diffeomorphisms, In: *Symmetry and Perturbation Theory, Proceedings SPT98*, (eds. A. Degasparis and G. Gaeta), World Sci. Publ., 1999, pp. 159–166.
- [2] M.-C. Ciocci and A. Vanderbauwhede, Bifurcation of periodic points in reversible diffeomorphisms, I: *New Progress in Difference Equations, Proceedings ICDEA2001*, (eds. S. Elaydi, G. Ladas and B. Aulbach), Chapman & Hall, CRC Press, 2004, pp. 75–93.
- [3] M.-C. Ciocci, Subharmonic branching at a reversible 1 : 1 resonance, *J. Difference Equ. Appl.*, **11** (2005), 1119–1135.
- [4] R. Cushman and L. Bates, *Global aspects of Classical Integrable Systems*, Birkhäuser Verlag, Basel, 1997.
- [5] J. Knobloch and A. Vanderbauwhede, A general reduction method for periodic solutions in conservative and reversible systems, *J. Dynam. Differential Equations*, **8** (1996), 71–102.
- [6] F. J. Muñoz-Almaraz, E. Freire, J. Galán and A. Vanderbauwhede, Branches of invariant tori and rotation numbers in symmetric Hamiltonian systems: an example, In: *Symmetry and Perturbation Theory*,

- Proceedings SPT2002, (eds. S. Abenda, G. Gaeta and S. Walcher), World Sci. Publ., 2003. pp. 267–276.
- [7] F. J. Muñoz-Almaraz, E. Freire and J. Galán, Bifurcation behavior of the Furuta pendulum, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **17** (2007), 2571–2578.
- [8] A. Vanderbauwhede, Branching of periodic solutions in time-reversible systems, In: *Geometry and Analysis in Non-Linear Dynamics*, (eds. H. Broer and F. Takens), Pitman Research Notes in Math., **222** (1992), 97–113.
- [9] A. Vanderbauwhede, Subharmonic bifurcation at multiple resonances, In: *Proceedings of the Mathematics Conference, Birzeit, 1998*, (eds. S. Elaydi et. al.), World Sci. Publ., 2000, pp. 254–276.

*Department of Pure Mathematics and Computer Algebra
University of Ghent
Belgium*

E-mail address: `avdb@cage.ugent.be`