Advanced Studies in Pure Mathematics 53, 2009
Advances in Discrete Dynamical Systems
pp. 347–361

Time evolution with and without remote past

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Abstract.

We usually discuss the time evolution from the present to the future or from the past, precisely, from some fixed initial time in the past, to the present or to the future. But we sometimes consider the time evolution from the remote past to the remote future as in the theory of stationary stochastic processes or dynamical systems. In the present paper we consider time evolutions governed by noise driven automorphism on locally compact abelian groups and give a necessary and sufficient condition for the time evolution to admit remote past. It turns out that *to admit the remote past is fairly restrictive.*

§1. Introduction

Let G be a locally compact group and φ be an automorphism of G. Consider the random time evolution governed by a stochastic equation

$$\eta_n = \xi_n \varphi(\eta_{n-1})(n \in \underline{Z}) \tag{1.1}$$

on G where (ξ_n) is a noise in the sense that
(a) the random variables ξ_n are mutually independent and subject to a common probability distribution, say μ, and
(b) for each n the random variable ξ_n is independent of the random variables η_k with $k < n$.

The point is that ξ_n and η_n are indexed for all integer n, including negative n. We say that the time evolution admits remote past if there exists a solution (ξ_n, η_n) of (1.1).

Received October 12, 2007.
2000 *Mathematics Subject Classification.* Primary 60G50; Secondary 37H10, 60J05.
Key words and phrases. random time evolution, stochastic difference equations, random walks on groups, remote past, automorphisms of groups, non-strong solutions.

It is immediate (cf. [1]) to see that the above equation (1.1) is reduced to the convolution equation

$$\lambda_n = \mu_n * \lambda_{n-1} \qquad (n \in \mathbb{Z}) \tag{1.2}$$

where λ_n's are unknown probability measures on G which stand for the probability distribution of $\varphi^{-n}\eta_n$ and μ_n's are known probability measures which come from $\varphi^{-n}\xi_n$. The equation (1.2) for arbitrarily given μ_n's will be discussed in Section 3. We remark that the equation (1.2) always admits the *trivial solution* (λ_n) where each λ_n is the normalized Haar measure ω_G on G and that the set of the solutions (λ_n) of (1.2) is a convex set in the infinite product space of copies of the space of probability Borel measures on G.

If the automorphism is the identity, i.e., if $\varphi = \text{id.}$, then the solution of (1.1) is nothing but a random walk on the group G with remote past. There is a long history in the study of random walks on groups (cf., e.g., [3]) and the asymptotic behavior as time goes to infinity is studied in detail. Due to [3] the theory of random walks on groups goes back to the paper [4] by Y. Kawada and K. Itô and our Theorem 1.1 stated below may be regarded as its improvement from the viewpoint of "with remote past".

On the other hand, our motivation of the study of the time evolution with remote past has the background in the theory of stochastic differential equation which was invented by Kiyosi Itô in 1942. He gave the definition of Itô's stochastic integral and found the Itô's formula. He solved the stochastic differential equation by generalizing the successive iteration method. Those solutions he and his pupils gave on the first stage are now called "strong" solutions because the solution process is adapted to the given Brownian filtration: in other words, the solution up to time t is a functional of the noise up to time t. In 1960's "pathological" solutions are found: there exist nonstrong solutions. A typical example is Tanaka's stochastic differential equation:

$$dX(t) = \operatorname{sgn}(X(t))dB(t),\ \operatorname{sgn}(x) = 1(x \geq 0); = -1(x < 0).$$

It has various solutions:
1) $X(t) = |B(t)|$ is a solution(strong solution).
2) At any zero of $B(t)$ one can switch $\operatorname{sgn}(X(t))$ to obtain a new solution which are not strong because extra randomness is put in there. Recently, Le Jan et al. characterized the solution set of Tanaka's equation (cf., [5]).

In 1975 Tsirelson ([7]) has constructed examples of nonstrong solutions by reducing stochastic differential equations into stochastic difference equations on the one dimensional torus $G = \mathbb{T}^1$. Later he studies the isomorphism problem of noises: white noise, black noise, etc.

In 1990's M. Yor ([8]) formulated Tsirelson's argument in the form (1.1) and studied the equation on $G = \mathbb{T}^1$ for general noise. In 2006 Akahori et al. ([1]) have studied it on general compact groups G and showed the structure of the extremal solution set, i.e., the extremal set of the solution set which is a compact convex set:

$$ex(\{solutions\}) \cong G/H \quad \text{for some subgroup} \quad H \subset G.$$

Now we state the results in the random walk case on a locally compact abelian group G. We denote the characteristic group of G by Γ.

Theorem 1.1. *Assume that the noise is stationary and the automorphism is the identity:*

$$\mu_n = \mu \textit{ for any } n \in \mathbb{Z} \textit{ and } \varphi = \mathrm{id}. \tag{1.3}$$

Set

$$\Gamma_\mu = \{\chi \in \Gamma : |\mu(\chi)| = 1\} \tag{1.4}$$

and

$$G_\mu = \{g \in G : \chi(g) = 1 \textit{ for all } \chi \in \Gamma_\mu\}. \tag{1.5}$$

Then there exists a unique element $\alpha(\mu)$ in G/G_μ such that solutions (λ_n) of (1.2) *are characterized by the following two properties:*
(a) *Each λ_n is G_μ-invariant.*
(b) *The projections $\widehat{\lambda}_n$ of λ_n to G/G_μ evolves by the Weyl transformation (or the translation) $\tau_{\alpha(\mu)}$ by $\alpha(\mu)$:*

$$\widehat{\lambda}_n = \tau_{\alpha(\mu)}\widehat{\lambda}_{n-1} \qquad (n \in \mathbb{Z}) \tag{1.6}$$

where $\tau_\alpha\beta = \beta + \alpha$ for $\alpha, \beta \in G/G_\mu$.

To illustrate the idea, we give examples in the case where $G = \mathbb{T}^1$. Here we identify $\mathbb{T}$ with the interval $[0, 1)$.

Example 1.2 (cf. [8, Section 7, Lemma 5]). For $p = 1, 2, \ldots$, we denote $\mathbb{Z}_p = \{k/p : k = 0, 1, \ldots, p-1\}$.
(i) Assume that $\mu(x + \mathbb{Z}_p) < 1$ for any $p = 1, 2, \ldots$ and any $x \in \mathbb{T}$. Then $G_\mu = \mathbb{T}$. In this case the solution of (1.2) is only the trivial solution.
(ii) Assume that $\mu(\{x\}) = 1$ for some $x \in \mathbb{T}$. Then $G_\mu = \{0\}$ and $\alpha(\mu) = \{x\}$. In this case every solution (λ_n) of (1.2) evolves by the translation by x.
(iii) Otherwise, we can choose $x \in \mathbb{T}$ and $p = 2, 3, \ldots$ such that p is

minimum among the pairs (x,p) with $\mu(x+\mathbb{Z}_p)=1$. Then $G_\mu=\mathbb{Z}_p$ and $\alpha(\mu)=x+\mathbb{Z}_p$.

In particular, we consider the case where the support of μ consists of two points: $\mu=p_1\delta_{x_1}+p_2\delta_{x_2}$ for some $p_1,p_2>0$, $p_1+p_2=1$ and $x_1,x_2\in\mathbb{T}$, $x_1\neq x_2$.
(i)$'$ Assume that x_2-x_1 is rational under identification $G\simeq[0,1)$. We can express $x_2-x_1=r/p$ for some $p=2,3,\dots$ and $r\in\mathbb{N}$ where p and r are coprime. Then $G_\mu=\mathbb{Z}_p$ and $\alpha(\mu)=x_1+\mathbb{Z}_p$ $(=x_2+\mathbb{Z}_p)$.
(ii)$'$ Otherwise, $G_\mu=\mathbb{T}$.

The following result also shows that the existence of nontrivial solution is fairly restrictive.

Theorem 1.3. *Assume that G has a countable basis. Let μ be a probability measure on G and φ an automorphism of G. Set*

$$\Gamma_\mu=\left\{\chi\in\Gamma:\prod_{k=m}^{\infty}|\mu(\chi\circ\varphi^k)|>0 \text{ for some } m\in\mathbb{Z}\right\} \tag{1.7}$$

and

$$G_\mu=\{g\in G:\chi(g)=1 \text{ for all } \chi\in\Gamma_\mu\}. \tag{1.8}$$

Then there exists an element $\alpha(\mu)\in G/G_\mu$ such that $\mu(\cap_{\chi\in\Gamma_\mu}W^s(a,\chi,\varphi))=1$ for any $a\in\alpha(\mu)$ where $W^s(a,\chi,\varphi)$ is the "stable set of a in direction χ":

$$W^s(a,\chi,\varphi)=\left\{x\in G:\lim_{k\to\infty}\chi(\varphi^k x)/\chi(\varphi^k a)=1\right\}. \tag{1.9}$$

Theorem 1.4. *Under the same assumption and notations as in Theorem 1.3, the following statements hold:*
(a) *Every solution (λ_n) of (1.2) consists of G_μ-invariant measures λ_n.*
(b) *There exists a sequence (ν_n) of G_μ-invariant probability measures on G such that every extremal solution (λ_n) of the convolution equation (1.2) corresponds to a unique element $\gamma\in G/G_\mu$ by the relation*

$$\lambda_n=\nu_n*\delta_{-\sum_{j=0}^{-n-1}\varphi^j a*\delta_c}\qquad(n\in\mathbb{Z}) \tag{1.10}$$

where a and c are arbitrary elements of the cosets $\alpha(\mu)$ and γ, respectively, and the sum $\sum_{j=0}^{-n-1}$ is interpreted as 0 for $n=0$ and $-\sum_{j=-n}^{-1}$ for positive n.

In the case of nonabelian groups we use the unitary representation theory. We keep Tannaka's duality theorem in mind, though it will not be explicitly stated below. We denote the totality of unitary representations of a compact group G by $\widehat{G}$. Thus, each $\rho \in \Gamma_\mu$ is a unitary operator acting on some finite dimensional Hilbert space $U(\rho)$.

Theorem 1.5. *For a given noise probability distribution μ set*

$$\Gamma_\mu = \{\rho \in \widehat{G}; \|\mu(\rho)\|_{op} = 1\} \tag{1.11}$$

where $\mu(\rho) = \int_G \mu(dx)\rho(x)$ as before and $\|\cdot\|_{op}$ denotes the operator norm. Then each $\rho \in \Gamma_\mu$ has the component ρ_{11} of operator norm exactly 1 which is necessarily the identity operator on some subspace $U_1(\rho)$ of $U(\rho)$. Set

$$G_\mu = \{g \in G; \rho_{11}(g) = \mathrm{id} \quad \textit{for all} \quad \rho \in \Gamma_\mu\}. \tag{1.12}$$

Then G_μ is a subgroup of G and, if (λ_n) is a solution of (1.1), each λ_n is G_μ-invariant and is obtained by n times translation of λ_0 by some element $\alpha(\mu)$ in G/G_μ.

The proofs of Theorem 1.1-4 is given below following the joint work with K. Yano [6] where G is assumed to be compact but the proof of Theorem 1.5 will be published elsewhere.

In Section 5 we give a further property in the special case where G is a finite-dimensional torus and $\Gamma_\mu = \Gamma$.

We must also refer to a work of Brossard and Leuridan ([2]). They have studied rather general Markov chains and investigate the uniqueness problem and the behavior of sample paths at the remote past. But they imposed a restrictive assumption that the one-step transition probability is absolute continuous with respect to a measure. Consequently, the case which involves the Weyl transform is excluded.

§2. Random walks on abelian groups

In this section we assume that G is abelian, the noise is stationary and the automorphism is the identity:

$$\mu_n = \mu \text{ for any } n \in \mathbb{Z} \text{ and } \varphi = \mathrm{id}. \tag{2.1}$$

Then the convolution equation (1.2) takes the form

$$\lambda_n = \mu * \lambda_{n-1} \qquad (n \in \mathbb{Z}) \tag{2.2}$$

and we obtain

$$\lambda_n(\chi) = \mu(\chi)^m \lambda_{n-m}(\chi) \qquad n \in \mathbb{Z},\ m \in \mathbb{N},\ \chi \in \Gamma. \tag{2.3}$$

Definition 2.1. Set

$$\Gamma_\mu = \{\chi \in \Gamma : |\mu(\chi)| = 1\}. \tag{2.4}$$

Lemma 2.2. *If $|\mu(\chi)| < 1$, then $\lambda_n(\chi) = 0$ for any $n \in \mathbb{Z}$.*

Proof. By (2.3), we have $|\lambda_n(\chi)| = |\mu(\chi)|^m |\lambda_{n-m}(\chi)|$. Since $|\lambda_{n-m}(\chi)| \leq \lambda_{n-m}(|\chi|) \leq 1$, we obtain $|\lambda_n(\chi)| \leq |\mu(\chi)|^m$ for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Letting m tend to infinity, we obtain $\lambda_n(\chi) = 0$. Q.E.D.

Lemma 2.3. *Let λ be a probability measure on G and Γ_0 be a subset of Γ. Assume that $\lambda(\chi) = 0$ whenever $\chi \notin \Gamma_0$. Then, the measure λ is G_0-invariant where G_0 is the annihilator of Γ_0:*

$$G_0 = \{x \in G : \chi(x) = 1 \textit{ for all } \chi \in \Gamma_0\}. \tag{2.5}$$

Proof. Let T_g be the translation by $g \in G_0$. Then, $(T_g\lambda)(\chi) = \chi(g)\lambda(\chi) = \lambda(\chi)$ if $\chi \in \Gamma_0$ by the definition of G_0. Otherwise, $(T_g\lambda)(\chi) = \chi(g)\lambda(\chi) = 0 = \lambda(\chi)$ by the assumption on λ. Hence, $T_g\lambda = \lambda$. Q.E.D.

Let us denote the annihilator of Γ_μ by G_μ.

Lemma 2.4. *If $|\mu(\chi)| = 1$, then $\chi(x)$ is constant μ-a.e. In particular, $\chi(x) = \mu(\chi)$ for μ-a.e. x.*

Proof. Since $|\chi(x)| = 1$ μ-a.e., one obtains

$$\begin{aligned} 0 &\leq \int_X |\chi(x) - \mu(\chi)|^2 \mu(dx) \\ &\leq \int_X \int_X |\chi(x) - \chi(y)|^2 \mu(dx)\mu(dy) = 2(1 - |\mu(\chi)|^2). \end{aligned} \tag{2.6}$$

Hence, if $|\mu(\chi)| = 1$, then $\chi(x) = \mu(\chi)$ μ-a.e. Q.E.D.

The above lemma can have various versions, for instance, for vectors in Hilbert spaces, for unitary operators etc.

Proposition 2.5. *The following statements hold:*
(i) *Γ_μ is a subgroup of the character group Γ.*
(ii) *For any $\chi_1, \chi_2 \in \Gamma_\mu$,*

$$\mu(\chi_1\overline{\chi_2}) = \mu(\chi_1)\overline{\mu(\chi_2)}. \tag{2.7}$$

In other words, the restriction $\mu|_{\Gamma_\mu}$ is a character of Γ_μ.
(iii) *There exists a unique element α_μ in G/G_μ such that $\mu(\chi) = \chi(a)$ for any $a \in \alpha_\mu$ and any character $\chi \in \Gamma_\mu$.*

Proof. Let $\chi_1, \chi_2 \in \Gamma_\mu$. By Lemma 2.4, we see that $\chi_1(x) = \mu(\chi_1)$ and $\chi_2(x) = \mu(\chi_2)$ for μ-a.e. $x \in G$. Then we have $(\chi_1\overline{\chi_2})(x) = \mu(\chi_1)\overline{\mu(\chi_2)}$ for μ-a.e. $x \in G$, and, hence, we obtain $\mu(\chi_1\overline{\chi_2}) = \mu(\chi_1)\overline{\mu(\chi_2)}$. This implies (ii) and also (i).

Note that Γ_μ is identified with the character group of G/G_μ. By Pontryagin's duality theorem, the character $\mu|_{\Gamma_\mu}$ of Γ_μ obtained in Proposition 2.5 can be identified with an element of G/G_μ. We identify it with a coset $\alpha(\mu)$. Then, for any $a \in \alpha(\mu)$ and any character $\chi \in \Gamma_\mu$, we obtain $\mu(\chi) = \chi(a)$. Q.E.D.

Proof of Theorem 1.1. We already proved (a) in Lemmas 2.2 and 2.3.

It then follows from Proposition 2.5 (iii) and from a similar argument in the proof of Lemma 2.3 that $\lambda_n(\chi) = \chi(a)\lambda_{n-1}(\chi)$ for all $n \in \mathbb{Z}$, $a \in \alpha(\mu)$ and $\chi \in \Gamma$. Consequently, $\lambda_n = T_a\lambda_{n-1}$. Since each λ_n is G_μ-invariant, we obtain (b). Q.E.D.

Remark 2.6. Assume, in addition, that G has a countable basis. Since each λ_n is G_μ-invariant, there exists a probability measure $\widehat{\lambda}_n$ on the quotient group G/G_μ such that

$$\int_G \lambda_n(dx)f(x) = \int_{G/G_\mu} \widehat{\lambda}_n(dh) \int_{G_\mu} \nu(dy)f(h.y) \tag{2.8}$$

for any continuous function f on G where ν is the normalized Haar measure on G_μ and $h.y$ stands for an element of $h \in G/G_\mu$ identified with a coset.

§3. Nonstationary noise on abelian groups

In this section we continue to assume that $\varphi = \mathrm{id}$ but we consider the case where μ_n may depend on n. Now the convolution equation (1.2) takes the original form

$$\lambda_n = \mu_n * \lambda_{n-1} \qquad n \in \mathbb{Z} \tag{3.1}$$

and, hence, we obtain

(3.2)
$$\lambda_n(\chi) = \mu_n(\chi)\mu_{n-1}(\chi)\cdots\mu_{n-m+1}(\chi)\lambda_{n-m}(\chi) \quad n \in \mathbb{Z},\ m \in \mathbb{N},\ \chi \in \Gamma.$$

Lemma 3.1. *If* $\prod_{k=1}^{\infty} \mu_{-k}(\chi) = 0$, *then* $\lambda_n(\chi) = 0$ *for any* $n \in \mathbb{Z}$.

Proof. By (3.2), we have $|\lambda_n(\chi)| = \prod_{k=1}^{m-1} |\mu_{n-k}(\chi)||\lambda_{n-m}(\chi)|$. Since $|\lambda_{n-m}(\chi)| \leq \lambda_{n-m}(|\chi|) \leq 1$, we obtain $|\lambda_n(\chi)| \leq \prod_{k=1}^{m-1} |\mu_{n-k}(\chi)|$ for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Letting m tend to infinity, we obtain $\lambda_n(\chi) = 0$. Q.E.D.

Definition 3.2. Set

$$\Gamma_\mu = \left\{ \chi \in \Gamma : \prod_{k=m}^{\infty} |\mu_{-k}(\chi)| > 0 \text{ for some } m \right\}. \tag{3.3}$$

Remark 3.3. In the case considered in Section 2 the two definitions of Γ_μ given by (2.4) and (3.3) coincide.

Lemma 3.4. *The inequality $\prod_{k=m}^{\infty} |\mu_{-k}(\chi)| > 0$ holds if and only if $\mu_{-k}(\chi) \neq 0$ for any $k \geq m$ and*

$$\sum_{k=m}^{\infty} \int_G \int_G \mu_{-k}(dx)\mu_{-k}(dy)|\chi(x) - \chi(y)|^2 < \infty. \tag{3.4}$$

Proof. Notice that

$$\int_G \int_G \mu_{-k}(dx)\mu_{-k}(dy)|\chi(x) - \chi(y)|^2 = 2(1 - |\mu_{-k}(\chi)|^2). \tag{3.5}$$

Hence the assertion follows from the fact that the infinite product $\prod_{k=1}^{\infty} c_k$ of $0 \leq c_k \leq 1$ converges to a positive limit if and only if c_k's are positive and $\sum_{k=1}^{\infty}(1 - c_k) < \infty$. Q.E.D.

Proposition 3.5. Γ_μ *is a subgroup of the character group* Γ.

Proof. Let $\chi_1, \chi_2 \in \Gamma_\mu$. Then it follows from Lemma 3.4 that, for sufficiently large m,

(3.6)

$$\begin{aligned}
&\sum_{k=m}^{\infty} \int_G \int_G \mu_{-k}(dx)\mu_{-k}(dy)|(\chi_1\overline{\chi_2})(x) - (\chi_1\overline{\chi_2})(y)|^2 \\
&\leq \sum_{k=m}^{\infty} \int_G \int_G \mu_{-k}(dx)\mu_{-k}(dy)2\left\{|\chi_1(x) - \chi_1(y)|^2 + |\chi_2(x) - \chi_2(y)|^2\right\} \\
&< \infty.
\end{aligned}$$

Moreover,

(3.7)

$$\sum_{k=m}^{\infty} \int_G \mu_{-k}(dx)|(\chi_1\overline{\chi_2})(x) - \mu_{-k}(\chi_1\overline{\chi_2})|^2 < \infty.$$

Hence, $\mu_{-k}(\chi_1\overline{\chi_2}) \neq 0$ except for finitely many k. Consequently, again by Lemma 3.4 we conclude that $\chi_1\overline{\chi_2} \in \Gamma_\mu$. Since we assume that G is compact, the character group Γ is discrete. Hence, an algebraic subgroup of Γ is a (topological) subgroup. Q.E.D.

Remark 3.6. Lemma 2.3 in the previous section works here, too. Thus, each λ_n of a solution (λ_n) is G_μ-invariant.

Here we stop the preliminary discussion on nonstationary noises and we proceed in the next section to the case where the noise is stationary and the automorphism is arbitrary.

§4. Noise-driven automorphisms of locally compact abelian group

Let φ be an automorphism of a compact abelian group G. We assume that the noise (ξ_n) is stationary so that the random variables ξ_n's are independent and subject to a common probability distribution μ. So we consider the stochastic equation (1.1) stated in Section 1 where μ_n is the probability distribution of $\varphi^{-n}\xi_n$ and λ_n is the probability distribution of $\varphi^{-n}\eta_n$. We denote by $\widetilde{\lambda}_n$ the probability distribution of η_n itself.

The automorphism φ of G induces an automorphism φ^* of the character group Γ: $\varphi^*\chi(x) = \chi(\varphi x)$, $x \in G$, $\chi \in \Gamma$. For $\chi \in \Gamma$ define

(4.1)
$$W_2^s(\chi, \varphi) = \left\{ (x, y) \in G \times G : \sum_{k=0}^{\infty} |(\varphi^{*n}\chi)(x) - (\varphi^{*n}\chi)(y)|^2 < \infty \right\}$$

and, for $x \in G$,

$$(4.2) \quad \begin{aligned} W_2^s(x; \chi, \varphi) &= \{y \in G : (x, y) \in W_2^s(\chi, \varphi)\} \\ &= \left\{ y \in G : \sum_{k=0}^{\infty} |(\varphi^{*n}\chi)(y) - (\varphi^{*n}\chi)(x)|^2 < \infty \right\}. \end{aligned}$$

We may call $W_2^s(x; \chi, \varphi)$ the *ℓ^2-stable set of x in the direction χ with respect to φ.*

Remark 4.1. We have the obvious relation $W_2^s(x; \chi, \varphi) \subset W^s(x; \chi, \varphi)$ where $W^s(x; \chi, \varphi)$ is defined in (1.9).

Lemma 4.2. *The set $W_2^s(0; \chi, \varphi)$ is a φ-invariant subgroup of G.*

Proof. Obvious. Q.E.D.

Now Lemma 3.4, Proposition 3.5 and Remark 3.6 can be restated as follows.

Proposition 4.3. *Assume that* (λ_n) *solves the equation* (1.2). *Then the following statements hold.*
(i) *If* $\chi \in \Gamma_\mu$, *then,* $(\mu \otimes \mu)(W_2^s(\chi, \varphi)) = 1$.
(ii) Γ_μ *is* φ^**-invariant.*
(iii) G_μ *is a* φ*-invariant subgroup.*
(iv) λ_n *is* G_μ*-invariant and so is* $\widetilde{\lambda}_n$.

Proof. (ii)-(iv) are obvious restatements. To see (i) it suffices to note that

$$\int_G \int_G \mu(dx)\mu(dy) \sum_{k=m}^{\infty} |(\varphi^{*k}\chi)(x) - (\varphi^{*k}\chi)(y)|^2 = \sum_{k=m}^{\infty} \int_G \int_G \mu_{-k}(dx)\mu_{-k}(dy)|\chi(x) - \chi(y)|^2. \tag{4.3}$$

Q.E.D.

Remark 4.4. If $\varphi = \mathrm{id}$, then, $W_2^s(x; \chi, \varphi) = \{y : \chi(y) = \chi(x)\}$. Hence, if we assume, in addition, that G is metrizable or that Γ_μ is countable, then we can apply Fubini's theorem and the assertion (i) of Proposition 4.2 shows

$$(\mu \otimes \mu)\{(x, y) \in G : \chi(x) = \chi(y) \text{ for all } \chi \in \Gamma_\mu\} = 1. \tag{4.4}$$

This implies that the support of μ consists of a single coset in G/G_μ, which is nothing but the element $\alpha(\mu)$ introduced in Section 2.

Proof of Theorem 1.2. Obvious from (i) of Proposition 4.2. Q.E.D.

Proof of Theorem 1.3. Let (λ_n) be a solution of (1.2) and $a \in \alpha(\mu)$. Set

$$\mathring{\mu}_n = T_{-\varphi^{-n}a}\mu_n \qquad (n \in \mathbb{Z}) \tag{4.5}$$

and

$$\mathring{\lambda}_n = T_{\sum_{j=0}^{-n-1} \varphi^j a}\lambda_n \qquad (n \in \mathbb{Z}) \tag{4.6}$$

for each n. Here we interpret $\sum_{j=0}^{-1} = 0$ and $\sum_{j=0}^{-n-1} = -\sum_{j=-n}^{-1}$ for positive n. Then they satisfy

$$\mathring{\lambda}_n = \mathring{\mu}_n * \mathring{\lambda}_{n-1} \qquad (n \in \mathbb{Z}), \tag{4.7}$$

and, hence,

$$(4.8) \qquad \lambda_n^\circ = \mu_n^\circ * \mu_{n-1}^\circ * \cdots * \mu_{n-k}^\circ * \lambda_{n-k-1}^\circ \qquad (n \in \mathbb{Z},\ k \in \mathbb{N}).$$

Recall that the totality of probability measures on a compact metrizable space is compact in the weak topology. Here we say that $\mu_n \to \mu$ weakly if $\mu_n(f) \to \mu(f)$ for any continuous function f. This topology is called the *weak* topology* in the context of the functional analysis.

Now we can choose an increasing sequence of integers $m_j \to \infty$ such that the weak limit

$$(4.9) \qquad \nu_n^\circ = \lim_{j\to\infty} \mu_n^\circ * \mu_{n-1}^\circ * \cdots * \mu_{n-m_j}^\circ \qquad (n \in \mathbb{Z})$$

exists. Since

$$(4.10) \qquad \nu_n^\circ(\chi) = \lim_{j\to\infty} \prod_{i=0}^{m_j} \mu_{n-i}^\circ(\chi) \qquad (n \in \mathbb{Z})$$

for any $\chi \in \Gamma$, we see that $\nu_n^\circ(\chi)$ for each $n \in \mathbb{Z}$ is not equal to 0 for any $\chi \in \Gamma_\mu$ and is equal to 0 for any $\chi \notin \Gamma_\mu$. Thus we conclude that ν_n° is G_μ-invariant by Lemma 2.3. Note that (ν_n) is not uniquely determined from (μ_n), but, for each choice of a sequence m_j, the limit $(\nu_n(\chi))$ is uniquely determined up to a multiplicative constant of modulus 1.

Take a limit point of the sequence $(\lambda_{-m_j}^\circ)$ and denote it by $\lambda_{-\infty}^\circ$. Recall that $\mu_n \to \mu$ and $\nu_n \to \nu$ weakly imply $\mu_n * \nu_n \to \mu * \nu$ weakly. In fact, it is obvious that the product measure $\mu_n \otimes \nu_n \to \mu \otimes \nu$ weakly and that the pullback of any continuous function under the map $(x, y) \mapsto x + y$ is again a continuous function on the product space. Letting k tend to infinity in (4.8), we obtain

$$(4.11) \qquad \lambda_n^\circ = \nu_n^\circ * \lambda_{-\infty}^\circ.$$

Then (λ_n) is expressed as

$$(4.12) \qquad \begin{aligned} \lambda_n &= T_{-\sum_{j=0}^{-n-1} \varphi^j a}(\nu_n^\circ * \lambda_{-\infty}^\circ) \\ &= \nu_n * T_{-\sum_{j=0}^{-n-1} \varphi^j a}(\lambda_{-\infty}^\circ). \end{aligned}$$

Here we denote $\nu_n = T_{-\sum_{j=0}^{-n-1} \varphi^j a}(\nu_n^\circ)$, which is also G_μ-invariant.

Consequently, an extremal solution (λ_n) is expressed as

$$(4.13) \qquad \lambda_n = \nu_n * \delta_{-\sum_{j=0}^{-n-1} \varphi^j a + c}$$

for some $c \in G$ and the element c is unique modulo G_μ. Q.E.D.

§5. The case of toral automorphisms

By the definition the probability measures ν_n° satisfy the equation

$$\nu_n^\circ = \mu_n^\circ * \nu_{n-1}^\circ \qquad n \in \mathbb{Z}. \tag{5.1}$$

Consequently, if we take a sequence of independent random variables ξ_n° subject to μ_n°, then, we may formally understand that each ν_n° is the probability distribution of the infinite sum $\sum_{k=-\infty}^n \xi_k^\circ$.

In some special cases the convergence of $\sum_{k=-\infty}^n \xi_k^\circ$ is justified and an explicit formula for the solution (η_n) of the equation (1.1) is obtained.

Theorem 5.1. *Let G be a finite dimensional torus, say $G = \mathbb{T}^d$ and assume that $\Gamma_\mu = \mathbb{Z}^d$. Let d be a distance in $\mathbb{T}^d$. Then*

$$\sum_{k=-n}^{\infty} \varphi^k(\xi_{-k}^\circ) \text{ converges almost surely} \tag{5.2}$$

and

$$E\left[\sum_{k=-n}^{\infty} d(\varphi^k(\xi_{-k}^\circ), 0)^2\right] < \infty \tag{5.3}$$

for each $n \in \mathbb{Z}$. Moreover, the extremal solution (η_n) is given by the formula

$$\varphi^{-n}(\eta_n) = c + \sum_{k=-n}^{\infty} \varphi^k(\xi_{-k}^\circ) + \sum_{k=0}^{-n-1} \varphi^k(a) \quad \text{for} \quad n \in \mathbb{Z} \tag{5.4}$$

with $c \in \mathbb{T}^d$ and $a \in \alpha(\mu)$ where $\alpha(\mu)$ is defined in Theorem 1.4.

Lemma 5.2. *There exists a constant r with $0 < r < 1$ such that*

$$\begin{aligned} \cap_{\chi\in\Gamma} W^s(a;\chi,\varphi) &= \cap_{\chi\in\Gamma} W_2^s(a;\chi,\varphi) \\ &= \{x \in G : d(\varphi^k(x), \varphi^k(a)) \le Cr^n \\ &\qquad \text{for any } k \in \mathbb{N} \text{ and for some constant } C\}. \end{aligned} \tag{5.5}$$

Proof. Let us identify $\mathbb{T}^d$ with the unit cube $[-1/2, 1/2)^d$ in $\mathbb{R}^d$ and measures on $\mathbb{T}^d$ with those on $[-1/2, 1/2)^d$. Then the automorphism φ is regarded as an automorphism on $[-1/2, 1/2)^d$ and is defined by a matrix A as

$$\varphi(x) \quad = \quad Ax \bmod \mathbb{Z}^d. \tag{5.6}$$

Under the identification stated above, $\varphi^k(x) \to 0$ as $k \to \infty$ in $\mathbb{T}^d$ if and only if $A^k x \to 0$ as $k \to \infty$ in $\mathbb{R}^d$. Since A is a finite dimensional matrix, it means that the vector x in $\mathbb{R}^d$ belongs to the linear span of eigenvectors of A corresponding to eigenvalues of modulus less than 1. Take a constant r which is less than 1 and is greater than the maximum modulus of such eigenvalues. Then for any norm $\|\cdot\|$ there holds the inequality $\|A^k x\| \le Cr^k$ for some constant C. Consequently, for any distance d on $\mathbb{T}^d$ there holds the inequality $d(0, \varphi^k(x)) \le Cr^k$ for some constant C depending on x (which may be different from the previous C). Hence follows the desired assertion. Q.E.D.

Remark 5.3. If $x \neq 0$ and $A^k x \to 0$ as $k \to \infty$, then $\|A^{-k}x\| \to \infty$ as $k \to \infty$ but the converse is not true.

To prove Theorem 5.1 we want to apply a well-known convergence theorem: if X_k, $k = 0, 1, \ldots$, are independent $\mathbb{R}^d$-valued random variables and if they are square integrable with $\sum_{k=0}^{\infty} E[\|X_k - E[X_k]\|^2] < \infty$, then $\sum_{k=0}^{\infty}(X_k - E[X_k])$ converges almost surely and in L^2 sense. Here appear two obstacles:
(a) Absence of the notion of mean for group elements.
(b) The sequence $\chi(\varphi^k(x)) - \int_G \mu(dy)\chi(\varphi^k(y))$ is square summable for μ-a.e. x but is generally not summable.

Lemma 5.2 above shows that the assumptions of Theorem 5.1 eliminates (b). Indeed, the sequence $\chi(\varphi^k(x)) - \int_G \mu(dy)\chi(\varphi^k(y))$ decreases exponentially.

Lemma 5.4. *Under the identification of $\mathbb{T}^d$ with $[-1/2, 1/2)^d$, we obtain*

$$E\left[\sum_{k=0}^{\infty} \|\varphi^k(\xi^{\circ}_{-k})\|^2\right] < \infty. \tag{5.7}$$

Proof. We start with the following restatement of (3.4):

$$\int_G \mu(dx) E\left[\sum_{k=0}^{\infty} |\chi(\varphi^k(\xi_{-k})) - \chi(\varphi^k(x))|^2\right] < \infty. \tag{5.8}$$

Thus, for μ-a.e. x,

$$E\left[\sum_{k=0}^{\infty} |\chi(\varphi^k(\xi_{-k})) - \chi(\varphi^k(x))|^2\right] < \infty. \tag{5.9}$$

Therefore, $\xi^\circ_{-k} = \xi_{-k} - \varphi^k(a)$ satisfies

$$E\left[\sum_{k=0}^{\infty} |\chi(\varphi^k(\xi^\circ_{-k})) - 1|^2\right] < \infty. \tag{5.10}$$

Now let χ_j, $j = 1, 2, \ldots, d$, be the standard generators of Γ:

$$\chi_j(x) = \exp(2\pi\sqrt{-1}x_j) \text{ for } x = (x_1, \ldots, x_d) \in [-1/2, 1/2)^d. \tag{5.11}$$

Note that $|\exp(2\pi\sqrt{-1}x) - 1| \geq c|x|$ for $x \in [-1/2, 1/2)$. Hence it follows from (5.10) with $\chi = \chi_j$, $j = 1, 2, \ldots, d$, in $\mathbb{T}^d$ that

$$E\left[\sum_{k=0}^{\infty} \|\varphi^k(\xi^\circ_{-k})\|^2\right] < \infty \tag{5.12}$$

in $\mathbb{R}^d$. Q.E.D.

Proof of Theorem 5.1. It follows from Lemma 5.4 that

$$E\left[\sum_{k=0}^{\infty} \left\|\varphi^k(\xi^\circ_{-k}) - E[\varphi^k(\xi^\circ_{-k})]\right\|^2\right] < \infty. \tag{5.13}$$

Hence, the sum

$$\sum_{k=0}^{\infty} \left\{\varphi^k(\xi^\circ_{-k}) - E[\varphi^k(\xi^\circ_{-k})]\right\} \tag{5.14}$$

converges almost surely. On the other hand, we already know that

$$\sum_{k=0}^{\infty} (E[\varphi^k(\xi^\circ_{-k})] - 1) \tag{5.15}$$

converges absolutely. Consequently, the sum $\sum_{k=0}^{\infty} \varphi^k(\xi^\circ_{-k})$ converges almost surely. Q.E.D.

Added in proofs

The author would like to dedicate this article to the memory of Kiyosi Itô who passed away on Nov. 10, 2008.

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