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On asymptotic stability of linear stochastic Volterra difference equations with respect to a fading perturbation

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Abstract.

The paper concerns necessary and sufficient conditions on the fading intensity of a state–independent stochastic perturbation for the asymptotic stability of a linear stochastic Volterra difference equation. In broad terms, it is shown here that the results obtained in the deterministic case are robust to fading stochastic perturbations which are independent of the state, once it is known that these perturbations fade more rapidly than an identifiable critical rate.

§1. Introduction

This note can be considered as an extension of the continuous–time approaches and results of the paper [1] to the discrete case, in which necessary and sufficient conditions for the stability of linear Itô–Volterra equations with deterministically fading noise intensity is obtained. Similar conditions on the noise intensity are established in the continuous case in [2]. Almost sure asymptotic stability results for autonomous scalar non-linear difference equations without delays were obtained in [3] and in [4].

The outline of the note is as follows. Section 2 lists and comments upon the general results obtained. Section 3 shows how the general

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results can be applied to specific problems, and shows the connections between conditions required for the almost sure asymptotic stability of linear Volterra difference equations and corresponding continuous time Itô–Volterra equations driven by Brownian motion. Finally, Section 4 contains proofs of the results.

§2. Results

The following standard notation is employed. $\mathbb{N}$ denotes the natural numbers, $\mathbb{Q}^+$ the positive rational numbers, $\mathbb{R}$ the real numbers, and $\mathbb{R}^r$ r–dimensional real space, for $r \in \mathbb{N}$. If E is a subset of $\mathbb{R}$ then the characteristic function $\chi_E : \mathbb{R} \to \mathbb{R}$ is defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

The standard basis vectors in $\mathbb{R}^r$ are denoted by $\mathbf{e}_1, \dots, \mathbf{e}_r$. The standard innerproduct of x and y in $\mathbb{R}^r$ is denoted by $\langle x, y \rangle$. The $d \times d$ real valued identity matrix is denoted by I_d. We say that a real sequence $a = \{a(n) : n \in \mathbb{N}\}$ obeys $a \in \ell^1(\mathbb{N}; \mathbb{R})$ if $\sum_{n \in \mathbb{N}} |a(n)| < \infty$. We say that the $d \times r$ matrix–valued sequence $a = \{a(n) = (a(n))_{ij} : n \in \mathbb{N}\}$ is in $\ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^r)$ if each entry $a_{ij} \in \ell^1(\mathbb{N}; \mathbb{R})$, $i = 1, \dots, d$, $j = 1, \dots, r$. The convolution of two sequences $f = \{f(n) : n \in \mathbb{N}\}$ and $g = \{g(n) : n \in \mathbb{N}\}$, $f * g$, is a sequence defined by

$$(f * g)(n) = \sum_{k=0}^{n} f(n-k)g(k), \quad n \in \mathbb{N}.$$

The $*$ notation is also used here for the convolution of distribution functions; however, the type of convolution being used will be clear from the context.

We consider the stochastic Volterra difference equation

(2.1)
$$X(n+1) = X(n) + \sum_{k=0}^{n} A(n-k)X(k) + \sigma(n)\xi(n+1), \quad n \in \mathbb{N}; \quad X(0) = \zeta.$$

Here $A \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$, $\sigma(n) \in \mathbb{R}^d \times \mathbb{R}^r$ for all $n \in \mathbb{N}$. Each $\xi(n)$ is a random vector in $\mathbb{R}^r$. We make the following standing assumption about ξ through the paper:

(2.2)
$\xi = \{\xi(n) : n \in \mathbb{N}\}$ is a sequence of independent $\mathbb{R}^r$–valued random vectors;

(2.3)
$\xi_j(n) := \langle \xi(n), \mathbf{e}_j \rangle$ have distribution function F, $\forall n \in \mathbb{N}, \forall j = 1, \dots, r$;

(2.4) $\mathbb{E}[\xi_j(n)] = 0 \quad \mathbb{E}[\xi_j(n)^2] = 1, \quad \forall n \in \mathbb{N}, \forall j = 1, \dots, r$;

(2.5) for fixed n, $\{\xi_j(n)\}_{j=1}^r$ are independent random variables.

Let R be the resolvent defined by

$$R(n+1) = R(n) + \sum_{k=0}^{n} A(n-k)R(k), \quad n \in \mathbb{N}; \quad R(0) = I_d, \tag{2.6}$$

so that each $R(n) \in \mathbb{R}^d \times \mathbb{R}^d$. The significance of the resolvent in this context is that it allows X to be written purely in terms of the perturbation, according to the variation of parameters formula

$$X(n) = R(n)\zeta + \sum_{k=1}^{n} R(n-k)\sigma(k-1)\xi(k), \quad n = 1, 2, \dots \tag{2.7}$$

Everywhere in the paper we suppose that for A and R (defined by (2.6)) the following condition holds true:

$$A \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^r), \quad R \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d). \tag{2.8}$$

The importance of the process Σ defined by $\Sigma(n+1) := \sigma(n)\xi(n+1)$ is that its almost sure asymptotic stability is equivalent to the almost sure asymptotic stability of X, provided the underlying deterministic resolvent R is summable. This result is proven in Proposition 1 below. Consequently, it is unnecessary to study the correlated *unbounded* delay equation for X directly; instead it is enough to determine the conditions under which an independent white noise process with *zero* lag tends to zero. Moreover, an explicit formula is not generally available for X, because it depends on R, which is in general not known in closed form. On the other hand, however, the formula for Σ could hardly be simpler.

Proposition 1. *Suppose that the sequence of random variables* $\xi = \{\xi(n) : n \in \mathbb{N}\}$ *obeys* (2.2)–(2.5). *Let* A *and* R *obey (2.8) and let* X *be defined by* (2.1). *Then the following are equivalent:*

(a) $\lim_{n\to\infty} \sigma(n)\xi(n+1) = 0$, *a.s.*

(b) $\lim_{n\to\infty} X(n) = 0$, *a.s.*

There is a simple *sufficient* condition which ensures $\sigma(n)\xi(n+1) \to 0$ as $n \to \infty$ a.s.

Proposition 2. *Suppose that* $\xi = \{\xi(n) : n \in \mathbb{N}\}$ *is a* $\mathbb{R}^r$*-valued sequence of random vectors which obeys* (2.4)*, and* $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ *is a* $\mathbb{R}^d \times \mathbb{R}^r$*-valued sequence. If* $\sum_{n=1}^{\infty} |\sigma(n)|^2 < \infty$*, then* $\sigma(n)\xi(n+1) \to 0$ *as* $n \to \infty$*, a.s.*

The merit of this summability condition on σ is that it does not require independence of the stochastic sequence; however, it is not clear how close it comes to being optimal in ensuring almost sure stability of X. In the rest of the paper, we concern ourselves with establishing necessary and sufficient conditions on σ and the random sequence ξ which give stability.

Next, we need to connect the condition on the stochastic equation (2.1) to the conditions on the data. This is achieved by the following result.

Lemma 3. *Let* $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ *be a* $\mathbb{R}^d \times \mathbb{R}^r$*-valued sequence. Suppose that the sequence of random variables* $\xi = \{\xi(n) : n \in \mathbb{N}\}$ *obeys* (2.2)–(2.5)*. Define the distribution function* $F_{n,i,j}$ *by*

$$F_{n,i,j}(x) = \begin{cases} F(x/|\sigma_{ij}(n)|), & \sigma_{ij}(n) > 0, \\ 1 - F(-x/|\sigma_{ij}(n)|), & \sigma_{ij}(n) < 0, \\ \chi_{[0,\infty)}(x), & \sigma_{ij}(n) = 0, \end{cases} \tag{2.9}$$

and $F_{n,i}$ *as the convolution of the distributions* $F_{n,i,j}$ *for* $j = 1, \dots, r$*. Then*

$$\sum_{i=1}^{d}\sum_{n=1}^{\infty}\left[1 - F_{n,i}(\varepsilon) + F_{n,i}(-\varepsilon)\right] < \infty \quad \textit{for all } \varepsilon \in \mathbb{Q}^+ \tag{2.10}$$

is equivalent to $\sigma(n)\xi(n+1) \to 0$ *as* $n \to \infty$ *a.s.*

In the scalar case ($d = r = 1$), this simplifies to give

Lemma 4. *Let* $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ *be a real sequence. Suppose that* $\xi = \{\xi(n) : n \in \mathbb{N}\}$ *is a sequence of independent random variables with the same distribution function* F *with* $\mathbb{E}[\xi(n)] = 0$ *and* $\mathbb{E}[\xi(n)^2] = 1$*.*

(a) *The following statements are equivalent:*

$$\sum_{n=0}^{\infty} 1 - F(\varepsilon/|\sigma(n)|) + F(-\varepsilon/|\sigma(n)|) < \infty, \quad \textit{for all } \varepsilon \in \mathbb{Q}^+; \tag{2.11}$$

$$\lim_{n\to\infty} \sigma(n)\xi(n+1) = 0, \quad \textit{a.s.} \tag{2.12}$$

(b) *The following statements are equivalent:*

$$\sum_{n=0}^{\infty} 1 - F(\varepsilon/|\sigma(n)|) + F(-\varepsilon/|\sigma(n)|) = \infty, \quad \textit{for some } \varepsilon \in \mathbb{Q}^+; \tag{2.13}$$

$$\limsup_{n\to\infty} |\sigma(n)\xi(n+1)| > \varepsilon, \quad \textit{a.s.} \tag{2.14}$$

In the general case, by combining Lemma 3 and Proposition 1, we get the following necessary and sufficient conditions on F and $\sigma(n)$ such that $X(n) \to 0$ a.s. as $n \to \infty$.

Theorem 5. *Let $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ be a $\mathbb{R}^d \times \mathbb{R}^r$–valued sequence. Suppose the sequence of random variables $\xi = \{\xi(n) \in \mathbb{R}^r : n \in \mathbb{N}\}$ obeys* (2.2)– (2.5). *Let A and R obey (2.8) and let X be defined by* (2.1). *Then the following are equivalent:*

(a) (2.10) *holds, where $F_{n,i}$ and $F_{n,i,j}$ are as defined in Lemma 3;*
(b) $\lim_{n\to\infty} X(n) = 0$, *a.s.*

In the scalar case, by combining Lemma 4 and Proposition 1, we may establish a more easily–expressed result which characterises asymptotic stability.

Theorem 6. *Let $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ be a real sequence. Suppose $\xi = \{\xi(n) : n \geq 1\}$ is a sequence of independent and identically distributed random variables with the same distribution function F such that $\mathbb{E}[\xi(n)] = 0$, $\mathbb{E}[\xi(n)^2] = 1$. Let A and R obey (2.8) and let X be defined by* (2.1).

(a) *If F and σ obey* (2.11), *then* $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] = 1$.
(b) *If F and σ do not obey* (2.11), *then* $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] = 0$.

§3. Applications and extensions of main results

If we additionally assume some monotonicity in $n \mapsto \sigma^2(n)$, we can recover the critical rate of convergence in the case when the white noise process is Gaussian. We first state the result in the scalar case and then indicate how the result can be generalized to the finite dimensional case.

Corollary 7. *Let $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ be a real sequence. Let $\xi = \{\xi(n) : n \in \mathbb{N}\}$ be a sequence of independently and identically distributed standard normal random variables. Let A and R obey (2.8) and let X be defined by* (2.1).

(a) *If*

$$\sum_{n=1}^{\infty} \sigma(n) \exp\left(-\frac{\varepsilon^2}{2\sigma^2(n)}\right) < \infty, \quad \textit{for all } \varepsilon \in \mathbb{Q}^+, \tag{3.1}$$

then $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] = 1$.

(b) *If* (3.1) *does not hold, then* $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] = 0$.

If, moreover, $n \mapsto \sigma^2(n)$ *is non–increasing, then the following are equivalent:*

(A) $\lim_{n\to\infty} \sigma^2(n) \log n = 0$;
(B) $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] > 0$;
(C) $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] = 1$.

A proof of the following finite–dimensional analogue of Corollary 7 is not given, but left to the interested reader as an exercise. The key observations in proving this result are that (i) the i–th component of $\sigma(n)\xi(n+1)$, which is $Y^{(i)}(n) = \sum_{j=1}^{r} \sigma_{ij}(n)\xi_j(n+1)$ is normally distributed $\mathcal{N}\left(0, \sum_{j=1}^{r} \sigma_{ij}^2(n)\right)$, and (ii) for each $i = 1, \dots, d$, $Y^{(i)} = \{Y^{(i)}(n) : n \in \mathbb{N}\}$ is a sequence of independent random variables.

Corollary 8. *Let* $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ *be a* $\mathbb{R}^d \times \mathbb{R}^r$*–valued sequence. Let* $\xi = \{\xi(n) \in \mathbb{R}^r : n \in \mathbb{N}\}$ *be a sequence of random vectors obeying* (2.2)–(2.5) *such that* $\xi_j(n) = \langle \xi(n), \mathbf{e}_j \rangle$ *are standard normal random variables,* $j = 1, \dots, r$. *Define* $\sigma_i(n) \geq 0$ *by* $\sigma_i^2(n) = \sum_{j=1}^{r} \sigma_{ij}^2(n)$, $i = 1, \dots, d$. *Let* A *and* R *obey (2.8) and let* X *be defined by* (2.1).

(a) *If*

$$\sum_{i=1}^{d} \sum_{n=1}^{\infty} \sigma_i(n) \exp\left(-\frac{\varepsilon^2}{2\sigma_i^2(n)}\right) < \infty, \quad \text{for all } \varepsilon \in \mathbb{Q}^+, \tag{3.2}$$

then $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] = 1$.

(b) *If* (3.2) *does not hold, then* $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] = 0$.

If, moreover, each of the maps $n \mapsto \sigma_i^2(n)$, $i = 1, \dots, d$, *is non–increasing, then the following are equivalent:*

(A) $\lim_{n\to\infty} \|\sigma(n)\|^2 \log n = 0$;
(B) $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] > 0$;
(C) $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] = 1$.

These two results are analogous to those found in [1], which studies the asymptotic behaviour of linear continuous–time stochastic Volterra equations of Itô–type where the driving semimartingale is Brownian motion. The necessary and sufficient conditions on the rate of decay of the noise intensity σ given in (A) is the same as that required in [1]. The rate of decay $\sigma^2(n) \log n \to 0$ has also been shown to be necessary and sufficient for almost sure asymptotic stability in a class of nonlinear stochastic delay–differential equations studied in [2]. This logarithmic decay condition was shown to be necessary for a class of non–delay gradient dynamical system in Chan and Williams [5].

Now give a result for a scalar equation in which the distribution of ξ has power–law decay in the tails, with the same rate of decay being present in each tail.

Corollary 9. *Let* $\xi = \{\xi(n) : n \in \mathbb{N}\}$ *be a sequence of independently and identically distributed random variables such that* $\mathbb{E}[\xi(n)] = 0$, $\mathbb{E}[\xi(n)^2] = 1$ *and whose common distribution function* F *obeys*

$$\lim_{x \to -\infty} F(x)|x|^\beta = c_1, \quad \lim_{x \to \infty} [1 - F(x)]x^\beta = c_2,$$

for some $\beta > 2$, *and positive* c_1 *and* c_2. *Let* A *and* R *obey (2.8) and let* X *be defined by* (2.1).

(a) *If* $\sum_{n=1}^{\infty} |\sigma(n)|^\beta < \infty$, *then* $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] = 1$.
(b) *If* $\sum_{n=1}^{\infty} |\sigma(n)|^\beta = \infty$, *then* $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] = 0$.

To prove the results in this section, the following auxiliary results are needed.

Lemma 10. *Let* $\xi = \{\xi(n) : n \in \mathbb{N}\}$ *be a sequence of independently and identically distributed random variables, with common distribution function* F *which does not have compact support. If* (2.12) *holds, then* $\lim_{n\to\infty} \sigma(n) = 0$.

Lemma 11. *Suppose that* $(a(n))_{n\geq 0}$ *is a non–negative and non–increasing sequence such that* $\sum_{n=0}^{\infty} a(n) < \infty$. *Then* $na(n) \to 0$ *as* $n \to \infty$.

§4. Proofs

4.1. Proof of Proposition 1

We show first that (a) implies (b). If $\Sigma(n+1) := \sigma(n)\xi(n+1) \to 0$ as $n \to \infty$, a.s., it follows immediately from the fact that $R \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$ that $(R * \Sigma)(n) \to 0$ as $n \to \infty$, a.s. Moreover, because $R \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$, we have $R(n) \to 0$ as $n \to \infty$. Therefore by (2.7) we have that $X(n) \to 0$ as $n \to \infty$, a.s.

To show (b) implies (a), suppose $X(n) \to 0$ as $n \to \infty$ a.s. Rewriting (2.1) gives $\sigma(n)\xi(n+1) = X(n+1) - X(n) - \sum_{k=0}^{n} A(n-k)X(k)$. Since $A \in \ell^1(\mathbb{R}^{d\times d})$, we have $(A * X)(n) \to 0$ as $n \to \infty$, a.s. Therefore, $\sigma(n)\xi(n+1) \to 0$ as $n \to \infty$, a.s.

4.2. Proof of Proposition 2

Consider $Y^{(i)}(n) = \sum_{j=1}^{r} \sigma_{ij}(n)\xi_j(n+1)$, $i = 1, \ldots, d$. Then $\sigma(n)\xi(n+1) \to 0$ as $n \to \infty$, a.s. once $Y^{(i)}(n) \to 0$ a.s. as $n \to \infty$. Evidently $\mathbb{E}[Y^{(i)}(n)] = 0$, while

$$\mathbb{E}[Y^{(i)}(n)^2] \le \sum_{j=1}^{r}\sum_{l=1}^{r} |\sigma_{ij}(n)||\sigma_{il}(n)| = \left(\sum_{j=1}^{r} |\sigma_{ij}(n)|\right)^2 .$$

This estimate, Chebyshev's inequality, and $|\sigma(n)\xi(n+1)|_1 = \sum_{i=1}^{d} |Y^{(i)}(n)|$ give

$$\sum_{n=1}^{\infty} \mathbb{P}[|\sigma(n)\xi(n+1)|_1 > \varepsilon] \le \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\sum_{i=1}^{d}\left(\sum_{j=1}^{r} |\sigma_{ij}(n)|\right)^2, \quad \text{for any } \varepsilon > 0.$$

The righthand side can be bounded by an n–independent multiple of $\sum_{n=1}^{\infty} |\sigma(n)|_F^2$, which is finite by hypothesis. The result now follows from the Borel–Cantelli lemma.

4.3. Proof of Lemma 3

Fix $\varepsilon \in \mathbb{Q}^+$. Define $Y^{(i)}(n) = \sum_{j=1}^{r} \sigma_{ij}(n)\xi_j(n+1)$, $i = 1, \ldots, d$. Proving $\sigma(n)\xi(n+1) \to 0$ as $n \to \infty$ a.s. is equivalent to proving that $Y^{(i)}(n) \to 0$ as $n \to \infty$ a.s. for each $i = 1, \ldots, d$. Define $A_{n,\varepsilon}^{(i)} = \{\omega : |Y^{(i)}(n)| > \varepsilon\}$. Then $\mathbb{P}[\bar{A}_{n,\varepsilon}^{(i)} \text{ ev.}] = 1$ for every $\varepsilon \in \mathbb{Q}^+$ is equivalent to $\lim_{n\to\infty} Y^{(i)}(n) = 0$, a.s. Indeed, since for each fixed i the sequence $Y^{(i)} = \{Y^{(i)}(n) : n \in \mathbb{N}\}$ is an independent sequence, by the Borel–Cantelli lemma $\mathbb{P}[\bar{A}_{n,\varepsilon}^{(i)} \text{ ev.}] = 1$ for all $\varepsilon \in \mathbb{Q}^+$ is equivalent to $\sum_{n=1}^{\infty} \mathbb{P}[A_{n,\varepsilon}^{(i)}] < \infty$, for all $\varepsilon \in \mathbb{Q}^+$. Hence $\sigma(n)\xi(n+1) \to 0$ as $n \to \infty$ a.s. is equivalent to $\sum_{n=1}^{\infty} \mathbb{P}[|Y^{(i)}(n)| > \varepsilon] < \infty$ for all $\varepsilon \in \mathbb{Q}^+$. Let $F_{n,i}$ be the distribution function of $Y^{(i)}(n)$, F the distribution function of ξ_i, and $F_{n,i,j}$ the distribution function of $\sigma_{ij}(n)\xi_j(n+1)$. Then

$$F_{n,i,j}(x) = \mathbb{P}[\sigma_{ij}(n)\xi_j(n+1) \le x] = \begin{cases} F(x/|\sigma_{ij}(n)|), & \sigma_{ij}(n) > 0, \\ 1 - F(-x/|\sigma_{ij}(n)|), & \sigma_{ij}(n) < 0. \end{cases}$$

If $\sigma_{ij}(n) = 0$, $F_{n,i,j}(x) = 1$ if $x \ge 0$ and $F_{n,i,j}(x) = 0$ if $x < 0$. Thus $F_{n,i,j}(x) = \chi_{[0,\infty)}(x)$. Since for each fixed n, $(\xi_j(n))_{j=1}^{r}$ are independent random variables, $F_{n,i} = F_{n,i,1} * \ldots * F_{n,i,r}$, so $\mathbb{P}[|Y^{(i)}(n)| > \varepsilon] = 1 - F_{n,i}(\varepsilon) + F_{n,i}(-\varepsilon)$, completing the proof.

4.4. Proof of Lemma 4

Define $Y(n) = \sigma(n)\xi(n+1)$ and $A_n^\varepsilon = \{\omega \in \Omega : |Y(n,\omega)| > \varepsilon\}$. Thus, if $\sigma(n) \neq 0$, we have $\mathbb{P}[A_n^\varepsilon] = 1 - F(\varepsilon/|\sigma(n)|) + F(-\varepsilon/|\sigma(n)|)$, and $\mathbb{P}[A_n^\varepsilon] = 0$ if $\sigma(n) = 0$. To prove part (a), by the Borel–Cantelli Lemma and independence of the A_n^ε, $\mathbb{P}[\bar{A}_n^\varepsilon \text{ ev.}] = 1$ is equivalent to $\sum_{n=1}^\infty \mathbb{P}[A_n^\varepsilon] < \infty$. Therefore

$$\limsup_{n\to\infty} |Y(n)| \leq \varepsilon \text{ on } \Omega_\varepsilon, \text{ with } \mathbb{P}[\Omega_\varepsilon] = 1 \text{ is equivalent to } \sum_{n=1}^\infty \mathbb{P}[A_n^\varepsilon] < \infty,$$

so by considering this equivalence for all $\varepsilon \in \mathbb{Q}^+$, we conclude that (2.11) is equivalent to (2.12). To prove part (b), the independence of the A_n^ε and the Borel–Cantelli lemma imply that $\mathbb{P}[\bar{A}_n^{\varepsilon_0} \text{ ev.}] = 0$ for some $\varepsilon > 0$ is equivalent to $\sum_{n=1}^\infty \mathbb{P}[A_n^{\varepsilon_0}] = \infty$ for some $\varepsilon_0 > 0$. Thus $\mathbb{P}[A_n^{\varepsilon_0} \text{ i.o.}] = 1$ for some $\varepsilon_0 > 0$ is equivalent to $\sum_{n=1}^\infty \mathbb{P}[A_n^{\varepsilon_0}] = \infty$ for some $\varepsilon_0 > 0$. Therefore we conclude that (2.13) is equivalent to (2.14).

4.5. Proof of Corollary 7

Let the common distribution function be F. Then

$$\lim_{x\to-\infty} F(x)|x|e^{\frac{1}{2}x^2} = (2\pi)^{-1/2}, \lim_{x\to-\infty} [1 - F(x)]|x|e^{\frac{1}{2}x^2} = (2\pi)^{-1/2}.$$

To prove part (a), notice by Lemma 10 that we must have $\sigma(n) \to 0$ as $n \to \infty$. Now, let ε be a fixed positive rational number. Therefore, the estimates

$$\lim_{n\to\infty} \frac{F(-\varepsilon/|\sigma(n)|)}{\frac{|\sigma(n)|}{\varepsilon} \cdot e^{-\frac{1}{2}\frac{\varepsilon^2}{\sigma^2(n)}}} = \frac{1}{\sqrt{2\pi}}, \quad \lim_{n\to\infty} \frac{1 - F(\varepsilon/|\sigma(n)|)}{\frac{|\sigma(n)|}{\varepsilon} \cdot e^{-\frac{1}{2}\frac{\varepsilon^2}{\sigma^2(n)}}} = \frac{1}{\sqrt{2\pi}},$$

hold and so

$$\lim_{n\to\infty} [1 - F(\varepsilon/|\sigma(n)|) + F(-\varepsilon/|\sigma(n))]\varepsilon|\sigma(n)|^{-1} e^{\frac{1}{2}\frac{\varepsilon^2}{\sigma^2(n)}} = 2(2\pi)^{-1}. \tag{4.1}$$

Thus

$$\sum_{n=1}^\infty |\sigma(n)|\varepsilon^{-1} \exp\{-2^{-1}\varepsilon^2\sigma^{-2}(n)\} < \infty \tag{4.2}$$

implies $\mathbb{P}[X(n) \to 0 \text{ as } n \to \infty] = 1$. To prove part (b), if the sum in (4.2) is infinite for some $\varepsilon \in \mathbb{Q}^+$, then $\limsup_{n\to\infty} |\sigma(n)\xi(n+1)| > 0$ a.s., and so a brief perusal of the proof of Proposition 1 reveals that $\mathbb{P}[\lim_{n\to\infty} X(n) = 0] = 0$. It is easily seen that the implication holds under the hypothesis (a) above.

Now suppose that $n \mapsto \sigma^2(n)$ is non–increasing. First, note that $\sigma^2(n)\log n \to 0$ as $n \to \infty$ implies (3.1). By part (a), this implies that $X(n) \to 0$ a.s. as $n \to \infty$. Thus (A) implies (C), and (C) obviously implies (B). Finally, we show that (B) implies (A). According to (B) and the proof of Proposition 1 it follows that $\mathbb{P}[\lim_{n\to\infty}\sigma(n)\xi(n+1) = 0] > 0$. However, as $(\xi(n))_{n=1}^{\infty}$ is a sequence of independent random variables, this event is in the tail σ–field of the ξ and as it is of positive probability, by the Kolmogorov zero–one law it is almost sure. Thus by Lemma 4, and (4.1), (4.2) holds for every $\varepsilon \in \mathbb{Q}^+$. Now, fix $\varepsilon \in \mathbb{Q}^+$. Then the sequence $(a_\varepsilon(n))_{n\ge 0}$ defined by $a_\varepsilon(n) = \varepsilon^{-1}|\sigma(n)|\exp\left(-\varepsilon^2/(2\sigma^2(n))\right)$, $n > N_1(\varepsilon)$ is non–increasing. By Lemma 11, $\lim_{n\to\infty} n a_\varepsilon(n) = 0$, or $\lim_{n\to\infty} n|\sigma(n)|\exp\left(-\varepsilon^2/(2\sigma^2(n))\right) = 0$. Thus

$$\lim_{n\to\infty}\sigma^{-2}(n)\left(\sigma^2(n)\log n + 2^{-1}\sigma^2(n)\log\sigma^2(n) - 2^{-1}\varepsilon^2\right) = -\infty.$$

Since $\lim_{x\to\infty} x\log x = 0$, we have $\limsup_{n\to\infty}\left[\sigma^2(n)\log n - \frac{1}{2}\varepsilon^2\right] \le 0$. However, this inequality holds for every $\varepsilon \in \mathbb{Q}^+$, and so it follows that $\lim_{n\to\infty}\sigma^2(n)\log n = 0$.

4.6. Proof of Corollary 9

To do this suppose first that $\sum_{n=1}^{\infty}|\sigma(n)|^\beta < \infty$. Then $\lim_{n\to\infty}\sigma(n) = 0$. Let ε be any number in $\mathbb{Q}^+$. Since $\varepsilon/\lim_{n\to\infty}\sigma(n) = \infty$,

$$\begin{aligned}\lim_{n\to\infty}|\sigma(n)|^{-\beta}F(-\varepsilon/|\sigma(n)|) &= c_1\varepsilon^{-\beta},\\ \lim_{n\to\infty}|\sigma(n)|^{-\beta}[1 - F(\varepsilon/|\sigma(n)|)] &= c_2\varepsilon^{-\beta},\end{aligned} \tag{4.3}$$

and so (2.11) holds. Then by Theorem 6, $\lim_{n\to\infty}X(n) = 0$ a.s. On the other hand, consider the case when $\sum_{n=1}^{\infty}|\sigma(n)|^\beta = \infty$. Suppose that $\limsup_{n\to\infty}|\sigma(n)| > 0$. Then $\limsup_{n\to\infty}|\sigma(n)\xi(n+1)| = \infty$ a.s., and so $\mathbb{P}[\lim_{n\to\infty}X(n) = 0] = 0$. If $\limsup_{n\to\infty}|\sigma(n)| = 0$, we can repeat the calculations above to get (4.3). Since the denominator is not summable, we have $\sum_{n=0}^{\infty}[F(-\varepsilon/|\sigma(n)|) + 1 - F(\varepsilon/|\sigma(n)|)] = \infty$. Thus by Theorem 6, we have $\mathbb{P}[\lim_{n\to\infty}X(n) = 0] = 0$, whence the result.

4.7. Proof of Lemma 10

Suppose to that $\limsup_{n\to\infty}\sigma^2(n) = c_0 > 0$. Then, for every $c \in (0, c_0)$ there is a sequence $m_n^{(c)} \to \infty$ such that $\sigma^2(m_n^{(c)}) > c$ for all $n > N(c)$. Since $\limsup_{n\to\infty}|\xi(m_n^{(c)}+1)| = \infty$, a.s., we have $\limsup_{n\to\infty}\sigma^2(m_n^{(c)})\xi^2(m_n^{(c)}+1) = \infty$ a.s., which contradicts the assumption. Hence $\sigma^2(n) \to 0$ as $n \to \infty$.

4.8. Proof of Lemma 11

By the monotonicity of a, the inequalities $2^{j-1}a(2^j) \le \sum_{l=1}^{2^{j-1}} a(2^{j-1} +l)$ hold for each j, so we get

$$\sum_{j=1}^{\infty} a(j) \ge a(1) + \sum_{j=1}^{\infty} 2^{j-1}a(2^j) = a(1) + \frac{1}{2}\sum_{j=1}^{\infty} 2^j a(2^j).$$

Therefore $2^j a(2^j) \to 0$ as $j \to \infty$. This means that for every $\varepsilon > 0$ there is $J = J(\varepsilon)$ such that $j > J$ implies $2^j a(2^j) < \varepsilon$. Consider now $N = 2^{J+1}$. Let $n > N$. Clearly, there is $j > 0$ such that $2^j \le n < 2^{j+1}$. Then $2^j > n/2 > N/2 = 2^J$, so $j > J$. Hence $na(n) < 2^{j+1}a(2^j) < 2\varepsilon$, for all $n > N$.

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