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Analysis of an evolutionary variational inequality arising in elasticity quasi-static contact problems

Nicolae Pop

Abstract.

In this paper, we present the strong and the variational form of a dynamic contact problem with friction. We derive the result and obtain an incremental formulation by space discretization with finite element method and by time discretization with finite difference method of the dynamic contact problem. As well we describe the solution strategies for spatially discrete system and the condition for stability of the solution. Dificulties caused by the discontinuity of the Coulomb's friction law (passage from sliding to adhesion), are tried to be passed over using the Newton-Raphson iterative techniques, and their success depends also on the small value of the parameter , from the regularized friction law.

§1. Introduction

The present paper is a continuation of the analysis presented in [10], which consists in a numerical analysis of a dynamic contact problem with friction presented in [11]. The problem is intended to model the physical situation of two elastically deforming bodies that come into contact with friction and are obeying the normal compliance law, where the normal stress is depending on statistical parameters of the contact surface's profile.

First we give a classical and a variational formulation of the continuous dynamic contact problem. Afterwards, we derive the result and

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obtain an incremental formulation by space discretization with finite element method and by time discretization with finite difference method of the dynamic contact problem and we present the solution strategies for spatially discrete system and the condition for stability of the solution.

In the present paper we compare the quasi-static contact problem with the dynamic contact problem. In the case of the quasi-static contact problem the inertial forces are neglected, but for the dynamic contact problem case, the inertial terms cannot be neglected.

§2. Formulation of the problem

Let us consider two linear elastic bodies that at a given time $t = 0$ occupy domains Ω^1 and $\Omega^2 \subset R^d$, respectively, where $d = 2$ or $d = 3$. The boundary of each body, is divided into three subregions such that

$$\partial\Omega^1 = \Gamma^1 = \overline{\Gamma}_U^1 \cup \overline{\Gamma}_N^1 \cup \overline{\Gamma}_C^1 \text{ and } \partial\Omega^2 = \Gamma^2 = \overline{\Gamma}_U^2 \cup \overline{\Gamma}_N^2 \cup \overline{\Gamma}_C^2,$$

which are topologically open, and disjoint, only Γ_C^1 and Γ_C^2 being accepted to have common points, and $mes(\Gamma_N^i) > 0$, $i = 1, 2$.

The displacement $\overline{u}(t, x)$ will be prescribed on $\Gamma_U = \overline{\Gamma}_U^1 \cup \overline{\Gamma}_U^2$ and traction $\overline{h}(t, x)$ is to be given on $\Gamma_N = \Gamma_N^1 \cup \Gamma_N^2$. For the beginning, the boundary $\Gamma_N = \Gamma_N^1 \cup \Gamma_N^2$ is considered without tensions. At the same time the stress vector $\sigma^{(n)}(u)$ is defined, oriented towards the exterior of the boundary $\partial\Omega = \partial\Omega^1 \cup \partial\Omega^2$. The initial displacement $u(0, x) = u_0(x)$, the initial velocity $\dot{u}(0, x) = u_1(x)$ and the density of body force f are also given.

So long as the two bodies do not touch each other, the field of the displacements will be the solution of a boundary value problem of the differential equations of elastodynamics. If the two bodies touch one another, then in the contact boundary there are forces strong enough to prevent the interaction (penetration) of the two bodies. The condition that needs to be expressed in order to describe this process is called *the contact condition.* Beside these forces there may appear in the contact area friction forces as well, which are described by a law of friction. The contact problem in a time interval $[0, t_E]$ with $t_E > 0$, has the following form:

The elastodynamic equation on $\Omega = \Omega_1 \cup \Omega_2$

$$\rho\ddot{u}(t, x) - \sigma_{ij,j}(u(t, x)) = f(t, x) \text{ on } [0, t_E] \times \Omega. \tag{2.1}$$

The boundary conditions

$$u(t, x) = \bar{u}(t, x) \text{ on } [0, t_E] \times \Gamma_U \tag{2.2}$$

$$\sigma^{(n)}(u)(t,x) = \bar{h}(t,x) \text{ on } [0,t_E] \times \Gamma_N \tag{2.3}$$

and the initial conditions are $u(0,x) = u_0(x)$, the initial velocity $\dot{u}(0,x) = u_1(x)$. The condition of non-penetrating bodies will be understood as a geometric *contact condition.* It is approximated by the inequation $u_N^R(x,t) \leq g(x)$, where $g(x)$ is the initial gap between the bodies and u_N^R is the relative displacement between the contact boundaries. We shall use the following notations for the normal and tangential components of the displacements and of the stress vector:

$u_N = u \cdot n = u_i n_i^1$, $u_T = u - u_N n^1$, $\sigma_N = \sigma_{ij} n_i^1 n_j^1$, $1 \leq i,j \leq d$ and n^1 is the outward normal unit vector on Γ^1.

In the normal interface response case, we have

$$\sigma_N(u) = -c_N(u_N^R - g)_+^{m_N}, \text{ on} \Gamma_N \times (0,t_E) \tag{2.4}$$

if

$$\dot{u}_N^R \leq g \Rightarrow \quad |\sigma_T(u)| \leq c_T(u_N^R - g)_+^{m_T}$$

else

$$\sigma_T(u) = c_T(u_N^R - g)_+^{m_T}$$

Where c_N, m_N, c_T, m_T are material constants depending on the interface properties and b_+={b,0}. The following steps, similar with those of Oden and Martins [9], the nonlinear elastodynamics problem can be formally equivalent to the following variational problem:

Problem P1. *Find the function u:[0,t]→ V s.t.*

$$\langle \ddot{u}(t), v - \dot{u}(t) \rangle + a(u(t), v - \dot{u}(t)) + \langle P(u(t)), v - \dot{u}(t) \rangle + \tag{2.5}$$

$$+ j(u(t), v) - j(u(t), \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle, \ \forall v \in V$$

with the initial conditions:

$$u(0,x) = u_0(x) \, and \dot{u}(0,x) = u_1(x). \tag{2.6}$$

We have assumed here, for simplicity, that $\rho \equiv 1$. The following notations and definitions were also used:

$$V = \left\{ v \in \left[H^1(\Omega)\right]^d \mid v = 0 \ a.e. \ on \ \Gamma_C \right\}, \tag{2.7}$$

the space of admissible displacements (velocities);

$$a : V \times V \to \mathrm{R}, \quad a(u,v) = \int_\Omega C_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) dx, \tag{2.8}$$

the virtual work produced by the action of the stress $\sigma_{ij}(u)$ on the strains $\varepsilon_{ij}(v)$;

$$P : V \to V', \quad \langle P(u), v\rangle = \int_{\Gamma_C} c_N (u_N^R - g)_+^{m_N} v_N ds, \tag{2.9}$$

the virtual work produced by the normal pressure on the displacement (velocity) v;

$$j : V \times V \to \mathrm{R}, \quad j(u,v) = \int_{\Gamma_C} c_T \left(u_N^R - g\right)_+^{m_T} |v_T^R| ds, \tag{2.10}$$

the virtual power produced by the frictional force on the velocity v ;

$$f(t) \in V', \ \langle L, v\rangle \equiv \langle f(t), v\rangle = \int_\Omega f(t)\, v dx + \int_{\Gamma_C} \overline{h}\gamma(v)\, ds, \tag{2.11}$$

the virtual work produced by the external forces.

Here $\langle \cdot, \cdot \rangle$ denotes duality pairing on $V \times V'$ where V' is the topological dual of of V; γ is the trace operator mapping from $\left[H^1(\Omega)\right]^d$ onto $\left[H^{\frac{1}{2}}(\Gamma)\right]^d$ which may be decomposed into normal component $\gamma_N(v)$ and tangential component $\gamma_T(v)$. For simplicity of notation, the later are denoted as v_N and v_T, respectively. We also observe that the boundary integrals on Γ_C are well defined for $1 \leq m_N,\ m_T \leq 3$ if d=3 and for $1 \leq m_N,\ m_T < \infty$ if d=2. These conditions on m_N and m_T basically com from the embedding theorem which states that for $v \in \left[H^1(\Omega)\right]^d$,$\gamma(v) \in \left[L^q(\Gamma_C)\right]^d$ with $2 \leq q \leq \infty$ for d=2, and with $2 \leq q \leq 4$ for d=3.

One of the directions for the approximation of the Problem P1 is a family of regularized problems which lead insteads to of a variational equality. For this, it approximates the functional j:$V \times V \to \mathrm{R}$ which is nondifferentiable in the second argument (velocity) by a family of functionals j_ε convex and differentiable on the second argument:

$$j_\varepsilon : V \times V \to \mathrm{R}, \quad j_\varepsilon(u,v) = \int_{\Gamma_C} c_T \left(u_N^R - g\right)_+^{m_T} \Psi_\varepsilon\left(v_T^R\right) ds, \tag{2.12}$$

where the function Ψ_ε:$[L^q(\Gamma_C)]^d \to L^q(\Gamma_C)$, is an approximation of the function $|\cdot|$: $[L^q(\Gamma_C)]^d \to L^q(\Gamma_C)$ and is defined for $\varepsilon > 0$, $\xi \in [L^q(\Gamma_C)]^d$ and a.e. $x \in \Gamma_C$, according to

$$\Psi_\varepsilon(\xi) = \begin{cases} \varepsilon \left|\frac{\xi}{\varepsilon}\right|^2 \left(1 - \frac{1}{3}\left|\frac{\xi}{\varepsilon}\right|\right) & if \ \ |\xi(x)| \leq \varepsilon \\ \varepsilon\left(\left|\frac{\xi}{\varepsilon}\right| - \frac{1}{3}\right) & if \ \ |\xi(x)| > \varepsilon \end{cases} \tag{2.13}$$

The regularized form of Problem P1, is:

Problem P1ε. *Find the functions* $u_\varepsilon : [0, t_E] \to V$ *s.t.*

$$(2.14)\ \langle \ddot{u}_\varepsilon(t), v\rangle + a(u_\varepsilon(t), v) + \langle P(u_\varepsilon(t)), v\rangle + \langle j_\varepsilon(u_\varepsilon(t), \dot{u}_\varepsilon(t)), v\rangle = \langle f(t), v\rangle, \ \forall v \in V$$

with the initial conditions:

$$(2.15)\qquad u_\varepsilon(0, x) = u_0(x) \ \text{ and } \ \dot{u}_\varepsilon(0, x) = u_1(x).$$

We observe that now we have a variational equation instead of a variational inequality. However, the regularized friction conditions are now the form:

$$(2.16)\qquad \sigma_T = -c_T \left(u_N^R - g\right)_+^{m_T} F_\varepsilon\left(\dot{u}_T^R\right)$$

with

$$F_\varepsilon\left(\dot{u}_T^R\right) = \begin{cases} \left(2 - \left|\frac{\dot{u}_T^R}{\varepsilon}\right|\right)\frac{\dot{u}_T^R}{\varepsilon} & if\ \dot{u}_T^R \le \varepsilon \\ \frac{\dot{u}_T^R}{\left|\dot{u}_T^R\right|} & if\ \dot{u}_T^R > \varepsilon \end{cases}.$$

where F_ε is the friction coefficient. The choice of ε will be dictated only by the desired proximity of the solutions of the Problems P1 and P1ε and the corresponding computational costs associated.

§3. Spatially semi-discrete approximations of the evolutionary variational equality

Two types of semi-discrete approximation scheme can be developed. In the first type, when we replace the infinite-dimensional space V by a finite-dimensional subspace V_h (the element finite spaces), leading to a finite-dimensional system of ordinary differential equations (3.3). Such approximation schemes can be termed as spatially semi-discrete scheme. In the second type, we replace the time derivatives by finite differences that lead to elliptic variational equations over infinite-dimensional space V at each time step. Such approximation schemes can be termed at temporally semi-discrete schemes. In this paper, we consider the first type, spatially semi-discrete schemes, because after a spatially semi-discrete discretization, we can employ some ODE solvers for mathematical solution, see [8].

Using standard finite element procedures, approximate version of Problem P1ε can be constructed in finite-dimensional subspaces V_h ($\subset V$

$\subset V'$). For certain (h) the approximate displacements, velocities and accelerations at each time t are elements of

$$V_h, \quad r^h(t), \quad \dot{r}^h(t), \quad \ddot{r}^h(t) \in V_h.$$

Within each element Ω_h^e $(e = 1, \ldots, N_h)$ the components of the displacements, velocities and accelerations are expressed in the form:

$$\begin{gathered} r_k^h(t,x) = \sum_{I=1}^{N_e} r_k^I(t) N_I(x), \; \dot{r}_k^h(t,x) = \sum_{I=1}^{N_e} \dot{r}_k^I(t) N_I(x), \\ \ddot{r}_k^h(t,x) = \sum_{I=1}^{N_e} \ddot{r}_k^I(t) N_I(x), \end{gathered} \tag{3.1}$$

where $k = 2$ *or* 3, N_e =number of node of the element, $r_k^I(t)$, $\dot{r}_k^I(t)$, $\ddot{r}_k^I(t)$ are the nodal values of the displacements, velocities and accelerations, at the time t and N_I is the element shape function associated with the nodal point. The finite element version of the Problem P21ε with N_h^Ω representing the number of the nodes of finite element mesh of Ω, is equivalent to the following matrix problem:

Problem MP31ε^h. *Find the function* $r : [0, t_E] \to \mathrm{R}^{d \times N_h^\Omega}$, *s.t.*

$$M\ddot{r}(t) + Kr(t) - P(r(t)) + J(r(t), \dot{r}(t)) = F(t), \tag{3.2}$$

with the initial conditions $r(0) = p_0, \quad \dot{r}(0) = p_1$.

Here we have introduced the following matrix notations:

$r(t)$, $\dot{r}(t)$, $\ddot{r}(t)$: the column vectors of nodal displacements, velocities and accelerations, respectively;

M : standard mass matrix, K: standard stiffness matrix, $F(t)$: consistent nodal exterior forces vector, $P(r(t))$: vector of consistent nodal forces on Γ_C $J(r(t), \dot{r}(t))$: vector of consistent nodal friction forces on Γ_C, p_0, p_1: initial nodal displacement, velocity. The components of the element vector ${}^{(e)}P$ are of the form: ${}^{(e)}P = -\int_{(e)\Gamma_C^2} \sigma_N \cdot n \cdot N_I ds$ and the components of the element vector ${}^{(e)}J$ are of the form: ${}^{(e)}J = -\int_{(e)\Gamma_C^2} \sigma_T \cdot N_I ds$.

In order to obtain the components of the element vector P and J is used a contact finite element, see [10], [11].

The algorithms that we shall use for solving the discrete dynamical system involve variants of standard schemes used in nonlinear structural dynamics calculations: the Newmark-type algorithm or the central-difference scheme.

Using the Newton-Raphson method to solve the variational equation obtained at time t_k introduced, into the variational equation P31ε^h,

the relations which define the Newmark-type algorithm or the central-difference scheme the following system of algebraic linear equations is obtained to solve at each iteration.

Let us consider a partition of the time interval $I = \bigcup_{k=1}^{N} [t_{k-1}, t_k]$ with $0 = t_0 < t_1 < \cdots < t_N = t_E$, we denote $\Delta t = t_k - t_{k-1}$ for the length of the sub-interval $[t_{k-1}, t_k]$. Two general classes are generally of interest. In the quasistatic case the inertial loads are taken to be insignificant in comparison with the applied loads and internal forces. In the dynamic case, the inertial terms are restored and issued associated with temporal accuracy and stability that come to the fore (must analysis).

If we denote in ordinary differential equation (3.2), $K_N \equiv K - P + J$ (where the matrices P and J are nonlinear), $d_k \equiv r(t_k)$, $v_k \equiv \dot{r}(t_k)$, $a_k \equiv \ddot{r}(t_k)$, then (3.2) becomes:

$$Ma(t) + K_N(d(t)) = F(t) \tag{3.3}$$

For the analysis of this problem, must begin with *quasistatic case*:

$$K_N(d(t)) = F(t) \tag{3.4}$$

subject to only one initial condition $d(0) = d_0$.

Note that the time variable t may correspond to real time, but doesn't need to have physical meaning for rate independent behavior. For example, it is common for t to be taken as a generic parameterization for the applied loading on the system. With the above notation, the incremental load approach attempts (in each increment Δt) to find d_{k+1}, given the solution d_k at time level t_k, such that

$$K_N(d_{k+1}) = F(t_{k+1}). \tag{3.5}$$

By introducing the concept of a residual vector $R(d_{k+1})$:

$$R(d_{k+1}) := F(t_{k+1}) - K_N(d_{d+1}) \tag{3.6}$$

Solution of (3.5) therefore amounts to the finding of the root of the equation

$$R(d_{k+1}) = 0. \tag{3.7}$$

The physical meaning that is starting with an initial equilibrium state at t_k, so that $R(d_k) = 0$, we introduce a prescribed load increment $\Delta F = F(t_{k+1}) - F(t_k)$, and attempt to find that displacement increment $d_{k+1} - d_k$ that will restore equilibrium (eq. (3.5)).

In *dynamic case* inertial terms can not be neglected, and we note that it is possible in contrast to the quasi-static problem, the variable t in the dynamic case has the interpretation of real time.

Problem P32. Find approximations d_{k+1}, v_{k+1}, a_{k+1} at time t_{k+1} from equation (3.5), with given displacement vector d_k, velocity v_k and acceleration a_k at time t_k.

$$Ma_{k+1} + K_N\left(d_{k+\alpha}\right) = F\left(t_{k+\alpha}\right) \tag{3.8}$$

$$d_{k+\alpha} = \alpha d_{k+1} + (1-\alpha)\, d_k,$$

$$d_{k+1} = d_k + \Delta t v_k + \frac{\Delta t^2}{2}\left[(1-2\beta)\, a_k + 2\beta a_{k+1}\right],$$

$$v_{k+1} = v_k + \Delta t\left[(1-\gamma)\, a_k + \gamma a_{k+1}\right],$$

where α, β and γ are algorithmic parameters that define the stability and accuracy characteristics of the method. In particular, when $\alpha = 1$, the algorithm reduces to the classical Newmark algorithm.

A wide range of algorithms exists corresponding to the different available choices of these parameters, we illustrate two methods: *implicit methods*, which is simply member of the Newmark family obtained by setting $\alpha = 1$, $\beta = 1/4$ and $\gamma = 1/2$ in (3.8). In this case, the integrator is second order accurate and unconditionally stable for linear problems, meaning that the spectral radii of the integrator remains less than 1 in modulus, for any time step Δt. For *explicit methods*, let us take $\alpha = 1$, $\beta = 0$ and $\gamma = 1/2$, substitute into (3.8) and examining the central difference algorithm. This integrator, for explicit method, is second order accurate and only conditionally stable, meaning that linearized stability is only retained when Δt is less than some critical limit. This limit, sometimes called the *Courant stability limit*, can be shown to be as follows $\Delta t \leq {}^{2}/_{\omega_{\max}}$, where $\omega_{\max}$ is the highest modal natural frequency in the mesh. This frequency can be estimated: $\omega_{max} \approx 2\left({}^{c}/_{h}\right)_{\max}$ where c and h are the sound speed and characteristic mesh size, respectively. Finally, we find that $\Delta t \leq \left({}^{h}/_{c}\right)_{\min}$.

This meaning that the time step may by no larger than the amount of time required for a sound wave to traverse the element in the mesh having the smallest transit time. This fact tells us that explicit methods are appropriate only for those problems featuring very high frequency response, for problems featuring low frequency response, the implicit methods are is highly desirable, albeit at the cost of explicit updates in each increment. The implicit and explicit methods are valid only for linear or linearized problems. In this section we give a general framework for solving the nonlinear discrete equations associated with computation

of an unknown state at step t_{k+1}, in either context of a quasi-static (3.7) and dynamic contact problem formulation as in (3.8). In either case, the equation to be solved takes the form $R(d_{k+1}) = 0$, where the R is considered to be a nonlinear function of the solution vector d_{k+1}.

The general concept of a *Newton-Raphson iterative solution technique* consists in the linearization of a nonlinear equation. This concept is applied to equation (3.8) and is defined in iteration j by $R\left(d_{k+1}^{j}\right) + \left[\partial R/\partial d\right]_{d_{k+1}^{j}} \Delta d_k^j = 0$, followed by the update $d_{k+1}^{j+1} = d_{k+1}^{j} + \Delta d_k^j$.

Iteration on j typically continue until the Euclidian norm $\left\|\Delta d_k^j\right\|$ is smaller than some tolerance.

The residual at iteration j , from (3.8) is of the form

$$
\begin{aligned}
(3.9) \qquad R\left(d_{k+1}^{j}\right) &:= F(t_{k+1}) - K_N\left(d_{k+1}^{j}\right) - \frac{4}{\Delta t^2} M d_{k+1}^{j} \\
&+ M\left(a_k + \Delta t v_k + \frac{4}{\Delta t^2} d_k^j\right) = 0
\end{aligned}
$$

which take the form $\left[(4/\Delta t)M + K_L\left(d_{k+1}^{j}\right)\right] \Delta d_k^j = R\left(d_{k+1}^{j}\right)$, where the stiffness matrix $K_L\left(d_{k+1}^{j}\right)$ is given as

$$
K_L\left(d_{k+1}^{j}\right) = \left(\partial K_N/\partial d\right)_{d_{k+1}^{j}} .
$$

We note that a variety of iterative procedures exist as alternatives to the Newton-Raphson nonlinear solution procedure (quasi-Newton, secant methods etc.).

The scheme of solving the linearized dynamic contact problems is the following:

(i) initialization the set of the iterative count $t_k = 0$, $\Delta t = 0$, $k = 0$, $j = 0$, $d_k^j = 0$;

(ii) compute the mass matrix M, the standard stiffness matrix K and a dynamic residual R;

(iii) compute the contact nodal forces P and the contact friction forces J;

(1) compute the normal gap g_N^j;

(2) check for contact finite element status:

IF g_N^j >TOL then out of contact

ELSE in contact. Check for frictional stick or slip contact status

ENDIF

(3) compute total matrix K_L and residual R, this involves to compute

the vectors $\left\{a_{k+1}^j,\ v_{k+1}^j,\ d_{k+1}^j\right\}$, from k=0 to N, (k=N=>$t_{k+1}$=$t_E$);
(IV) check for convergence:
IF$\left|\Delta d_{k+1}^j - \Delta d_{k+1}^{j+1}\right|$ <TOL1 then converge and exit.
ELSE go to step (VI)
(V) update the displacement field $d_{k+1}^{j+1} = d_{k+1}^j + \Delta d_k^j$;
(VI) set j=j+1 and go back to (ii);

§4. Conclusions

1) It is known that the matrix K_N is ill conditioned, if we split the normal and tangential stress from contact boundary, in blocks diagonal matrices, these block matrices contain coefficients of the same size order, and with this procedure we obtain a better conditioned matrix.

2) The discontinuity of the Coulomb's friction law at zero sliding velocity is a major source of computational difficulties in friction problems. Even though, in the algorithms described in this and the previous sections, a regularized form of that law is used, those difficulties cannot be completely avoided. The situation which may arise when using the methods described here with a constant step is the following: in unloading situations (passage from sliding to adhesion) the Newton-Raphson iterative techniques may fail to converge if ε is very small and the step too large. For small values of ε the radius of converge of the iterative scheme used is very small due to the step change in Ψ_ε on the interval $[-\varepsilon, \varepsilon]$.

3) The critical situations arise in transitions from sliding to adhesion, because these are the most important (dramatic) changes. In order to avoid these difficulties that appear in the iterative solutions sequence, we decrease the time step until two successive solutions are not too far apart.

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North University of Baia Mare
Romania

E-mail address: nicpop@gmail.com