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Asymptotic stability conditions for a delay difference system

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Abstract.

In this paper we obtain some necessary and sufficient conditions for the asymptotic stability of the zero solution of a delay difference system

$$x_n = A(x_{n-k} + x_{n-k-l}), \quad n = 0, 1, 2, \ldots,$$

where A is a 2×2 real constant matrix and k and l are positive integers.

§1. Introduction

The purpose of this paper is to establish some necessary and sufficient conditions for the zero solution of a delay difference system

$$x_n = A(x_{n-k} + x_{n-k-l}), \quad n = 0, 1, 2, \ldots, \tag{1}$$

to be asymptotically stable. Here, A is a 2×2 real constant matrix, and k and l are positive integers. In the scalar case, Kaewong et al. [3] showed the following result.

Theorem A. *Let $A = a$ be a real number. Then the zero solution of* (1) *is asymptotically stable if and only if*

$$a^-_{\max} < a < \frac{1}{2}, \tag{2}$$

where $a^-_{\max}$ is the negative maximum value of $1/(2\cos\frac{2kp\pi}{2k+l})$ for $p = 0, 1, \ldots, 2k+l-1$.

By the transformation $x_n = Py_n$ with a nonsingular matrix P, the system (1) can be written as

$$y_n = P^{-1}AP(y_{n-k} + y_{n-k-l}), \quad n = 0, 1, 2, \ldots.$$

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Consequently, we have only to consider (1) where the matrix A is either of the following two matrices in Jordan form:

$$\text{(I)}\quad A = a\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \text{(II)}\quad A = \begin{pmatrix} d_1 & b \\ 0 & d_2 \end{pmatrix},$$

where a, b, d_1, d_2 and θ are real numbers with $|\theta| \le \pi/2$.

§2. Main results

The following theorems are our main results. For simplicity, we put

$$a_p = \frac{1}{2\cos\frac{2kp\pi - l|\theta|}{2k+l}} \tag{3}$$

for $p = 0, 1, \ldots, 2k+l-1$.

Theorem 1. *Suppose that the matrix A is given by* (I)*. Then the zero solution of* (1) *is asymptotically stable if and only if*

$$a^-_{\max} < a < a^+_{\min}, \tag{4}$$

where $a^+_{\min}$ and $a^-_{\max}$ are the positive minimum value of a_p and the negative maximum value of a_p for $p = 0, 1, \ldots, 2k+l-1$, respectively.

Theorem 2. *Suppose that the matrix A is given by* (II)*. Then the zero solution of* (1) *is asymptotically stable if and only if*

$$a^-_{\max} < d_j < \frac{1}{2} \quad \textit{for } j = 1, 2, \tag{5}$$

where $a^-_{\max}$ is the negative maximum value of a_p with $\theta = 0$ for $p = 0, 1, \ldots, 2k+l-1$.

Our results are proved by using the fact that the zero solution of (1) is asymptotically stable if and only if all the roots of its associated characteristic equation

$$F(\lambda) \equiv \det(\lambda^{k+l} I - \lambda^l A - A) = 0 \tag{6}$$

are inside the unit disk. Here, I is the 2×2 identity matrix. In this paper we only give the proof of Theorem 1 because Theorem 2 is proved in a similar way. Thus we consider the matrix A given by (I). An easy calculation yields that

$$
\begin{aligned}
F(\lambda) &= \begin{vmatrix} \lambda^{k+l} - a(\lambda^l+1)\cos\theta & a(\lambda^l+1)\sin\theta \\ -a(\lambda^l+1)\sin\theta & \lambda^{k+l} - a(\lambda^l+1)\cos\theta \end{vmatrix} \\
&= \{\lambda^{k+l} - a(\lambda^l+1)\cos\theta\}^2 - \{ia(\lambda^l+1)\sin\theta\}^2 \\
&= \{\lambda^{k+l} - ae^{i\theta}(\lambda^l+1)\}\{\lambda^{k+l} - ae^{-i\theta}(\lambda^l+1)\}.
\end{aligned}
$$

Hence, all the roots of the characteristic equation (6) are inside the unit disk if and only if all the roots of the equation

$$
f(a,\lambda) \equiv \lambda^{k+l} - ae^{i\theta}(\lambda^l+1) = 0 \tag{7}
$$

are inside the unit disk, since $F(\lambda) = f(a,\lambda)\overline{f(a,\bar{\lambda})}$ where $\bar{\lambda}$ denotes the complex conjugate of any complex λ. Consequently, the following proposition holds.

Proposition 1. *The zero solution of* (1) *is asymptotically stable if and only if all the roots of* (7) *are inside the unit disk.*

Note that, in case $a = 0$, all the roots of (7) are 0 (multiplicity $k+l$). Now, we will discuss the location of the roots of (7) as a varies. The first two lemmas deal with the value of a and the roots of (7) on the unit circle. Hereafter, we may assume that $\theta \geq 0$.

Lemma 1. *Let λ be a root of* (7) *on the unit circle. Then the root λ and the real number a are expressed as*

$$
\lambda = e^{i\omega_p}, \quad \textit{where } \omega_p = \frac{2p\pi + 2\theta}{2k+l}, \tag{8}
$$

and

$$
a = \frac{1}{2\cos\frac{2kp\pi - l\theta}{2k+l}} \equiv a_p \tag{9}
$$

for some $p = 0, 1, \ldots, 2k+l-1$. Conversely, if $a = a_p$, then $\lambda = e^{i\omega_p}$ is a root of (7).

Proof. Let λ be a root of (7) on the unit circle. Then $\lambda^l + 1 \neq 0$ and (7) becomes

$$
a = \frac{\lambda^{k+l}}{e^{i\theta}(\lambda^l+1)}. \tag{10}
$$

Since a is real and $\bar{\lambda} = 1/\lambda$, we have

$$
a = \frac{\bar{\lambda}^{k+l}}{e^{-i\theta}(\bar{\lambda}^l+1)} = \frac{e^{i\theta}/\lambda^{k+l}}{1/\lambda^l+1} = \frac{e^{i\theta}}{\lambda^k(1+\lambda^l)}. \tag{11}
$$

From (10) and (11), we obtain $\lambda^{2k+l} = e^{i2\theta}$, which implies that (8) is valid for some integer p. Then it follows from (10) that

$$a = \frac{e^{i2\theta}\lambda^{-k}}{e^{i\theta}(e^{i2\theta}\lambda^{-2k}+1)} = \frac{1}{e^{i\theta}\lambda^{-k}+e^{-i\theta}\lambda^{k}} = \frac{1}{2\cos(k\omega_p - \theta)} \equiv a_p.$$

Conversely, if $a = a_p$, it is clear that $\lambda = e^{i\omega_p}$ is a root of (7). The proof is complete. Q.E.D.

Lemma 2. *The roots of* (7) *on the unit circle are simple.*

Proof. Let λ be given by (8). Note that $\lambda^l + 1 \neq 0$. By (10), we have

$$\frac{\partial f(a,\lambda)}{\partial \lambda} = (k+l)\lambda^{k+l-1} - lae^{i\theta}\lambda^{l-1} = \frac{\lambda^{k+l-1}(k\lambda^l + k + l)}{\lambda^l + 1}.$$

Suppose that $\partial f(a,\lambda)/\partial\lambda = 0$. Then

$$k\lambda^l + k + l = 0. \tag{12}$$

In view of $\bar{\lambda} = 1/\lambda$, (12) yields $k/\lambda^l + k + l = 0$, that is,

$$(k+l)\lambda^l + k = 0. \tag{13}$$

By adding (12) and (13), we obtain $(2k+l)(\lambda^l+1) = 0$, which contradicts that $\lambda^l + 1 \neq 0$. Hence $\partial f(a_p, e^{i\omega_p})/\partial\lambda \neq 0$, and therefore, the assertion of this lemma holds. Q.E.D.

Next, we will observe how the roots of (7) cross the unit circle when the real number a varies.

Lemma 3. *The absolute values of the roots of* (7) *at* $\lambda = e^{i\omega_p}$ *increase as* $|a|$ *increses.*

Proof. By virtue of Lemma 2, we may regard the root λ of (7) as a holomorphic function of a in a neighborhood of $a = a_p$, and thus, the implicit function theorem shows that

$$\frac{d\lambda}{da} = -\frac{\frac{\partial f(a,\lambda)}{\partial a}}{\frac{\partial f(a,\lambda)}{\partial \lambda}}. \tag{14}$$

Let λ be expressed in the polar form, that is, $\lambda = re^{i\omega}$. Then we have

$$\frac{d\lambda}{da} = \frac{\lambda}{r}\left(\frac{dr}{da} + ir\frac{d\omega}{da}\right). \tag{15}$$

Hence it follows from (14) and (15) that

$$(16)\quad \frac{dr}{da} = \mathrm{Re}\left\{\frac{r}{\lambda}\cdot\frac{d\lambda}{da}\right\} = \frac{r}{\left|\frac{\partial f(a,\lambda)}{\partial\lambda}\right|^2}\mathrm{Re}\left\{-\frac{1}{\lambda}\cdot\frac{\partial f(a,\lambda)}{\partial a}\cdot\overline{\frac{\partial f(a,\lambda)}{\partial\lambda}}\right\}.$$

By using the fact that

$$\frac{\partial f(a,\lambda)}{\partial a} = -e^{i\theta}(\lambda^l+1) = -\frac{\lambda^{k+l}}{a},$$

we find

$$(17)\qquad \mathrm{Re}\left\{-\frac{1}{\lambda}\cdot\frac{\partial f(a,\lambda)}{\partial a}\cdot\overline{\frac{\partial f(a,\lambda)}{\partial\lambda}}\right\}$$

$$= \frac{1}{2}\left\{-\frac{1}{\lambda}\cdot\frac{\partial f(a,\lambda)}{\partial a}\cdot\overline{\frac{\partial f(a,\lambda)}{\partial\lambda}} - \lambda\cdot\overline{\frac{\partial f(a,\lambda)}{\partial a}}\cdot\frac{\partial f(a,\lambda)}{\partial\lambda}\right\}$$

$$= \frac{1}{2}\left\{\frac{(k+l)\lambda^l+k}{a(\lambda^l+1)} + \frac{k\lambda^l+k+l}{a(\lambda^l+1)}\right\} = \frac{2k+l}{2a}.$$

From (16) and (17), we therefore obtain

$$\left.\frac{dr}{da}\right|_{a=a_p} = \frac{(2k+l)r}{2a_p\left|\frac{\partial f(a,\lambda)}{\partial\lambda}\right|^2},$$

which, together with Lemma 1, implies that the absolute values of the roots of (7) at $\lambda = e^{i\omega_p}$ increase as $|a|$ increases. This completes the proof. Q.E.D.

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. By virtue of Proposition 1, we verify that the condition (4) is the necessary and sufficient condition for all the roots of (7) to be inside the unit disk. Recall that in case $a = 0$, all the roots of (7) lie on the origin. This, together with the continuity of the roots with respect to a, implies that if $|a|$ is sufficiently small, then all the roots of (7) lie inside the unit disk. In addition, Lemma 1 asserts that $a^+_{\min}$ (resp. $a^-_{\max}$) is the positive minimum (resp. negative maximum) value of a such that a root of (7) intersects the unit circle as a increases (resp. decreases) from 0. Lemma 3 also shows that if $a \geq a^+_{\min}$ (resp. $a \leq a^-_{\max}$), then there exists a root λ^* of (7) such that $|\lambda^*| \geq 1$. Thus, we conclude that all the roots of (7) are inside the unit disk if and only if the condition (4) holds. The proof of Theorem 1 is complete. Q.E.D.

§3. The case of $l = mk$

Our asymptotic stability conditions (4) and (5) in Theorems 1 and 2, unfortunately, involve constrained maximization and minimization problems concerning a_p. However, when l is a multiple of k, we can settle these problems. In the following, we consider the case of $l = mk$ where m is a positive integer. By virtue of Lemma 1, we notice that if a root of (7) lies on the unit circle, the real number a can be written as

$$a = \frac{1}{2\cos\xi_p} \equiv \alpha_p, \quad \text{where } \xi_p = \frac{2p\pi - m\theta}{m+2}, \tag{18}$$

for some $p = 0, 1, \ldots, m+1$. Thus, we have only to deal with α_p instead of a_p to obtain explicit asymptotic stability conditions. Throughout the remainder of this paper, we denote by $\alpha^+_{\min}$ and $\alpha^-_{\max}$ the positive minimum value of α_p and the negative maximum value of α_p, respectively.

Remark 1. In case $m = 1$, it follows from (18) that $\alpha_p = 1/(2\cos\frac{2p\pi-\theta}{3})$ for $p = 0, 1, 2$, and hence, we can immediately conclude that $\alpha^+_{\min} = \alpha_0$ and $\alpha^-_{\max} = \alpha_2$.

In case $m \geq 2$, the explicit values of $\alpha^+_{\min}$ and $\alpha^-_{\max}$ are given by the next lemma. For the sake of convenience, we divide the interval $[0, \pi/2]$ into intervals I_j defined by

$$\begin{aligned}
I_0 &= \left[0, \frac{\pi}{m}\right], \\
I_j &= \left(\frac{j}{m}\pi, \frac{j+1}{m}\pi\right] \quad \text{for } j = 1, 2, \ldots, \left[\frac{m}{2}\right] - 1, \\
I_{[m/2]} &= \begin{cases} \emptyset & \text{if } m \text{ is even,} \\ \left(\left[\frac{m}{2}\right]\frac{\pi}{m}, \frac{\pi}{2}\right] & \text{if } m \text{ is odd and } m \geq 3, \end{cases}
\end{aligned}$$

where $[\cdot]$ denotes the greatest integer function.

Lemma 4. *Let $m \geq 2$. Suppose that $\theta \in I_{2h}$ or $\theta \in I_{2h-1}$ for some integer h. Then*

$$\alpha^+_{\min} = \alpha_h \quad \textit{and} \quad \alpha^-_{\max} = \begin{cases} \alpha_{h+\frac{m+2}{2}} = -\alpha_h & \textit{if } m \textit{ is even}, \\ \alpha_{h+\frac{m+3}{2}} & \textit{if } m \textit{ is odd and } \theta \in I_{2h}, \\ \alpha_{h+\frac{m+1}{2}} & \textit{if } m \textit{ is odd and } \theta \in I_{2h-1}. \end{cases}$$

Proof. In view of $\theta \in I_{2h}$ or $\theta \in I_{2h-1}$, we notice that $(2h-1)\pi < m\theta \le (2h+1)\pi$. This, together with (18), yields that

$$\frac{(2p-2h-1)\pi}{m+2} \le \xi_p < \frac{(2p-2h+1)\pi}{m+2}. \tag{19}$$

First, we find the explicit value of $\alpha^+_{\min}$. By the definition of ξ_p and (19), we have

$$-\pi < \xi_0 < \cdots < \xi_{h-1} < -\tfrac{\pi}{m+2} \le \xi_h < \tfrac{\pi}{m+2} < \xi_{h+1} < \cdots < \xi_{m+1} < 2\pi + \xi_0 < 2\pi,$$

which implies $\alpha^+_{\min} = \alpha_h$.

Next, we investigate the explicit value of $\alpha^-_{\max}$. There are two cases to consider.

Case 1: m is even. By the definition of ξ_p and (19), we have $-\pi < \xi_{m+1} - 2\pi < \xi_0$ and

$$-\pi < \xi_0 < \cdots < \xi_{h+\frac{m}{2}} < \tfrac{(m+1)\pi}{m+2} \le \xi_{h+\frac{m+2}{2}} < \tfrac{(m+3)\pi}{m+2} \le \xi_{h+\frac{m+4}{2}} < \cdots < \xi_{m+1} < 2\pi,$$

which yield that

$$\alpha^-_{\max} = \alpha_{h+\frac{m+2}{2}} = \frac{1}{2\cos\xi_{h+\frac{m+2}{2}}} = \frac{1}{2\cos(\xi_h + \pi)} = -\frac{1}{2\cos\xi_h} = -\alpha_h.$$

Case 2: m is odd. By the definition of ξ_p and (19), we have

$$-\pi < \xi_{m+1} - 2\pi < \xi_0 < \cdots < \xi_{h+\frac{m+1}{2}} < \pi \le \xi_{h+\frac{m+3}{2}} < \cdots < \xi_{m+1} < 2\pi,$$

which implies $\alpha^-_{\max} = \alpha_{h+\frac{m+1}{2}}$ or $\alpha_{h+\frac{m+3}{2}}$. Note that

$$\gamma \equiv \left|\pi - \xi_{h+\frac{m+3}{2}}\right| - \left|\pi - \xi_{h+\frac{m+1}{2}}\right| = \xi_{h+\frac{m+3}{2}} + \xi_{h+\frac{m+1}{2}} - 2\pi = \frac{2(2h\pi - m\theta)}{m+2}. \tag{20}$$

If $\theta \in I_{2h}$, then $2h\pi < m\theta \le (2h+1)\pi$ and therefore $\gamma < 0$ by (20). This implies $\alpha^-_{\max} = \alpha_{h+\frac{m+3}{2}}$. On the other hand, if $\theta \in I_{2h-1}$, then $(2h-1)\pi < m\theta \le 2h\pi$ and thus $\gamma \ge 0$ by (20). This yields $\alpha^-_{\max} = \alpha_{h+\frac{m+1}{2}}$. The proof is complete. Q.E.D.

By virtue of Remark 1 and Lemma 4, we have the following result by Theorem 1.

Theorem 3. *Suppose that $l = mk$ and the matrix A is given by* (I) *where $|\theta| \in I_{2h}$ or $|\theta| \in I_{2h-1}$ for some integer h. Then the following statements hold:*

(i) *If* $m=1$, *then the zero solution of* (1) *is asymptotically stable if and only if* $\alpha_2 < a < \alpha_0$.
(ii) *If* m *is even, then the zero solution of* (1) *is asymptotically stable if and only if* $-\alpha_h < a < \alpha_h$.
(iii) *If* m *is odd,* $m \geq 3$ *and* $|\theta| \in I_{2h}$, *then the zero solution of* (1) *is asymptotically stable if and only if* $\alpha_{h+\frac{m+3}{2}} < a < \alpha_h$.
(iv) *If* m *is odd,* $m \geq 3$ *and* $|\theta| \in I_{2h-1}$, *then the zero solution of* (1) *is asymptotically stable if and only if* $\alpha_{h+\frac{m+1}{2}} < a < \alpha_h$.

In case $\theta = 0$, it follows from (18) that $\alpha_p = 1/(2\cos\frac{2p\pi}{m+2})$ for $p = 0, 1, \ldots, m+1$. Then Lemma 4 and $\theta \in I_0$ assert that

$$\alpha_{\max}^{-} = \begin{cases} -\alpha_0 = -1/2 & \text{if } m \text{ is even,} \\ \alpha_{\frac{m+3}{2}} = -1/(2\cos\frac{\pi}{m+2}) & \text{if } m \text{ is odd,} \end{cases}$$

and hence, we have the following result by Theorem 2.

Theorem 4. *Suppose that* $l = mk$ *and the matrix* A *is given by* (II). *Then the following statements hold:*

(i) *If* m *is even, then the zero solution of* (1) *is asymptotically stable if and only if* $-1/2 < d_j < 1/2$ *for* $j = 1, 2$.
(ii) *If* m *is odd, then the zero solution of* (1) *is asymptotically stable if and only if* $-1/(2\cos\frac{\pi}{m+2}) < d_j < 1/2$ *for* $j = 1, 2$.

§4. Concluding remarks

We have established several results on the asymptotic stability of (1). In paticular, we have also obtained the explicit asymptotic stability conditions for (1) with $l = mk$.

Finally, we state some remarks on other related works for the asymptotic stability of delay difference systems. In [1], Dannan presented asymptotic stability conditions for a delay difference equation

$$x_n = ax_{n-k} + bx_{n-k-l}, \quad n = 0, 1, 2, \ldots, \tag{21}$$

where a and b are real numbers. In [4], the second author gave explicit asymptotic stability conditions for a delay difference system

$$x_n = ax_{n-1} + Bx_{n-k}, \quad n = 0, 1, 2, \ldots, \tag{22}$$

where a is a real number and B is a 2×2 real constant matrix. So it is interesting to extend these results. Indeed, the asymptotic stability problem for a delay difference system

$$x_n = Ax_{n-k} + Bx_{n-k-l}, \quad n = 0, 1, 2, \ldots, \tag{23}$$

where A and B are 2×2 real constant matrices, has not yet been solved.

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