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3-dimensional i.i.d. binary random vectors governed by Jacobian elliptic space curve dynamics

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Abstract.

Sufficient conditions have been recently given for a classs of ergodic maps of an interval onto itself: $I = [0,1] \subset R \to I$ and its associated binary function to generate a sequence of independent and identically distributed (i.i.d.) binary random variables. Jacobian elliptic Chebyshev map, its derivative and second derivative induce Jacobian elliptic space curve. A mapping of the space curve with its coordinates, e.g., X, Y and Z, onto itself is introduced which defines 3 projective onto mappings, represented in the form of rational functions of $\{x_n, y_n, z_n\}_{n=0}^{\infty}$. Such mappings with their absolutely continuous invariant measures as functions of elliptic integrals and their associated binary function can generate a 3-dimensional sequence of i.i.d. binary random vectors.

§1. Introduction

Bernoulli shift and its associated binary function can produce a sequence of independent and identically distributed (i.i.d.) binary random variables (BRVs) [1], [2]. Tent map [3], closely related to the Bernoulli map, and its associated binary function can also generate a sequence of i.i.d. BRVs. Ulam and von Neumann[4] showed that the logistic map is topologically conjugate to the tent map via the homeomorphism $h^{-1}(\omega) = \dfrac{2}{\pi} \sin^{-1} \sqrt{\omega}$. They also pointed out that the logistic map is a strong candidate for pseudo-random number generation (PRNG) even though it has a non-uniform absolutely continuous invariant (ACI) measure. A number of analog chaos techniques, which use a chaotic real-valued trajectory itself, have also been proposed [5],[6]. Binary sequences

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using chaos, however, play an important role in several applications such as spreading spectrum codes [7], [8], [9] , [10], pseudo-random number generators [11] and cryptosystems [12], [13].

Motivated by this situation, we have shown that a class of ergodic maps with its unique ACI measure satisfying equi-distributivity property (EDP) can generate a sequence of i.i.d. binary random variables if its associated binary function satisfies constant summation property (CSP) [14]. Fortunately, many well-known 1-dimensional maps, which are topologically conjugate to the tent map via homeomorphism [3], satisfy EDP. The Bernoulli map, logistic map and Chebyshev polynomial are good examples [15]. These maps are governed by duplication formulae. In other words, *a duplication formula gives chaotic dynamics.* It is well known that elliptic functions satisfy an addition theorem [16]. We introduced a Jacobian elliptic Chebyshev rational map as a rational function version of Chebyshev polynomial [17]. This map as well as the other well known maps mentioned above are mappings from an interval onto itself.

Modern cryptosystems, however, need more and more pseudo-random numbers. In fact, to break DES of 64 bits, it takes 2^{43} steps and the success rate is 85% if 2^{43} pairs (plaintext, ciphertext) are known [18]. Furthermore, the size of new block ciphers such as AES becomes large, e.g., 512 bits.

This situation motivated us to discuss a closed smooth space curve defined by an algebraic relation between the Jacobian elliptic function, its derivative and second derivative. These duplication formulae give a real-valued sequence $\{x_n, y_n, z_n\}_{n=0}^{\infty}$ generated by 3-dimensional dynamics with Cartesian coordinates, e.g., X, Y and Z. Such 3-dimensional dynamics forces us to define a mapping from such a space curve onto itself and three projection mappings from an interval onto itself associated with coordinates, i.e., $x_{n+1} = \tau_x(x_n)$, $y_{n+1} = \tau_y(y_n)$ and $z_{n+1} = \tau_z(z_n)$, respectively. The former $\tau_x(\cdot)$, being single-valued, is the same as the Jacobian elliptic Chebyshev rational map. On the contrary, the latter two $\tau_y(\cdot)$ and $\tau_z(\cdot)$, being multi-valued, consist of single-valued mappings, each of which is a rational function of x_n, y_n, z_n and have their ACI measures with EDP. Hence, every bit of binary expansion of real-valued vector (x_n, y_n, z_n) satisfies CSP. This implies that the mapping from the space curve onto itself governed by duplication formulae gives a sequence of i.i.d. 3-dimensional binary random vectors.

§2. Related theories

We will begin by describing some of the related theories which play an important role in evaluating statistical properties of a sequence of binary random variables generated by a real-valued sequence.

2.1. EDP and CSP

Perhaps the simplest mathematical object that can display chaotic behavior is a class of one-dimensional maps $\omega_{n+1} = \tau(\omega_n)$, where $\omega_n = \tau^n(\omega_0) \in I = [d, e]$, $n = 0, 1, 2, \ldots$ and $\tau(\cdot) : I \to I$.
Consider a piecewise monotonic (PM) onto ergodic map $\tau(\cdot)$ that satisfies the following properties:

- **i):** there is a (trvial) partitition $d = d_0 < \cdots < d_{N_\tau} = e$ of I such that for each integer $i = 1, \cdots, N_\tau$, $(N_\tau > 2)$ the restriction of $\tau(\cdot)$ to the interval $I_i = [d_{i-1}, d_i)$, denoted by $\tau_i(\omega)$, is a C^2 function; as well as
- **ii):** $\tau(I_i) = (d, e)$, that is, τ has N_τ monotonic onto maps τ_i;
- **iii):** τ has a unique ACI measure, denoted by $f^*(\omega)d\omega$.

The following four definitions are important to evaluate statistical properties of $\{\omega_n\}_{n=0}^{\infty}$.

Definition 1 (Perron-Frobenius operator [19]). *The Perron-Frobenius operator P_τ acting on function of bounded variation $F(\omega) \in L^\infty$ for $\tau(\omega)$ is defined as*

$$P_\tau F(\omega) = \frac{d}{d\omega} \int_{\tau^{-1}([d,\omega])} F(y)dy = \sum_{i=0}^{N_\tau - 1} |g_i'(\omega)| F(g_i(\omega)),$$

where $g_i(\omega)$ is the i-th preimage of ω and N_τ denotes the number of preimages.

The ACI measure $f^*(\omega)d\omega$ satisfies

$$P_\tau f^*(\omega) = f^*(\omega). \tag{1}$$

Birchoff Individual Ergodic Theorem [19] tells us that for a stationary real-valued sequence $\{F(\omega_n)\}_{n=0}^{\infty}$, the time average of $\{F(\omega_n)\}_{n=0}^{\infty}$, defined by

$$\langle F \rangle = \lim_{T \to \infty} (1/T) \sum_{n=0}^{T-1} F(\omega_n) \tag{2}$$

is equal almost everywhere to the expectation of $F(\omega)$, defined by

$$\mathbf{E}_\omega[F(\tau^n)] = \int_I F(\tau^n(\omega))f^*(\omega)d\omega. \tag{3}$$

From the stationarity of process, we denote $\mathbf{E}_\omega[F(\tau^n)]$ by $\mathbf{E}_\omega[F]$. Consider two sequences $\{G(\tau^n(\omega))\}_{n=0}^\infty$ and $\{H(\tau^n(\omega))\}_{n=0}^\infty$, where $G(\omega)$, $H(\omega) \in L^\infty$. The second-order cross-covariance function between these sequences from a seed $\omega = \omega_0$ is defined by

$$\rho(\ell, G, H) = \int_I (G(\omega) - \mathbf{E}_\omega[G]) \cdot (H(\tau^\ell(\omega)) - \mathbf{E}_\omega[H])f^*(\omega)d\omega, \tag{4}$$

where $\ell = 0, 1, 2, \cdots$. The operator P_τ is useful in evaluating correlation functions because it has the following important property:

$$\int_I G(\omega)P_\tau\{H(\omega)\}d\omega = \int_I G(\tau(\omega))H(\omega)d\omega. \tag{5}$$

Using this property, we get

$$\rho(\ell, G, H) = \int_I P_\tau^\ell\{(G(\omega) - \mathbf{E}_\omega[G])f^*(\omega)\}(H(\omega) - \mathbf{E}_\omega[H])d\omega. \tag{6}$$

Bernoulli map with its uniform ACI measure $f^*(\omega)d\omega = d\omega$ is defined as

$$\tau_B(\omega) = 2\omega (\text{mod } 1) = \begin{cases} 2\omega, & 0 < \omega < \frac{1}{2}, \\ 2\omega - 1, & \frac{1}{2} \leqslant \omega < 1. \end{cases} \tag{7}$$

If ω is represented by its binary expansion as $\omega = 0.d_1(\omega)d_2(\omega)\cdots$, then the binary expansion of $\tau_B(\omega)$ is given by $\tau_B(\omega) = 0.d_2(\omega)d_3(\omega)\cdots$. This implies that $\tau_B(\cdot)$ shifts the digits one place to the left. The functions $d_k(\cdot)$, called Rademacher functions, furnish us with a model of independent tosses of a fair coin [2]. A sequence $\{d_k(\omega)\}_{k=0}^\infty$ can be regarded as a sequence of i.i.d. BRVs in the sense that for almost every ω, $d_k(\omega)$ can imitate coin tossing.

Another map and its associated binary function are as follows. Consider piecewise linear map of p branches with $f^*(\omega)d\omega = d\omega$, given by [3] $(N_\tau = p)$,

$$N_p(\omega) = (-1)^{\lfloor p\omega \rfloor} p\omega (\text{mod } p), \ \omega \in [0, 1]. \tag{8}$$

In particular, $N_2(\omega)$ is referred to as the tent map. Introduce its associated BRV defined as

$$a_k = \begin{cases} 0, & \text{for } N_2^k(\omega) \leqslant \frac{1}{2}, \\ 1, & \text{for } N_2^k(\omega) > \frac{1}{2}. \end{cases} \tag{9}$$

Then for $\omega = 0.d_1(\omega)d_2(\omega)\cdots$, we get $a_0(\omega) = d_1(\omega)$, $a_k(\omega) = d_k(\omega) \oplus d_{k+1}(\omega)$, $k \geqslant 1$, where $\oplus$ denotes a modulo 2 addition (or exclusiveor) operation. Hence $N_2(\omega)$ and its associated binary functions $a_k(\cdot)$ can generate a sequence of i.i.d. BRVs.

Naturally, the important question arises, that can any other map and its associated binary function generate a sequence of i.i.d. BRVs? We have got an affirmative answer to this question [14], [15], which is firstly, the map should satisfy EDP and secondly, the binary function should satisfy CSP.

Definition 2 (EDP [14]). *If a piecewise-monotonic onto map $\tau(\omega)$ satisfies*

$$|g_i'(\omega)|f^*(g_i(\omega)) = \frac{1}{N_\tau} f^*(\omega), \quad 0 \le i \le N_\tau - 1, \tag{10}$$

then the map is said to satisfy equi-distributivity property (EDP).

Definition 3 (CSP [14],[15]). *For a class of maps with EDP, if its associated function $G(\cdot)$ satisfies*

$$\frac{1}{N_\tau} \sum_{i=0}^{N_\tau - 1} G(g_i(\omega)) = \mathbf{E}_\omega[G] \ \ or \ \ P_\tau\{G(\omega)f^*(\omega)\} = \mathbf{E}_\omega[G]f^*(\omega) \tag{11}$$

then $G(\cdot)$ is said to satisfy constant summation property (CSP).

CSP guarantees no-correlation between two functions $G(\cdot)$ and ${}^\forall H(\cdot)$, i.e., $\rho(\ell, G, H) = 0$, $\ell > 0$ [15]. Fortunately, EDP is satisfied by many well-known maps and is invariant under topological conjugation.

Definition 4 (topological conjugation [19]). *Two transformations $\bar{\tau} : \bar{I} \to \bar{I}$ and $\tau : I \to I$ on intervals $\bar{I}$ and I are called* topological conjugate *if there is a homeomorphism $h : \bar{I} \overset{onto}{\to} I$ as $\tau(\omega) = h \circ \bar{\tau} \circ h^{-1}(\omega)$.*

Suppose $\tau(\cdot)$ and $\bar{\tau}(\cdot)$ have their ACI measures $f^*(\omega)d\omega$ and $\bar{f}^*(\bar{\omega})d\bar{\omega}$ respectively. Then, under the topological conjugation, these ACI measures have the relation

$$f^*(\omega) = \left|\frac{dh^{-1}(\omega)}{d\omega}\right| \bar{f}^*(h^{-1}(\omega)). \tag{12}$$

The relation between $\tau(\cdot)$ and $\bar{\tau}(\cdot)$ via h is represented diagrammatically as follows :

$$\begin{array}{ccc} I & \xrightarrow{\tau} & I \\ {\scriptstyle h^{-1}}\downarrow & & \uparrow{\scriptstyle h} \\ \bar{I} & \xrightarrow{\bar{\tau}} & \bar{I} \end{array} \tag{13}$$

Remark 1. If we take $N_2(\overline{\omega})$ as $\overline{\tau}(\overline{\omega})$, then $f^*(\omega)$ is simply represented by the derivative of $h^{-1}(\omega)$. Hence, if $h(\overline{\omega})$ can be given in an inverse function form, then its integrand gives an ACI measure within normalization factor. Most famous example of inverse functions is sin function, *i.e.*, $\omega = \int_0^{\sin\omega} \frac{du}{\sqrt{1-u^2}}$.
This remark provides a starting point for discussion. In fact, Ulam and von Neumann [4] gave the logistic map

$$L_2(\omega) = 4\omega(1-\omega),\ \omega \in [0,1] \tag{14}$$

with $f^*(\omega)d\omega = \frac{d\omega}{\pi\sqrt{\omega(1-\omega)}}$ which is topologically conjugate to $N_2(\overline{\omega})$ using $h^{-1}(\omega) = \dfrac{2}{\pi}\sin^{-1}\sqrt{\omega}$.

2.2. Binary function

In our previous study [14], we proposed methods to obtain binary sequences from chaotic real-valued sequences $\{\tau^n(\omega)\}_{n=0}^{\infty}$. We define a (non-trivial) partition $d = t_0 < t_1 < \cdots < t_{2M} = e$ of $[d,e]$ and T denotes the set of thresholds $\{t_r\}_{r=0}^{2M}$. Then the following binary function is obtained

$$C_T(\omega) = \sum_{r=0}^{2M} (-1)^r \Theta_{t_r}(\omega), \tag{15}$$

where $\Theta_t(\omega)$ is the threshold function such that

$$\Theta_t(\omega) = \begin{cases} 0, & \text{for } \omega < t \\ 1, & \text{for } \omega \geq t. \end{cases} \tag{16}$$

§3. Duplication formula gives chaos

The example mentioned above shows that duplication formula gives chaos. To observe it, several examples are listed as follows.
(1) logistic map: Transformation $x = \sin^2\theta$ gives $\left(\frac{dx}{d\theta}\right)^2 = 4x(1-x)$. Let $x_n = \sin^2\theta_n, \theta_{n+1} = 2\theta_n$. Then we get 2-dimensional sequences $\{(x_n, y_n)\}_{n=0}^{\infty}$, given by

$$\begin{aligned} x_{n+1} &= L_2(x_n) = 4x_n(1-x_n), \\ y_{n+1}^2 &= \left(\tfrac{1}{2}\cdot\tfrac{d\,L_2(x_n)}{d\theta_n}\right)^2 = 4L_2(x_n)(1-L_2(x_n)). \end{aligned} \tag{17}$$

(2) Chebyshev map of degree 2: Grossmann and Thomae [3] observed that Chebyshev polynomial maps of degree p ($p = 2, 3, \cdots$) [20] with its

ACI measure $f^*(\omega)d\omega = \dfrac{d\omega}{\pi\sqrt{1-\omega^2}}$, defined by

(18) $$T_p(\omega) = \cos(p\cos^{-1}\omega),\ \omega \in [-1,1]$$

is topologically conjugate to $N_p(\omega)$ via $h(\overline{\omega}) = \cos\pi\overline{\omega}$. Transformation $x = \cos\theta$ gives $\left(\frac{dx}{d\theta}\right)^2 = 1 - x^2$. Let $x_n = \cos\theta_n$, $\theta_{n+1} = 2\theta_n$. Then we get 2-dimensional sequences $\{(x_n, y_n)\}_{n=0}^{\infty}$, given by

(19)

$$x_{n+1} = T_2(x_n) = 2x_n^2 - 1,\ y_{n+1}^2 = \left(\frac{1}{2}\cdot\frac{dT_2(x_n)}{d\theta_n}\right)^2 = 1 - (T_2(x_n))^2.$$

(3) Schröder and Böttcher map: [1] Schröder [22] and Böttcher [23] gave a rational function version of $L_2(\cdot)$ with parameter k, defined as

(20) $$R_2^{\mathrm{sn}^2}(\omega, k) = \frac{4\omega(1-\omega)(1-k^2\omega)}{(1-k^2\omega^2)^2},\ \omega \in [0,1]$$

with its ACI measure

(21) $$f^*(\omega, k)d\omega = \frac{d\omega}{2K(k)\sqrt{\omega(1-\omega)(1-k^2\omega)}}$$

via $h^{-1}(\omega) = \dfrac{1}{K(k)}\mathrm{sn}^{-1}(\sqrt{\omega}, k)$, where $\mathrm{sn}(\omega, k)$ is the inverse function of the elliptic integral with modulus k ($|k| < 1$) and $K(k)$ is the complete elliptic integral, each of which is given respectively as

(22) $$u = \int_0^{\mathrm{sn}(u,k)} \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}},\ K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

Transformation $x = \mathrm{sn}^2 u$ gives $\left(\frac{dx}{du}\right)^2 = 4x(1-x)(1-k^2x)$. Let $x_n = \mathrm{sn}^2 u_n, u_{n+1} = 2u_n$. Then we get 2-dimensional sequences $\{(x_n, y_n)\}_{n=0}^{\infty}$, given by

(23) $$x_{n+1} = R_2^{\mathrm{sn}^2}(x_n, k) = \frac{4x_n(1-x_n)(1-k^2x_n)}{(1-k^2x_n^2)^2}$$

(24) $$\begin{aligned} y_{n+1}^2 &= \left(\frac{1}{2}\cdot\frac{dR_2^{\mathrm{sn}^2}(x_n, k)}{du_n}\right)^2 \\ &= 4R_2^{\mathrm{sn}^2}(x_n, k)(1 - R_2^{\mathrm{sn}^2}(x_n, k))(1 - k^2R_2^{\mathrm{sn}^2}(x_n, k)). \end{aligned}$$

[1] see [21] for a historical review of rational maps.

§4. Jacobian elliptic space curve and 3-dimensional dynamics

We know that the Jacobian elliptic function $\mathrm{cn}(u,k)$ [2] is an inverse function of an elliptic integral of the first kind in the Legendre-Jacobi normal form [16]

$$u = \int_{\mathrm{cn}(u,k)}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2+k^2t^2)}}. \tag{25}$$

Kohda and Fujisaki [17] introduced the Jacobian elliptic Chebyshev rational map with positive integer p

$$R_p^{\mathrm{cn}}(\omega,k) = \mathrm{cn}(p\,\mathrm{cn}^{-1}(\omega,k),k), \quad \omega \in [-1,1] \tag{26}$$

which is topologically conjugate to the tent map $N_p(u)$ via homeomorphism $h^{-1}(\omega,k) = \frac{\mathrm{cn}^{-1}(\omega,k)}{2K(k)}$ and has its ACI measure

$$f^*(\omega,k)d\omega = \frac{d\omega}{2K(k)\sqrt{(1-\omega^2)(1-k^2+k^2\omega^2)}}. \tag{27}$$

This map is a rational function version of the Chebyshev polynomial

$$T_p(\omega) = \cos(p\cos^{-1}\omega), \quad \omega \in [-1,1]. \tag{28}$$

We know that $R_p^{\mathrm{cn}}(\omega,k)$ satisfies the semi-group property

$$R_r^{\mathrm{cn}}(R_s^{\mathrm{cn}}(\omega,k),k) = R_{rs}^{\mathrm{cn}}(\omega,k) \tag{29}$$

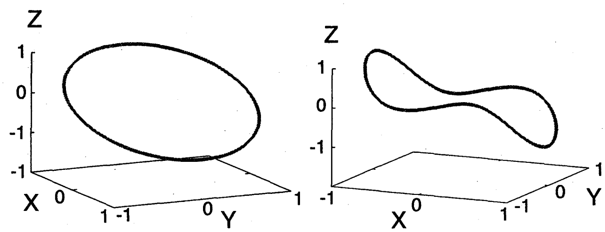
for integers r, s and when $p=2$,

$$R_2^{\mathrm{cn}}(\omega,k) = \frac{1-2(1-\omega^2)+k^2(1-\omega^2)^2}{1-k^2(1-\omega^2)^2}. \tag{30}$$

Let us concentrate on the Jacobian real elliptic function with $p=2$ [16]. As shown in Fig. 1, the Jacobian elliptic function $X = \mathrm{cn}(u,k)$, its derivative $Y = \frac{d}{du}\mathrm{cn}\,u = -\mathrm{sn}\,u\,\mathrm{dn}\,u$ and the second derivative $Z = \frac{d^2}{du^2}\mathrm{cn}\,u$ give the Jacobian elliptic space curve, given by

$$Y^2 = (1-X^2)(1-k^2+k^2X^2), \; Z = X(-1+2k^2(1-X^2)). \tag{31}$$

[2] $\mathrm{cn}(u,0)$ simply reduces to $\cos u$.



(a) $k = 0.1$ (b) $k = 0.9$

Fig. 1. Two Jacobian elliptic space curves (X, Y, Z).

Let $u_{n+1} = 2u_n$, $x_n = \operatorname{cn} u_n, y_n = \frac{dx_n}{du_n}$ and $z_n = \frac{d^2x_n}{du_n^2}$. Then we get a 3-dimensional dynamics, given by

(32)

$$\begin{cases} x_{n+1} & = R_2^{\mathrm{cn}}(x_n, k) = \tau_x(x_n, k), \\ y_{n+1}^2 & = (\frac{1}{2}\frac{dx_{n+1}}{du_n})^2 = (1 - x_{n+1}^2)(1 - k^2 + k^2x_{n+1}^2) = \tau_y^2(y_n, k), \\ z_{n+1} & = \frac{1}{4}\frac{d^2x_{n+1}}{du_n^2} = \tau_z(z_n(x_n), k) = \tau_z(x_n, k) \\ & = \frac{k^2-1+2(1-k^2)x_n^2+k^2x_n^4}{1-k^2(1-x_n^2)^2}\{1 - 2(\frac{1-k^2+k^2x_n^4}{1-k^2(1-x_n^2)^2})^2\}. \end{cases}$$

This gives a mapping from such a space curve onto itself which induces three projective onto mappings associated with coordinates,e.g.,X, Y, Z, denoted by $\tau_x(\cdot), \tau_y(\cdot), \tau_z(\cdot)$. The first one is shown in Fig.2(a), which has a symmetric ACI measure, defined by

$$f_X^*(x, k)dx = \frac{dx}{2K(k)\sqrt{(1 - x^2)(1 - k^2 + k^2x^2)}}$$

in Fig.3(a).

In addition, it has been shown [24] that the projective onto map τ_y is symmetric and has a symmetric ACI measure as shown in Figs.2(b) and 3(b), respectively. (see Appendix A for theoretical expression of τ_y) Its associated symmetric binary function, e.g., binary expansion of real-valued orbit $\{x_n\}_{n=0}^{\infty}$ or $\{y_n\}_{n=0}^{\infty}$ can generate a sequence of i.i.d. binary random variables [24].

Here we consider the map τ_z and examine whether it has its symmetric ACI measure [25]. Squaring the second expression of Eq.(31) with $k \neq 0$ gives the relation

$$X^6 - \frac{1}{k^2}(-1 + 2k^2)X^4 + \frac{1}{4k^4}(-1 + 2k^2)^2X^2 - \frac{Z^2}{4k^4} = 0 \tag{33}$$

which implies that for a given Z, X^2 has the following three real-valued solutions at most.

$$(34)\quad X^2(Z) = \begin{cases} \xi_1^2(Z), & \text{for } k \le \sqrt{1/2}\ (R(Z,k) > 0) \\ \xi_1^2(Z), & \text{for } k > \sqrt{1/2} \text{ and } R(Z,k) > 0 \\ \xi_i^2(Z),\ 2 \le i \le 4, & \text{for } k > \sqrt{1/2} \text{ and } R(Z,k) < 0, \end{cases}$$

where $R(Z,k) = \frac{b^2(Z,k)}{4} + \frac{a^3(k)}{27}$, $a(k) = -\frac{1}{12k^4}(-1+2k^2)^2, b(Z,k) = \frac{1}{4\cdot 27}\{\frac{(-1+2k^2)^3}{k^6} - \frac{27}{k^4}Z^2\}$.

On the space curve, 3-dimensional dynamics has a unique ACI measure with respect to each coordinate. Fig. 3(c) shows comparison between the marginal distribution taken from experiments and theoretical calculations, where the theoretical distributions of τ_z is given as follows

(35)

$$f_Z^*(z,k)dz = \begin{cases} \frac{1}{2K(k)} f_Z(\xi_1(Z),k)dz, & \text{for } 0 < k \le \sqrt{1/2} \\ \frac{1}{2K(k)} f_Z(\xi_1(Z),k)dz, & \text{for } k > \sqrt{1/2},\ r(k) \le |z| < 1 \\ \frac{1}{2K(k)} \sum_{\ell=2}^4 f_Z(\xi_\ell(Z),k)dz & \text{for } k > \sqrt{1/2}, |z| \le r(k) \end{cases}$$

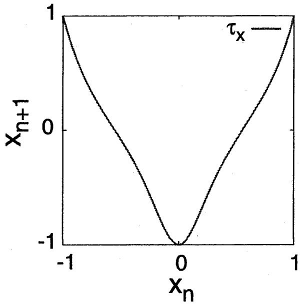
where

$$(36)\quad \begin{aligned} r(k) &= \sqrt{\frac{2}{27}(-1+2k^2)^3}, \\ & f_Z(\xi_\ell(Z),k)dz \\ &= \frac{dz}{\sqrt{(1-\xi_\ell^2(Z))(1-k^2+k^2\xi_\ell^2(Z))}\,|-6k^2\xi_\ell^2(Z)+2k^2-1|}. \end{aligned}$$

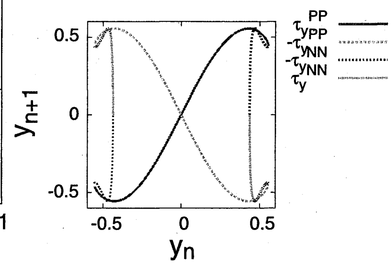
Finally, we notice that theoretical distribution $f_X^* dx$ is also given by integrand of elliptic integral for inverse function $\mathrm{cn}^{-1}(u,k)$ (see Eq.(25)). The same is true for $f_Y^* dy$. In fact, inverse function $\left(\frac{d\,\mathrm{cn}(u,k)}{du}\right)^{-1} = (-\mathrm{sn}(u,k)\,\mathrm{dn}(u,k))^{-1}$ is defined by Eq.(45) and Eq.(46) (see Appendix B). Similarly $f_Z^* dz$ is expressed in the inverse function form, as given by Eq.(49) and Eq.(50) (see Appendix C).

§5. I.I.D. binary random vectors

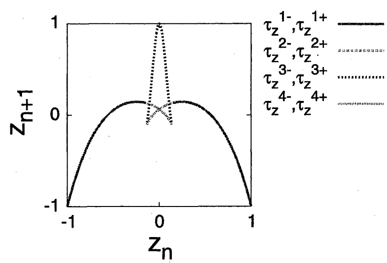
We shall now look into the relation between (z_n, z_{n+1}). Eqs.(33) and (34) tell us that the relation $z_{n+1} = \tau_z(\xi_1(z_n))$ is one-to-one when $k < \sqrt{1/2}$ but the graph of z_n versus z_{n+1} is *one-to-many* when $k >$



(a) $x_{n+1} = \tau_x(x_n)$

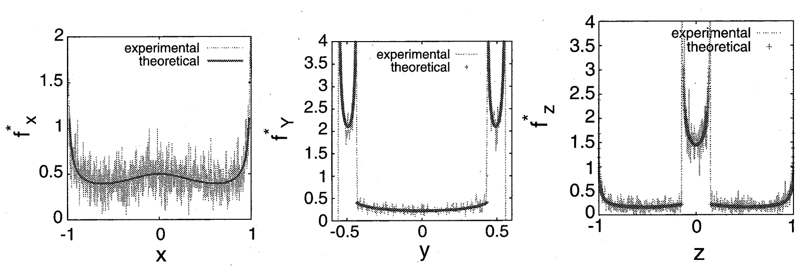


(b) $y_{n+1} = \tau_y(y_n)$



(c) $z_{n+1} = \tau_z(z_n)$

Fig. 2. Three projection mappings when $k = 0.9$.



(a) $f_X^*(x)dx$ (b) $f_Y^*(y)dy$ (c) $f_Z^*(z)dz$

Fig. 3. Three marginal distributions when $k = 0.9$

$\sqrt{1/2}$. Namely, the latter case gives a closed curve as shown in Fig. 2(c). Suppose that $k > \sqrt{1/2}$ and that $X_1(x)$ is the first bit of normalized x in binary representation, such as

$$\frac{x+1}{2} = 0.X_1(x)X_2(x)\cdots X_i(x)\cdots, X_i(x) \in \{0,1\}.$$

We denote $X_1(x)$ by X_1 and $1 - X_1(x)$ by $\overline{X_1}$. Similarly $Z_1(z)$ and $1 - Z_1(z)$ are denoted by Z_1 and $\overline{Z_1}$ respectively. In addition, $D(\frac{dz}{dx})$ and $1 - D(\frac{dz}{dx})$ are represented by D_z and $\overline{D_z}$ respectively, where $D(\frac{dz}{dx}) = 0(\text{or } 1)$ when $\frac{dz}{dx} < 0$ (or when $\frac{dz}{dx} \geq 0$).

Then, we can obtain a piecewise-monotonic onto map τ_z defined by

$$\begin{aligned} (37) \quad \tau_z &= X\,\overline{Z}\,\overline{D}\tau_z^{1-} + \overline{X}Z\overline{D}\tau_z^{1+} + X\overline{Z}\,\overline{D}\tau_z^{2-} + \overline{X}\,Z\overline{D}\tau_z^{2+} \\ &+ \overline{X}\,\overline{Z}\,D\tau_z^{3-} + XZD\tau_z^{3+} + \overline{X}\,\overline{Z}\,\overline{D}\tau_z^{4-} + XZ\overline{D}\tau_z^{4+} \end{aligned}$$

where $\tau_z^{i-} = \tau_z(-\xi_i(z)), \tau_z^{i+} = \tau_z(\xi_i(z))$, $1 \leq i \leq 4$ and where $\xi_i^2(z)$ is defined by Eq.(34).

It can be shown that for uniform ACI measure $f_U^*(u)du = du$,

$$(38) \quad \left.\begin{array}{lll} P_{\tau_x}\{C_{T_x}(x)f_X^*(x)\} & = \mathbf{E}_u[C_{T_x}]f_X^*(x), & x = \operatorname{cn} u \\ P_{\tau_y}\{C_{T_y}(y)f_Y^*(y)\} & = \mathbf{E}_u[C_{T_y}]f_Y^*(y), & y = -\operatorname{sn} u \operatorname{dn} u \\ P_{\tau_z}\{C_{T_z}(z)f_Z^*(z)\} & = \mathbf{E}_u[C_{T_z}]f_Z^*(z), & z = \dfrac{d(-\operatorname{sn} u \operatorname{dn} u)}{du} \end{array}\right\}$$

holds, where $\{C_{T_x}(x_n)\}_{n=0}^{\infty}, \{C_{T_y}(y_n)\}_{n=0}^{\infty}$ and $\{C_{T_z}(z_n)\}_{n=0}^{\infty}$ are symmetric binary sequences with their sets of symmetric thresholds T_x, T_y and T_z associated with real-valued sequences $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$.

This implies that $\rho(\ell, C_{T_x}, C_{T_x}) = \rho(\ell, C_{T_y}, C_{T_y}) = \rho(\ell, C_{T_z}, C_{T_z}) = 0$, for $\ell \geq 0$. [14]

It should be noted that $C_{T_x}(x)$, $C_{T_y}(\tau_y^{\ell}(y))$, $C_{T_z}(\tau_z^m(z))$ are not always independent each other for $\ell = m = 0$, that is, e.g., $\mathbf{E}_u[C_{T_x}C_{T_y}] \neq \mathbf{E}_u[C_{T_x}]\mathbf{E}_u[C_{T_y}]$ even if each of them is a sequence of i.i.d. BRVs. This is inevitable as long as these sequences are generated from a single seed $u = u_0$. However, we can design appropriate sets of thresholds T_x, T_y, T_z satisfying $\mathbf{E}_u[C_{T_x}C_{T_y}] = \mathbf{E}_u[C_{T_x}]\mathbf{E}_u[C_{T_y}]$ (see [14] for details).

§6. Conclusion

We discussed a real-valued dynamics on the Jacobian elliptic space curve between Jacobian elliptic function, its derivative and second derivative, governed by their duplication formulae. Furthermore, we showed that a mapping of the space curve onto itself: $R^3 \to R^3$ which defines 3 projective onto mappings with their ACI measures satisfying EDP and can generate sequences of 3-dimensional i.i.d. binary random vectors when using their associated symmetric binary functions, e.g., bits of binary expansions of these real-valued x_n, y_n, z_n as shown in Fig. 4.

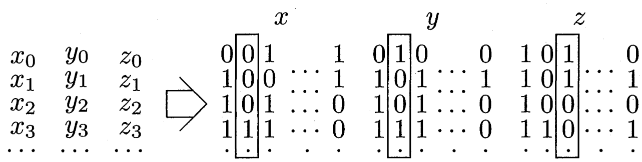


Fig. 4. Method of generating multidimensional i.i.d. binary vectors

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§**Appendix A.** Derivation of the theoretical expression of τ_y

The first expression in Eq.(31) gives

$$y_n^2 = (1 - x_n^2)(1 - k^2 + k^2 x_n^2). \tag{39}$$

Solving Eq.(39), we get for $k \neq 0$

$$x_n^2 = \frac{2k^2 - 1 \pm \sqrt{1 - 4k^2 y_n^2}}{2k^2}. \tag{40}$$

Eq.(30) and Eq.(32) give

$$R_2^{\mathrm{cn}}(x_n, k) = \frac{1 - 2(1 - x_n^2) + k^2(1 - x_n^2)^2}{1 - k^2(1 - x_n^2)^2} \tag{41}$$

and

$$y_{n+1} = \sqrt{(1 - (R_2^{\mathrm{cn}}(x_n, k))^2)\,(1 - k^2 + k^2(R_2^{\mathrm{cn}}(x_n, k))^2)}. \tag{42}$$

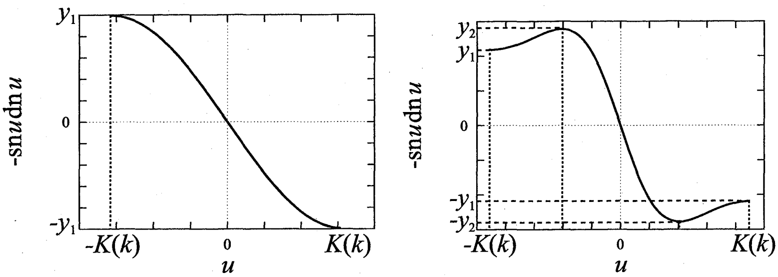
Substituting Eq.(40) and Eq.(41) into Eq.(42), we have

$$\begin{aligned} y_{n+1} &= \frac{2\sqrt{2}ky\sqrt{2k^2 - 1 \pm \sqrt{1 - 4k^2y^2}}}{(2k^2 - 1 + 2k^2y^2 \pm \sqrt{1 - 4k^2y^2})^2} \\ &\times \left\{1 - 2k^2y^2 \pm (2k^2 - 1)\sqrt{1 - 4k^2y^2}\right\}. \end{aligned} \tag{43}$$

where three $\pm$ signs on R.H.S. are either $+$ or $-$. Denote two maps by $\tau_y^{PP}(y)$ and $\tau_y^{NN}(y)$ when $+$ and $-$ are chosen on the R.H.S. of Eq.(43), respectively. Then

$$\begin{aligned} \tau_y(y) &= X_1(D \oplus Y_1)\tau_y^{PP}(y) + \overline{X_1}\overline{(D \oplus Y_1)}(-\tau_y^{PP}(y)) \\ &+ X_1\overline{(D \oplus Y_1)}\tau_y^{NN}(y) + \overline{X_1}(D \oplus Y_1)(-\tau_y^{NN}(y)) \end{aligned} \tag{44}$$

§Appendix B. Inverse function Y [24]



(a) when $k \leq \sqrt{1/2}$ (b) when $k > \sqrt{1/2}$

Fig. 5. $y = -\operatorname{sn} u \operatorname{dn} u$ ($y_1 = \sqrt{1-k^2}$ and $y_2 = 1/2k$, $k \neq 0$).

When $0 < k \leq \sqrt{1/2}$,

$$u = \int_{-\operatorname{sn} u \operatorname{dn} u}^{0} f_Y^+(y)dy. \tag{45}$$

When $k > \sqrt{1/2}$,

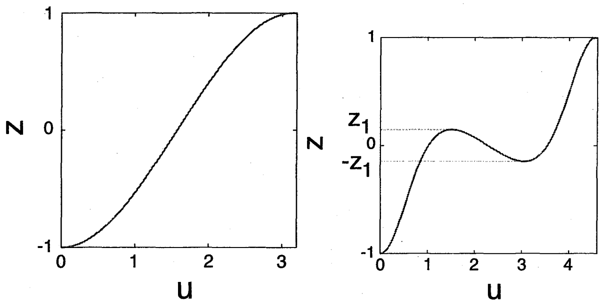
$$u = \begin{cases} \displaystyle\int_{-\operatorname{sn} u \operatorname{dn} u}^{0} f_Y^+(y)dy, \quad \text{for } |u| \leq \operatorname{cn}^{-1}\sqrt{\frac{2k^2-1}{2k^2}} \\ \displaystyle\int_{\frac{1}{2k}}^{0} f_Y^+(y)dy - \int_{-\operatorname{sn} u \operatorname{dn} u}^{\frac{1}{2k}} f_Y^-(y)dy, \\ \qquad \text{for } -K(k) \leq u < -\operatorname{cn}^{-1}\sqrt{\frac{2k^2-1}{2k^2}} \\ \displaystyle\int_{-\frac{1}{2k}}^{0} f_Y^+(y)dy - \int_{-\operatorname{sn} u \operatorname{dn} u}^{-\frac{1}{2k}} f_Y^-(y)dy, \\ \qquad \text{for } \operatorname{cn}^{-1}\sqrt{\frac{2k^2-1}{2k^2}} < u \leq K(k) \end{cases} \tag{46}$$

where[24]

$$f_Y^{\pm}(y)dy = \frac{\sqrt{2}k}{\sqrt{(2k^2 - 1 \pm \sqrt{1-4k^2y^2})(1-4k^2y^2)}}dy$$

where the $\pm$ sign on R.H.S is either $+$ or $-$ and is to be decided on the basis whether there is f_Y^+ or f_Y^- on the L.H.S.

§**Appendix C.** Inverse function Z [25]



(a) $k \leq \sqrt{1/2}$ (b) $k > \sqrt{1/2}$

Fig. 6. $z = \operatorname{cn} u(-1 + 2k^2 - 2k^2 \operatorname{cn}^2 u)$, $(z_1 = r(k))$

When $k \leq \sqrt{1/2}$ (see Fig.6(a)), simple differential calculation gives

$$\text{(47)} \qquad \frac{d(\operatorname{cn} u(-1 + 2k^2 - 2k^2\operatorname{cn}^2 u))}{du} = \sqrt{(1 - \operatorname{cn}^2 u)(1 - k^2 + k^2\operatorname{cn}^2 u)} \times \{6k^2\operatorname{cn}^2 u - 2k^2 + 1\}.$$

Integrating each side of Eq.(47) over u, we have

$$\text{(48)} \qquad u = \int_{-1}^{\operatorname{cn} u(-1+2k^2-2k^2\operatorname{cn}^2 u)} \frac{dZ}{\sqrt{(1-X^2(Z))(1-k^2+k^2X^2(Z))}\{6k^2X^2(Z)-2k^2+1\}},$$

where $X^2(Z)$ is given by Eq.(34). ACI measure of the map τ_z is defined in the form of inverse of elliptic functions, i.e., elliptic integral.

$$\text{(49)} \qquad u(z) = \int_{-1}^{z} f_Z(\xi_1(Z))dZ, \qquad \text{for } -1 \leq z \leq 1, k \leq \sqrt{1/2}.$$

The same discussion applies to $k > \sqrt{1/2}$ case with care to constants of integration (see Fig.6(b)).

(50)

$$\begin{cases} u_1(z) = \int_{-1}^{z} f_Z(\xi_1(Z))dZ, & \text{for} \quad -1 \le z < -r(k) \\ u_2(z) = u_1(-r(k)) + \int_{-r(k)}^{z} f_Z(\xi_2(Z))dZ, & \text{for} \; -r(k) \le z < 0 \\ u_3(z) = u_2(0) + \int_{0}^{z} f_Z(\xi_4(Z))dZ, & \text{for} \quad 0 \le z < r(k) \\ u_4(z) = u_3(r(k)) - \int_{r(k)}^{z} f_Z(\xi_3(Z))dZ, & \text{for} \quad r(k) \ge z > -r(k) \\ u_5(z) = u_4(-r(k)) + \int_{-r(k)}^{z} f_Z(\xi_4(Z))dZ, & \text{for} \quad -r(k) \le z < 0 \\ u_6(z) = u_5(0) + \int_{0}^{z} f_Z(\xi_2(Z))dZ, & \text{for} \quad 0 \le z < r(k) \\ u_7(z) = u_6(r(k)) + \int_{r(k)}^{z} f_Z(\xi_1(Z))dZ, & r(k) \le z \le 1 \end{cases}$$

where $f_Z(X_i(Z))$ is given by Eq.(35) and

$$r(k) = \sqrt{\frac{2}{27}(-1+2k^2)^3}.$$

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