

Indeterminacy in a monetary economy with heterogeneous agents

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§1. Introduction

It is well known that indeterminacy and chaotic behaviour of equilibria can arise in a monetary economy. To the best of our knowledge, the first work analysing the relationship between a monetary economy and the indeterminacy of equilibria is Brock (1974), who shows that there are multiple equilibrium paths in a discrete-time version of a monetary model with a single agent and elastic labour supply. Gray (1984) and Obstfeld (1984) show that indeterminacy of monetary equilibria may arise in a model with a nonseparable utility function in real money holdings and consumption in continuous-time frameworks. In addition, Mino (1984) studies indeterminacy in connection with several endogenized money supply rules. Matsuyama (1991) finds that chaotic behaviour of equilibria also arises in a discrete-time framework. Fukuda (1993) demonstrates that these results also hold in a model with separable utility function. However, all of the above studies mainly concern indeterminacy and chaotic behaviour of equilibria in connection with the monetary policy or preferences of a single agent in an economy. In contrast, this paper focuses on heterogeneity of real asset holdings and its relationship to indeterminacy of monetary equilibria.

The linkage between indeterminacy and heterogeneity of agents has been investigated in several recent studies. Using an overlapping-generations model with heterogeneous agents, Ghigliano and Tvede (1995) show that heterogeneity may generate indeterminacy and cycles. Ghigliano and Olszak-Duquenne (2001) and Ghigliano and Soger (2002) demonstrate that these results also hold in the discrete-time version of a two-sector

Received October 12, 2007.

¹ The author is grateful to Takuma Kunieda, John Stachurski, Makoto Yano, and all other participants of the conference, some of whom provided him with helpful comments. Especially, he is grateful to Atsumasa Kondo, Kazuo Nishimura, and Akihisa Shibata for their generous encouragement and suggestions and to anonymous referee for his helpful suggestions.

model with Leontief-type production and in a continuous-time version of a one-sector model with externalities and elastic labour supply¹. In a similar spirit, this paper investigates indeterminacy of equilibria in connection with wealth distribution in a standard model of money-in-the-utility function.

§2. Model

In the economy, there are J types of household, indexed by $j = 1, \dots, J$. Each household has additive separable preferences between periods and between goods and money holdings. The households also have the same positive discount factor, denoted by β . In specific terms, the problem to be solved is as follows:

$$\max \sum_{t=0}^{\infty} \beta^t (u_j(c_{jt}) + v_j(m_{jt})) \quad j = 1, \dots, J$$

(1)

$$\text{s.t. } P_t y_j + (1 + i_t) Q_t a_{jt} + M_{jt} + \psi_j X_t = P_t c_{jt} + Q_{t+1} a_{j,t+1} + M_{j,t+1},$$

where i_t denotes the nominal interest; P_t , the price of goods; Q_t , the price of the capital asset, and X_t , an aggregate nominal transfer to households in period t ². Further, ψ_j and y_j denote respectively an exogenous income received by and the share of the nominal transfer to household j , and they are assumed to be independent of periods. Finally, c_{jt} denotes the consumption of goods; M_{jt} , money holdings; m_{jt} , real money holdings, that is, $m_{jt} = M_{jt}/P_t$; and a_{jt} , the non-produced capital asset, such as land, of household j in period t . The capital asset is assumed to be initially supplied to each household at an amount $\theta_j \bar{k}$, or $a_{j0} = \theta_j \bar{k}$, where \bar{k} is the aggregate amount of capital and θ_j is the initial share of the endowment of household j . The capital asset is assumed to yield a fixed outcome of r per unit³. This implies that the nominal

¹See Ghiglinò (2005) and Ghiglinò and Olszak-Duquenne (2005) for other works studying the linkage between indeterminacy and heterogeneity.

²The existence of nominal transfers, or negative inflation taxes, will be the source of indeterminacy of equilibria in the model of this paper. In general, the distortion tax, as well as externalities, is known to be one of the sources of indeterminacy.

³The assumptions that the exogenous income and the interest rate are ensured by the assumptions that the amounts of labour and the initial endowment of capital are exogenous and they cannot employ other inputs. Suppose that the aggregate production function takes a Cobb-Douglas form: $y = \bar{k}^\alpha l^{1-\alpha}$, where

interest rate satisfies $(1 + i_t) Q_t = P_{t+1}r + Q_{t+1}$. The left-hand side of this equation presents the gross nominal revenue obtained by giving up a unit of the capital asset. The right-hand side of the equation is the gross nominal revenue of a unit of the capital asset since a unit of the capital asset in period t yields r units of outcome in period $t + 1$ and its price is Q_{t+1} . It follows that the nominal interest rate can be written as $1 + i_t = (r + q_{t+1})(1 + \pi_t)/q_t$, where q_t — the relative price of capital to output — Q_t/P_t and the inflation rate π_t equals $(P_{t+1} - P_t)/P_t$. Hence, it follows from (1) that the lifetime budget constraint can be written as

$$(2) \quad \begin{aligned} & (r + q_1) P_1 \theta_j \bar{k} + \sum_{t=0}^{\infty} \prod_{s=1}^t \frac{1}{1 + i_s} (P_t y_j + M_{jt} + \psi_j X_t) \\ & = \sum_{t=0}^{\infty} \prod_{s=1}^t \frac{1}{1 + i_s} (P_t c_{jt} + M_{jt+1}), \end{aligned}$$

where θ_j denotes the share of household j in the aggregate capital stock. This implies that $\sum_{j=1}^J n_j \theta_j = 1$, where n_j is the number of household j . Moreover, we assume that $\prod_{s=1}^0 1/(1 + i_s) = 1$ for tractability. Therefore, the Lagrangian of this problem can be written as

$$(3) \quad \begin{aligned} \mathcal{L}_j &= \sum_{t=0}^{\infty} \beta^t (u_j(c_{jt}) + v_j(m_{jt})) \\ &+ \lambda_j \left[(1 + q_1) P_1 \theta_j \bar{k} + \sum_{t=0}^{\infty} \prod_{s=1}^t \frac{1}{1 + i_s} (P_t y_j + M_{jt} + \psi_j X_t - P_t c_{jt} - M_{jt+1}) \right]. \end{aligned}$$

The first-order conditions of this problem are as follows:

$$\beta^t u'_j(c_{jt}) = \lambda_j P_t \prod_{s=1}^t \frac{1}{1 + i_s} \quad \text{and} \quad \beta^t v'_j(m_{jt}) = \lambda_j i_{t+1} P_t \prod_{s=1}^{t+1} \frac{1}{1 + i_s}.$$

y and l are the aggregate amounts of output and labour employed, respectively. In this case, the interest rate, r , and the wage rate, w , are determined independently of periods since the amounts of capital and labour are fixed over periods: $r = \alpha \bar{k}^{\alpha-1} l^{1-\alpha}$ and $w = (1 - \alpha) k^{\alpha} l^{-\alpha}$, respectively. The latter of the two equations also implies the income of household j is constant through periods: $y_j = w l_j$, where l_j is the amount of labour supplied by household j .

Moreover, the market-clearing conditions of this economy are

$$(4) \quad y + r\bar{k} = c_t \quad \text{and} \quad M_t = \sum_{j=1}^J n_j M_{jt},$$

where y and c_t denote the aggregate amount of income and consumption in period t , respectively; $y = \sum_{j=1}^J n_j y_j$ and $c_t = \sum_{j=1}^J n_j c_{jt}$. The market-clearing conditions implies Walras's law that yields

$$(5) \quad X_t = (\mu - 1)M_t,$$

where M_t denotes the aggregate money supply in period t and μ denotes the gross growth rate of money supply; $\mu = M_{t+1}/M_t$. To be well defined the problem, we assume that $\mu > \beta$. This also implies that

$$(6) \quad \pi_t = m_t/m_{t+1}\mu - 1.$$

We then proceed to consider the steady-state equilibria, the following must hold: $c_j^* = c_{jt}$, $m_j^* = m_{jt}$ and $q^* = q_t$ for all t . It follows from (6) that these conditions imply that $P_{t+1} = \mu P_t$, $Q_{t+1} = \mu Q_t$ and $1 + i^* = (r/q^* + 1)\mu$. Thus, the first-order conditions can be rewritten as

$$(7) \quad u'_j(c_j^*) = \lambda_j P_0 \left[\frac{1}{\beta \left(\frac{r}{q^*} + 1 \right)} \right]^t \quad \text{and} \quad v'_j(m_j^*) = \lambda_j P_0 i^* \left[\frac{1}{\beta \left(\frac{r}{q^*} + 1 \right)} \right]^{t+1}.$$

For these two conditions to be well defined, the equality $q^* = r\beta/(1 - \beta)$ must hold. Substituting it back into (7), we have

$$(8) \quad u'_j(c_j^*) = \lambda_j P_0 \quad \text{and} \quad v'_j(m_j^*) = \lambda_j P_0 \left(\frac{\mu}{\beta} - 1 \right).$$

Here, these two equations show c_j^* and m_j^* to be decreasing functions of the Lagrange multiplier λ_j . Moreover, it follows from (2) that the consumption of household j in a steady state can be written as

$$(9) \quad c_j^* = y_j + r\theta_j \bar{k} + (\mu - 1)(\psi_j m^* - m_j^*),$$

where

$$(10) \quad m^* = \sum_{j=1}^J n_j m_j^*.$$

Moreover, (9) can be rewritten as

$$(11) \quad c_j^* + (\mu - 1)m_j^* = y_j + r\theta_j\bar{k} + (\mu - 1)\psi_j m^*.$$

To ensure the uniqueness of the steady state, we assume that the demand of each household for money holdings is equal to the amount of monetary transfer; thus, $m_j^* = \psi_j m^*$ ⁴. In this case, (11) is simplified as

$$(12) \quad c_j^* = y_j + r\theta_j\bar{k}.$$

Combining equations (7) and (12), we have

$$(13) \quad \frac{v_j'(m_j^*)}{u_j'(y_j + r\theta_j\bar{k})} = \frac{\mu}{\beta} - 1.$$

This equation implies that the real money holdings of household j are increasing in its income and capital endowment.

§3. Aggregate behaviour

As shown in Negishi (1960) and in Kehoe, Levine, and Romer (1992), the aggregate behaviour of the economy can be characterized by the following problem:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t W(c_t, m_t; \alpha_1, \dots, \alpha_J) \\ \text{s.t.} \quad & c_t = y + r\bar{k} + m_t - (1 + \pi_t)m_{t+1} + \frac{X_t}{P_t}, \end{aligned}$$

where, $W(\cdot)$ is a Negishi function that is defined as follows:

$$(14) \quad W(c_t, m_t; \alpha_1, \dots, \alpha_J) = \max_{\{c_{jt}, m_{jt}\}} \sum_{j=1}^J \alpha_j n_j (u_j(c_{jt}) + v_j(m_{jt})).$$

Here, α_j is the reciprocal of the Lagrange multiplier of household j weighted by its population, $\alpha_j = 1/(\lambda_j n_j)$. To characterize the Negishi

⁴Equations (10) and (11) determine the value of λ_j , and thus, that of c_j^* and m_j^* from (8). However, these equations imply the possibility of multiple steady states since both hand sides of (10) are decreasing in λ_j from (8) and (10). Although the existence of multiple steady states is an interesting issue, we only address the case of a unique steady state in this paper.

function, we define the Lagrangian of this problem as

$$(15) \quad \mathcal{L}_t = \sum_{j=1}^J \alpha_j n_j (u_j(c_{jt}) + v_j(m_{jt})) \\ + \lambda_{ct} \left(c_t - \sum_{j=1}^J n_j c_{jt} \right) + \lambda_{mt} \left(m_t - \sum_{j=1}^J n_j m_{jt} \right).$$

The necessary conditions of this problem can be written as

$$(16) \quad \alpha_j n_j u'_j(c_{jt}^*) = \lambda_{ct} \quad \text{and} \quad \alpha_j n_j v'_j(m_{jt}^*) = \lambda_{mt} \quad \text{for all } j,$$

where c_{jt}^* and m_{jt}^* denote the optimal consumption and money holdings of household j , respectively⁵. Since, from (16), both c_{jt} and m_{jt} are monotonically decreasing in λ_{ct} and λ_{mt} , respectively, c_{jt} and m_{jt} are uniquely determined if c_t and m_t are given. Therefore, we define Negishi functions for goods and money holdings as $\hat{u}(c_t) = \sum_{j=1}^J \alpha_j n_j u_j(c_{jt}^*)$ and $\hat{v}(m_t) = \sum_{j=1}^J \alpha_j n_j v_j(m_{jt}^*)$, respectively. Using these notations, we can express the Bellman equation for the intertemporal problem as⁶

$$(17) \quad \hat{V}_t(m_t) = \max \hat{u} \left(y + r\bar{k} + m_t - (1 + \pi_t) m_{t+1} + \frac{X_t}{P_t} \right) \\ + \hat{v}(m_t) + \beta \hat{V}_{t+1}(m_{t+1}),$$

where \hat{V}_t is the social value function. Therefore, the necessary and envelope conditions for maximization yields

$$(18) \quad \hat{u}'(c_t) - \beta \frac{P_t}{P_{t+1}} (\hat{u}'(c_{t+1}) + \hat{v}'(m_{t+1})) = 0.$$

Using (4), (5) and (6), equation (18) can be rewritten as

$$(19) \quad m_t = \frac{\beta}{\mu} m_{t+1} \left(1 + \frac{\hat{v}'(m_{t+1})}{\hat{u}'(y + r\bar{k})} \right).$$

⁵Note that the first-order conditions (16) are the same as the above equations of (4) if $\lambda_{ct} = \beta^{-t} P_t \prod_{s=1}^t 1/(1+i_s)$ and $\lambda_{mt} = \beta^{-t} P_t i_{t+1} \prod_{s=1}^{t+1} 1/(1+i_s)$, which imply that the solution of the problem in this section represents equilibria of the market economy considered in Section 2.

⁶As in Kehoe, Levine, and Romer (1992) and in Ghigliino and Olszak-Duquenne (2001), we can call the solution of this problem to be *pseudo*-Pareto optimum in the sense that it is the solution to the maximization of a Negishi function under given nominal transfers.

The above equation describes the aggregate behavior of the real stock of money. Moreover, in a steady state, since the amount of real money holdings is constant over periods, (19) can be rewritten as follows:

$$(20) \quad \frac{\hat{v}'(m^*)}{\hat{u}'(y + rk)} = \frac{\mu}{\beta} - 1.$$

This equation determines the aggregate real stock of money in a steady state.

§4. Indeterminacy

This section considers a condition for indeterminacy of equilibria. From (19), we know equilibria are locally indeterminate if the absolute value of the gradient of the right-hand side of (19) with respect to m_{t+1} is greater than 1 around the steady state. Therefore, it follows from (20) that this condition can be written as

$$(21) \quad \hat{\eta}(m^*) < \frac{1}{2} \left(1 - \frac{\beta}{\mu} \right),$$

where $\hat{\eta}$ denotes a social intertemporal elasticity of substitution in money holdings; that is, $\hat{\eta}(m^*) = -\hat{v}'(m^*) / (\hat{v}''(m^*) m^*)$. Thus, (21) suggests that a lower social intertemporal elasticity of substitution tends to generate indeterminacy. To investigate this condition in greater detail, we calculate the social intertemporal elasticity of substitution in a manner similar to that of Ghigliano (2005), that yields

$$(22) \quad \hat{\eta}(m^*) = \sum_{j=1}^J n_j \eta_j(m_j^*) \frac{m_j^*}{m^*},$$

where $\eta_j(m_j^*)$ denotes the intertemporal elasticity of substitution of the individual utility of money holdings: $\eta_j(m_j^*) = -v'_j(m_j^*) / (m_j^* v''_j(m_j^*))$. This equation, together with (21), implies that wealth distribution may cause the indeterminacy of monetary equilibria since (13) suggests that m_j^* depends on the distribution of income and the initial shares of the capital asset among agents.

§5. Examples

Here, we present a few examples with specific forms of utility functions. In the following examples, we assume throughout that there are only two types of households with the same population 1/2 and income flow y and that their utility functions with respect to consumption take an identical logarithmic form: $u_j(c_{jt}) = \ln c_{jt}$.

■ **CIES Utility** When $v_j(m_{jt}) = \eta_j / (\eta_j - 1) m_{jt}^{\eta_j / (\eta_j - 1)}$, the social intertemporal elasticity of substitution can be written as

$$\hat{\eta}_{\text{CIES}} = \frac{\sum_{j=1}^2 \left(\frac{\mu}{\beta} - 1\right)^{\eta_j - 1} (y + r\theta_j \bar{k})^{1 - \eta_j} \eta_j}{\sum_{j=1}^2 \left(\frac{\mu}{\beta} - 1\right)^{\eta_j - 1} (y + r\theta_j \bar{k})^{1 - \eta_j}}.$$

The above expression implies that the difference of the individual intertemporal elasticity of substitutions plays an important role in indeterminacy of monetary equilibria. However, the heterogeneity of the initial share of capital asset holdings plays no role in the occurrence of indeterminacy if the individual intertemporal elasticity of substitution is identical over households.

■ **CARA Utility** In contrast to the first example, the following two examples are more interesting since wealth distribution has a crucial role in indeterminacy even if the utility functions are identical across households. When $v_j(m_{jt}) = -1/a \exp(-am_{jt})$, it can be written as

$$\hat{\eta}_{\text{CARA}} = \frac{1}{\frac{1}{2} \sum_{j=1}^2 \ln(y + r\theta_j \bar{k}) - \ln\left(\frac{\mu}{\beta} - 1\right)}.$$

In this case, wealth distribution is crucial even if the preferences of agents are identical. Figure 1 illustrates that indeterminacy tends to arise in a highly egalitarian economy.

■ **Quadratic Utility** When $v_j(m_{jt}) = -b/2 (m_{jt} - \bar{m})^2$, it can be written as

$$\hat{\eta}_{\text{QD}} = \frac{1}{\frac{\beta}{\mu - \beta} b \bar{m} \left(\sum_{j=1}^2 \frac{1}{y_j + r\theta_j \bar{k}} \right)^{-1} - 1}.$$

This case also derives a result similar to that in the case of CARA utility.

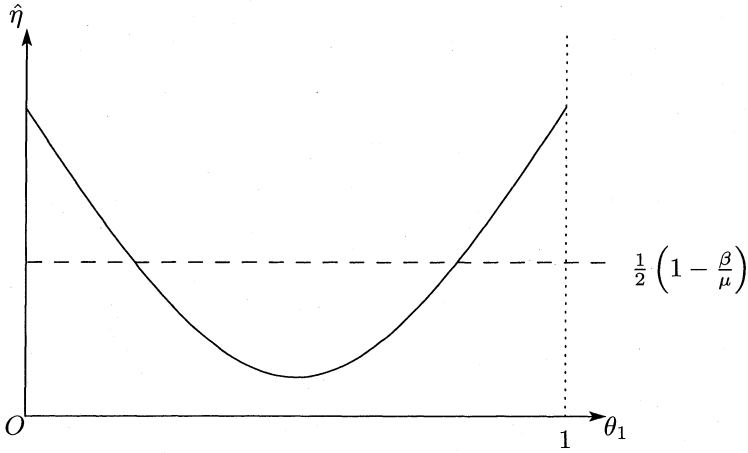


Fig. 1. CARA and Quadratic Cases

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