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On the existence of almost periodic solutions to a nonlinear Volterra difference equation

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Abstract.

The existence of almost periodic solutions of a nonlinear Volterra difference equation with infinte delay is obtained by using some restrictive conditions of the equation.

§1. Introduction

For ordinary differential equations and integrodifferential equations, the existence of almost periodic solutions of almost periodic systems have been studied by many authors [2,3,6,7]. Hamaya [3] has investigated the existence of almost periodic solutions of functional difference equations with infinite delay by using stability properties of bounded solutions. Y. Song, C.T.H. Baker and H. Tian [6] have studied the existence of periodic and almost periodic solutions for nonlinear Volterra difference equations by means of certain stability conditions. However, it is not easy to know whether or not there exists a bounded solutions for nonlinear Volterra difference equations. In this paper, we discuss the existence of nonlinear Volterra difference equations motivated by C. Feng [1] and S. Kato and M. Imai [5]. Our restrictive conditions to guarantee the existence of almost periodic solutions are weaker than that of Hamaya [3], and Y. Song, C.T.H. Baker and H. Tian [6], and moreover, this paper is based on [4].

We first introduce an almost periodic function $f(n,x) : Z \times D \to R^l$, where D is an open set in R^l, and $f(n,\cdot)$ be continuous for each $n \in Z$.

Definition 1. $f(n,x)$ is said to be almost periodic in n uniformly for $x \in D$, if for any $\epsilon > 0$ and any compact set K in D, there exists a

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positive integer $L^*(\epsilon, K)$ such that any interval of length $L^*(\epsilon, K)$ contains an integer τ for which

$$(1) \qquad |f(n+\tau, x) - f(n, x)| \leq \epsilon$$

for all $n \in Z$ and all $x \in K$. Such a number τ in (1) is called an ϵ-translation number of $f(n, x)$.

In order to formulate a property of almost periodic functions, which is equivalent to the above definition, we discuss the concept of the normality of almost periodic functions. Namely, Let $f(n, x)$ be almost periodic in n uniformly for $x \in D$. Then, for any sequence $\{h_k^{'}\} \subset Z$, there exists a subsequence $\{h_k\}$ of $\{h_k^{'}\}$ such that the sequence of functions $\{f(n + h_k, x)\}$ converges uniformly on $Z \times S$, where S is a compact set in D. In other words, for any sequence $\{h_k^{'}\} \subset Z$, there exists a subsequence $\{h_k\}$ of $\{h_k^{'}\}$ and function $g(n, x)$ such that

$$f(n + h_k, x) \to g(n, x)$$

uniformly on $Z \times K$ as $k \to \infty$, where K is any compact set in D. There are many properties of the discrete almost periodic functions [6], which are corresponding properties of the continuous almost periodic function $f(t, x) \in C(R \times D, R^l)$ [cf.2,7].
The following concept of asymptotic almost periodicity was introduced by Frechet in the case of continuous function (cf.[2,7]).

Definition 2. $u(n)$ is said to be asymptotically almost periodic if it is a sum of an almost periodic function $p(n)$ and a function $q(n)$ defined on $I^* = [a, \infty) \subset Z^+$ which tends to zero as $n \to \infty$, that is,

$$u(n) = p(n) + q(n).$$

$u(n)$ is asymptotically almost periodic if and only if for any sequence $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$ there exists a subsequence $\{n_{k_j}\}$ for which $u(n + n_{k_j})$ converges uniformly on I^*.

We shall consider the following Volterra difference equation

$$(2) \quad \Delta x(n) = f(n, x(n)) + \sum_{m=-\infty}^{n} F(n, m)x(m) + p(n), \quad n \geq 0,$$

where $f(n, x)$ is almost periodic in n uniformly for x, $p(n)$ is an almost periodic function and $F(n, m)$ is almost periodic, that for any $\epsilon > 0$

and any compact set K, there exists an integer $L^{**} = L^{**}(\epsilon, K) > 0$ such that any interval of length L^{**} contains an τ for which

$$|F(n+\tau, m+\tau) - F(n,m)| \leq \epsilon.$$

Let R^l be Euclidean space with norm denoted by $\|\cdot\|$. We define the functional $[\cdot,\cdot] : R^l \times R^l \to R$ by

$$[x,y] = h^{-1}(\|x+hy\| - \|x\|) \quad \text{for } h > 0.$$

Let x, y and z be in R^l, from [5], the functional $[\cdot,\cdot]$ has the following properties:

(*i*) $|[x,y]| \leq \|y\|$;

(*ii*) $[x, y+z] \leq [x,y] + [x,z]$;

(*iii*) $\Delta_h^+ \|u(n)\| = [u(n), \Delta_h u(n)]$,
where u is a function and $\Delta_h^+\|u(n)\|$ denotes the h-differences of $\|u(n)\|$, that is $\Delta_h^+\|u(n)\| = h^{-1}(\|u(n+h)\| - \|u(n)\|)$.

§2. Main results

We assume the following conditions:

(I) $\|p(n) + f(n,0)\| \leq L$ for all $n \in Z$, where L is a positive constant.

(II) There exists a positive constant γ such that

$$\lim_{n-m\to\infty} \frac{1}{n-m} \sum_{j=m}^{n-1} e(j) = -\gamma < 0,$$

where the function $e(n) : Z \to R$.
(III) for all $(n,x) \in Z \times R^l$,

$$[x-y, f(n,x) - f(n,y) + \sum_{m=-\infty}^{n} F(n,m)x(m) - \sum_{m=-\infty}^{n} F(n,m)y(m)] \leq e(n)\|x-y\|.$$

We have two preliminary lemmas.

Lemma 1. Suppose that $e(n)$ is an almost periodic function. Then, for any $b \in Z$, the following equality holds

$$\lim_{n-m\to\infty} \frac{1}{n-m} \sum_{j=m}^{n-1} e(j) = \lim_{n-m\to\infty} \frac{1}{n-m} \sum_{j=m}^{n-1} e(j+b).$$

By discrete analogy of Corollary 3.2 in [2], we can prove this lemma. So, we will omit it.

Lemma 2. Suppose that (II) is satisfied. Then, for any $b \in Z$, the following inequality holds

$$\exp(\sum_{j=m}^{n-1} e(j+b)) \le \beta \exp(-\alpha(n-m)). \tag{3}$$

where $e(n)$ is an almost periodic function, α and β are positive constants independent for b.

Proof. From Lemma 1, for $\epsilon = \frac{\gamma}{4} > 0$, there exists a sufficiently large $T = T(\epsilon)$ such that $|\frac{1}{n-m}\sum_{j=m}^{n-1} e(j+b) - (-\gamma)| < \epsilon$, namely,

$$\sum_{j=m}^{n-1} e(j+b) < -\frac{3\gamma(n-m)}{4}, \quad \text{for } n-m \ge T. \tag{4}$$

Since $e(n)$ is an almost periodic function, there exists a positive constant c such that inequality $|e(j+b)| < c$ holds for any $j, b \in Z$. Thus, when $|n-m| \le T$, we have

$$|\sum_{j=m}^{n-1} e(j+b)| \le \sum_{j=m}^{n-1} |e(j+b)| \le cT. \tag{5}$$

Let $\alpha = \frac{3\gamma}{4}$ and $\beta = \exp(cT + \alpha T)$, inequality (4) and (5) imply that (3) holds. Q.E.D.

Now, we have main result of this paper.

Theorem. Suppose that assumptions (I),(II) and (III) hold, then equation (2) has a unique almost periodic solution.

Proof. We first show that equation (2) has a bounded solution.

Let $x(n)$ be a solution of equation (2), then we have

$$\begin{aligned}\Delta^+\|x(n)\| &= [x(n), \Delta x(n)] \\ &= [x(n), f(n,x(n)) + \sum_{m=-\infty}^{n} F(n,m)x(m) + p(n)] \\ &\leq [x(n), f(n,x(n)) - f(n,0) + \sum_{m=-\infty}^{n} F(n,m)x(m) - 0] \\ &+ [x(n), f(n,0) + p(n)] \\ &\leq [x(n), f(n,x(n)) - f(n,0) + \sum_{m=-\infty}^{n} F(n,m)x(m)] \\ &+ \|f(n,0) + p(n)\| \\ &\leq e(n)\|x(n)\| + L. \qquad (6)\end{aligned}$$

Solving this difference inequality, from Lemma 2 and

$$\prod_{i=0}^{n-1}(1+|e(i)|) \leq \prod_{i=0}^{n-1}\exp(|e(i)|) = \exp(\sum_{i=0}^{n-1}|e(i)|) \leq \beta\exp(-\alpha n),$$

we obtain

$$(7) \qquad \|x(n)\| \leq \beta\|x(0)\| + \frac{L\beta}{\alpha^*} \quad n \geq 0,$$

where $\alpha^* = \dfrac{\exp(\alpha) - 1}{\exp(\alpha)}$.

This implies that $x(n)$ $(n \geq 0)$ is bounded. Since equation (2) is almost periodic equation, from our assumptions, there exists subsequence $\{n_k\}, n_k \to +\infty$ as $k \to +\infty$ such that $x(n+n_k)$ locally uniformly converges to $\hat{x}$ for $n \geq 0$. Moreover, $x(n+n_k)$ is a solution of the following equation

$$\begin{aligned}\Delta x(n+n_k) &= f(n+n_k, x(n+n_k)) \\ &+ \sum_{m=-\infty}^{n} F(n+n_k, m+n_k)x(m+n_k) + p(n+n_k).\end{aligned}$$

Let $k \to \infty$, we have

$$\Delta\hat{x}(n) = f(n,\hat{x}(n)) + \sum_{m=-\infty}^{n} F(n,m)\hat{x}(m) + p(n),$$

that is, $\hat{x}(n)$ is a solution of equation (2). On the other hand, $x(n+n_k) \le \beta\|x(0)\| + \frac{L\beta}{\alpha^*}$ for $n > -n_k$. Let $k \to \infty$ we have $\hat{x}(n) \le \beta\|x(0)\| + \frac{L\beta}{\alpha^*}$ for $n \in Z$. This means that equation (2) has a bounded solution.

Now, we will show that equation (2) has a unique bounded solution. Suppose that $\hat{x}_1(n)$ and $\hat{x}_2(n)$ for $n \in Z$ are two bounded solutions of equation (2), then we have

$$\begin{aligned} \Delta^+\|\hat{x}_1(n) - \hat{x}_2(n)\| &= [\hat{x}_1(n) - \hat{x}_2(n), \Delta(\hat{x}_1(n) - \hat{x}_2(n))] \\ &\le e(n)\|\hat{x}_1(n) - \hat{x}_2(n)\|. \end{aligned}$$

Solving this difference inequality, we obtain

$$\|\hat{x}_1(n) - \hat{x}_2(n)\| \le \|\hat{x}_1(0) - \hat{x}_2(0)\| \exp(\sum_{i=0}^{n-1} e(i)).$$

Then $\|\hat{x}_1(n) - \hat{x}_2(n)\| \to 0$ as $n \to \infty$, this implies that there is a unique bounded solution of equation (2). According to [3,6,7], if there exists an asymptotically almost periodic solution for an almost periodic difference equation, then there exists an almost periodic solution of the equation. Therefore, in order to show that there exists an almost periodic solution of the equation (2), we prove that there exists an asymptotically almost periodic solution of this equation. Suppose that $x(n)$ is a bounded solution of equation (2), then for sequence $\{r_k\}, r_k \to \infty$ as $k \to \infty$, $\{x(r_k)\}$ is a bounded sequence. Thus, there exists a convergent subsequence $\{x(n_k)\}$ of $\{x(r_k)\}$. Consider sequence $\{x(n+n_k)\}$, then we have

$$\begin{aligned} &\Delta^+\|x(n+n_k) - x(n+n_j)\| \\ &\le [x(n+n_k) - x(n+n_j), \Delta(x(n+n_k) - x(n+n_j))] \\ &\le [x(n+n_k) - x(n+n_j), f(n+n_k, x(n+n_k)) \\ &- f(n+n_j, x(n+n_j)) + \sum_{m=-\infty}^{n} F(n+n_k, m+n_k)x(m+n_k) \\ &- \sum_{m=-\infty}^{n} F(n+n_j, m+n_j)x(m+n_j)] + \|f(n+n_k, 0) \\ &- f(n+n_j, 0)\| + \|p(n+n_k) - p(n+n_j)\| \\ &\le e(n)\|x(n+n_k) - x(n+n_j)\| + \|f(n+n_k, 0) - f(n+n_j, 0)\| \\ &+ \|p(n+n_k) - p(n+n_j)\|. \qquad (8) \end{aligned}$$

Solving difference inequality (8), we obtain

$$\begin{aligned}
&\|x(n+n_k) - x(n+n_j)\| \\
&\leq \beta\|x(n_k) - x(n_j)\| + \frac{\beta}{\alpha^*}\|f(n+n_k, 0) - f(n+n_j, 0)\| \\
&+ \frac{\beta}{\alpha^*}\|p(n+n_k) - p(n+n_j)\|.
\end{aligned}$$

Since sequence $\{x(n_k)\}$ is convergent, so there exists N_1 sufficently large such that for any $\epsilon > 0$, if $k, j > N_1$, then

$$\|x(n_k) - x(n_j)\| < \frac{\epsilon}{3\beta}.$$

Note that $f(n, x(n))$ is almost periodic in n uniformly for x, and $p(n)$ is a almost periodic function. Therefore, there exists N_2 sufficently large, such that if $k, j > N_2$, we have $\|f(n+n_k, 0) - f(n+n_j, 0)\| < \frac{\epsilon\alpha^*}{3\beta}$ and similar, if $k, j > N_3$, we have $\|p(n+n_k) - p(n+n_j)\| < \frac{\epsilon\alpha^*}{3\beta}$. Let $N_0 = \max\{N_1, N_2, N_3\}$. Then if $k, j > N_0$, we have

$$\|x(n+n_k) - x(n+n_j)\| < \beta \cdot \frac{\epsilon}{3\beta} + \frac{\beta}{\alpha^*} \cdot \frac{\epsilon\alpha^*}{3\beta} + \frac{\beta}{\alpha^*} \cdot \frac{\epsilon\alpha^*}{3\beta} = \epsilon.$$

This means $x(n)$ is an asymptotically almost periodic solution of equation (2). Note that almost periodic solution is bounded, equation (2) has one and only one bounded solution, thus, there exists a unique almost periodic solution of equation (2). The proof is completed.

Q.E.D.

The next example can be adapt to Theorem by slight modification for equation (2) with term $g(n, x(n-\tau))$ (cf. [4]).

Example. We consider the following almost periodic equation

$$\begin{aligned}
\Delta x(n) &= -\left(\frac{2}{3} + \frac{x(n)\sin n}{\sqrt{1+x^2(n)}}\right)x(n) \\
&+ \frac{1}{4}\sum_{m=-\infty}^{n} x(m)e^{-(n-m)}\cos n + \sqrt{|x(n-k_0)|}\sin\sqrt{2}n + 2\sin n. \qquad (9)
\end{aligned}$$

We do not know whether or not this equation (9) has a bounded solutiona . Even if suppose that equation (9) has a bounded solution, since $\sqrt{|x(n-k_0)|}\sin\sqrt{2}n$ can not be sufficently small as $n \to \infty$, so the Song, Baker and Tian's theorem [6] also fails for equation (9). We

note that $F(n,m) = \frac{1}{4}\exp(-(n-m))\cos n$ and
$\frac{1}{4}\sum_{m=-\infty}^{n} |F(n,m)| < \infty$. Obviously, equation (9) satisfies all assumptions of our theorem, so there exists a unique almost periodic solution of equation (9).

References

[1] C. Feng, On the existence of almost periodic solutions for some nonlinear delayed integrodifferential equations, to appear.

[2] A. M. Fink, Almost Periodic Differential Equations, Lecture Notes in Math., **377**, Springer-Verlag, Berlin, 1974.

[3] Y. Hamaya, Existence of almost periodic solutions in a difference equation with infinite delay, J. Difference Equ. Appl., **9** (2003), 227–237.

[4] Y. Hamaya, On the existence of almost periodic solutions of a nonlinear Volterra difference equation, Int. J. Difference Equ., **2** (2007), 187–196.

[5] S. Kato and M. Imai, Remarks on the existence of almost periodic solutions of systems of nonlinear differential equations, Nonlinear Anal., **25** (1995), 490–415.

[6] Y. Song and H. Tian, Periodic and almost periodic solutions of nonlinear discrete Volterra equations with unbounded delay, J. Comput. Appl. Math., **205** (2007), 859–870.

[7] T. Yoshizawa, Stability theory and the existence of periodic solutions and almost periodic solutions, Applied Mathematical Science, **14**, Springer, 1975.

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