Advanced Studies in Pure Mathematics 53, 2009
Advances in Discrete Dynamical Systems
pp. 41–50

Oscillation theory of symplectic difference systems

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Abstract.

Recent results in the oscillation theory of symplectic difference systems are presented. A particular attention is devoted to Sturmian and oscillation theorems for these systems and for the so-called reversed symplectic systems. Some open problems and conjectures are formulated.

§1. Introduction

We deal in this contribution with oscillatory properties of solutions of symplectic difference systems (further SDS)

$$z_{k+1} = \mathcal{S}_k z_k, \tag{1}$$

where $z_k \in \mathbb{R}^{2n}$ and the $2n \times 2n$-matrices $\mathcal{S}_k$ are *symplectic*, i.e.,

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J} \quad \text{with} \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

If we split a solution $z \in \mathbb{R}^{2n}$ of (1) into two n-dimensional vectors x, u, i.e. $z = \binom{x}{u}$, and $\mathcal{S}$ is considered in the form $\mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ with $n \times n$ matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, then system (1) takes the form

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k \tag{2}$$

and symplecticity of $\mathcal{S}$ is equivalent to the identities

$$\mathcal{A}^T \mathcal{C} = \mathcal{C}^T \mathcal{A}, \quad \mathcal{B}^T \mathcal{D} = \mathcal{D}^T \mathcal{B}, \quad \mathcal{A}^T \mathcal{D} - \mathcal{C}^T \mathcal{B} = I. \tag{3}$$

Revised October 12, 2007.

Key words and phrases. Oscillation, Symplectic difference system, Discrete eigenvalue problem, principal solution, focal point, Sturmian theorem, Oscillation theorem.

Oscillatory properties of (1) are closely related to positivity/nonnegativity of the associated discrete quadratic functional

$$(4)\quad \mathcal{F}(z)=\mathcal{F}(x,u)=\sum_{k=0}^{N}\{x_k^T\mathcal{A}_k^T\mathcal{C}_k x_k+2x^T\mathcal{C}_k^T\mathcal{B}_k u_k+u_k^T\mathcal{B}_k^T\mathcal{D}_k u_k\}$$

considered over the class of sequences satisfying the first equation in (2) (the so-called *admissible sequences*, the first equation in (1) is usually called the *equation of motion*) and certain boundary conditions at the endpoints $k=0$ and $k=N+1$. We refer to the paper [6] where basic oscillatory properties of (1) are established and to the subsequent papers [8, 7, 11, 13, 14] (and references given therein), where various aspects of oscillation theory of (1) are investigated, including the problem of positivity and nonnegativity of (4).

Symplectic difference systems (1) cover a large variety of difference equations and systems, among them also the *linear Hamiltonian difference system*

$$(5)\qquad \Delta x_k=A_k x_{k+1}+B_k u_k,\quad \Delta u_k=C_k x_{k+1}-A_k^T u_k,$$

where the $n\times n$-matrices B_k and C_k are symmetric and $I-A_k$ is nonsingular, as discussed e.g. in the monograph by Ahlbrandt and Peterson [3]. Essentials of the oscillation theory of (5) where the matix B is allowed to be singular are established in the fundamental paper [5].

This means, in turn, that systems (1) also cover the *higher order Sturm-Liouville difference equation*

$$\sum_{\mu=0}^{n}(-\Delta)^\mu\{r_k^{[\mu]}\Delta^\mu y_{k+n-\mu}\}=0\quad\text{with}\quad r_k^{[n]}\neq 0,$$

in particular, its special case, the *Sturm-Liouville second order difference equation*

$$(6)\qquad \Delta(r_k\Delta x_k)+p_k x_{k+1}=0\quad\text{with}\quad r_k\neq 0,$$

which are well studied in the recent literature, see [1, 2, 12, 15].

The aim of this paper is to present some recent results of the oscillation theory of (1) which are complemented here by some new results and observations. A particular attention is devoted to the Sturmian theory for (1) and to the formulation of related open problems and conjectures.

§2. Sturmian and Oscillation theorems for SDS

The symplecticity of the matrices $\mathcal{S}_k$ in (1) means that a fundamental matrix $Z_k\in\mathbb{R}^{2n\times 2n}$ of (1) is symplectic, whenever it is symplectic at one particular index, say $k = 0$. If this is the case and we write Z in the form $Z = \binom{X\ \tilde{X}}{U\ \tilde{U}}$, where $\binom{X}{U}$, $\binom{\tilde{X}}{\tilde{U}}$ are $2n \times n$ solutions of (1), then each solution is called a *conjoined basis* and together form the so-called pair of *normalized* conjoined bases. Consequently, a conjoined basis $\binom{X}{U}$ of (1) is a $2n \times n$ matrix solution satisfying

$$\operatorname{rank}\begin{pmatrix} X \\ U \end{pmatrix} = n, \quad X^T U = U^T X.$$

Oscillatory properties of (1) are defined using the concept of a focal point as follows. We say that a conjoined basis $\binom{X}{U}$ has no *focal point* in the interval $(k, k+1]$ if

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{and} \quad X_k X_{k+1}^{\dagger} \mathcal{B}_k \geq 0 \tag{7}$$

holds. Here † and $\geq$ denote the Moore-Penrose generalized inverse and nonnegative definiteness of the matrix indicated, respectively. Note that if the first condition in (7) holds then the matrix $X_k X_{k+1}^{\dagger} \mathcal{B}_k$ is really symmetric, (cf. [6]), and it equals the matrix P_k given below by (8) since $T_k = I$ in this case.

The following matrices were introduced in [16]:

$$M_k = (I - X_{k+1} X_{k+1}^{\dagger}) \mathcal{B}_k, \quad T_k = I - M_k^{\dagger} M_k, \quad P_k = T_k^T X_k X_{k+1}^{\dagger} \mathcal{B}_k T_k, \tag{8}$$

for $k \in \{0, \dots, N\}$. Then obviously $M_k T_k = 0$ and it can be shown (see, e.g., [16]) that the matrix P_k is symmetric. The *multiplicity* of a focal point in the interval $(k, k+1]$ is defined as the number

$$\operatorname{rank} M_k + \operatorname{ind} P_k,$$

where ind stands for the index, i.e., the number of negative eigenvalues of the matrix indicated.

The next statement is proved in [9], it is a Sturmian type theorem for (1) and it can be regarded as a discrete version of [17, Lemma 7.1, p. 357] which concerns the linear Hamiltonian differential system

$$\dot{x} = A(t)x + B(t)u, \quad \dot{u} = C(t)x - A^T(t)u. \tag{9}$$

Proposition 1. *Suppose that there exists a conjoined basis of* (1) *with no focal point in* $(0, N+1]$. *Then any other conjoined basis of this system has at most n focal points in* $(0, N+1]$, *each focal point counted a number of times equal to its multiplicity.*

Proof. We will present a general idea of the proof only, details can be found in [9]. The proof is based on the fact that the existence of a conjoined basis of (1) with no focal point in $(0, N+1]$ implies positivity of the associated quadratic functional (4). Hence, by contradiction, suppose that there exists a conjoined basis $\binom{X}{U}$ of (1) having at least $n+1$ focal points in $(0, N+1]$, including multiplicities. Then it is possible to construct admissible sequences $\{z_k^{[i]}\}_{k=0}^{N+1} = \binom{x_k^{[i]}}{u_k^{[i]}}$, $i = 1, \dots, n+1$, such that $x_{N+1}^{[i]} = 0$, $\mathcal{F}(z^{[i]}) = (x_0^{[i]})^T u_0 + d^{[i]}$, where $d^{[i]}$ are certain nonpositve numbers (which can be specified explicitly, but this specification is rather technical), and

$$\begin{aligned}\mathcal{F}(z^{[i]}; z^{[j]}) \quad &:= \quad \sum_{k=0}^{N} \Big\{ (x_k^{[i]})^T \mathcal{A}_k^T \mathcal{C}_k x_k^{[j]} + 2(x_k^{[i]})^T \mathcal{C}_k^T \mathcal{B}_k u_k^{[j]} \\ &\qquad\qquad + (u_k^{[i]})^T \mathcal{B}_k^T \mathcal{D}_k u_k^{[j]} \Big\} \\ &= \quad 0\end{aligned}$$

for $i \neq j$. Now, there exists a nontrivial linear combination

$$z = \begin{pmatrix} x \\ u \end{pmatrix} := \sum_{i=1}^{n+1} \alpha_i \begin{pmatrix} x^{[i]} \\ u^{[i]} \end{pmatrix}$$

such that $x_0 = 0$, $\{x_k\}_{k=1}^{N} \not\equiv 0$, and $\mathcal{F}(x, u) \leq 0$. This contradicts positivity of the functional $\mathcal{F}$. Q.E.D.

When $n = 1$ and (1) is rewritten Sturm-Liouville second order difference equation (6), the previous statement is the Sturm separation theorem for generalized zeros of solutions of (6).

Now we turn our attention to the so-called *oscillation theorem* for symplectic difference systems. It concerns the eigenvalue problem associated with (1)

$$\text{(10)} \qquad \begin{cases} x_{k+1} = & \mathcal{A}_k x_k + \mathcal{B}_k u_k, \\ u_{k+1} = & \mathcal{C}_k x_k + \mathcal{D}_k u_k - \lambda \mathcal{W}_k x_{k+1}, \quad 0 \leq k \leq N \\ & x_0 = 0 = x_{N+1}, \end{cases}$$

where $N \in \mathbb{N}$, $\lambda \in \mathbb{R}$ is the eigenvalue parameter, and where we assume that the matrices $\mathcal{W}$ are symmetric and nonnegative definite. The number λ is an *eigenvalue* of (10) if there exists a nontrivial solution $z = \binom{x}{u} = \binom{x_k}{u_k}_{k=0}^{N+1}$, a corresponding *eigenvector* of (10), i.e., z solves (10) and there exists $k \in \{1, \dots N\}$ with $(x_k, u_k) \neq (0, 0)$.

In the next statement, usually referred to as the *oscillation theorem*, we will use the following notation:

$$\begin{aligned} n_1(\lambda) &:= \text{the number of focal points of a conjoined basis } \binom{X}{U} \\ & \text{in the interval } (0, N+1], \text{ including multiplicities;} \\ n_2(\lambda) &:= \text{the number of eigenvalues of (10),} \\ & \text{which are less than or equal to } \lambda. \end{aligned}$$

Proposition 2. *Let* $Z = \binom{X}{U} = \binom{X_k(\lambda)}{U_k(\lambda)}_{k\in\mathbb{Z}}$ *be the principal solution at* $k = 0$ *of* (1), *i.e., the solution given by the initial condition* $X_0 = 0$, $U_0 = I$, *and suppose that*

$$\lim_{\lambda\to-\infty} n_1(\lambda) = 0 \quad \textit{and} \quad \lim_{\lambda\to-\infty} n_2(\lambda) = 0. \tag{11}$$

Then

$$n_1(\lambda) = n_2(\lambda) \quad \text{for all} \quad \lambda \in \mathbb{R}. \tag{12}$$

Proof. The proof is technically rather complicated and it is based on the so-called *local oscillation theorem* which deals with the local dependece on λ of the quantities which appeared in the definition of a focal point, and states that

$$\begin{aligned} \operatorname{rank} M_k(\lambda+) - \operatorname{rank} M_k(\lambda-) &= \operatorname{ind} P_k(\lambda+) - \operatorname{ind} P_k(\lambda-) \\ &= \operatorname{rank} X_k(\lambda) - \operatorname{rank} X_k(\lambda-) - \operatorname{rank} X_{k+1}(\lambda) + \operatorname{rank} X_k(\lambda+) \end{aligned}$$

(the matrices M, P are defined by (8), now these matrices depend on λ) for all $\lambda \in \mathbb{R}$. This local result then leads to the equality

$$n_1(\lambda) - n_1(\lambda-) = n_2(\lambda) - n_2(\lambda-)$$

which togeteher with (11) gives the required result. We refer to [7] and [10] for technical details. Q.E.D.

The previous proposition is a combination of [7, Theorem 2] and [10, Theorem 1]. In Theorem 2 of [7], a certain "exceptional" set of (finitely many) λ's appeared, where (12) may fail to hold since oscillation theorem presented in that paper does not take into consideration the concept of *multiplicity* of a focal point (which did not exist when that paper was prepared). Theorem 1 of [10] already reflects this concept, but assumption (11) is not supposed in that paper and this fact required a substantial modification of the concept of eigenvalue of (10). We prefer here the above given combination of [7] and [10] since it is more understandable for the presentation given in this paper.

§3. Reversed symplectic difference system

System (1) can be written in the equivalent form $z_k = \mathcal{S}_k^{-1} z_{k+1}$ and this system we call the *reversed symplectic difference system.* In components x, u this system takes the form

$$(13) \qquad x_k = \mathcal{D}_k^T x_{k+1} - \mathcal{B}_k^T u_{k+1}, \quad u_k = -\mathcal{C}_k^T x_{k+1} + \mathcal{A}_k^T u_{k+1},$$

here we have used the fact that $S^{-1} = \begin{pmatrix} \mathcal{D}^T & -\mathcal{B}^T \\ -\mathcal{C}^T & \mathcal{A}^T \end{pmatrix}$ which follows e.g. from (3). In accordance with the terminology for "non-reversed" symplectic system (1), we say that a conjoined basis $\binom{X}{U}$ of (1) has no *reversed focal point* (shortly *r-focal point*) in the interval $[k, k+1)$, if

$$(14) \qquad \operatorname{Ker} X_k \subseteq \operatorname{Ker} X_{k+1} \quad \text{and} \quad \hat{P}_k = X_{k+1} X_k^\dagger \mathcal{B}_k^T \geq 0.$$

The following statement is a "reversed" analogue of [6, pp. 711–712].

Lemma 1. *Let $\binom{X}{U}$ be a conjoined basis of* (1). *First condition in* (14) *holds if and only if* $\operatorname{Ker} X_k^T \subseteq \operatorname{Ker} \mathcal{B}_k$ *and this is equivalent to the identity* $X_k X_k^\dagger \mathcal{B}_k^T = \mathcal{B}_k^T$. *Also, if* (14) *holds, the matrix* $\hat{P}_k$ *is symmetric.*

Proof. First of all recall that the standard result of the theory od Moore-Penrose generalized inverses claims that $WV^\dagger V = W$ holds for a pair of matrices V, W if and only if $\operatorname{Ker} V \subseteq \operatorname{Ker} W$, see [4]. Hence $\mathcal{B}_k (X_k^\dagger)^T X_k^T = \mathcal{B}$, i.e. $X_k X_k^\dagger \mathcal{B}_k^T = \mathcal{B}_k^T$ if and only if the first condition in (14) holds. Suppose now that this condition in (14) holds. Let $\binom{\tilde{X}}{\tilde{U}}$ be a conjoined basis which together with $\binom{X}{U}$ forms a pair of normalized conjoined bases, i.e., the matrix $Z = \begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix}$ is symplectic. Then

$$\mathcal{S}_k = Z_{k+1} Z_k^{-1} = \begin{pmatrix} X_{k+1} & \tilde{X}_{k+1} \\ U_{k+1} & \tilde{U}_{k+1} \end{pmatrix} \begin{pmatrix} \tilde{U}_k^T & -\tilde{X}_k \\ -U_k^T & X_k^T \end{pmatrix}$$

and hence

$$\mathcal{B}_k = \tilde{X}_{k+1} X_k^T - X_{k+1} \tilde{X}_k^T.$$

Now, let $\alpha \in \operatorname{Ker} X_k^T$. Since the identity $Z^T \mathcal{J} Z = \mathcal{J}$ which defines symplecticity of the matrix Z is equivalent to the identity $Z \mathcal{J} Z^T = \mathcal{J}$, we also have $\tilde{X}_k X_k^T = X_k \tilde{X}_k^T$, i.e., $\tilde{X}_k X_k^T \alpha = 0$ implies $X_k \tilde{X}_k^T \alpha = 0$. Hence we have $\tilde{X}_k^T \alpha \in \operatorname{Ker} X_k \subseteq \operatorname{Ker} X_{k+1}$. Then $\mathcal{B}_k \alpha = (\tilde{X}_{k+1} X_k^T - X_{k+1} \tilde{X}_k^T) \alpha = -X_{k+1} \tilde{X}_k \alpha = 0$, which means that $\operatorname{Ker} X_k^T \subseteq \operatorname{Ker} \mathcal{B}_k$. Conversely, let $\operatorname{Ker} X_k^T \subseteq \operatorname{Ker} \mathcal{B}_k$ and $\alpha \in \operatorname{Ker} X_k$. Then

$$\begin{aligned} X_{k+1} \alpha &= (\mathcal{A}_k X_k + \mathcal{B}_k U_k) \alpha = \mathcal{B}_k U_k \alpha = \mathcal{B}_k (X_k^T)^\dagger X_k^T U_k \alpha \\ &= \mathcal{B}_k (X_k^T)^\dagger U_k^T X_k \alpha = 0, \end{aligned}$$

i.e., $\alpha \in \operatorname{Ker} X_{k+1}$. As for the symmetry of the matrix $\hat{P}$, we have

$$\begin{aligned}\hat{P}_k &= X_{k+1} X_k^{\dagger} \mathcal{B}_k^T = (\mathcal{A}_k X_k + \mathcal{B}_k U_k) X_k^{\dagger} \mathcal{B}^T \\ &= \mathcal{A}_k \mathcal{B}_k^T - \mathcal{B}_k (X_k^{\dagger})^T X_k^T U_k X_k^{\dagger} \mathcal{B}_k^T,\end{aligned}$$

here we have used the identity $\mathcal{B}_k (X_k^{\dagger})^T X_k^T = \mathcal{B}_k$ and the fact that $\binom{X}{U}$ is a conjoined basis. Q.E.D.

Next, following the previous section, we introduce the notation

(15)
$$\hat{M}_k = (I - X_k X_k^{\dagger}) \mathcal{B}_k^T, \quad \hat{T}_k = I - \hat{M}_k^{\dagger} \hat{M}_k, \quad \hat{\mathcal{P}}_k = \hat{T}_k^T X_{k+1} X_k^{\dagger} \mathcal{B}_k^T \hat{T}_k.$$

Using this notation, a conjoined basis $\binom{X}{U}$ has the r-focal point of *multiplicity* m at an integer k if $m = \operatorname{rank} \hat{M}_k$, and this conjoined basis has the r-focal point of multiplicity p in the open interval $(k, k+1)$ if $p = \operatorname{ind} \hat{\mathcal{P}}_k$. The number $m+p$ defines the multiplicity of the r-focal point of $\binom{X}{U}$ in the interval $[k, k+1)$. Our main result of this section is the following complement of Proposition 1.

Theorem 1. *Suppose that there exists a conjoined basis of* (1) *with no r-focal point in* $[0, N+1)$. *Then any other conjoined basis of* (1) *has at most* n *r-focal points in* $[0, N+1)$, *counting multiplicities.*

Proof. To prove the statement, we proceed similarly as in [6], we "reflect" the interval $[0, N+1]$ by the substitution $k \longmapsto N+1-k$. Let $z = \{z_k\}_{k=0}^{N+1}$ be a solution of (1). Denote

$$\tilde{z}_k = \mathcal{K} z_{N+1-k}, \quad \tilde{\mathcal{S}}_k = \mathcal{K} \mathcal{S}_{N-k}^{-1} \mathcal{K}, \quad \text{where } \mathcal{K} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

We have (taking into account that $\mathcal{K}^{-1} = \mathcal{K}$)

$$\tilde{z}_{k+1} = \mathcal{K} z_{N-k} = \mathcal{K} \mathcal{S}_{N-k}^{-1} z_{N+1-k} = \mathcal{K} \mathcal{S}_{N-k}^{-1} \mathcal{K} \tilde{z}_k = \tilde{\mathcal{S}}_k \tilde{z}_k, \tag{16}$$

where we have denoted

$$\tilde{\mathcal{S}}_k = \begin{pmatrix} \tilde{\mathcal{A}}_k & \tilde{\mathcal{B}}_k \\ \tilde{\mathcal{C}}_k & \tilde{\mathcal{D}}_k \end{pmatrix} := \mathcal{K} \mathcal{S}_{N-k}^{-1} \mathcal{K} = \begin{pmatrix} \mathcal{D}_{N-k}^T & \mathcal{B}_{N-k}^T \\ \mathcal{C}_{N-k}^T & \mathcal{A}_{N-k}^T \end{pmatrix},$$

i.e., in particular, $\tilde{\mathcal{B}}_k = \mathcal{B}_{N-k}^T$. Next we show that a conjoined basis $\binom{X}{U}$ of (1) has an r-focal point of multiplicity $m+p$ at $[N-k, N+1-k)$ for some $k \in \{0 \ldots, N\}$ (here $m = \operatorname{rank} \hat{M}_{N-k}$ is the multiplicity at $N-k$ and $p = \operatorname{ind} \hat{P}_{N-k}$ is the multiplicity in the open interval $(N-k, N-k+$

1)), if and only if the solution $\binom{\tilde X_k}{\tilde U_k} := \mathcal{K}\binom{X_{N+1-k}}{U_{N+1-k}}$ has the ("normal") focal point of multiplicity $m+p$ at $(k, k+1]$. We have

$$\hat M_{N-k} = (I - X_{N-k}X_{N-k}^{\dagger})\mathcal{B}_{N-k}^T = (I - \tilde X_{k+1}\tilde X_{k+1}^{\dagger})\tilde{\mathcal{B}}_k = \tilde M_k$$

hence $\hat T_{N-k} = \tilde T_k$, and

$$\hat P_{N-k} = \hat T_{N-k}X_{N+1-k}X_{N-k}^{\dagger}\hat T_{N-k} = T_k\tilde X_k\tilde X_{k+1}^{\dagger}\tilde{\mathcal{B}}_kT_k = \tilde P_k.$$

Here the matrices $\tilde M_k, \tilde T_k, \tilde P_k$ define the multiplicity of a focal point of $\binom{\tilde X}{\tilde U}$ in $[k, k+1)$. This implies the claimed statement about multiplicity of the r-focal point of $\binom{X}{U}$ at $[N-k, N+1-k)$ and the "normal" focal point of $\binom{\tilde X}{\tilde U}$ at $(k, k+1]$. To finish the proof, it suffices to apply Proposition 1. Q.E.D.

Remark 1. (i) We start with the open problem formulated in [9], which concerns the number of focal points of conjoined bases of (1). Based on the statement of Proposition 1 and on a known result (see [17, Chap. VII]) for conjoined bases of the linear Hamiltonian differential system (9) (which is a continuous counterpart of (1)), it is conjectured in [9] that the numbers of focal points (including multiplicities) of any pair of conjoined bases of (1) in a given discrete interval differ by at most n, where n is the dimension of square matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ in (1). It seems that the crucial role in the proof of this statement could be played by the relationship between the number of focal point of the principal solution of (1) in the interval $(0, N+1]$ and the index and nullity of the quadratic form defined by the functional (4). We conjecture here that this relationship can be obtained using Proposition 2 and the proof of the above mentioned conjecture of [9] is a subject of the present investigation.

(ii) Another open problem concerns the number and the *location* of focal points and r-focal points of *one* conjoined basis of (1) in a given discrete interval. We conjecture that the numbers of focal and r-focal points of a conjoined basis $\binom{X}{U}$ of (1) in the discrete interval $[0, N+1]$ *are the same* and that the location of focal points and r-focal point is essentially the same. In particular, we conjecture that

$$\operatorname{rank} M_k = \operatorname{rank} \hat M_{k+1} \quad \text{and} \quad \operatorname{ind} P_k = \operatorname{ind} \hat P_k,$$

where the matrices $M_k, P_k, \hat M_k, \hat P_k$ are defined by (8) and (15).

Acknowledgement. This research is supported by the grant 201/04/0580 of the Grant Agency of Czech Republic and the Research Project MSM00 22162409 of the Czech Ministry of Education.

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Theory, Methods, and Applications. Second ed., Monogr. Textbooks Pure Appl. Math., **228**, Marcel Dekker, Inc., New York, 2000.

[2] R. P. Agarwal, M. Bohner, S. R. Grace and D. O'Regan, Discrete Oscillation Theory, Hindawi Publ. Corporation, New York, 2005.

[3] C. D. Ahlbrandt and A. C. Peterson, Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations, Kluwer Academic Publ., Boston, 1996.

[4] A. Ben-Israel and E. N. Greville, Generalized Inverses: Theory and Applications, Springer-Verlag, New York, 2003.

[5] M. Bohner, Linear Hamiltonian difference systems: disconjugacy and jacoby-type conditions, J. Math. Anal. Appl., **199** (1996), 804–826.

[6] M. Bohner and O. Došlý, Disconjugacy and transfromations for symplectic difference systems, Rocky Mountain J. Math., **27** (1997), 707–743.

[7] M. Bohner, O. Došlý and W. Kratz, An oscillation theorem for discrete eigenvalue problems, Rocky Mountain J. Math., **33** (2003), 1233–1260.

[8] M. Bohner, O. Došlý and W. Kratz, Positive semidefiniteness of discrete quadratic functionals, Proc. Edinburg Math. Soc., **46** (2003), 227–236.

[9] O. Došlý and W. Kratz, A Sturmian separation theorem for symplectic difference systems, J. Math. Anal. Appl., **325** (2007), 333–341.

[10] O. Došlý and W. Kratz, Oscillation theorems for symplectic difference systems, J. Difference Equ. Appl., **13** (2007), 585–605.

[11] O. Došlý, R. Hilscher and V. Zeidan, Nonnegativity of discrete quadratic functionals corresponding to symplectic difference systems, Linear Algebra Appl., **375** (2003), 21–44.

[12] S. Elaydi, An Introduction to Difference Equations. Third ed., Undergrad. Texts Math., Springer, New York, 2005.

[13] R. Hilscher and V. Zeidan, A remark on discrete quadratic functionals with separable endpoints, Rocky Mountain J. Math., **33** (2003), 1337–1351.

[14] R. Hilscher and V. Zeidan, Equivalent conditions to the nonnegativity of a quadratic functional in discrete optimal control, Math. Nachr., **266** (2004), 48–59.

[15] W. G. Kelley and A. C. Peterson, Difference Equations. An Introduction with Applications. Second ed., Harcourt/Academic Press, San Diego, 2001.

[16] W. Kratz, Discrete oscillation, J. Difference Equ. Appl., **9** (2003), 135–147.

[17] W. T. Reid, Ordinary Differential Equations, John Wiley & Sons, New York, 1971.

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