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Asymptotic behaviour of a nonlinear stochastic difference equation modelling an inefficient financial market

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Abstract.

This note studies the asymptotic behaviour of linear and nonlinear stochastic difference equations whose structure is motivated by a financial market model. The asymptotic results show that the models can produce behaviour consistent with random walk efficient markets as well as bubbles or crashes.

§1. Motivation and background material

In recent years, much attention in financial economics has focussed on the trading strategies of investors. Classical models of financial markets assume that agents are rational, have homogeneous preferences, and do not use historical market data in framing their investment decisions. An important and seminal collection of papers which summarise this position is [5].

Econometric evidence of market returns (see e.g., [9]) and analysis of the behaviour of traders in real markets reveal a more complex picture. Traders often employ rules of thumb which do not conform to notions of rational behaviour based on knowledge of the empirical distribution of returns (see e.g., [8]). Moreover, many traders use past prices as a guide to the evolution of the price in the future (see e.g., [10]). Linear continuous–time stochastic models of markets which involve agents using past prices to determine their demand, but in which the traders discount past returns with a simple type of exponentially fading memory, include [1] and [7].

In this paper, we present a stochastic difference equation model of an inefficient financial market. The model is informationally inefficient,

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in the sense that past movements of the stock price have an influence on future movements. We assume that there is trading at intervals of one time unit, with prices fixed in the intervening period. The inefficiency stems from the presence of trend–following speculators, whose demand for the asset depends on the difference between a weighted average over the last N periods of the cumulative return on the stock and the current cumulative return. More precisely, if $X(n)$ is the cumulative return up to time n, the planned excess demand just before trading at time $n+1$ is $g(X(n)) - \sum_{j=1}^{N} w(j)g(X(n-j))$ where $\sum_{j=1}^{N} w(j) = 1$ and g is an increasing function. Speculators react to other random stimuli— "news"— which is independent of past returns. This news arrives at time $n+1$, adding a further $\xi(n+1)$ to the traders' excess demand. Prices increase when there is excess demand (resp. fall when there is excess supply), with the rise (resp. fall) being larger the greater the excess demand (resp. supply). Hence, the price adjustment at time $n+1$ is given by

$$X(n+1) = X(n) + g(X(n)) - \sum_{j=1}^{N} w(j)g(X(n-j)) + \xi(n+1). \tag{1}$$

We study the almost sure asymptotic behaviour as $n \to \infty$ of solutions of (1).

This paper shows three things: first, if the trend following speculators do not behave very aggressively to the difference between current returns and historical returns, or do not discount prices quickly, then returns behave very similarly to a simple random walk, in that they have the same size of large fluctuations. This is characteristic of an efficient market. This occurs once g is linear (Theorem 3), or obeys $g(x) \sim \beta x$ as $x \to \infty$ for some $\beta \geq 0$ (Theorem 5). Moreover, in the case when g is linear the returns follow a random walk plus a stationary mean–reverting process (Theorem 3). Also when g is linear, and the trend–following speculators behave aggressively, the returns will tend to plus or minus infinity exponentially fast: this is a mathematical realisation of a stock market bubble (Theorem 4).

The distinction between traders who are "aggressive with long memory" or "less aggressive with short memory" depends on whether $\beta \sum_{j=1}^{N} jw(j)$ is greater than or less than unity. Large values of β correspond to aggressive behaviour; if $g(x) = \beta x$ for example, the planned excess demand of traders is β multiplied by the difference between the current returns and a weighted average of returns. Therefore, for larger β, a smaller signal from the market produces a given response from the traders. The term $M = \sum_{j=1}^{N} jw(j)$ is in $[1, N]$, and the greater weight that traders give to returns further back in time, the larger M becomes. Therefore, M is a measure of the effective length of memory of the traders; indeed, if traders make their decisions based only on a comparison of current returns with returns j periods ago, then $M = j$, $j = 1, \dots, N$. A fuller treatment of the economic interpretation of the results, the correlation of the returns, and the effect on the market of contrarian (or negative feedback traders) will be presented in a later paper. Due to restrictions on space, economic interpretation of the results is restricted to this introduction.

The paper has the following structure. Section 2 gives notation and supporting results. The asymptotic behaviour of the linear equation is presented in Section 3 and the nonlinear equation is considered in Section 4. Results employed from the theory of deterministic difference equations are standard, see e.g., [6]. Definitions and results from discrete–time martingale theory may all be found in [11].

§2. Background material

$\mathbb{N}$ denotes the integers $0, 1, 2, \dots$, and $\mathbb{R}$ the real line. A real sequence $a = \{a(n) : n \in \mathbb{N}\}$ obeys $a \in \ell^1(\mathbb{N}; \mathbb{R})$ if $\sum_{n \in \mathbb{N}} |a(n)| < \infty$. The convolution of $f = \{f(n) : n \in \mathbb{N}\}$ and $g = \{g(n) : n \in \mathbb{N}\}$, $f * g$, is a sequence defined by $(f * g)(n) = \sum_{k=0}^{n} f(n-k)g(k)$, $n \in \mathbb{N}$.

Let $\beta > 0$, $N \in \mathbb{N}$, and suppose $w = \{w(n) : n = 1, \dots, N\}$ obeys

$$w(n) \geq 0, \quad n = 1, \dots, N; \quad \sum_{n=1}^{N} w(n) = 1 \tag{2}$$

The resolvent $r = \{r(n) : n \geq -N\}$ is a scalar sequence defined by

$$r(n+1) = r(n) + \beta \left(r(n) - \sum_{j=1}^{N} w(j) r(n-j) \right), \quad n \in \mathbb{N} \tag{3a}$$

$$r(0) = 1, \quad r(n) = 0, \quad n < 0. \tag{3b}$$

Lemma 1. *Let $\beta > 0$, w obey (2), and r be defined by (3).*

(a) *r is a non–decreasing sequence with $r(n) > 0$ for $n \in \mathbb{N}$.*

(b) *If $\beta \sum_{j=1}^{N} j w(j) < 1$, then*

$$\lim_{n \to \infty} r(n) = \frac{1}{1 - \beta \sum_{n=1}^{N} j w(j)} =: r^*, \tag{4}$$

and $\delta = \{\delta(n) : n \geq -N\}$ defined by $\delta(-N) = 0$ and $\delta(n+1) = r(n+1) - r(n)$ for $n \geq -N$ obeys $\delta \in \ell^1(\mathbb{N}; \mathbb{R}^+)$.

(c) *If* $\beta\sum_{j=1}^{N} jw(j) > 1$, *there exists* $\alpha \in (0,1)$ *defined by*

$$\alpha^{-1} = 1 + \beta\left(1 - \sum_{k=1}^{N} \alpha^k w(k)\right) \tag{5}$$

such that $\lim_{n\to\infty} \alpha^n r(n) = R^*$, *where* $R^* > 0$ *is given by*

$$R^* = \frac{1}{(1-\alpha)(1+\beta\alpha\sum_{j=1}^{N-1} j\alpha^j W(j))}, \tag{6}$$

and W *is defined by* $W(j) = \sum_{k=j+1}^{N} w(k)$, $j = 0, \dots, N-1$.

Proof. If $N = 1$, $r(n+1) = r(n) + \beta(r(n) - r(n-1))$, so $\delta(n) = \beta^n$, and the results are trivial. Assume $N \ge 2$. Part (a) follows by induction. For (b), putting $r(n) = \sum_{j=-N}^{n} \delta(j)$ into (3) and using (2) gives

$$\begin{aligned}\delta(n+1) = \beta\delta(n) &+ \beta \sum_{j=-N}^{n-N} \left(\sum_{k=1}^{N} w(k) - \sum_{k=1}^{(n-j)\wedge N} w(k)\right)\delta(j)\\ &+ \beta \sum_{j=n-N+1}^{n-1} \left(\sum_{k=1}^{N} w(k) - \sum_{k=1}^{(n-j)\wedge N} w(k)\right)\delta(j).\end{aligned}$$

Rearrange the righthand side and set $W(j) = \sum_{k=j+1}^{N} w(k)$, $j = 0, \dots,$ $N-1$ to get

$$\delta(n+1) = \beta\sum_{j=0}^{N-1} W(j)\delta(n-j), \quad n \in \mathbb{N}; \quad \delta(0) = 1, \delta(n) = 0, \quad n < 0. \tag{7}$$

If $\beta\sum_{k=1}^{N} kw(k) < 1$, $\sum_{n=-N}^{\infty} \delta(n) =: r^*$ is finite, and $r^* = 1 + \beta r^* \sum_{j=0}^{N-1} W(j)$. Since $\sum_{j=0}^{N-1} W(j) = \sum_{j=1}^{N} jw(j)$, (4) holds. Thus $r^* - r(n) = \sum_{j=n+1}^{\infty} \delta(j) =: \Delta(n+1)$. Then $\Delta(n) = r^*$ for $n = -N, \dots, 0$, $\Delta(1) = r^* - 1$, and from (7)

$$\Delta(n+1) = \beta\sum_{j=0}^{N-1} W(j)\Delta(n-j), \quad n \in \mathbb{N}. \tag{8}$$

For part (c), multiplying across (7) by α^{n+1}, where α is given by (5), we get $\delta_\alpha(n+1) = \sum_{j=0}^{N-1} \beta\alpha W_\alpha(j)\delta_\alpha(n-j)$ where $\delta_\alpha(n) = \alpha^n\delta(n)$,

$W_\alpha(n) = \alpha^n W(n)$. Then, the discrete time renewal theorem or z–transform techniques imply

$$\lim_{n\to\infty} \alpha^n \delta(n) = \lim_{n\to\infty} \delta_\alpha(n) = \frac{1}{1+\beta\alpha\sum_{j=1}^{N-1} j\alpha^j W(j)} > 0.$$

Therefore $\lim_{n\to\infty} \alpha^n r(n) = R^*$, where $R^* > 0$ is given by (6). Q.E.D.

Next we find the growth rate of a moving average of a sequence.

Lemma 2. *Let γ be positive and increasing with $\gamma(n-N)/\gamma(n) \to 1$, as $n \to \infty$, for all $N \in \mathbb{N}$. If $k = \{k(n) : n \in \mathbb{N}\}$ is non–negative with $\sum_{n=0}^\infty k(n) \in (0,\infty)$, then $\lim_{n\to\infty}(k*\gamma)(n)/\gamma(n) = \sum_{n=0}^\infty k(n)$.*

Proof. Without loss of generality, let $\sum_{n=0}^\infty k(n) = 1$. For every $\varepsilon > 0$ there is $N > 0$ such that $\sum_{j=N+1}^\infty k(j) < \varepsilon/2$. For $n \geq N+1$, we have

$$\begin{aligned}\frac{(k*\gamma)(n)}{\gamma(n)} - \sum_{j=0}^{n} k(j) &= \sum_{j=0}^{N} k(j)\left(\frac{\gamma(n-j)}{\gamma(n)} - 1\right)\\ &\quad + \sum_{j=N+1}^{n} k(j)\left(\frac{\gamma(n-j)}{\gamma(n)} - 1\right),\end{aligned}$$

which, using monotonicity of γ, and the fact that $\sum_{n=0}^\infty k(n) = 1$ gives

$$\left|\sum_{j=0}^{n} \frac{k(j)\gamma(n-j)}{\gamma(n)} - \sum_{j=0}^{n} k(j)\right| \leq \left(1 - \frac{\gamma(n-N)}{\gamma(n)}\right) + 2\sum_{j=N+1}^{\infty} k(j).$$

Using $\gamma(n-N)/\gamma(n) \to 1$, and then letting $\varepsilon \to 0$ yields the result. Q.E.D.

§3. Linear stochastic difference equation

We consider the linear stochastic difference equation

(9a)

$$Y(n+1) = Y(n) + \beta\{Y(n) - \sum_{j=1}^{N} w(j)Y(n-j)\} + \xi(n+1),\ n \geq 0$$

(9b)
$$Y(n) = \phi(n), \quad n \leq 0,$$

where $\xi = \{\xi(n) : n \in \mathbb{N}\}$ is a sequence of random variables obeying

(10a) ξ is a sequence of independent, identically distributed r.vs;

(10b) $\mathbb{E}[\xi(n)] = 0, \quad \mathbb{E}[\xi(n)^2] = \sigma^2, \quad$ for some $\sigma > 0$, and all $n \in \mathbb{N}$.

If $\beta \sum_{j=1}^N jw(j) < 1$, Y behaves asymptotically as a random walk. For instance, if ξ obeys (10) the process S given by $S(n) = \sum_{j=1}^n \xi(j)$ is a random walk and obeys *the Law of the Iterated Logarithm*:

$$\limsup_{n\to\infty} \frac{S(n)}{\sqrt{2n\log\log n}} = -\liminf_{n\to\infty} \frac{S(n)}{\sqrt{2n\log\log n}} = \sigma, \quad \text{a.s.}$$

Theorem 3. *Let w obey* (2), $0 < \beta \sum_{n=1}^N nw(n) < 1$, *$\xi$ obey* (10), *and Y obey* (9). *Then*

(11)
$$\limsup_{n\to\infty} \frac{Y(n)}{\sqrt{2n\log\log n}} = -\liminf_{n\to\infty} \frac{Y(n)}{\sqrt{2n\log\log n}} = \frac{|\sigma|}{1-\beta\sum_{j=1}^N jw(j)}.$$

Proof. For $n \geq 1$, $Y(n) = y(n) + \sum_{j=1}^n r(n-j)\xi(j)$, where $y(n+1) = y(n) + \beta(y(n) - \sum_{j=1}^N w(j)y(n-j))$, $n \geq 0$ and $y(n) = \phi(n)$, $n \leq 0$. With $U(n) = \sum_{j=1}^n \Delta(n-j)\xi(j)$, we get $Y(n) = y(n) - U(n) + r^* \sum_{j=1}^n \xi(j)$, $n \geq 1$. Since $r(n) \to r^*$, $\lim_{n\to\infty} y(n)$ exists. By the law of the iterated logarithm, we need only show $\lim_{n\to\infty} |U(n)|/\sqrt{2n\log\log n} = 0$ a.s. Let $b(x) = \sqrt{x}$, $x \geq 0$. Then $b : [0,\infty) \to [0,\infty)$ is increasing and $b^{-1}(x) = x^2$. If ξ is a random variable with the same distribution as $\xi(n)$, by Corollary 4.1.3 in [4], we have

$$\sum_{n=1}^{\infty} \mathbb{P}[|\xi(n)| > \sqrt{n}] \leq \mathbb{E}[b^{-1}(|\xi|)] = \mathbb{E}[\xi^2] < \infty.$$

By the Borel–Cantelli lemma, $\limsup_{n\to\infty} |\xi(n)|/\sqrt{n} \leq 1$, a.s. which implies that $\lim_{n\to\infty} |\xi(n)|/\sqrt{2n\log\log n} = 0$ a.s. Thus, there is an a.s. event Ω^* such that for all $\omega \in \Omega^*$, and all $\varepsilon > 0$, there is $C(\varepsilon,\omega) > 0$ such that

$$|\xi(n,\omega)| < C(\varepsilon,\omega) + \varepsilon\sqrt{2n\log\log(n+e^e)} =: \gamma(n,\omega), \quad n \in \mathbb{N}.$$

By (8) and $\beta\sum_{j=1}^N jw(j) < 1$, $\Delta \in \ell^1(\mathbb{N};\mathbb{R})$, so by Lemma 2

$$\limsup_{n\to\infty} \frac{|U(n,\omega)|}{\gamma(n,\omega)} \leq \limsup_{n\to\infty} \frac{\sum_{j=1}^n |\Delta(n-j)|\gamma(\omega,j)}{\gamma(n,\omega)} = \sum_{j=0}^{\infty} |\Delta(j)|;$$

thus $\limsup_{n\to\infty} |U(n,\omega)|/\sqrt{2n\log\log n} < \varepsilon \sum_{j=0}^{\infty} |\Delta(j)|$, hence the result. Q.E.D.

When $\beta\sum_{j=1}^{N} jw(j) < 1$, then $U(n+1) = \beta\sum_{k=0}^{N-1} W(k)U(n-k) + \zeta(n+1)$, $n \geq N$, where $\zeta(n+1) := \sum_{k=0}^{N-1}\theta(k)\xi(n-k+1)$ and θ is a deterministic sequence which depends on Δ. In this case, $y - U$ is an asymptotically stationary ARMA process, so Y is the sum of an asymptotically stationary process and a random walk. ARMA (autoregressive moving average) processes are used widely in financial econometrics (see e.g., [2]). When $\beta\sum_{j=1}^{N} jw(j) > 1$, we now prove $\alpha^n Y(n) \to Y^*$ as $n \to \infty$ where $\alpha \in (0,1)$ and Y^* is a random variable given explicitly in terms of ξ. Hence $Y(n)$ tends to $\pm\infty$ according to the sign of Y^*.

Theorem 4. *Let w obey (2), $\beta\sum_{j=1}^{N} jw(j) > 1$, ξ obey (10), and Y obeys (9). If $\alpha \in (0,1)$ is given by (5), and R^* by (6), then*

$$\lim_{n\to\infty} \alpha^n Y(n) = R^*\left(\phi(0) - \beta\sum_{j=0}^{N-1}\sum_{k=j+1}^{N}\alpha^{j+1}w(k)\phi(j-k) + \sum_{j=1}^{\infty}\alpha^j\xi(j)\right).$$

Proof. Let $y(n+1) = y(n) + \beta(y(n) - \sum_{k=1}^{N} w(k)y(n-k))$, $n \geq 0$ and $y(n) = \phi(n)$, $n \leq 0$. Then $y(n) = r(n)\phi(0) - \beta(\tilde{\phi} * r)(n-1)$, $n \geq 1$, where

$$\tilde{\phi}(n) = \begin{cases} \sum_{j=n+1}^{N} w(j)\phi(n-j), & n = 0,1,\dots,N-1 \\ 0, & n = N, N+1, \dots, \end{cases}$$

so $y(n) = r(n)\phi(0) - \beta\sum_{j=0}^{N-1} r(n-1-j)\tilde{\phi}(j)$, $n \geq N$. As $\lim_{n\to\infty} r(n)\alpha^n = R^*$,

$$\lim_{n\to\infty} \alpha^n y(n) = R^*\left(\phi(0) - \beta\sum_{j=0}^{N-1}\sum_{k=j+1}^{N}\alpha^{j+1}w(k)\phi(j-k)\right).$$

Next, for $n \geq 1$, we have

$$\alpha^n Y(n) = \alpha^n y(n) + \sum_{j=1}^{n}(\alpha^{n-j}r(n-j) - R^*)\alpha^j\xi(j) + R^*\sum_{j=1}^{n}\alpha^j\xi(j). \tag{12}$$

Let $M(n) = R^*\sum_{j=1}^{n}\alpha^j\xi(j)$, $n \geq 0$. Since $\alpha \in (0,1)$ M is martingale with finite quadratic variation, so by the martingale convergence theorem $\lim_{n\to\infty} M(n)$ is finite a.s. Since $\mathbb{E}[|\xi(j)|]^2 \leq \mathbb{E}[\xi(j)^2] = \sigma^2$, $\mathbb{E}\sum_{j=1}^{n}\alpha^j|\xi(j)| = \sum_{j=1}^{n}\alpha^j\mathbb{E}[|\xi(j)|] \leq \alpha\sigma(1-\alpha)^{-1}$, so $\xi_\alpha(n) := \alpha^n\xi(n) \in \ell^1(\mathbb{N};\mathbb{R})$, a.s. But $r_1(n) := \alpha^n r(n) - R^* \to 0$, so $(r_1 * \xi_\alpha)(n) \to 0$ as $n \to \infty$. We finish by letting $n \to \infty$ in (12). Q.E.D.

§4. Nonlinear stochastic difference equation

We now study the nonlinear stochastic difference equation

(13a)

$$X(n+1) = X(n) + g(X(n)) - \sum_{j=1}^{N} w(j) g(X(n-j)) + \xi(n+1), \quad n \in \mathbb{N},$$

(13b)

$$X(n) = \phi(n), \quad n = -N, -N+1, \ldots, 0.$$

$g : \mathbb{R} \to \mathbb{R}$ is presumed to have the following properties

$$g \in C(\mathbb{R};\mathbb{R}), \ \lim_{x\to\infty} \frac{g(x)}{x} = \lim_{x\to-\infty} \frac{g(x)}{x} = \beta \text{ for some } \beta \geq 0. \tag{14}$$

We now show if the conditions of Theorem 3 hold, the a.s. partial extrema of the solution of (13) grow exactly as those of the solution of (9), which are consistent with the extrema of a random walk. The proof of this result is partly inspired by work in [3].

Theorem 5. *Let w obey (2), $0 < \beta \sum_{n=1}^{N} n w(n) < 1$, g obey (14), ξ obey (10), and Y obey (9). Then the solution of (13) obeys*

$$\lim_{n\to\infty} \frac{|X(n) - Y(n)|}{\sqrt{2n \log\log n}} = 0, \tag{15}$$

(16)

$$\limsup_{n\to\infty} \frac{X(n)}{\sqrt{2n \log\log n}} = -\liminf_{n\to\infty} \frac{X(n)}{\sqrt{2n \log\log n}} = \frac{|\sigma|}{1 - \beta \sum_{j=1}^{N} j w(j)}.$$

Proof. Set $Z(n) = X(n) - Y(n)$, $\gamma(x) = g(x) - \beta x$, and $G(n+1) = \gamma(X(n)) - \sum_{j=1}^{N} w(j)\gamma(X(n-j))$, so $Z(n+1) - Z(n) = G(n+1) + \beta[Z(n) - \sum_{j=1}^{N} w(j) Z(n-j)]$. Therefore $Z(n) = \sum_{j=0}^{n-1} r(n-1-j) G(j+1)$, $n \geq 1$. Let $n \geq 2$, $n \geq N+1$, so

$$\begin{aligned} Z(n) &= \sum_{j=0}^{n-1} r(n-1-j)\gamma(X(j)) - \sum_{k=1}^{N} w(k) \sum_{j=0}^{n-1} r(n-1-j)\gamma(X(j-k)) \\ &= \sum_{j=0}^{n-1} r(n-1-j)\gamma(X(j)) - \sum_{k=1}^{N} w(k) \sum_{l=-k}^{n-k-1} r(n-k-l-1)\gamma(X(l)) \end{aligned}$$

Hence

$$Z(n) = \sum_{j=0}^{n-1} r(n-1-j)\gamma(X(j))$$
$$- \sum_{j=-N}^{n-2} \left(\sum_{k=-j\vee 1}^{(n-j-1)\wedge N} w(k)r(n-k-j-1) \right) \gamma(X(j))$$
$$= \sum_{l=1}^{n-1} \left(r(l) - \sum_{k=-(n-l-1)\vee 1}^{l\wedge N} w(k)r(l-k) \right) \gamma(X(n-l-1))$$
$$+ \gamma(X(n-1)) - \sum_{j=-N}^{-1} \left(\sum_{k=-j\vee 1}^{(n-j-1)\wedge N} w(k)r(n-k-j-1) \right) \gamma(X(j)).$$

Thus

$$Z(n) = \gamma(X(n-1)) + \sum_{l=N}^{n-1} \delta(l+1)\frac{1}{\beta}\gamma(X(n-l-1))$$
$$+ \sum_{l=1}^{N-1} \left(r(l) - \sum_{k=1}^{l} w(k)r(l-k) \right) \gamma(X(n-1-l)) - f_1(n),$$

where $f_1(n) = \sum_{j=-N}^{-1} \left(\sum_{k=-j}^{N} w(k)r(n-k-j-1) \right) \gamma(X(j))$. Set $u(0) = 1$, $u(l) = r(l) - \sum_{k=1}^{l} w(k)r(l-k)$, $l = 1, .., N-1$, $f_2(n) := |f_1(n)| + L(\varepsilon)[\sum_{l=0}^{N-1} |u(l)| + \beta^{-1} \sum_{l=N}^{n-1} \delta(l+1)]$. As $\lim_{|x|\to\infty} \gamma(x)/x = 0$, for each $\varepsilon > 0$ there is $L(\varepsilon) > 0$ such that $|\gamma(x)| \le L(\varepsilon) + \varepsilon|x|$, $x \in \mathbb{R}$. As $\lim_{n\to\infty} f_1(n)$ exists,

$$|Z(n)| \le \sum_{l=0}^{N-1} |u(l)| \left(L(\varepsilon) + \varepsilon|Z(n-1-l)| + \varepsilon|Y(n-1-l)| \right)$$
$$+ \frac{1}{\beta} \sum_{j=0}^{n-1-N} \delta(n-j) \left(L(\varepsilon) + \varepsilon|Z(j)| + \varepsilon|Y(j)| \right) + |f_1(n)|$$

and so there is an f_2 tending to a finite limit such that

$$|Z(n)| \le \sum_{l=0}^{N-1} |u(l)|\varepsilon|Y(n-1-l)| + \frac{1}{\beta}\sum_{l=N}^{n-1} \delta(l+1)\varepsilon|Y(n-l-1)|$$
$$+ \sum_{l=0}^{N-1} |u(l)|\varepsilon|Z(n-1-l)| + \frac{1}{\beta}\sum_{l=N}^{n-1} \delta(l+1)\varepsilon|Z(n-l-1)| + f_2(n).$$

Since $\delta \in \ell^1(\mathbb{N};\mathbb{R}^+)$, there is a summable κ such that

$$|Z(n)| \leq f_3(n) + \varepsilon \sum_{j=0}^{n-1} \kappa(n-1-j)|Y(j)| + \varepsilon \sum_{j=0}^{n-1} \kappa(n-1-j)|Z(j)|,$$

where f_3, which tends to a finite limit, has been introduced so that this estimate holds for $n \geq 0$ too. Fix $\varepsilon > 0$ so that $\varepsilon \sum_{n=0}^{\infty} \kappa(n) < 1/2$. Define ρ by $\rho(0) = 1$, $\rho(n+1) = \varepsilon \sum_{j=0}^{n} \kappa(n-j)\rho(j)$, $n \in \mathbb{N}$, and z by

$$z(n+1) = f_3(n+1) + \varepsilon \sum_{j=0}^{n} \kappa(n-j)|Y(j)| + \varepsilon \sum_{j=0}^{n} \kappa(n-j)z(j), \quad n \in \mathbb{N},$$

where $z(0) = 0$. Therefore $|Z(n)| \leq z(n)$ and

$$z(n) = \sum_{j=1}^{n} \rho(n-j) \left(f_3(j) + \varepsilon \sum_{k=0}^{j-1} \kappa(j-1-k)|Y(k)| \right).$$

As $\rho \in \ell^1(\mathbb{N};(0,\infty))$, there is an f_4 obeying $\lim_{n\to\infty} f_4(n) = 0$ and

$$|Z(n)| \leq f_4(n) + \varepsilon \sum_{k=0}^{n-1} (\rho * \kappa)(n-k-1)|Y(k)|.$$

Therefore, from (11) and Lemma 2 it follows that

$$\limsup_{n\to\infty} \frac{|Z(n)|}{\sqrt{2n \log\log n}} \leq \varepsilon c' \sum_{k=0}^{\infty} (\rho * \kappa)(k) = \varepsilon c' \sum_{k=0}^{\infty} \kappa(k) \frac{1}{1 - \varepsilon \sum_{k=0}^{\infty} \kappa(k)},$$

where $c' > 0$ is the righthand side of (11). Since ε can be taken as small as required, and the last inequality holds pathwise, we have (15). (16) is an immediate consequence of (15) and Theorem 3. Q.E.D.

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