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Mapping class actions on surface group completions

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Abstract.

Recent projects on pronilpotent and profinite completions of surface groups are discussed, and common threads are then compared in the context of general group completions. In particular, a profinite version of the Torelli groups and a pronilpotent version of the punctured solenoid are introduced.

§1. Introduction

I would like to describe here both the union and then the intersection of two recent projects:

- on pronilpotent completions of surface groups, which is joint work with Shigeyuki Morita [14];
- on profinite completions of surface groups, which is joint work with Dragomir Šarić [19] and with Dragomir Šarić and Sylvain Bonnot [3].

Background for both of these projects is the decorated Teichmüller theory of punctured surfaces [17, 18], which I shall first briefly recall.

§2. Decorated Teichmüller Theory

Let F_g^s denote a fixed smooth oriented surface of genus g with $s \geq 1$ punctures, where $2g - 2 + s > 0$. The *mapping class group* $MC(F_g^s)$ of isotopy classes of orientation-preserving diffeomorphisms of F_g^s acts on the *Teichmüller space* $\mathcal{T}(F_g^s)$ of isotopy classes of hyperbolic metrics on F_g^s by push-forward of metric with quotient *Riemann's moduli space*

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$\mathcal{M}(F_g^s)$. There is a trivial $\mathbb{R}^s_{>0}$-bundle over $\mathcal{T}(F_g^s)$ called the *decorated Teichmüller space* $\tilde{\mathcal{T}}(F_g^s)$, where the fiber over a point is identified with the collection of all s-tuples of horocycles in F_g^s, one horocycle about each puncture. Indeed, since horocycles in F_g^s are closed curves, we may take the coordinate on the fiber to be simply the tuple of hyperbolic lengths of the (not necessarily embedded) horocycles. $MC(F_g^s)$ acts on $\tilde{\mathcal{T}}(F_g^s)$ by permuting these numbers, and we thus have the diagram

$$\begin{array}{c} MC(F_g^s) \circlearrowright \tilde{\mathcal{T}}(F_g^s) - \mathbb{R}^s_{>0} \\ \downarrow \\ MC(F_g^s) \circlearrowright \mathcal{T}(F_g^s) \\ \downarrow \\ \mathcal{M}(F_g^s) \end{array}$$

relating these spaces and mapping class group actions.

Uniformize in Minkowski three-space with its pairing

$$< (x,y,z),(x',y',z') > = xx' + yy' - zz',$$

where (x,y,z) are the usual Cartesian coordinates, and with its group $SO(2,1)$ of isometries. We may regard a point of $\mathcal{T}(F_g^s)$ as the conjugacy class of a discrete and faithful representation of the fundamental group of F_g^s in the component $SO^+(2,1) \approx PSL_2(\mathbb{R})$ of the identity, where the representation is required to map peripheral elements of the fundamental group to parabolic elements of $SO^+(2,1)$. Letting

$$\mathbb{H} = \{w = (x,y,z) : < w,w > = -1 \text{ and } z > 0\}$$

denote the upper sheet of the hyperboloid and

$$L^+ = \{w = (x,y,z) : < w,w > = 0 \text{ and } z > 0\}$$

denote the subspace of isotropic vectors with positive height, we may identify the collection of horocycles in $\mathbb{H}$ with L^+ via affine duality $L^+ \ni v \leftrightarrow h(v) = \{w \in \mathbb{H} :< v,w > = -2^{-\frac{1}{2}}\}$.

There are two basic ingredients to the decorated Teichmüller theory, the first of which gives coordinates on $\tilde{\mathcal{T}}(F_g^s)$, as follows. Given a pair of horocycles $h(u), h(v)$, a natural invariant is given by

$$\sqrt{- < u,v >} = \exp \delta/2,$$

where δ is the signed hyperbolic distance between $h(u), h(v)$ taken with positive sign if and only if the horocycles are disjoint. We promote this invariant to the setting of decorated hyperbolic surfaces in the natural way: If $\tilde{G} \in \tilde{\mathcal{T}}(F_g^s)$ is a decorated hyperbolic structure on F_g^s and α is an isotopy class of arcs connecting punctures of F_g^s, then a lift to $\mathbb{H}$ of α is asymptotic to a pair of rays in L^+, each of which contains a well-defined point determined by the decoration, say these points are $u, v \in L^+$. Define the *lambda length* of α with respect to $\tilde{G}$ to be $\lambda(\alpha; \tilde{G}) = \sqrt{- < u, v >}$, which is independent of the choice of lift since the inner product is invariant by $SO^+(2,1)$.

The second main ingredient of decorated Teichmüller theory is the "convex hull construction" [6], which is described as follows. A point $\tilde{G} \in \tilde{\mathcal{T}}(F_g^s)$ determines not only (the conjugacy class of) a discrete group $G < SO^+(2,1)$ of isometries, but it also determines $s \geq 1$ many G-orbits of points in L^+, namely, the set $\mathcal{B} \subset L^+$ of points corresponding via affine duality to the collection of horocycles in the decoration of F_g^s. The closed convex hull of $\mathcal{B}$ in the underlying vector space structure of Minkowski three-space is a G-invariant convex body since $G < SO^+(1,2)$ acts linearly, and the *convex hull construction* associates to $\tilde{G}$ the projection $\Delta(\tilde{G})$ to F_g^s of the edges in the frontier of this closed convex hull of $\mathcal{B}$ which meet L^+ only in their endpoints, and we may straighten these arcs connecting punctures to geodesics for the hyperbolic structure underlying $\tilde{G}$.

Define an *ideal triangulation* Δ of F_g^s to be (the isotopy class of) a collection of disjointly embedded arcs connecting punctures which decompose F_g^s into triangles with vertices at the punctures. Given an arc $\alpha \in \Delta$ so that α triangulates a quadrilateral complementary to $\Delta - \{\alpha\}$ in F_g^s, let β denote the other diagonal of this quadrilateral, and define the *flip* along α in Δ to produce the ideal triangulation $\Delta \cup \{\beta\} - \{\alpha\}$. Finally, define an *ideal cell decomposition* (i.c.d) of F_g^s to be (the isotopy class of) a collection of disjointly embedded arcs connecting punctures which decompose F_g^s into ideal polygons with vertices at the punctures. We have the following "omnibus" theorem:

Theorem 2.1. [17, 18] **Part 1** *If Δ is an ideal triangulation of F_g^s, then*

$$\begin{aligned} \tilde{\mathcal{T}}(F_g^s) &\to \mathbb{R}_{>0}^{\Delta} \\ \tilde{G} &\mapsto (\alpha \mapsto \lambda(\alpha; \tilde{G})) \end{aligned}$$

is a real-analytic surjective homeomorphism. The action of $MC(F_g^s)$ *on these global coordinates is given by permutation followed by finite compositions of "Ptolemy transformations"* $ef = ac + bd$, *which describe the lambda lengths* e, f *of the pair of diagonals of a quadrilateral involved in a flip whose opposite sides have lambda lengths* a, c *and* b, d. *Furthermore, the Weil-Petersson Kähler two-form on* $\mathcal{T}(F_g^s)$ *pulls back to*

$$2\sum d\log\, a \wedge d\log\, b + d\log\, b \wedge d\log\, c + d\log\, c \wedge d\log\, a,$$

where the sum is over all triangles in F_g^s *complementary to* Δ *with lambda lengths* a, b, c *in this clockwise cyclic order as determined by the orientation of* F_g^s.

Part 2 *The convex hull construction* $\tilde{G} \mapsto \Delta(\tilde{G})$ *provides a canonical i.c.d.* $\Delta(\tilde{G})$ *for each* $\tilde{G} \in \tilde{\mathcal{T}}(F_g^s)$. *Furthermore,*

$$\{\{\tilde{G} \in \mathcal{T}(F_g^s) : \Delta(\tilde{G}) \text{ is isotopic to } \Delta\} : \Delta \text{ is an i.c.d of } F_g^s\}$$

is an $MC(F_g^s)$*-invariant ideal cell decomposition of* $\tilde{\mathcal{T}}(F_g^s)$. *If* Δ' *is any ideal triangulation containing the i.c.d.* Δ, *then the corresponding cell in decorated Teichmüller space is described in lambda length coordinates on* Δ' *by the coupled inequalities*

$$0 \leq \frac{a^2 + b^2 - e^2}{abe} + \frac{c^2 + d^2 - e^2}{cde} = E,$$

where the inequality is strict for each arc in Δ, *equality holds for each arc in* $\Delta' - \Delta$, *and the notation is as above with the edge of lambda length* e *separating those with lengths* a, b *from those with lengths* c, d.

The quantity E is called the *simplicial coordinate* of the edge with lambda length e, and it turns out that $2^{\frac{3}{2}}abcdE$ is the signed volume of the corresponding tetrahedron in Minkowski three-space, which is taken with a positive sign if the edge of lambda length e lies below that with lambda length f. (Note that elements of $SO^+(2,1)$ have unit determinant and hence preserve signed Euclidean volume.)

As a first corollary, notice that taking all the lambda lengths on an ideal triangulation to have value one gives positive simplicial coordinates, so every isotopy class of ideal triangulation of F_g^s actually arises from the convex hull construction. Since $\tilde{\mathcal{T}}(F_g^s)$ is connected, it follows from general position that finite compositions of flips act transitively on ideal triangulations of F_g^s, thus giving a new proof of this classical fact due to Whitehead.

This leads to two faithful representations of $MC(F_g^s)$. The first algebraic one given by the action on coordinates with respect to a fixed ideal triangulation follows from the evident naturality of lambda lengths $\lambda(\alpha, \tilde{G}) = \lambda(\phi(\alpha), \phi(\tilde{G}))$, for any $\phi \in MC(F_g^s)$, and represents an element of $MC(F_g^s)$ as a permutation followed by a finite composition of Ptolemy transformations. By the way, one easily directly checks that the two-form described in Part 1 of the Omnibus Theorem is invariant by Ptolemy transformations, which in particular shows directly that it is invariant under $MC(F_g^s)$.

For the second combinatorial representation of $MC(F_g^s)$, let us consider *labeled ideal triangulations*, by which we mean an ideal triangulation Δ of F_g^s together with an enumeration of the arcs in Δ by $1, 2, \ldots, 6g - 6 + 3s$. Define the *mapping class groupoid* $MD(F_g^s)$ to be the category whose objects are $MC(F_g^s)$-orbits of labeled ideal triangulations and whose morphisms are pairs of labeled ideal triangulations modulo the diagonal action of $MC(F_g^s)$ together with the obvious composition. The mapping class group $MC(F_g^s)$ is then just the group of self-morphisms in $MD(F_g^s)$ of any object.

Consider the flip on an arc α in an ideal triangulation supported in some ideal quadrilateral. Given a labeling on the ideal triangulation where α has label j, the corresponding *labeled flip* f_j along α simply assigns the labeling of α to the other diagonal of this quadrilateral and leaves invariant the labeling of the other arcs in the ideal triangulation.

Corollary 2.2. *The mapping class groupoid* $MD(F_g^s)$ *of* $F_g^s \neq F_0^3$ *admits the following presentation. Generators are given by labeled flips and permutations* (i, j) *of labels* i *and* j. *Relations are given by those of the symmetric group together with the following:*

Involutivity *for any labeled flip* f_i, *we have* $f_i \circ f_i = \text{identity}$;

Commutativity *for any labeled flips* f_i, f_j *where the arcs with labels* i, j *do not lie in the frontier of a common complementary triangle, we have* $f_i \circ f_j = f_j \circ f_i$;

Pentagon *for any labeled flips* f_i, f_j *where the arcs with labels* i, j *triangulate a pentagon with frontier in the ideal triangulation, we have* $f_i \circ f_j \circ f_i \circ f_j \circ f_i = (i, j)$;

Naturality *for any labeled flip* f_i *and any permutation* σ, *we have* $\sigma \circ f_i = f_{\sigma(i)} \circ \sigma$.

We have included this result here because of its independent interest and because it is a paradigm for several of the subsequent results. The main point of its proof is that a homotopy of paths in $\tilde{\mathcal{T}}(F_g^s)$ can be put into general position with respect to the codimension-two skeleton of our ideal cell decomposition, and there are precisely two kinds of codimension-two cells: either five points in $\mathcal{B} \subset L^+$ become coplanar, which leads to the pentagon relation, or two sets of four points in $\mathcal{B} \subset L^+$ become coplanar, which leads to the commutativity relation. In the context of unlabeled ideal triangulations, there are thus corresponding relations among unlabeled flips (where the right-hand side of the pentagon relation is replaced by the identity), and we shall continue to refer to these identities as involutivity, commutativity, and the pentagon relation.

In fact, it is convenient in the sequel to reformulate the combinatorics of i.c.d.'s as follows. The Poincaré dual of an i.c.d. of F_g^s is a *fatgraph* embedded as spine of F_g^s, namely, a graph in the usual sense of the term together with a cyclic ordering on the half-edges about each vertex, where the cyclic ordering is determined by the clockwise cyclic ordering in F_g^s. It is often more convenient to employ the fatgraph formalism rather than the entirely equivalent formalism of i.c.d.'s. The dual of a flip on an ideal triangulation is a *Whitehead move* on the dual trivalent fatgraph, namely, contract an edge of the fatgraph with distinct endpoints (corresponding to removing its dual arc), and then expand the resulting four-valent vertex differently (corresponding to adding the other diagonal of the corresponding quadrilateral). We shall also in this context of Whitehead moves refer to the identities discussed above as involutivity, commutativity, and the pentagon relation.

§3. Pronilpotent Case

We shall think of the surface F_g^1 as the unpunctured surface F_g^0 of genus g with a basepoint $*$ that plays the role of the puncture. Define the *lower central series* of the fundamental group recursively by setting $\Gamma_0 = \pi_1(F_g^0, *)$ and $\Gamma_{k+1} = [\Gamma_k, \Gamma_0]$, where the square brackets denote the commutator group. There are the corresponding *kth nilpotent quotients* defined by $N_k = \Gamma_0/\Gamma_k$ related by the basic exact sequence

$$0 \to \Gamma_k/\Gamma_{k+1} \to N_{k+1} \to N_k \to 1,$$

and in particular, N_1 is the integral first homology group of F_g^0. The mapping class group $MC(F_g^1)$ is identified with the mapping class group of F_g^0 which fixes $*$, there are induced actions of $MC(F_g^1)$ on each N_k, and we define the *kth Torelli group*

$$MC_{g,*}[k] \ = \ ker(MC(F_g^1) \to Aut(N_k))$$

to be those mapping classes that act identically on N_k. Dennis Johnson defined homomorphisms

$$\tau_k : MC_{g,*}[k] \to \mathrm{Hom}(N_1, \Gamma_k/\Gamma_{k+1}),$$

which arise naturally from the basic exact sequence.

Johnson himself proved that $MC_g[1] = ker(MC(F_g^0) \to Sp(2g, \mathbb{Z}))$ and that $MC_{g,*}[1]$ is finitely generated [7] by an explicit set of homeomorphisms for $g \geq 3$, he calculated $ker(\tau_1)$ in [8], and he computed the abelianization of $MC_{g,*}[1]$ in [9]. Geoff Mess proved [12] that $MC_2[1]$ is not finitely generated. Daniel Biss and Benson Farb have shown [2] that $MC_{g,*}[2]$ are not finitely generated for $g \geq 2$ together with the analogous result for closed surfaces.

Here we shall give infinite presentations of all of the groups $MC_{g,*}[k]$ and indeed finite presentations for certain related groupoids and groups. These results should be compared with recent work by Andrew Putman, who has given different infinite presentations for $MC_g[1]$ and $MC_{g,*}[1]$ when $g \geq 2$ which have a more efficient and geometrical generating set but more complicated relations. We shall also describe an explicit combinatorial cocycle expression for the first Johnson homomorphism.

The elementary but potent remarks are that the cell decomposition of $\tilde{\mathcal{T}}(F_g^1)$ descends to an ideal cell decomposition of $\mathcal{T}(F_g^1)$ itself (since changing the unique horocycle in the decoration simply scales the set $\mathcal{B}$ of points in Minkowski space and the convex hull construction is linear), and furthermore, this decomposition of $\mathcal{T}(F_g^1)$ is invariant under any subgroup of the mapping class group, and in particular is invariant under each $MC_{g,*}[k]$. Thus, there is a "Torelli tower"

$$\mathcal{T}(F_g^1) \to \cdots \to T_{k+1} \to T_k \to T_{k-1} \to \cdots \to T_1 \to \mathcal{M}(F_g^1),$$

where the *kth* "Torelli space" defined by

$$T_k = \mathcal{T}(F_g^1)/MC_{g,*}[k], \text{ for each } k \geq 1,$$

is a manifold and an Eilenberg-MacLane space $K(MC_{g,*}[k], 1)$, each mapping $T_{k+1} \to T_k$ is an unbranched covering of manifolds, and each $T_k \to \mathcal{M}(F_g^1)$ is an orbifold covering. The point is that the ideal cell decomposition of $\mathcal{T}(F_g^1)$ descends to each of these spaces and gives a combinatorial realization of this tower of spaces and maps.

Just as it is useful to consider the Poincaré dual of the cell decomposition of a surface to get a fatgraph, so too is it useful to pass to Poincaré duals here. Define $\hat{\mathcal{G}}$ to be the dual cell complex to the ideal cell decomposition of $\mathcal{T}(F_g^1)$, so $\hat{\mathcal{G}}$ is an *honest* cell complex (not an ideal cell complex) of dimension $4g-3$, and is contractible since its dual is. We shall also consider the quotients $\hat{\mathcal{G}}_k = \hat{\mathcal{G}}/MC_{g,*}[k]$, for $k \geq 1$.

Define an N_k*-marking* on a fatgraph Γ to be a function

$$\mu : \{\text{oriented edges of } \Gamma\} \to N_k$$

so that:

- $\mu(\bar{e}) = [\mu(e)]^{-1}$ if $e, \bar{e}$ denote the two orientations on a common underlying edge of Γ;
- if $a_1, \ldots, a_k$ are oriented edges pointing towards a common vertex of Γ in this counter-clockwise cyclic ordering, then

$$\mu(a_1)\mu(a_2)\cdots\mu(a_k) = 1 \in N_k$$

  where we compose from left to right by convention;
- the image of μ generates N_k.

Suppose that the trivalent fatgraph Γ' arises from the trivalent fatgraph Γ by a Whitehead move along the edge e of Γ producing the edge f of Γ'. Suppose that the oriented edge e points from the vertex with incident oriented edges $c, d, \bar{e}$ towards the vertex with incident oriented edges a, b, e in these correct counter-clockwise cyclic orders, where the oriented edges a, b, c, d point towards the corresponding vertex. Identify the oriented edges of Γ' with those of Γ in the natural way, where the oriented edge f points from the vertex with incident oriented edges $d, a, \bar{f}$ towards the vertex with incident oriented edges b, c, f in these correct counter-clockwise cyclic orders with these oriented edges pointing towards the corresponding vertices as before. The N_k-marking μ' on Γ' induced by an N_k-marking μ on Γ agrees with μ on oriented edges other than e, f, and

$$\mu'(f) = \mu(d)\mu(a) = [\mu(b)\mu(c)]^{-1}.$$

Theorem 3.1. [14] **Part 1** *There is a $MC(F_g^1)$-invariant cocycle*

$$j \in Z^1(\hat{\mathcal{G}}; \Lambda^3 N_1)$$

defined for a Whitehead move in the notation of the previous paragraph by $j(\Gamma \to \Gamma') = a \wedge b \wedge c = c \wedge d \wedge a$*, where* $\Lambda^3 N_1$ *denotes the third exterior power of* N_1 *and* $\wedge$ *denotes the exterior product, and the associated group homomorphism*

$$[j] \in H^1(\hat{\mathcal{G}}_1; \Lambda^3 N_1) \;\approx\; \mathrm{Hom}(MC_{g,*}[1], \Lambda^3 N_1)$$

coincides with $6\tau_1$*.*

Part 2 *For each* $k \geq 1$*, cells in* $\hat{\mathcal{G}}_k$ *are indexed by isotopy classes of suitable* N_k*-marked fatgraph spines of* F_g^1*, and coordinates on a given cell are provided by simplicial coordinates. Furthermore,* $MC_{g,*}[k]$ *is generated (in fact enumerated) by sequences of Whitehead moves beginning and ending on the same* $MC(F_g^1)$*-orbit which leave invariant the* N_k*-marking. Relations in* $MC_{g,*}[k]$ *are given by involutivity, commutativity, and the pentagon relation as before.*

In fact, one can use Part 2 together with Johnson's generating set for $MC_{g,*}[1]$ to give *finite presentations* for the fundamental path *groupoid* of the Torelli spaces T_1 and T_2 and also finite presentations for the "level N classical Torelli *groups*", i.e., the subgroup of $MC(F_g^1)$ that acts trivially on homology with coefficients mod N, for any $N \geq 1$.

Furthermore, one can use Part 1 with the Alexander-Whitney approximation to the diagonal together with results of Morita [13] and Kawazumi-Morita [10, 11] to give new cocycles generating the tautological algebra of $\mathcal{M}(F_g^1)$. Please see [14] for details on Theorem 3.1 as well as these further applications.

In fact, one uses the description of relations in Part 2 to show that the expression in Part 1 for the Johnson homomorphism is well-defined. The more general point is that there is a kind of "machine" here for producing $MC(F_g^1)$-invariant cocycles with coefficients in general modules by solving for coefficients so that the required relations of involutivity, commutativity, and the pentagon relation hold. Together with Shigeyuki Morita and Alex Bene, we have produced roughly ten or so such cocycles with values in various modules, so far without success in writing down an explicit combinatorial formula for the second Johnson homomorphism. We take this opportunity just to mention one further very simple such cocycle with values in the second symmetric power of the second exterior power of N_1, namely in the notation of Part 1, assigning the expression

$$(a \wedge c) \otimes (b \wedge d) \;+\; (b \wedge d) \otimes (a \wedge c)$$

to a Whitehead move gives another invariant cocycle as one can check.

Let us finally mention that recent joint work with Alex Bene and Nariya Kawazumi has shown that the higher Johnson homomorphisms lift to the mapping class groupoid in the sense of Part 1 of Theorem 3.1 for the case of surfaces with a single boundary component. Furthermore, this work also provides an algorithm for the explicit calculation of representative combinatorial cocycles in analogy to the cocyle j, and explicit formulae for the first several Johnson homomorphisms have been obtained.

§4. Profinite Case

Consider the collection of all finite pointed unbranched covers of some fixed surface $F = F_g^s$ of negative Euler characteristic with $s \geq 1$. Put another way, we may consider all finite pointed branched covers, where the branching occurs only at the removed punctures of F. This collection of covers is naturally inverse directed, where $\pi_1 : \tilde{F}_1 \to F$ is greater than or equal to $\pi_2 : \tilde{F}_2 \to F$ if there is a finite pointed unbranched cover $\pi_{12} : \tilde{F}_1 \to \tilde{F}_2$ so that $\pi_1 = \pi_2 \circ \pi_{12}$ (here reading composition functionally from right to left).

Define the *punctured solenoid* S to be the inverse limit of this collection of covers, and note that the homeomorphism type of S is independent of the initial choice of surface F subject to the stated conditions since any two such surfaces admit a common finite unbranched cover. For an explicit model of a space homeomorphic to S, take the classical modular group $G = PSL_2(\mathbb{Z})$, or any finite-index subgroup of it, let $\hat{G}$ denote its profinite completion, i.e., $\hat{G}$ is the inverse limit of G/H over all normal subgroups $H < G$ of finite index, and set

$$S_G = (\mathbb{D} \times \hat{G})/G, \text{ where } \gamma(z,t) = (\gamma z, t\gamma^{-1}) \text{ for } \gamma \in G,$$

$\mathbb{D}$ denoting the Poincaré disk with its usual action of G by hyperbolic isometries. Thus, $S \approx S_G$ has local neighborhoods modeled on the product of an open set in $\mathbb{D}$ with a Cantor set, and furthermore, these local foliations by open sets in $\mathbb{D}$ combine to give a canonical foliation of S_G by leaves which are identified with $\mathbb{D}$, here using that surface groups are residually finite.

Following Sullivan's work [20] in the analogous case of closed surfaces, we may define a Teichmüller theory of S in the spirit of Ahlfors-Bers theory [1], namely: We say that a homeomorphism from S_G is *quasiconformal* if it is quasiconformal in the usual sense in the $\mathbb{D}$ direction and varies continuously (technical point: for both the $\mathcal{C}^\infty$ and quasiconformal topologies) in the transverse Cantor set direction.

The *Teichmüller space* $\mathcal{T}(S)$ is then defined in perfect analogy to the classical case, namely, the space of equivalence classes of quasiconformal homeomorphisms from S_G, where two such $S_G \to S_i$, for $i = 1, 2$ are equivalent if there is a conformal map (or in other words, hyperbolic isometry) $S_1 \to S_2$, which is defined analogously, so that the obvious diagram commutes up to a homotopy (technical point: the homotopy must be bounded for the hyperbolic metric). By Sullivan's work in the closed case, $\mathcal{T}(S)$ has the natural structure of a Banach space in analogy to the classical case. It is easy to see that there are embeddings of $\mathcal{T}(F_g^s) \subset \mathcal{T}(S)$ by demanding that the covering maps over some F_g^s are conformal. Sullivan called these the "TLC" for transversely locally constant structures on S, and he proved [20] that these TLC structures on S are dense in $\mathcal{T}(S)$. Thus, the solenoid is a natural universal object in the sense that its Teichmüller space is a natural completion of all of the classical ones.

One may then define the *mapping class group* $MC(S)$ of the punctured solenoid as the group of all such equivalence classes of quasiconformal self-homeomorphisms of S_G. This is a very mysterious uncountable group which contains the more tractable discrete "baseleaf preserving" subgroup $MC_{BLP}(S) < MC(S)$ comprised of those mapping classes which fix some leaf (called the "baseleaf") of the natural foliation of S. Essentially by definition *as a topological space*, we have

$$MC(S) \approx MC_{BLP}(S) \times_G \hat{G},$$

but really nothing is known about $MC(S)$ as a group. There is the feeling that perhaps it is related to the absolute Galois group.

Chris Odden showed in his thesis [15, 16] that $MC_{BLP}(S)$ is isomorphic to the "virtual automorphism group" of G, namely, equivalence classes of isomorphisms between finite-index subgroups of G, which may have different indexes, where two such are equivalent if they agree on a common finite-index subgroup of their domains. Furthermore, Sullivan showed that the "Ehrenpreis Conjecture" (namely, for any points of $\mathcal{T}(F_{g_1}^{s_1})$ and $\mathcal{T}(F_{g_2}^{s_2})$, there are finite unbrached covers of $F_{g_1}^{s_1}$ and $F_{g_2}^{s_2}$ by a common surface F_g^s so that the induced structures on $\mathcal{T}(F_g^s)$ are arbitrarily close) is equivalent to the statement that $MC_{BLP}(S)$ has dense orbits on $\mathcal{T}(S)$.

The *Farey tesselation* Δ_* is the $PSL_2(\mathbb{Z})$-orbit of any ideal triangle in $\mathbb{D}$, and in a sense we shall make precise in Theorem 4.1, Δ_* plays the role for the punctured solenoid of an ideal triangulation for a punctured surface. To apply the decorated Teichmüller theory in the current context, we note that there are natural *punctures* of S corresponding to equivalence class of wandering rays, where two rays are equivalent if they are asymptotic in their common leaf. We may furthermore decorate these punctures by choosing a horocycle in each leaf centered at each puncture and requiring these choices of horocycles to vary continuously in the Cantor set direction. There is thus a natural *decorated Teichmüller space* $\tilde{\mathcal{T}}(S)$ of the solenoid, a natural continuous forgetful mapping $\tilde{\mathcal{T}}(S) \to \mathcal{T}(S)$, and given a homotopy class of arc in S connecting punctures and a point of $\tilde{\mathcal{T}}(S)$, there is again a lambda length defined as before.

Say that an ideal triangulation Δ of $\mathbb{D}$ is TLC if it is the lift to $\mathbb{D}$ of some ideal triangulation on $\mathbb{D}/K$ for some finite-index torsion free group $K < G$. Given a TLC tesselation Δ corresponding to $K < G$ and any arc $e \in \Delta$ whose projection $\bar{e}$ to $\mathbb{D}/K$ triangulates a quadrilateral complementary to $\Delta/K - \{\bar{e}\}$, we may define the result of the *K-equivariant flip on e* to be the lift to $\mathbb{D}$ of the ideal triangulation of $\mathbb{D}/K$ that arises by flipping $\bar{e}$ in Δ/K.

Theorem 4.1. [19] **Part 1** *The assignment of lambda lengths*

$$\tilde{\mathcal{T}}(S) \to Cont^G(\hat{G}, \mathbb{R}_{>0}^{\Delta_*})$$

gives a homeomorphism onto the space of all G-equivariant continuous functions with the compact-open topology from the profinite completion $\hat{G}$ to $\mathbb{R}_{>0}$-valued functions on the Farey tesselation Δ_* with the strong topology. The same formula for Ptolemy transformations describes the action of $MC_{BLP}(S)$ on these global coordinates. Furthermore, the same formula for the Weil-Petersson two-form gives a $MC_{BLP}(S)$-invariant two-form on $\tilde{\mathcal{T}}(S)$, which descends to a non-degenerate two-form on the Teichmüller space $\mathcal{T}(S)$.

Part 2 *Again, there is a convex hull construction associating to a point $\tilde{G}' \in \tilde{\mathcal{T}}(S)$ a collection $\Delta(\tilde{G}')$ of disjointly embedded arcs connecting punctures in S. The subspace*

$$\cup_{\mathrm{TLC}\Delta}\ \{\tilde{G}' \in \tilde{\mathcal{T}}(S) : \Delta(\tilde{G}') \simeq \Delta \text{ on the baseleaf}\}$$

of $\tilde{\mathcal{T}}(S)$ is open and dense, where $\simeq$ means homotopic rel endpoints at infinity, and the quotient of this subspace by $MC_{BLP}(S)$ is Hausdorff. Furthermore, $PSL_2(\mathbb{Z})$ together with all K-equivariant flips generate $MC_{BLP}(S)$, and a complete set of relations for these generators of $MC_{BLP}(S)$ can be written [3].

Thus, whereas lambda length numbers on the arcs in an ideal triangulation of F_g^s give global coordinates for $\tilde{\mathcal{T}}(F_g^s)$, continuous G- equivariant lambda length *functions*

$$\lambda_e : \hat{G} \to \mathbb{R}_{>0}, \text{ for } e \in \Delta_*,$$

on the edges of the Farey tesselation give global coordinates for $\tilde{\mathcal{T}}(S)$.

The same formula for the Weil-Petersson two-form gives an invariant two-form on $\tilde{\mathcal{T}}(S)$ in two senses: either fix a fundamental domain for $K < G$ torsion free of finite index, sum over the finitely many triangles in this fundamental domain the formula from before, and then integrate against Haar measure on $\hat{K}$, or equivalently choose a sequence of groups G_n, for $n \geq 1$, that are cofinal in the set of all finite-index normal subgroups of G (see the next section for such a cofinal set), choose nested fundamental domains D_n for each G_n, and take the limit as $n \to \infty$ of the reciprocal of the index of G_n in G times the two-form defined by summing the formula from before only over triangles in D_n.

It is entirely unclear which polyhedral decompositions of the baseleaf arise from the convex hull construction on $\tilde{\mathcal{T}}(S)$, let alone what is the topology of the corresponding subset of $\tilde{\mathcal{T}}(S)$. The presentation of $MC_{BLP}(S)$ therefore cannot be derived in analogy to the classical case, and a different complex on which $MC_{BLP}(S)$ suitably acts must be introduced to derive this presentation; again, the relations are tantamount to involutivity, commutativity, and the pentagon relation plus the obvious further relation corresponding to the case of K_1- and K_2-equivariant flips when $K_1 < K_2$ is finite-index. See [3] for details.

Since the union over all TLC tesselations Δ of

$$\{\tilde{G}' \in \tilde{\mathcal{T}}(S) : \Delta(\tilde{G}') \simeq \Delta \text{ on the baseleaf}\}$$

is open dense with Hausdorff quotient by $MC_{BLP}(S)$, the Ehrehpreis conjecture is dramatically false for decorated structures on the punctured solenoid.

§5. Group Completions

The previous two sections have surveyed applications of the decorated Teichmüller theory to pronilpotent and profinite completions of surface groups. We now turn our attention briefly to general group completions citing [4] for generalities and [5] for details on the profinite case in particular.

In the general set-up given a group G, we may consider any family $\mathcal{N}$ of normal subgroups of G where the intersection of any two groups in $\mathcal{N}$ again contains a group in $\mathcal{N}$, so that $\mathcal{N}$ is naturally a directed set. The canonical projections $G/N \to G/M$ for $N \subset M$ with $M, N \in \mathcal{N}$ thus provide an inverse system of groups, and the inverse limit of this system is the *pro-$\mathcal{N}$-completion* of G, to be denoted $\hat{\mathcal{N}}(G)$, which is a totally disconnected and complete topological group. The canonical homomorphism $G \to \hat{\mathcal{N}}(G)$ has dense image with kernel $\cap\mathcal{N}$, the canonical projection $G \to G/N$ extends uniquely to a homomorphism $\hat{\mathcal{N}}(G) \to G/N$, for each $N \in \mathcal{N}$, and the right action of G on itself extends uniquely to a continuous right action of G on $\hat{\mathcal{N}}(G)$. In case $\mathcal{N}$ is countable, we may assume that $\mathcal{N}$ is given by a nested sequence $N_1 \supset N_2 \supset \cdots$ and define the functions $\nu_i : G \to \{0,1\}$ by $\nu_i(g) = 0$ if and only if $g \in N_i$. The topology on G with local neighborhoods of the identity given by the elements of $\mathcal{N}$ is induced by the metric $\mu : G \times G \to [0,1]$ given by $\mu(g_1, g_2) = \sum_{i \geq 1} 2^{-i}\nu_i(g_1^{-1}g_2)$, and $\hat{\mathcal{N}}(G)$ is the metric completion of G for this metric comprised of equivalence classes of convergent Cauchy sequences. In the special case that $\cap\mathcal{N}$ is trivial, the canonical homomorphism $G \to \hat{\mathcal{N}}(G)$ is injective, and $\hat{\mathcal{N}}(G)$ is Hausdorff.

For instance, the two examples we have considered correspond to fixing an appropriate surface group G and taking $\mathcal{N}$ to be all normal subgroups $\mathcal{N}_f$ with finite index, giving the profinite completion, or all normal subgroups $\mathcal{N}_n$ with nilpotent quotient, giving the pronilpotent completion. Since fundamental groups of surfaces are residually finite and residually nilpotent, we have $\cap\mathcal{N}_f = \{id\} = \cap\mathcal{N}_p$, so the group injects into its Hausdorff completion in these cases. Furthermore, in the profinite case, we have the cofinal set

$$G_k = \cap\{K < G : [G : K] \leq k\} \in \mathcal{N}_f,$$

of characteristic subgroups, for $k \geq 1$, and in the pronilpotent case, we have the cofinal set of characteristic subgroups given by the terms of the lower central series for G.

There are many other interesting possibilities for surface groups where little is known including the prosolvable completion, the proLie completion, and the extreme case of the pronormal completion (corresponding to all normal subgroups).

In addition to simply pointing out here that these seem to be interesting directions for future research, we wish also to remark that in each case of the described projects on pronilpotent and profinite completions, there are aspects that are more or less functorial and are applicable in other cases of completions of a surface group G. Specifically for any completion, we may construct a corresponding "pro-$\mathcal{N}$ solenoid"

$$S(\mathcal{N}) = (\mathbb{D} \times \hat{\mathcal{N}}(G))/G, \text{ where } \gamma(z,t) = (\gamma z, t\gamma^{-1}) \text{ for } \gamma \in G,$$

we may define quasiconformal mappings as before, and we may ask for a description of the corresponding mapping class group, or more explicitly the relationship between this mapping class group and the groups $MC(S)$ and $MC_{BLP}(S)$ for $S = S(\mathcal{N}_f)$ discussed in the previous section.

Likewise, for any completion with a cofinal set $\{U_k\}_{k\geq 1}$ of characteristic groups in $\mathcal{N}$, we may consider the "$\mathcal{N}$-Torelli groups" defined by

$$\mathcal{I}_{\mathcal{N}}[k] = \{\phi \in Aut(G) : \phi \text{ acts identically on } G/U_k\},$$

and ask the obvious questions, for instance, are these groups finitely generated?, what are their abelianizations?, and so on, in analogy to the case $MC_{g,*}[k] = \mathcal{I}_{\mathcal{N}_p}[k]$ discussed already for G the fundamental group of a closed surface with basepoint. In particular, it certainly seems viable to study the profinite Torelli groups and the pronilpotent solenoid again using the decorated Teichmüller theory.

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