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Parameterized Gromov-Witten invariants and topology of symplectomorphism groups

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Abstract.

In this note we introduce parameterized Gromov-Witten invariants for symplectic fiber bundles and study the topology of the symplectomorphism group. We also give sample applications showing the non-triviality of certain homotopy groups of some symplectomorphism groups.

§1. Introduction

Given a symplectic manifold (M, ω), one of the basic mathematical objects associated to (M, ω) is its automorphism group $\mathrm{Symp}(M, \omega)$. Since the group $\mathrm{Symp}(M, \omega)$ can be equipped with the C^∞ topology, we would like to know the homotopy type of this automorphism group. In this note we are interested in the following questions:

1) How large is the rank of the homotopy group $\pi_i(\mathrm{Symp}(M, \omega) \otimes \mathbf{Q})$?

2) What are the characteristic classes of $\mathrm{Symp}(M, \omega)$, that is, the cohomology ring of the classifying space $\mathrm{BSymp}(M, \omega)$.

Our approach to these problems uses symplectic fiber bundle setting (a similar setting is used in the study of homotopy type of diffeomorphism groups) and Gromov's technique of pseudoholomorphic curves. We would also like to remark that the Gromov technique of pseudoholomorphic curves has been developed and extended in different directions in the study of the topology of symplectomorphism groups. Recent developments in this direction can be found in, e.g., McDuff's survey [19].

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This note consists of four sections. In section 2 we recall the definition of symplectic fiber bundles and we introduce the notion of fiber-wise (vertical) stable maps. In section 3 we show that the basic properties of the moduli space of stable maps also hold in the fiber-wise (family) version. As an immediate consequence we construct parameterized Gromov-Witten invariants for symplectic fiber bundles, which is a family version of the usual Gromov-Witten invariants for symplectic manifolds. In section 4 we apply this construction to problems 1, 2 mentioned above. We associate to each element $\pi_i(\mathrm{Symp}(M,\omega))$ a symplectic fiber bundle over S^{i+1} which is the union of two trivial symplectic bundles over a disk D^{i+1} glued along the boundary ∂D^{i+1} by this element $\pi_i(\mathrm{Symp}(M,\omega))$. We re-interpret a result by Gromov [8], Theorem 2.4.C_2 on the existence of an element of infinite order in the symplectomorphism group of non-monotone $S^2 \times S^2$ in terms of parameterized Gromov-Witten invariants, see Theorem 4.3. We also slightly generalize Gromov's result in the following cases. We denote by (X_1^4, ω_1) a non-monotone symplectic manifold which is diffeomorphic to $S^2 \times S^2$, and by (X_2^4, ω_2) a symplectic manifold which is diffeomorphic to $\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$.

Theorem 4.5. a) *Let* $(M_1, \Omega_1) = (X_1^4 \times N^{2k}, \omega_1 \oplus \omega_0)$ *be a symplectic manifold with* (X_1^4, ω_1) *as above and* (N, ω_0) *a compact symplectic manifold. Then we have* $rk(\pi_1(\mathrm{Symp}(M_1, \Omega_1)) \otimes \mathbf{Q}) \geq 1$.

b) *We also have* $rk(\pi_1(\mathrm{Symp}(X_2^4, \omega_2))) \geq 1$.

There are intensive studies on cohomology groups and homotopy types of symplectomorphism groups of rationally ruled symplectic 4-manifolds such as Abreu [1], Abreu-McDuff [2], Anjos [3], etc. In fact, we can improve Theorem 4.5 for the case of (M_1, Ω_1) without using "hard machinery'.

Theorem 4.8. a) *The rank of the homomorphism* $i_* : \pi_1(\mathrm{Symp}(M_1, \Omega_1)) \to \pi_1(\mathrm{Diff}(M_1))$ *is at least* 1.

b) *The rank of the homomorphism* $i_* : \pi_3(\mathrm{Symp}(M_1, \Omega_1)) \to \pi_3(\mathrm{Diff}(M_1))$ *is greater than or equal* 2.

In section 4 we also construct characteristic classes of the group $\mathrm{Symp}(M,\omega)$ by formulating the Gromov-Witten invariants in a dual way. We also include an Appendix, which contains an alternative proof of Theorem 4.3, a special version of Theorem 4.5.a.

After the preliminary version of this note was written [14], we learned several works on the topology of symplectomorphism groups, [10], [22], see also references in [19]. Since some results in [14] have been quoted in some literature e.g. [4], [5], [19], we feel a need to revise the version [14] to correct some errors as well as to add details to missing arguments.

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§2. Symplectic fiber bundles and vertical stable maps

In this section we recall the notions of symplectic fiber bundles, stable maps and introduce the notion of vertical stable maps. We refer to [7] for more discussions on symplectic fiber bundles. The idea of counting fiber-wise holomorphic curves in symplectic fiber bundles is also suggested by Kontsevich (in his communication to us after a preliminary version of this note has been written in 1997) and by Lu-Tian[1].

2.1. Symplectic bundles and their fiber-wise compatible almost complex structures

A fibration $M \to E \xrightarrow{\pi} B$ is said to be a symplectic fiber bundle, if the fiber M is diffeomorphic to a symplectic manifold (M, ω) and the transition function takes its value in the group $\mathrm{Symp}(M, \omega)$. We denote by $\mathcal{J}_\pi(E)$ the associated bundle over B whose fiber is the space $\mathcal{J}(M)$ of smooth compatible almost complex structures on (M, ω). Since the fiber $\mathcal{J}(M)$ is contractible, the space of sections $J(E) : B \to \mathcal{J}_\pi(E)$ is also non-empty and contractible.

In what follows, we are interested in defining invariants which detect the non-triviality of symplectic fiber bundles. Associated to any symplectic fiber bundle $M \to E \xrightarrow{\pi} B$ we obtain the local system of fiberwise homology groups, resp. fiberwise cohomology groups, denoted by $\mathcal{H}_*(E)$, resp. $\mathcal{H}^*(E)$. In what follows, the coefficients of cohomology groups are in $\mathbf{R}$ or $\mathbf{Z}$, and the coefficients of homology groups are in $\mathbf{Z}$.

[1]We thank Dusa McDuff for informing us that they used this idea in order to construct equivariant Gromov-Witten invariants, which is a special case of the parameterized Gromov-Witten invariants for the symplectic fiber bundle associated with the Hamiltonian action of a compact Lie group on a symplectic manifold.

Clearly all the invariants of the associate local systems $\mathcal{H}_*(E), \mathcal{H}^*(E)$ are also invariants of symplectic fiber bundles. In particular, if a symplectic fiber bundle is trivial, then the associated local systems are simple, i.e., trivial. We also observe that for a symplectic fiber bundle E there is always a section $s^{[\omega]} : B \to \mathcal{H}^2(E)$ which takes a given value $[\omega] \in H^2(M; \mathbf{R})$ and there is also a section $s^{c_1} : B \to \mathcal{H}^2(E)$ which takes a given value $c_1(M, \omega) \in H^2(M; \mathbf{Z})$.

2.2. Stable maps and vertical stable maps.

Our notion of vertical stable maps is based on the notion of stable maps due to Kontsevich [12], [11], see also [6] whose exposition we follow closely.

Let g and m be nonnegative integers. A semistable curve with m marked points is a pair (Σ, z) of a connected space $\Sigma = \cup \pi_\nu(C_\nu)$, where C_ν is a Riemann surface and $\pi_\nu : C_\nu \to \Sigma$ is a continuous map, and $z = (z_1, \cdots, z_m)$ are m distinct points in Σ with the following properties.
(1) π_ν is the normalization of the irreducible component $\Sigma_\nu = \pi_\nu(C_\nu)$ of Σ for all ν.
(2) For each $p \in \Sigma$ we have $\sum_\nu \#\pi_\nu^{-1}(p) \leq 2$. Here $\#$ denotes the order of the set.
(3) $\sum_\nu \#\pi_\nu^{-1}(z_i) = 1$ for each z_i.
(4) The number of Riemann surfaces C_ν is finite.
(5) The set $\{p \in \Sigma | \sum_\nu \#\pi_\nu^{-1}(p) = 2\}$ is finite.

We denote by g_ν the genus of C_ν and by m_ν the number of points $\overline{p}$ on C_ν, which are the inverse image of nodes of Σ, i.e. $\sum_\gamma \#\pi_\gamma^{-1}(\pi_\nu(\overline{p})) = 2$, or marked points, i.e. $\pi_\nu(\overline{p}) = z_j$ for some j. The genus g of a semistable curve Σ is defined by by

$$g = \sum_\nu g_\nu + \dim H_1(T_\Sigma; \mathbf{Q}),$$

where T_Σ is a graph associated to Σ in the following way. The vertices of T_Σ correspond to the components of Σ. Denote by v_ν the vertex corresponding to Σ_ν. For a node $p \in \Sigma_\nu \cap \Sigma_{\nu'}$, we assign an edge e_p joining vertices v_ν and $v_{\nu'}$. (When p is a node of Σ_ν, the "edge" e_p becomes a loop based at v_ν.)

A homeomorphism $\theta : \Sigma \to \Sigma'$ between two semistable curves is called an isomorphism, if it restricts to a biholomorphic isomorphism $\theta_{\nu\nu'} : \Sigma_\nu \to \Sigma'_{\nu'}$ for each component Σ_ν of Σ and some component $\Sigma'_{\nu'}$. We also require that θ maps the marked points in Σ onto the corresponding marked points in Σ' bijectively.

Let $J(E)$ be a vertical compatible almost complex structure. A map $u : (\Sigma, z) \to E$ is called a vertical $J(E)$-stable map, if the composition map $\pi \circ u$ sends (Σ, z) to a point $b \in B$ and u is a stable map from (Σ, z) to $\pi^{-1}(b) = (M, J(E)|_{E_b})$. In other words, for each ν, the restriction of u to each component Σ_ν is either a non-constant map, or we have $m_\nu + 2g_\nu \geq 3$.

To define the moduli space of vertical stable maps, we assume first, for the sake of simplicity and a later application, that the local system $\mathcal{H}_2(E)$ is simple, i.e., the fundamental group $\pi_1(B)$ acts trivially on $\mathcal{H}_2(E)$ (e.g. it is the case if the base B is simply connected).

In this case, for a class $A \in H_2(M; \mathbf{Z})$, there is a global locally constant section $s_A : B \to \mathcal{H}_2(E)$ whose value is A at a reference fiber. We consider the moduli space of all vertical stable map $((\Sigma_g, z), u)$ such that (Σ_g, z) is of genus g with m marked points. We denote by $C\mathcal{M}_{g,m}(E, J(E), s_A)$ the moduli space of vertical stable maps representing the class $s_A(\pi(u))$:

$$C\mathcal{M}_{g,m}(E, J(E), s_A) := \cup_{b\in B}\{b\} \times C\mathcal{M}_{g,m}(E_b, J(E)|_{E_b}, s_A(b)),$$

which carries a Kuranishi structure in the sense of [6], see Lemma 3.1 below.

Here $C\mathcal{M}_{g,m}(E_b, J(E)|_{E_b}, s_A(b))$ is the moduli space of stable maps of genus g, with m marked points and representing the homology class $s_A(b)$. Here, two pairs $((\Sigma, z), h))$ and $((\Sigma', z'), h')$ are equivalent, if and only if there exists an isomorphism $\theta : (\Sigma, z) \to (\Sigma', z')$ satisfying $h' \circ \theta = h$.

If the action of $\pi_1(B)$ on the fiber $H_2(M; \mathbf{Z})$ is non-trivial, we can still define a notion of a moduli space of vertical stable maps by considering a multi-valued section $s_A : B \to \mathcal{H}_2(E)$ which is obtained by the locally constant continuation of A in a typical fiber to a multi-valued section. We note that the pairing $\langle s^{[\omega]}(b), s_A(b)\rangle$ as well as the pairing $\langle s^{c_1}(b), s_A(b)\rangle$ are constant functions on B, since they are locally constant functions and we assume that B is connected. The number $\langle s^{[\omega]}, s_A(b)\rangle$ is the "energy" of a holomorphic curve realizing any class in $s_A(b)$, and the second number $\langle s^{c_1}(b), s_A(b)\rangle$ enters in the expected dimension of the moduli space of stable maps representing any class in $s_A(b)$. Now using the Gromov compactness theorem it is easy to see that there is only a finite number of values of s_A in each fiber $H_2(M = \pi^{-1}(b); \mathbf{Z})$ such that there is a $J(E)|_{E_b}$-holomorphic curve representing a homology class in the set $s_A(b)$. The projection from $C\mathcal{M}_{g,m}(E, J(E), s_A)$ to B is proper. It follows from the Gromov compactness theorem: If u_i is a sequence of J_i-stable maps with energy bounded by a constant and J_i converges to J_∞ in the space of compatible almost complex structures, then there exists a subsequence u_{i_k}, which converges to a J_∞-stable map u_∞.

Finally we observe that any element in $s_A(b)$ induces the same class in $H_*(E;\mathbf{Z})$ by the inclusion.

Remark 2.1. It is sometimes more convenient to work with a tame almost complex structure (i.e. $\omega(X, JX) > 0$ for any non-zero tangent vector X). As in the non-parameterized case all the compactness and perturbation theorems for pseudo-holomorphic curves with respect to a compatible almost complex structure hold for a tame almost complex structure.

2.3. Examples of symplectic fiber bundles.

There are several ways to construct symplectic fiber bundles.

The first way is the associate bundle method. Suppose that a group G acts symplectically on a symplectic manifold (M, ω), i.e. there is a homomorphism $\rho : \mathrm{G} \to \mathrm{Symp}(M, \omega)$. Then we can associate to each principal G-bundle P over B a symplectic fiber bundle $P \times_{\mathrm{G}} (M, \omega)$. This symplectic fiber bundle is non-trivial, if and only if the image $\rho_*(\lambda)$ of the homotopy class $\lambda \in [B, \mathrm{BG}]$ defining the G-bundle is non-trivial in $[B, \mathrm{BSymp}(M, \omega)]$.

The second way is the pull-back method. Suppose that we are given a symplectic fiber bundle E over a base B. Then any map f from B' to B pulls the bundle E back to a symplectic fiber bundle E' over B'.

The third way is the reduction method. We begin with a differentiable fiber bundle $M \to E \to B$ with M being a symplectic manifold and ask if this fiber bundle also admits a structure of a symplectic fiber bundle. Of course, it is the case if the inclusion of $\mathrm{Symp}(M, \omega)$ to $\mathrm{Diff}^+(M)$ is a homotopy equivalence (e.g. if dim $M = 2$). In general, we can state the following criterion, see e.g., [7] for more information.

Lemma 2.2. *Let $\pi : E \to B$ be a fiber bundle and $\omega \in \Omega^2(E)$ be a closed form such that ω is non-degenerate along all fibers of E. Then $\pi : E \to B$ admits a structure of a symplectic fiber bundle, which is compatible with ω.*

The fourth way to construct symplectic fiber bundles is the gluing method. Suppose that we are given two symplectic bundles E_1 and E_2 over bases B_1 and B_2 respectively. Suppose that the restriction of E_1 over the boundary ∂B_1 is isomorphic to the restriction of E_2 over the boundary ∂B_2. Then we can glue the bundle E_1 with E_2 along the boundary $E_i|_{\partial B_i}$. In particular when the restriction of E_i over ∂E_i is trivial then the glued bundle is defined uniquely by a map $\partial B_1 \to \mathrm{Symp}(M, \omega)$. If B_i is closed, we can define the operation of fiber connected sum as follows. Choose a small disk D_i in B_i and take a trivialization of $E_i|_{D_i} \to D_i$. Then glue $E_i|_{B_i \setminus D_i}$, $i = 1, 2$, along ∂D_i.

Remark 2.3. Each element $g \in \pi_k(\mathrm{Symp}(M, \omega))$ defines a symplectic fiber bundle E with the fiber (M, ω) over a sphere S^{k+1} by gluing two trivial symplectic fiber bundles $D^{k+1} \times M$ along the boundary $M \times S^k$ by the element g. Conversely any symplectic fiber bundle over S^{k+1} is defined by such a method.

§3. Parameterized Gromov-Witten invariants

In this section we define parameterized Gromov-Witten invariants for symplectic fiber bundles over a closed oriented manifold B. The base B is assumed to be oriented in order to deal with the orientation of the moduli space of stable maps. The base B is also assumed to be a closed manifold in order to get the fundamental class of the moduli space of vertical stable maps.

3.1. Geometric picture

Recall that a semistable curve (Σ, z) with m marked points is called stable, if for all its component C_ν of the normalization of Σ we have $m_\nu + 2g_\nu \geq 3$. Let $C\mathcal{M}_{g,m}$ denote the Deligne-Mumford moduli space of stable curves, i.e. $C\mathcal{M}_{g,m}$ is the set of all isomorphism classes of stable curves with m marked points and of genus g. Let us denote by $E^{(m)}$ the "Whitney sum" (the multiple fiber product over B) of m copies of E. When $2g + m \geq 3$, as in the usual case (see e.g. [12], 2.4, [11], 1.5), there is the evaluation map

$$\Pi = pr \times ev_{g,m,s_A} : C\mathcal{M}_{g,m}(E, J(E), s_A) \to C\mathcal{M}_{g,m} \times E^{(m)},$$

$$((\Sigma, z), u) \mapsto ((\tilde{\Sigma}, \tilde{z}), u(z_1), \cdots, u(z_m)).$$

Here $(\tilde{\Sigma}, \tilde{z})$ is the stable curve with marked points obtained from (Σ, z) by consecutive contractions of non-stable components. When $2g + m = 0, 1$, we call ev_{g,m,s_A} the evaluation map.

We briefly recall the notion of Kuranishi structures, see [6], §5 for details. Roughly speaking a compact Hausdorff space X has a Kuranishi structure, if it is locally described as the zero set $s^{-1}(0)$ of a V-bundle over a V-manifold, namely for each $p \in X$, there exist a V-manifold U_p, a V-bundle E_p on it and a continuous section s of $E_p \to U_p$ such that the difference $\dim E_p - \dim U_p$ is independent of $p \in X$. Moreover, we assume that such local descriptions are compatible under coordinate changes $\{\phi_{pq}\}$ in a suitable sense. We also have the notion that X with Kuranishi structure has its tangent bundle. If a continuous map $f : X \to$

Y extends locally to $f_p : U_p \to Y$ for each p such that $f_p \circ \phi_{pq} = f_q$, we call f is a strongly continuous map. If X has an oriented Kuranishi structure and f is a strongly smooth map from X to a topological space Y, then we can define the image $f_*([X])$ of the fundamental class of X as the image $f_*[(s')^{-1}(0)]$ of the fundamental class of the zero set $(s')^{-1}(0)$ of a perturbed smooth multi-section s' of E which is transversal to zero. Taking into account of multiplicity in an appropriate way, this fundamental class gives a well-defined element in $H_*(Y;\mathbf{Q})$.

Lemma 3.1. *The space* $C\mathcal{M}_{g,m}(E, J(E), s_A)$ *has a Kuranishi structure with oriented tangent bundle. This space is compact and of dimension* $\dim B + 2m + 2\langle c_1(M), s_A\rangle + (6 - \dim M)(g-1)$. *Moreover,* Π *is strongly continuous map in the sense of Kuranishi structure.*

By this Lemma, we can define the virtual fundamental cycle of the moduli space of vertical stable maps. Denote by $\Pi_*([C\mathcal{M}_{g,m}(E, J(E), s_A)])$ the induced class in $H_*(C\mathcal{M}_{g,m} \times E^{(m)};\mathbf{Q})$. The map Π induces a map in cohomologies

$$I^E_{g,m,s_A} : H^*(E^{(m)};\mathbf{Q}) \to H^{*+\mu}(C\mathcal{M}_{g,m};\mathbf{Q}) \tag{1}$$

by

$$I^E_{g,m,s_A}(\gamma) = PD(\gamma \backslash \Pi_*(C\mathcal{M}_{g,m}(E, J(E), s_A))) \tag{2}$$

If $2g + m = 0, 1$, we define

$$I^E_{g,m,s_A}(\gamma) = \langle \gamma, (ev_{g,m,s_A})_*(C\mathcal{M}_{g,m}(E, J(E), s_A))\rangle \in \mathbf{Q}. \tag{3}$$

The parameterized Gromov-Witten invariants, as in the usual case, are the collection of maps I^E_{g,m,s_A} defined in (2), (3). The shift of grading is given by $\mu = -\langle 2c_1(M), A\rangle + (g-1)\dim M - \dim B$.

Remark 3.2. The parametrized Gromov-Witten invariant $I^E_{g,0,s_A}$ is said to be of relative degree 0, if the expected dimension of $C\mathcal{M}_{g,0}(E, J(E), s_A)$ is zero, i.e., $\mu = 6(g-1)$. For symplectic fiber bundles E with the local system $\mathcal{H}_2(E)$ being simple, we can interpret the invariants $I^E_{g,0,s_A}$ of relative degree 0 in term of "counting vertical holomorphic curves" of genus g representing any class in s_A. Here "counting" means the "order" of the space with Kuranishi structure of expected dimension 0. The invariant

3.2. Proof of Lemma 3.1

To define rigorously the parametrized Gromov-Witten invariants for all compact symplectic fiber bundles, we need to prove Lemma 3.1 and moreover, to show that the map I^E_{g,m,s_A} does not depend on the choice of $J(E)$. Since the base space B is assumed to be compact, the moduli space $C\mathcal{M}_{g,m}(E, J(E), s_A)$ is compact, cf. section 2.2.

Proof of Lemma 3.1. Our proof of Lemma 3.1 is an adaptation of the proof of the corresponding results concerning Gromov-Witten invariants [6], Theorems 7.10 and 7.11.

Let u be a vertical stable map over $b_0 \in B$. For simplicity, we assume that the domain of u is irreducible. (The general case is handled in a similar way.) Pick a finite dimensional space $\mathcal{E}_0 \subset L^p\Omega^1(u^*TE_{b_0})$ such that

$$\mathrm{Im}\ D_u\overline{\partial}_{J_{b_0}} + \mathcal{E}_0 = L^p\Omega^1(u^*TE_{b_0}).$$

If $b \in B$ is in a neighborhood D_0 of b_0, we can identify fibers E_b and E_{b_0} by a local trivialization of E. Thus J_b is considered as an almost complex structure on E_{b_0}. Then there is a smaller neighborhood $D_0' \subset D_0$ of b_0 such that

$$\mathrm{Im}\ D_u\overline{\partial}_{J_b} + \mathcal{E}_0 = L^p\Omega^1(u^*TE_{b_0}).$$

Using this observation, the argument in [6] implies that there exists a Kuranishi neighborhood $(U, \mathcal{E}, s)$ of u on $C\mathcal{M}_{g,m}(E, J(E), s_A)$ so that a V-manifold U is fibered over an open subset of B.

All arguments in [6] can be directly adapted to the parametrized case or they even imply the corresponding statements in the parametrized case. For details of the construction of Kuranishi structure, see [6], §12. Q.E.D.

In order to show that the usual Gromov-Witten invariants do not depend on the choice of compatible almost complex structures, perturbations, we needed to construct a bordism between the moduli spaces corresponding to two almost complex structures J_1 and J_2 (with perturbations). This bordism is a version of the moduli space of vertical stable pseudo-holomorphic curves parametrized by the interval $[0, 1]$.

3.3. Properties of parameterized Gromov-Witten invariants

The following theorem immediately follows from Lemma 3.1 applied to the case that the base space is $B \times [0, 1]$.

Theorem 3.3. *The parameterized Gromov-Witten invariants are invariants of symplectic fiber bundles.*

Remark 3.4. The bordism type invariants of the moduli space of pseudoholomorphic curves may have more informations than the (cohomological) Gromov-Witten invariants. Such examples of finer Gromov-invariants of symplectic manifolds can be found in [17].

In order to distinguish a symplectic fiber bundle from the trivial one by parameterized Gromov-Witten invariants, we need to compute those for trivial symplectic fiber bundles $E = B \times M$. By the Künneth formula, the algebra $H^*(B \times (M)^{(m)}; \mathbf{Q})$ is isomorphic to $H^*(B; \mathbf{Q}) \otimes (H^*(M; \mathbf{Q}))^{\otimes m}$. Let us denote by α_i elements in $H^*(M; \mathbf{Q})$ and by β an element in $H^*(B; \mathbf{Q})$.

Proposition 3.5. *The Gromov-Witten invariants of a trivial symplectic fiber bundle satisfy*

$$I^E_{g,m,s_A}(\beta \otimes \alpha_1 \otimes \cdots \otimes \alpha_m) = I^M_{g,m,A}(\alpha_1 \otimes \cdots \otimes \alpha_m) \int_B \beta. \tag{4}$$

Proof. We choose a vertical compatible almost complex structure $J(E = B \times M)$ such that it is constant in the B-direction. Clearly the moduli space of vertical stable maps $C\mathcal{M}_{g,m}(E = B \times M, \{J_b \equiv J\}, s_A)$ is the direct product $B \times C\mathcal{M}_{g,m}(M, J, A)$. We take a multi-valued perturbation of Kuranishi map for $C\mathcal{M}_{g,m}(M, J, A)$ to define the virtual fundamental cycle of $B \times C\mathcal{M}_{g,m}(M, J, A)$. Denote by Π^{pt} the evaluation map in section 3.1 for $C\mathcal{M}_{g,m}(M, J, A)$, i.e. the case with base $B = pt$. By (2), the left hand side of (4) equals

$$\begin{aligned} &PD(\beta \otimes \alpha_1 \otimes \cdots \otimes \alpha_m \backslash \Pi_*[B \times C\mathcal{M}_{g,m}(M, J, A)]) \\ =&PD(\beta \otimes \alpha_1 \otimes \cdots \otimes \alpha_m \backslash ([B] \times \Pi^{pt}_*[C\mathcal{M}_{g,m}(M, J, A)]) \end{aligned} \tag{5}$$

Clearly the right hand side of (4) equals the right hand side of (5). Q.E.D.

Now let us compute parameterized Gromov-Witten invariants of a pull-back symplectic fiber bundle. Let $p : B_1 \to B_2$ be a k-fold covering space and $E_2 \to B_2$ a symplectic fiber bundle. Then the pull-back $E_1 = p^* E_2 \to B_1$ is also a symplectic fiber bundle. For a single section s_A of $\mathcal{H}_2(E_2)$, denote by $p^* s_A$ its pull back. We get immediately the following

Proposition 3.6. *We have*

$$I^{E_1}_{g,m,p^*s_A}(p^*_{(m)}\alpha) = kI^{E_2}_{g,m,s_A}(\alpha),$$

where $p_{(m)} : E_1^{(m)} \to E_2^{(m)}$.

Parameterized Gromov-Witten invariants of relative degree 0 and without marked points satisfy the following additivity.

Proposition 3.7. *Let $E = E_1 \# E_2$ be a fiber connected sum of symplectic fiber bundles E_1 and E_2. Then we have the following formula for parameterized Gromov-Witten invariants of relative degree 0.*

$$I^E_{g,0,s_A} = I^{E_1}_{g,0,s_A} + I^{E_2}_{g,0,s_A}.$$

Proof. By the dimension assumption of the Gromov-Witten invariants, we take perturbation, if necessary, so that there is no vertical stable curves representing the class A over a small disk $D_i(\varepsilon)$ in the base B_i for $i = 1, 2$. We can assume further that our fiberwise almost complex structures on $D_i(\varepsilon)$, $i = 1, 2$, are constant and isomorphic each other. Now we perform the connected sum of symplectic fiber bundles using these disks. The almost complex structures on E_i can be glued together identifying their restrictions on $D_i(\varepsilon) \times M$. Hence we obtain the proposition. Q.E.D.

Remark 3.8. For symplectic fiber bundles $\pi_i : E_i \to B_i$ with simple local systems $\mathcal{H}^*(E_i)$, $\alpha_j \in H^{q_j}(M; \mathbf{Q})$ defines the locally constant sections of $\mathcal{H}^{q_j}(E_i)$, $i = 1, 2$, and $\mathcal{H}^{q_j}(E_1 \# E_2)$. Suppose that there exist cohomology classes $\widetilde{\alpha}_j^{(i)} \in H^{q_j}(E_i; \mathbf{Q})$ such that their restrictions to typical fibers coincide with α_j. Let $\widetilde{\alpha}_j \in H^{q_j}(E_1 \# E_2; \mathbf{Q})$ be a cohomology class, which is equal to $\widetilde{\alpha}_j^{(i)}$ after restricting to $\pi_i^{-1}(B_i \setminus D_i(\varepsilon))$. When $\mu + \sum_j q_j = 6(g-1)$, we have

$$I^E_{g,m,s_A}(\prod_j \widetilde{\alpha}_j) = I^{E_1}_{g,m,s_A}(\prod_j \widetilde{\alpha}_j^{(1)}) + I^{E_2}_{g,m,s_A}(\prod_j \widetilde{\alpha}_j^{(2)}).$$

At the end of this section we would like to suggest that many properties of the Gromov-Witten invariants (e.g. the Kontsevich-Manin axioms) should be valid in the family version. Specially interesting seems to us an analog of Taubes' theorem on the relation of Gromov-Witten invariants and Seiberg-Witten invariants in dimension 4. It would imply that the parametrized Gromov-Witten invariants also bring information on the homotopy type of the diffeomorphism group of 4-dimensional symplectic manifolds.

§4. Homotopy groups of symplectomorphism groups.

In this section we combine Remark 2.3, Propositions 3.5, 3.6, 3.7 and other observations to estimate the rank of homotopy groups of symplectomorphism groups.

Proposition 4.1. *Parameterized Gromov-Witten invariants of relative degree* 0 *and without marked points* $I^E_{g,0,s_A}$ *over sphere* S^{i+1} *define elements in* $Hom(\pi_i(\mathrm{Symp}(M,\omega)),\mathbf{Q})$, $i \ge 1$.

Proof. Denote by E_g the symplectic fiber bundle over S^{i+1} by gluing $D_1^{i+1}\times M \cup_g D_2^{i+1}\times M$ with the help of a map $g : \partial D_1^{i+1} \to \mathrm{Symp}(M)$, i.e. we identify a pair $(x,y) \in \partial D_1^{i+1}\times M$ with $(x, g(x)\cdot y) \in \partial D_2^{i+1}\times M$. Then we have $E_{g\cdot f} \cong E_g \# E_f$. Now we get Proposition 4.1 immediately from Remark 2.3 and Proposition 3.7. Q.E.D.

Remark 4.2. Taking into account Remark 3.8 we can get a similar statement for certain parameterized Gromov-Witten invariants. We use such invariants in the proof of Theorem 4.5.a.

As an application of Proposition 4.1, Remark 4.2, we shall prove Theorem 4.3 and Theorem 4.5.

Let $(M,\omega) = (S^2\times S^2, \omega^{(1)} \oplus \omega^{(2)})$ be a product of symplectic manifolds. We denote by A_i the generators of $H_2(M;\mathbf{Z})$ realizing by the i-th sphere S^2.

Theorem 4.3. *If* $\omega^{(1)}(A_1) > \omega^{(2)}(A_2)$ *then there is an* $S^2\times S^2$*-symplectic fiber bundle over* S^2 *with non-vanishing parameterized Gromov-Witten invariants of relative degree* 0 *and* $m = 0$. *In particular the rank of* $\pi_1(\mathrm{Symp}(S^2\times S^2,\omega))$ *is at least* 1.

We recall that the last statement in Theorem 4.3 is established in Theorem 2.4. C_2 in [8].

Proof of Theorem 4.3. There are several ways of describing the proof of this theorem (see also Appendix, which follows the original idea of Gromov [8]). We present here a proof following the idea of McDuff in [17], Lemma 3.1, which uses the deformation space of complex structures of Hirzebruch's surfaces of even degree, which is diffeomorphic to $S^2\times S^2$, see [16]. Thus we can apply technique in complex analytic geometry for our computation.

Denote by $\mathcal{O}(\ell)$ the holomorphic line bundle of degree ℓ on $\mathbf{C}P^1$. Let us recall that the Hirzebruch surface F_k is the projectivization of a rank 2 holomorphic vector bundle $W_k = \mathcal{O}(0)\oplus\mathcal{O}(k)$ with $k \ge 0$ over $\mathbf{C}P^1$. The line subbundles $\mathcal{O}(0)\oplus 0$ and $0\oplus\mathcal{O}(k)$ define sections of $F_k = \mathbf{P}(W_k) \to \mathbf{C}P^1$ with self-intersection number k and $-k$, respectively. We denote by J^{F_k} the complex structure on the Hirzebruch surface F_k. It is known that all Hirzebruch surfaces with even degree k are diffeomorphic to $S^2\times S^2$, and F_0 is biholomorphic to $\mathbf{C}P^1\times\mathbf{C}P^1$ (see e.g., [16], chapter 1).

We define a family of holomorphic vector bundles $\{V_a\}$ of rank 2 on $\mathbf{C}P^1$ as follows. Write $U_0 = \mathbf{C}P^1 \setminus \{\infty\}$ and $U_\infty = \mathbf{C}P^1\setminus\{0\}$. Consider the transition function $f_a : (U_0 \setminus \{0\}) \times \mathbf{C}^2 \to (U_\infty \setminus \{\infty\}) \times \mathbf{C}^2$ by

$$f_a(z, v_1, v_2) = (z, zv_1 + av_2, z^{-1}v_2).$$

Denote by X_a the projectivization of V_a and J_a the complex structure on it. Note that $V_{a=0}$ is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ and that $\mathbf{P}(V_{a=0})$ is isomorphic to $\mathbf{P}(W_2)$. Thus $\{X_a\}$ is a complex one-dimensional deformation of $X_0 = F_2$. The complex structure J_0 is the complex structure J^{F_2}. All J_a, $a \neq 0$, are isomorphic to the complex structure J^{F_0}, i.e., the product complex structure.

Note that

$$\mathcal{X} = \cup_{a \in \mathbf{C}^1} \{a\} \times X_a \to \mathbf{C}^1 \times \mathbf{C}P^1$$

is the projectivization of the vector bundle

$$\mathcal{V} = \cup_{a \in \mathbf{C}^1} \{a\} \times V_a \to \mathbf{C}^1 \times \mathbf{C}P^1.$$

Since the restriction of the vector bundle $\mathcal{V}$ to $(\mathbf{C} \setminus \{0\}) \times \mathbf{C}P^1$ is holomorphically trivialized by the following 2 sections σ_1 and σ_2, which are everywhere linearly independent:

$$\sigma_1(a, z) = \begin{cases} (z, 0, 1) \in U_0 \times \mathbf{C}^2, & \text{if } z \in U_0 \\ (z, a, \frac{1}{z}) \in U_\infty \times \mathbf{C}^2, & \text{if } z \in U_\infty \end{cases}$$

$$\sigma_2(a, z) = \begin{cases} (z, 1, -\frac{z}{a}) \in U_0 \times \mathbf{C}^2, & \text{if } z \in U_0 \\ (z, 0, -\frac{1}{a}) \in U_\infty \times \mathbf{C}^2, & \text{if } z \in U_\infty \end{cases}$$

Note that

$$f_a(z, 0, 1) = (z, a, \frac{1}{z}), \quad f_a(z, 1, -\frac{z}{a}) = (z, 0, -\frac{1}{a}).$$

Thus σ_1, σ_2 are well-defined holomorphic sections. Using this trivialization, we extend $\mathcal{V}$ and $\mathcal{X}$ across $\{\infty\} \times \mathbf{C}P^1$ to $\mathbf{C}P^1 \times \mathbf{C}P^1$. We denote these extensions by the same symbols. $\mathcal{X}$ is also considered as an $S^2 \times S^2$ bundle with the projection to the first factor $\mathbf{C}P^1$ parameterized by a. Denote this fiber bundle by $E \to \mathbf{C}P^1$. The complex structure on $\mathcal{X}$ induces the fiberwise complex structure $J(E)$.

Clearly the parameterized Gromov-Witten invariant $I_{0,0,A_1-A_2}$ is of relative degree 0. We shall show that its value $I^E_{0,0,A_1-A_2}$ is 1. Note that there is no J^{F_0}-holomorphic sphere realizing class $(A_1 - A_2)$, since $J^{F_0} = \mathbf{C}P^1 \times \mathbf{C}P^1$. Further, there is exactly one J^{F_2}-holomorphic sphere realizing class $(A_1 - A_2)$, which is the section of self-intersection number -2 in the $\mathbf{C}P^1$ bundle over $\mathbf{C}P^1$. Thus the moduli space $C\mathcal{M}_{0,0,A_1-A_2}(E, J(E))$ consists of one point at $a = 0$.

To prove the transversality of this moduli space, we argue as follows. Let N be the normal bundle of the unique J^{F_2}-holomorphic sphere in the class $(A_1 - A_2)$ in the total space of fibration. In order to show the transversality in the case of an integrable complex structure, it suffices to prove the following

Lemma 4.4. *We have* $H^1(\mathbf{C}P^1; \mathcal{O}(N)) = 0$. *Hence,* $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Proof. Consider the cohomology exact sequence associated to

$$0 \to \mathcal{O}(-2) \to N \to \mathcal{O}_D \to 0,$$

where $\mathcal{O}(-2)$ is the normal bundle of the (-2)-curve in the central fiber $X_{a=0} \subset \mathcal{X}$ with the complex structure J^{F_2}, and the third term $\mathcal{O}_D$ is the quotient, which is nothing but the pull back of the normal bundle of the origin in $\mathbf{C}P^1$ parameterized by a. We shall show that the connecting homomorphism $H^0(\mathcal{O}_D) \to H^1(\mathcal{O}(-2))$ is surjective, (hence isomorphism). As a consequence we get $H^1(N) = 0$.

Here we regard X_a as the projectivization of $\mathcal{O}(-1) \otimes V_a$. When $a = 0$, it is isomorphic to $\mathcal{O}(-2) \oplus \mathcal{O}(0)$. The vector bundle $\mathcal{O}(-1) \otimes V_a$ is written as the gluing $U_0 \times \mathbf{C}^2$ and $U_\infty \times \mathbf{C}^2$ by

$$(z, v_1, v_2) \mapsto (z, z^2 v_1 + azv_2, v_2).$$

Note that the (-2)-curve representing $A_1 - A_2$ is the image of $\{0\} \otimes \mathcal{O}(0)$, hence the image of the section $(0, 1)$ in $\mathbf{P}(\mathcal{O}(-2) \oplus \mathcal{O}(0))$ over $a = 0$.

Consider the restrictions of the section $(0, 1)$ of $\mathcal{O}(-1) \otimes V_{a=0}$ over U_0 and U_∞, respectively. For $a \in \mathbf{C}$, the pair $z \in U_0 \mapsto (z, 0, 1) \in U_0 \times \mathbf{C}^2$ and $z \in U_\infty \mapsto (z, 0, 1) \in U_\infty \times \mathbf{C}^2$ gives a deformation as a Cech 0-cocycle. Taking the Cech differential, we get a Cech 1-cocycle $z \in U_0 \cap U_\infty \subset U_\infty \mapsto (z, az, 0) \in (U_\infty \setminus \{\infty\}) \times \mathbf{C}^2$. Differentiate in a, then we find that the last component of the Cech 1-cocycle vanishes and obtain a Cech 1-cocycle $z \in U_0 \cap U_\infty \subset U_\infty \mapsto (z, z) \in (U_\infty \setminus \{\infty\}) \times \mathbf{C}$ of $\mathcal{O}(-2)$, which represents a non-zero element in $H^1(\mathbf{C}P^1; \mathcal{O}(-2))$. Hence we conclude that the connecting homomorphism $H^0(\mathcal{O}_D) \to H^1(\mathcal{O}(-2))$ is surjective. Q.E.D.

Let us generalize Theorem 4.3. We denote by (X_1^4, ω_1) a non-monotone symplectic manifold which is diffeomorphic to $S^2 \times S^2$ and by (X_2^4, ω_2) a symplectic manifold which is diffeomorphic to $\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$. We are going to prove the following

Theorem 4.5. a) *Let* $(M_1, \Omega_1) = (X_1^4 \times N^{2k}, \omega_1 \oplus \omega_0)$ *be a symplectic manifold with* (X_1, ω_1) *as above and* (N^{2k}, ω) *a compact symplectic manifold. Then we have* $rk(\pi_1(\mathrm{Symp}(M_1, \Omega_1)) \otimes \mathbf{Q}) \geq 1$.
b) *We also have* $rk(\mathrm{Symp}(M_2^4, \omega_2)) \geq 1$.

Proof. a) To prove the statement for (M_1, Ω_1) we shall construct a symplectic fiber bundle E with fiber (M_1, Ω_1) over $B = S^2$ and compute a certain parameterized Gromov-Witten invariant with one marked point of E. For a symplectic fiber bundle E with fiber (M_1, Ω_1), we consider the moduli space of vertical holomorphic mappings $f : S^2 \to E$ whose image represents the class $A := A_1 - A_2$. Note that the local system $\mathcal{H}_2(E)$ is simple for any symplectic fiber bundle over the base space $B = S^2$.

The dimension computation shows us that the moduli space of vertical stable maps of genus 0 in the class A on an (M, ω)-bundle over S^2 has dimension $2k = \dim N^{2k}$. We will compare symplectic fiber bundles over S^2, each of which is the product of a symplectic fiber bundle E' with the fiber X_1^4 over S^2 and $N = N^{2k}$, i.e., $E = E' \times N \to S^2$. We shall count the number of vertical pseudo-holomorphic spheres u with one marked point z in the class A so that $ev(u; z) \in \Gamma$. Here Γ is a cycle represented by a submanifold $E' \times \{y_0\} \subset E$, for some arbitrary chosen point $y_0 \in N^{2k}$. This number is the parameterized Gromov-Witten invariant $I^E_{0,1,A_1-A_2}(PD[\Gamma])$.

Note that the restriction of $PD[\Gamma] \in H^*(E)$ to $E|_b$, $b \in B$, is $PD[X_1^4 \times \{y_0\}] \in H^*(X_1^4 \times N)$. But this condition does not characterize $PD[\Gamma] \in H^*(E)$. Let $\Gamma' \in E$ be another cycle such that the restriction of $PD[\Gamma']$ to $E|_b$ is equal to $PD[X_1^4 \times \{y_0\}] \in H^*(X_1^4 \times N)$. Since the base space B is S^2, we find that $PD[\Gamma'] - PD[\Gamma] \in H^2(B) \otimes H^*(X_1^4 \times N)$. In other words, $[\Gamma'] - [\Gamma]$ is represented by some cycle contained in a single fiber $E|_{b_0}$. By dimensional counting argument, we can take a fiberwise almost complex structure such that there are no pseudo-holomorphic $(A_1 - A_2)$-spheres contained in $E|_{b_0}$. Thus we have $I^E_{0,1,A_1-A_2}(PD[\Gamma]) = I^E_{0,1,A_1-A_2}(PD[\Gamma'])$. Hence $I^E_{0,1,A_1-A_2}(PD[\Gamma])$ gives an invariant for symplectic $X_1^4 \times N$-bundles over S^2.

We claim that this number of the trivial bundle $S^2 \times (M_1, \Omega_1)$ equals zero. To show it we consider a vertical almost complex structure J^{prod} on the trivial bundle $(M, \omega) \times S^2$ such that on each fiber (M, ω) we have $J^{\mathrm{prod}} = J^0 \times J^N$, where J^0 is the standard product complex structure on $S^2 \times S^2$ and J^N is an almost complex structure on (N^{2k}, ω_0). Clearly the projection on the first factor of any J^{prod}-sphere is also a J^0-sphere in $S^2 \times S^2$. Hence the moduli space of J^{prod}-sphere in class $A_1 - A_2$ is empty.

Now we construct a symplectic (M_1, Ω_1)-bundle E over S^2 by gluing two trivial symplectic (M_1, Ω_1)-bundles, one over a disk B^2 and the other over a disk D^2, using the loop $\tilde{g}_t : S^1 \to \mathrm{Symp}(M_1, \Omega_1)$ of the form $\tilde{g}_t = (g_t \times \{Id\}) \in (\mathrm{Symp}(X_1^4, \omega_1) \times \{Id\}) \subset \mathrm{Symp}(M_1, \Omega_1)$. Here g_t is the transition function for $\mathcal{X} \to \mathbf{C}P^1$. (See also Appendix.)

To compute the Gromov-Witten invariant $I^E_{0,1,A_1-A_2}(PD[\Gamma])$, we construct a fiberwise compatible almost complex structure $J(E)$ by gluing two fiberwise compatible almost complex structures on the restriction of E to disks D_0^2 and D_1^2. The first fiberwise compatible almost complex structure is defined as follows: $J(z) = (J^0 \times J^N)$ for $z \in D_0^2$. The second one is defined as follows: $J(a) = (J_a \times J^N)$, $a \in D_1^2 \subset \mathbf{C}$. They are glued by using the symplectomorphism loop $\tilde{g}_t$.

We claim that the constructed vertical almost complex structure is $A = A_1 - A_2$-generic. Clearly outside the singular point $a = 0$ in the disk D_1^2 of the base S^2, where the vertical almost complex structure take value $(J^{F_2} \times J^N)$, the moduli space of $J(E)$-holomorphic spheres realizing A is empty. At the singular point $a = 0$ the moduli space is diffeomorphic to N^{2k}, namely it consists of maps $u_y(t) = \{u_1(t) \times y\}, y \in N^{2k}$, where $t \in S^2$ and u_1 is the J^{F_2}-holomorphic (-2)-sphere in X_1^4. Clearly the transversality of the constructed $J(E)$ is equivalent to the surjectivity of the linearization map $D_u\bar{\partial}_{J(E)} : T_{\pi(u)}S^2 \times L_1^p(u^*T_{ver}E) \to L^p(\Lambda^{0,1}S^2 \otimes_{J(E)} u^*T_{ver}E)$. Since $u = (u_1, y)$, we have $u^*T_{ver}E = (u_1^*TX_1^4) \times T_yN^{2k}$, so the surjectivity of $D\bar{\partial}_{J(E)}$ follows from the surjectivity of the linearization map considered in the proof of Theorem 4.3, see also the proof of Lemma 5.1 in Appendix. This proves the first statement in Theorem 4.5 for the case (M_1, Ω_1).

Now let us prove the statement b). Denote by $\mathrm{Symp}(\mathbf{C}P^2, \omega, pt)$ the subgroup of the symplectomorphisms of $(\mathbf{C}P^2, \omega)$ which preserve a point pt. Clearly Theorem 4.5.b follows from the following Lemmas 4.6, 4.7. Q.E.D.

Lemma 4.6. *The fundamental group of* $\mathrm{Symp}(\mathbf{C}P^2, \omega, pt)$ *contains a subgroup* $\mathbf{Z}$.

Lemma 4.7. *There is an injective homomorphism* α *from the infinite cyclic subgroup of* $\pi_1(\mathrm{Symp}(\mathbf{C}P^2, \omega, pt))$ *in Lemma* 4.6 *to* $\pi_1(\mathrm{Symp}(X_2^4, \omega_2))$.

Lemma 4.6 is a direct consequence of Gromov's theorem, which states that $\mathrm{Symp}(\mathbf{C}P^2, \omega)$ is homotopy equivalent to $PU(3)$, since the quotient space $\mathrm{Symp}(\mathbf{C}P^2, \omega)/\mathrm{Symp}(\mathbf{C}P^2, \omega, pt)$ is isomorphic to $\mathbf{C}P^2$. We can also see this fact as follows. It suffices to show that the inclusion $U(2) \to \mathrm{Symp}(\mathbf{C}P^2, \omega, pt)$ induces an injective homomorphism on the corresponding fundamental groups. To see it we consider the evaluation map $ev : \mathrm{Symp}(\mathbf{C}P^2, \omega, pt) \to Sp(4) : g \mapsto Dg(pt, v)$, where v is an element in the frame $Sp(4)$ over the fixed point in $\mathbf{C}P^2$. The restriction of this evaluation map to $U(2)$ is injective, and we know that the image $ev(U(2))$ is a deformation retract of $Sp(4)$. Hence follows the Lemma.

Proof of Lemma 4.7. Denote by E_a the symplectic fiber bundle over S^2 with fiber $(\mathbf{C}P^2, \omega, pt)$ corresponding to element $a \in \pi_1(U(2))(\cong \mathbf{Z}) \subset \pi_1(\mathrm{Symp}(\mathbf{C}P^2, \omega, pt))$. Note that each fiber has a base point, hence there is a canonical section s of E_a.

Pick a $U(2)$-invariant Darboux ball B such that (X_2^4, ω_2) is symplectomorphic to the symplectic manifold $\mathbf{C}P^2 \setminus B$ with symplectic reduction applied to the boundary (symplectic cutting construction). Then we have the homomorphism $\rho : \pi_1(U(2)) \subset \pi_1(\mathrm{Symp}(\mathbf{C}P^2, \omega, pt)) \to \pi_1(\mathrm{Symp}(X_2^4, \omega_2))$. Denote by $\widetilde{E}_a$ the symplectic fiber bundle corresponding to $\rho(a)$. Then $\widetilde{E}_a$ is the fiberwise blow-up of E_a along the section s.

In fact, E_a carries a symplectic structure and $\widetilde{E}_a$ can be obtained by symplectic blowing-up of E_a along the symplectic submanifold $s(S^2)$ as follows. Using the spectral sequence for E_a we see that there is a closed 2-form Ω such that the restriction of Ω to the fiber $\mathbf{C}P^2$ equals ω. Thus we can apply the Thurston construction (see e.g. [21], p. 193) to conclude that E_a is a symplectic manifold with a symplectic form Ω_K in a class $K\pi^*(\omega_0) + \Omega$, where ω_0 is a symplectic form on the base S^2 and K is a sufficiently large real number. Moreover, all the fibers $\mathbf{C}P^2$ are symplectic submanifolds of (E_a, Ω_K) and we may assume that $s(S^2)$ is a symplectic submanifold. Recall that $\widetilde{E}_a$ is the symplectic fiber bundle over S^2 by fiber-wise blowing-up a symplectic fiber bundle with the fiber $(\mathbf{C}P^2, \omega)$ at point $s(x), x \in S^2$. This is exactly the blow-up (E_a, Ω_K) along the submanifold $s(S^2)$. Clearly the fiber of this new fiber bundle is $\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$. Now we apply Lemma 2.2 to conclude that $\widetilde{E}_a$ is a symplectic fiber bundle with the fiber (X_2^4, ω_2).

Now assume that α is not injective. Then, for some integer $a \neq 0$, we can find a trivialization of symplectic fiber bundle $\widetilde{E}_a$ with a constant compatible complex structure $J^0_{\widetilde{E}_a}$. We denote by $\{J^t_{\widetilde{E}_a}\}_{0 \le t \le 1}$ a family of compatible almost complex structures on $\widetilde{E}_a$ with $J^1_{\widetilde{E}_a}$ being the almost complex structure resulting from the blow-up process along $s(S^2)$. Note that the space of compatible almost complex structures, which admit $(-k)$-curve, is of real codimension $2(k-1)$. (See Appendix for the proof in the case that $k = 2$. The argument can be generalized for general

$k > 2$.) Note also that $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ does not contain any cycle of self-intersection number -2. In our case, $\{J^t_{\widetilde{E}_a}\}$ gives a three parameter family of compatible almost complex structures on $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$. Thus we can take $\{J^t_{\widetilde{E}_a}\}_{0\le t\le 1}$ such that there are no $(-k)$-curves, $k \ge 2$, in the fibers of $(\widetilde{E}_a, J^t_{\widetilde{E}_a})$ over $\mathbf{C}P^1$. Then, for each $t \in [0,1]$ and $b \in S^2$, there is a unique (-1) embedded curve in each fiber $(\widetilde{E}_a|_b \cong X^4_2, \omega, J^t)$. Thus the families of $J^t_{\widetilde{E}_a}$-holomorphic $(A_1 - A_2)$-spheres in the fibers of $\widetilde{E}_a$ parametrized by $b \in S^2$ are isotopic, when t varies. Therefore the process of blowing-down of all $J^t_{\widetilde{E}_a}$-holomorphic $(A_1 - A_2)$-spheres in the fibers of $\widetilde{E}_a$ is unique up to isomorphisms of symplectic fiber bundles. Hence follows that our symplectic fiber bundle (E_a, Ω_K) is also trivial, which is a contradiction. Q.E.D.

We shall improve Theorem 4.5 in the following statement[2].

Theorem 4.8. a) *The rank of the homomorphism* $\pi_1(\mathrm{Symp}(M_1, \Omega_1)) \to \pi_1(\mathrm{Diff}(M_1))$ *is at least* 1.

b) *The rank of the homomorphism* $\pi_3(\mathrm{Symp}(M_1, \Omega_1)) \to \pi_3(\mathrm{Diff}(M_1))$ *is at least* 2.

Proof. a) It is enough to show that the symplectic fiber bundle constructed by the loop $(\tilde{g}_t)^k$, $k \neq 0$, in the proof of Theorem 4.5.a is a non-trivial differentiable fibration for all k. To do so, it suffices to compute the cohomology ring of this differentiable bundle. Since the loop $\widetilde{g}_t$ by our choice is the product of g_t and the identity element in $\mathrm{Symp}(N, \omega_0)$, our cohomology is also the tensor product of the corresponding rings.

Here, we consider the projectivization E of the bundle $\mathcal{V}$ over $\mathbf{C}P^1 \times \mathbf{C}P^1$, which is given in the proof of Theorem 4.3. We compute the cohomology ring of E. We regard E as a family of $\mathbf{C}P^1$-bundles on $\mathbf{C}P^1$ parametrized by $\mathbf{C}P^1$, namely a $\mathbf{C}P^1$-bundle over $S^2 \times S^2$.

Let us compute $H^*(E; \mathbf{Z})$ by using the Leray-Hirsch Theorem. To see that the first Chern class $c_1(\mathcal{V})$ vanishes, it suffices to compute its evaluation on $S^2 \times \{pt\}$ and $\{pt\} \times S^2$. The second Chern class $c_2(\mathcal{V})$ equals the Poincare dual of the class of the zero section of σ_1, which extends to a section on $\mathbf{C}P^1 \times \mathbf{C}P^1$. This zero locus consists of the only point $(z = \infty, a = 0)$. So we find that $c_2(\mathcal{V})$ is the generator $\{S^2 \times S^2\}$

[2]This proof was suggested by Professor A. Kono, when K.O. gave a proof of this result at his seminar in 1996.

of $H^4(S^2 \times S^2; \mathbf{Z})$. Applying the Leray-Hirsch-Theorem we get

$$H^*(E; \mathbf{Z}) = \frac{H^*(S^2 \times S^2; \mathbf{Z})[t]}{t^2 - \{S^2 \times S^2\}}.$$

In this ring t^2 cannot be divided by 2. But in the ring $H^*(S^2 \times S^2 \times S^2; \mathbf{Z})$ any element t^2 can be divided by 2. Hence E is not homotopic to $S^2 \times S^2 \times S^2$. That proves the non-triviality of the loop $\{g_t\}$ in the diffeomorphism group.

Similarly, we compare the modules consisting elements of degree 2 of $H^*(E \times N; \mathbf{Z})$ and $H^*(S^2 \times S^2 \times S^2 \times N; \mathbf{Z})$, whose square is divisible by 2 to conclude that the loop $\{\tilde{g}_t\}$ is not null homotopic in the diffeomorphism group.

To prove that the loop $\{g_t^n\}$, $n \neq 0$, also realizes a non-trivial element in the diffeomorphism group we proceed similarly. Namely the loop $\{g_t^n\}$ corresponds to the n-time fiber connected sum $E^{\langle n \rangle}$. We have the following formula

$$H^*(E^{\langle n \rangle}; \mathbf{Z}) = \frac{H^*(S^2 \times S^2; \mathbf{Z})[t]}{t^2 - n\{S^2 \times S^2\}}.$$

Now we note that the set of element $x \in H^*(E^{\langle n \rangle}; \mathbf{Z})$ such that $x^2 = 0$ is the union $\mathbf{Z}(\{S^2\} \times 1) \cup \mathbf{Z}(1 \times \{S^2\})$. In particular from any 3 elements in this set we can get 2 linearly dependent elements. This implies that $E^{\langle n \rangle}$ and $S^2 \times S^2 \times S^2$ not homotopic, because there are 3 linearly independent elements of the cohomology ring of $S^2 \times S^2 \times S^2$, namely the generators $\{S^2\} \times 1 \times 1$, $1 \times \{S^2\} \times 1$, $1 \times 1 \times \{S^2\}$, whose square vanish.

Similarly, the ranks of the modules generated by elements of square zero in $H^2(E \times N; \mathbf{Z})$ and $H^2(S^2 \times S^2 \times S^2 \times N; \mathbf{Z})$ are different. Hence $\{\tilde{g}_t\}$ is not null homotopic in the diffeomorphism group.

b) It suffices to show that the two subgroups $SO(3) \times Id \subset \mathrm{Symp}(M_1, \omega)$ and $Id \times SO(3) \subset \mathrm{Symp}(M_1, \omega)$ realize two linearly independent elements in $\pi_3(\mathrm{Diff}(M_1))$. We denote by E_1 and E_2 two differentiable bundles with fiber M_1 over S^4, which correspond to the elements in $\pi_3(\mathrm{Diff}(M_1))$ realized by these subgroups $SO(3)$. Let us consider the homotopy exact sequences of these fibration E_i, which give us two connecting homomorphisms $h_i : \pi_4(S^4) \to \pi_3(M_1)$. We observe that $\pi_4(S^4) = \mathbf{Z}$, $\pi_3(M_1) = \mathbf{Z} \oplus \mathbf{Z} \oplus \pi_3(N)$. Now it is easy to check that the homomorphism h_i are linearly independent, and hence the image of the two subgroups are also linearly independent elements in $\pi_3(E)$. Q.E.D.

We can also describe parametrized Gromov-Witten invariants in a different way using the Poincaré duality. Define

$$CGW^E_{g,m,s_A}: \quad H^*(C\mathcal{M}_{g,m};\mathbf{R}) \to H^{*+\nu}(E^{(m)};\mathbf{R})$$
$$\delta \mapsto PD(\delta\backslash\Pi_*(C\mathcal{M}_{g,m,s_A}(E,J(E))).$$

Here $\nu = m\dim M - 2\langle c_1(M),A\rangle + (\dim M - 6)(g-1) - 2m$.

Assume that the local system $\mathcal{H}_2(E)$ for the symplectic fiber bundle $p: E \to B$ is trivial. Using the moduli space with $m=0$, we define the following homomorphism

$$CGW_{g,A}(E): \quad H^*(C\mathcal{M}_{g,0};\mathbf{R}) \to H^{*+\nu}(B;\mathbf{R})$$
$$\delta \mapsto PD(p_*(\delta\backslash\Pi_*(C\mathcal{M}_{g,0,s_A}(E,J(E)))),$$

where $p: C\mathcal{M}_{g,0,s_A}(E,J(E))) \to B$ is the projection.

We call the image of the map $CGW_{g,A}(E)$ the Gromov-Witten characteristic classes.

Theorem 4.9. *The Gromov-Witten characteristic classes are invariants of symplectic fiber bundles E with simple local system $\mathcal{H}_2(E)$. All the Gromov-Witten characteristic classes with positive degree vanish for trivial symplectic fiber bundles.*

Proof. The first statement is obvious (cf. Theorem 3.3). To prove the second statement we compute the Gromov-Witten characteristic classes on a trivial bundle $B \times M$. Let $[T]$ be a cycle in B. As before we denote also by p the projection $E^{(m)} \to B$.

$$\begin{aligned}\langle CGW_{g,A}(E)(\delta),[T]\rangle &= \langle PD(p_*(\delta\backslash\Pi_*(B\times C\mathcal{M}_{g,0}(J,A)))),([T])\rangle \\ &= \langle PD([B]),[T]\rangle\langle\delta,\Pi_*(C\mathcal{M}_{g,0}(J,A))\rangle.\end{aligned}$$

Clearly $\langle PD([B]),[T]\rangle = 0$, if $\dim[T] \geq 1$. Q.E.D.

The word "characteristic class" is explained by the following functoriality of these classes. In particular we see that the Gromov-Witten characteristic classes are cohomology classes of the classifying space $\mathrm{BSymp}_0(M,\omega)$. (Note that $\mathrm{BSymp}_0(M,\omega)$ is simply connected, hence any local systems on it are simple. If we consider the moduli space of vertical stable maps with a fixed energy and a fixed Chern number, we can obtain characteristic classes for $\mathrm{Symp}(M,\omega)$-bundles.)

Theorem 4.10. *Let E be a symplectic fiber bundle over B with the simple local system $\mathcal{H}_2(E)$ and $f^*(E)$ be the induced symplectic fiber bundle by a map $f: B_1 \to B$. Then we find that*

$$\tilde{f}^*(CGW^E_{g,m,A}) = CGW^{f^*E}_{g,m,A},$$

where $\widetilde{f} : f^*E \to E$ is the tautological bundle map of symplectic fiber bundles and

$$CGW_{g,A}(f^*E) = f^* \circ (CGW_{g,A})(E).$$

Proof. By the construction, we have

$$C\mathcal{M}_{g,m,s_A}(f^*(E)) = (C\mathcal{M}_{g,m,s_A}(E))_p \times_f B_1,$$

where we take the fiber product in the sense of spaces with Kuranishi structures and p is the projection from $C\mathcal{M}_{g,m,s_A}(E))$ to B. Hence, we find that

$$\widetilde{f}^*(PD(\Pi_*(C\mathcal{M}_{g,m,s_A}(E)))) = PD(\Pi_*(C\mathcal{M}_{g,m,s_A}(f^*E))).$$

Hence we can see immediately that

$$(id. \otimes (\widetilde{f}^*)^{\otimes m})(CGW^E_{g,m,A}) = CGW^{f^*E}_{g,m,A}.$$

When $m = 0$, it implies that

$$CGW_{g,A}(f^*E) = f^* \circ CGW_{g,A}(E)$$

Q.E.D.

It is an easy exercise to interpret Theorem 4.3 and Theorem 4.5 in term of Gromov-Witten characteristic classes.

§5. Appendix. An alternative proof of Theorem 4.3

Let $\omega^{(1)}$, $\omega^{(2)}$ be symplectic forms on S^2 such that $\int_{S^2} \omega^{(1)} > \int_{S^2} \omega^{(2)}$. According to Proposition 4.1 it suffices to find a symplectic bundle E over S^2 with fiber $(S^2 \times S^2, \omega = \omega^{(1)} \oplus \omega^{(2)})$ and a parameterized Gromov-Witten invariant whose value on E is non-trivial. We shall construct the bundle E by finding its transition function g, i.e., a loop in $\mathrm{Symp}(S^2 \times S^2, \omega)$. The existence of such element g was shown by Gromov [8], and in what follows we shall give a detailed proof. First we need the following lemma (compare with [8], 2.4.C). Denote by A_1, resp. A_2 the homology classes $[S^2 \times \{pt\}]$, resp. $[\{pt\} \times S^2]$.

Lemma 5.1. *The subspace $\mathcal{J}_0$ of compatible almost complex structures J on $S^2 \times S^2$, for which there exists a J-holomorphic sphere in a class $A_1 - \ell A_2$ for some $\ell \geq 1$, is a non-empty closed subset of codimension 2 in $\mathcal{J}(S^2 \times S^2, \omega)$. For each $J \in (\mathcal{J}(S^2 \times S^2, \omega) \setminus \mathcal{J}_0)$ and for each point in $S^2 \times S^2$ there is a unique J-holomorphic sphere representing class A_i and passing through x. Moreover, these spheres are embedded.*

Proof. First of all, we note that for any $J \in \mathcal{J}(S^2 \times S^2, \omega)$ there exists a unique embedded J-holomorphic sphere representing the class A_2 and passing through each point x, cf. [18]. We include here the proof of this fact for the reader's convenience.

For a generic compatible almost complex structure J, there exists such a J-holomorphic sphere. For $J_\infty \in \mathcal{J}(S^2 \times S^2, \omega)$, pick a sequence $\{J_i\}$ of generic compatible almost complex structures, which converges to J_∞. Let u_i be the J_i-holomorphic sphere representing the class A_2 and passing through x. Suppose that there exists a subsequence u_{i_k} converging to a J_∞-holomorphic sphere u_∞. Since A_2 is a primitive class, the adjunction formula implies that u_∞ is embedded. Clearly it passes through x. Hence we obtain the desired existence. If it is not the case, a subsequence of $\{u_i\}$ converges to a J_∞-stable map and there appears a J_∞-holomorphic map v representing the class $kA_2 - \ell A_1$ for some integers k and ℓ such that k is positive and $(k, \ell) \neq (1, 0)$. If v is multiply covered, factorize it as $v = p \circ v'$, where v' is a simple map and p is a ramified covering of $\mathbf{C}P^1$. Replace v by v', if necessary, we may assume that v is simple. Since $\int_{A_1} \omega > \int_{A_2} \omega$ and v is a J_∞-holomorphic map with the symplectic area smaller than $\int_{A_2} \omega$, we have $k \geq \ell + 1$ and $\ell \geq 1$. By the adjunction formula, the virtual genus of $C = v(\mathbf{C}P^1)$ is

$$
\begin{aligned}
g_v(C) \quad &= 1 + \frac{1}{2}(C \cdot C - c_1(C)) \\
&= 1 - k\ell - k + \ell \\
&\leq 1 - (\ell + 1)\ell - (\ell + 1) + \ell \\
&< 0.
\end{aligned}
$$

However, the virtual genus $g_v(C)$ is a non-negative integer, which is a contradiction. Therefore we obtain the existence of J_∞-holomorphic sphere representing the class A_2 and passing through the given point x.

Next we prove that $\mathcal{J}_0$ is a non-empty closed subset of codimension 2. The subspace $\mathcal{J}_0$ is non-empty because the sphere $(x, -x)$ is symplectic. To prove the closedness of $\mathcal{J}_0$ we first notice that the energy of a holomorphic sphere in class $A_1 - \ell A_2$ is less than $\omega^{(1)}(A_1)$. Thus we can apply the Gromov compactness argument to the following situation. Let a sequence of J_i-holomorphic spheres u_i representing $A_1 - \ell A_2$. Suppose that J_i converges to a compatible almost complex structure J_∞. Then there is a subsequence $\{u_{i_k}\}$, which converges to a J_∞-stable map u_∞. If u_∞ is a J_∞-holomorphic sphere, we find that $J_\infty \in \mathcal{J}_0$. (In this case, by the adjunction formula, u_∞ is an embedding.) Otherwise, u_∞ consists of at least two irreducible components, which represent the classes $k_i A_1 - \ell_i A_2$ such that $\sum k_i = 1$ and $\sum \ell_i = \ell$. As we mentioned above, there is always a J_∞-holomorphic sphere in class A_2. Taking into account of posivity of intersection in dimension 4, we conclude that these J_∞-holomorphic spheres must be of type $A_1 - \ell_i A_2$ and $m_j A_2$ such that $\sum(-\ell_i) + \sum m_j = -\ell$. Since m_j, if exists, must be positive, we conclude that there must be a bubble of type $A_1 - \ell' A_2$, $\ell' \geq 1$, that proves the closedness of $\mathcal{J}_0$.

The codimension of $\mathcal{J}_0$ is at least 2 by a similar argument as in [13], [14]. (Namely for a fixed ℓ we consider the universal moduli space of the pairs (J, J-holomorphic sphere in class $A_1 - lA_2$). The Fredholm index of the projection of this moduli space on the first factor is equal $4 + 2(1 - \ell) - 6 \leq -2$.) To prove that the codimension is precisely 2, we use the uniqueness of J-holomorphic sphere in class $A_1 - A_2$ if it exists. (cf. with the argument in [9]. It follows that the kernel of the linearization of the projection from the universal moduli space to the first factor, i.e., the space of compatible almost complex structures equals zero.).

Finally we prove the existence of J-holomorphic sphere in class A_1 for $J \in (\mathcal{J} \setminus \mathcal{J}_0)$ and use again the bubbling-off argument. For a generic compatible almost complex structure J, there exist J-holomorphic spheres in the class A_1, see [18]. Note that such J-holomorphic spheres are automatically embedded by the adjunction formula and that the class A_1 is primitive. If there is no J_0-holomorphic curve in the class A_1, we pick a sequence of generic compatible almost complex structures converging to J_0. Then the bubbling-off argument implies that there must be a J_0-holomorphic sphere in a class $A_1 - \ell A_2$ for $\ell \geq 1$. Q.E.D.

Now let us find an element $g \in \mathrm{Symp}(S^2 \times S^2, \omega^{(1)} \oplus \omega^{(2)})$ by studying the action of the group of symplectomorphisms on $\mathcal{J} \setminus \mathcal{J}_0$. Since $\mathcal{J}_0$ is a closed subset of codimension 2 we can choose a small disk D in $\mathcal{J}(S^2 \times S^2)$ such that this disk intersects $\mathcal{J}_0$ transversally at exactly one interior point. By results of Gromov [8] and McDuff [18], for any compatible almost complex structure J_θ, $\theta \in \partial D$, $S^2 \times S^2$ is foliated by A_1-curves and A_2-curves, respectively. In particular, J_θ is pointwisely positive on these A_1-curves and A_2-curves. It implies that there is a loop g_t in the group $\mathrm{Symp}(S^2 \times S^2, x)$ such that the image $g_t(J_0)$ is homotopic to the loop ∂D in $\mathcal{J} \setminus \mathcal{J}_0$, where J_0 is the complex structure on $\mathbf{C}P^1 \times \mathbf{C}P^1$. Thus, we can deform D along the boundary so that D intersects $\mathcal{J}_0$ transversally at one point and $\partial D = \{g_t(J_0)\}$.

Now we construct our bundle E by gluing two trivial $S^2 \times S^2$ bundles over another disk D' using this loop g_t. Since the base space of E is S^2, which is simply connected, $\mathcal{H}_2(E)$ is a simple local system. We claim that the parametrized Gromov-Witten invariant $I^E_{0,0,A_1-A_2}$ is 1.

Since $c_1(A_1 - A_2) = 0$, the moduli space $C\mathcal{M}_{0,0}(E, J(E), A_1 - A_2)$ is 0-dimensional. To compute the invariant for our bundle E, we choose a generic fiberwise compatible almost complex structure $J(E)$ on E as follows. Note that it is the case of weakly monotone symplectic manifolds and we are working with Gromov-Witten invariants of genus 0, $c_1(A_1 - A_2) = 0$ and the dimension of the base of E is 2. Therefore it suffices to perturb J to get the fundamental class of the corresponding moduli space. We observe that the standard product complex structure J_0 on $S^2 \times S^2$ is a $(A_1 - A_2)$-regular. Then we take $J(E)$ being the gluing of the constant complex structure J_0 over D' and the compatible almost complex structure parametrized by D along the boundary ∂D by g_t. By the transversality of the intersection of $J(E)$ with $\mathcal{J}_0$, we find that the vertical almost complex structure $J(E)$ is $(A_1 - A_2)$-regular for the symplectic fiber bundle. By the construction the parametrized moduli space $C\mathcal{M}_{0,0,A_1-A_2}(E, J(E))$ consists of one point over the point $D \cap \mathcal{J}_0$. Therefore the value $I^E_{0,0,A_1-A_2} = 1$. By Proposition 4.1, this nontrivial parametrized Gromov-Witten invariant defines a non-trivial element in $Hom(\pi_2(\mathrm{BSymp}(M, \omega), \mathbf{Q})$.

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