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# Pontrjagin classes and higher torsion of sphere bundles

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**Abstract.**

By a classical result of Schafer and Kahn ([12], [6]), oriented topological sphere bundles have well defined rational Pontrjagin classes. In the smooth case, we show that the corresponding Pontrjagin character is proportional to the higher Franz-Reidemeister torsion invariant in each degree when the fiber is even dimensional and we discuss the relationship in the odd dimensional case.

This short paper is intended to answer a question about oriented smooth sphere bundles that Shigeyuki Morita and Dieter Kotschick asked me at the AIM (American Institute of Mathematics in Palo Alto) conference in March, 2005 on the moduli space of curves, namely: Can higher Franz-Reidemeister torsion be used to define Pontrjagin classes for smooth oriented odd-dimensional sphere bundles?

If a smooth oriented sphere bundle $E \to B$ has a section then the vertical tangent bundle of $E$ along the section can be used to define the Pontrjagin classes and therefore the Pontrjagin character of the bundle $E$. In the case when the fiber is an even dimensional sphere this Pontrjagin character is proportional to the higher Franz-Reidemeister torsion invariant. Therefore, Morita and Kotschick pointed out to me that (the appropriate scalar multiple of) this higher torsion invariant can be used as a generalization of the Pontrjagin character and therefore defines Pontrjagin classes for all oriented even dimensional smooth sphere bundles. In the case of an odd dimensional sphere bundle the analogous statement is false by a construction of Hatcher. If there is a section of the bundle, the higher Franz-Reidemeister torsion is equal to a multiple of the Pontrjagin character plus an exotic term which measures how far the bundle differs from the linear bundle given by the section. The question

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is: Can we specify this decomposition in general, even when there is no section?

## §1. Rational Pontrjagin classes

We recall that a *Euclidean bundle* is a fiber bundle with fiber $\mathbb{R}^n$ and structure group Homeo$(\mathbb{R}^n, 0)$, the group of homeomorphisms of $\mathbb{R}^n$ preserving the basepoint 0 with the compact open topology. Euclidean bundles are equivalent to topological microbundles by [7]. We recall the following classical result.

**Theorem 1.1** (Schafer[12], Kahn[6]). *Oriented Euclidean bundles over a finite cell complexes have natural and well defined rational Pontrjagin classes which agree, rationally, with the usual Pontrjagin classes if the bundle is a vector bundle.*

**Corollary 1.2.** *Oriented topological sphere bundles over finite complexes have well defined rational Pontrjagin classes.*

*Proof.* The fiberwise open cone of a topological sphere bundle is a Euclidean bundle. Q.E.D.

**Lemma 1.3.** *If an $n$-dimensional Euclidean bundle $E$ over a finite complex contains an embedded $n$-disk bundle associated to a vector bundle $V$ then $E$ is fiberwise homeomorphic to $V$.*

*Proof.* Since $\mathbb{R}^n$ is contractible, we can move the base point to the center of the $n$-disk. This reduces the structure group of the bundle to the subgroup of Homeo$(\mathbb{R}^n, 0)$ which is orthogonal in a small neighborhood of the origin. There is a deformation retraction of this group to the orthogonal group $O(n)$ by the one parameter family of continuous automorphisms $\phi_t$ given by $\phi_t(f)(x) = \frac{1}{t}f(tx)$ if $0 < t \leq 1$ and $\phi_0(f) = Df(0) \in O(n)$ is the derivative of $f$ at 0. Q.E.D.

**Corollary 1.4.** *If $p : E \to B$ is a smooth oriented sphere bundle over a compact manifold $B$ and $s : B \to E$ is a section, then the rational Pontrjagin classes of the sphere bundle $E$ agree with the usual Pontrjagin classes of the pull-back $s^*T^vE$ of the vertical tangent bundle $T^vE$ of $E$.*

*Proof.* The Euclidean bundle given by coning off each fiber of $E$ contains a linear disk bundle associated to the stabilization of $s^*T^vE$. By the lemma, these bundles are homeomorphic. So, they have the same rational Pontrjagin classes by the theorem. Q.E.D.

We will combine the Pontrjagin classes of a vector bundle $E$ into the *Pontrjagin character* $ph(E) = \sum ph_k(E)$ where

$$ph_k(E) = (-1)^k ch_{2k}(E \otimes \mathbb{C})$$

is, up to sign, the degree $4k$ part of the Chern character of the complexification of $E$. Since $ph(E)$ is a polynomial with rational coefficients in the Pontrjagin classes, oriented Euclidean bundles have well-defined Pontrjagin characters. Also, it is well-known and easy to verify that the Pontrjagin character determines the rational Pontrjagin classes.

We will denote the Pontrjagin character of an oriented topological sphere bundle $E$ by $ph^{top}(E)$.

## §2. Higher torsion of sphere bundles

We recall the higher torsion invariants defined in [8], [3], [5], [4]. Given any smooth bundle $p : E \to B$ where $\pi_1 B$ acts trivially on the rational homology of the fiber, there are higher torsion invariants

$$\tau_{2k}^{FR}(E) \in H^{4k}(B; \mathbb{R})$$

called the *higher Franz-Reidemeister (FR) torsion invariants* of $E$ which are invariants of the smooth bundle but, in general, are not topological invariants. Oriented smooth sphere bundles satisfy the trivial action assumption and therefore have well-defined higher FR-torsion invariants.

For oriented smooth bundles $E \to B$ with closed even dimensional fibers, the higher FR-torsion is proportional to *generalized Miller-Morita-Mumford classes* $M_{2k}(E)$ which can be defined rationally in terms of the Pontrjagin character as

$$M_{2k}(E) = tr_B^E \left( \frac{(-1)^k (2k)!}{2} ph_k(T^v E) \right) \in H^{4k}(B; \mathbb{Q})$$

where $tr_B^E : H^*(E) \to H^*(B)$ is the *transfer* [2]. The coefficients are chosen so that $M_{2k}(E)$ is equal to the usual Miller-Morita-Mumford classes, also called *tautological classes*, for oriented surface bundles ([11], [9], [10]).

**Theorem 2.1.** [3],[5] *If $E \to B$ is an oriented smooth bundle with closed even dimensional fibers so that $\pi_1 B$ acts trivially on the rational homology of the fiber then*

$$\tau_{2k}^{FR}(E) = \frac{(-1)^k \zeta(2k+1)}{2(2k)!} M_{2k}(E) = \frac{\zeta(2k+1)}{4} tr_B^E(ph_k(T^v E))$$

*where $\zeta(s) = \sum \frac{1}{n^s}$ is the Riemann zeta function.*

**Corollary 2.2.** *If $p : E \to B$ is an oriented smooth even dimensional sphere bundle then*

$$\tau_{2k}^{FR}(E) = \frac{1}{2}\zeta(2k+1)ph_k^{top}(E).$$

*In particular, $\tau_{2k}^{FR}(E)$ is a topological invariant.*

*Proof.* One of the basic properties of the transfer [2] is that the composition

$$H^*(B;\mathbb{Q}) \xrightarrow{p^*} H^*(E;\mathbb{Q}) \xrightarrow{tr_B^E} H^*(B;\mathbb{Q})$$

is equal to multiplication by the Euler characteristic of the fiber which in this case is 2. By naturality of $ph^{top}$ and Corollary 1.4, $p^*(ph^{top}(E)) = ph^{top}(p^*E) = ph(T^vE)$. Transferring down to $H^{4k}(B)$ we get:

$$2ph_k^{top}(E) = tr_B^E(ph_k(T^vE)) = \frac{4}{\zeta(2k+1)}\tau_{2k}^{FR}(E)$$

proving the formula. Q.E.D.

For smooth oriented odd dimensional sphere bundles, the situation is not so clear. If the bundle is linear then we have the formula:

**Theorem 2.3.** [3],[5] *If $E = S^{2n-1}(\xi)$ is the $S^{2n-1}$-bundle associated to an $SO(2n)$-bundle $\xi$ over $B$ then*

$$\tau_{2k}^{FR}(S^{2n-1}(\xi)) = \frac{-\zeta(2k+1)}{2}ph_k(\xi)$$

However, when the bundle is not linear, there is an exotic component to the higher torsion. Hatcher gave a family of examples of such bundles. The first example has the following properties (Theorem 6.4.2 in [3]).

**Theorem 2.4** (Hatcher's example). *There is a smooth bundle $E \to S^4$ with fiber $S^{13}$ which has a section along which the vertical tangent bundle is trivial but so that $\tau_2^{FR}(E)$ is equal to $\pm 24\zeta(5)$ times the generator of $H^4(S^4)$.*

## §3. Exotic torsion

The difference between the two expressions in Theorem 2.3 can be defined for all oriented smooth sphere bundles as follows.

**Definition 3.1.** For any oriented smooth $S^n$-bundle $E \to B$, we define the *exotic torsion* $\tau_{2k}^x(E) \in H^{4k}(B;\mathbb{R})$ by

$$\tau_{2k}^x(E) = \tau_{2k}^{FR}(E) - (-1)^n \frac{\zeta(2k+1)}{2} ph_k^{top}(E)$$

Exotic torsion is zero for all smooth oriented even dimensional sphere bundles by Corollary 2.2 and for all linear odd dimensional sphere bundles by Theorem 2.3. Therefore, it measures the extent to which $E$ is not a linear bundle.

In [4], the general theory of higher torsion invariants is discussed. But the following proposition tells us that exotic torsion does not fit into this theory.

**Proposition 3.2.** *Exotic torsion, as defined above, is not the restriction to sphere bundles of a higher torsion theory as defined in* [4].

*Proof.* Higher torsion theories in degree $4k$ have an even and an odd component, each of which is unique up to a scalar multiple. However, exotic torsion is zero on all linear even and odd dimensional sphere bundles. So, both even and odd components would be zero making it identically zero if it were a higher torsion theory. Q.E.D.

This implies that exotic torsion is not an absolute higher torsion theory. However, it might be an example of a *relative theory.* As I explained in my lecture at the conference in honor of Professor Morita, there are different definitions of higher relative torsion, three of which agree according to my joint work with Sebastian Goette.

## §4. Higher relative torsion

There are three definitions of relative smooth torsion: axiomatic relative torsion, higher relative Franz-Reidemeister (FR) torsion and relative Dwyer-Weiss-Williams (DWW) torsion. The axiomatic relative torsion is defined when we have a pair of smooth bundles $E \to B$, $E' \to B$ over the same base $B$ with compact smooth manifold fibers $M, M'$ and a fiber homotopy equivalence $f : E \to E'$. I.e., $f$ commutes with the projection to $B$ and induces a homotopy equivalence on fibers $M \simeq M'$. In this case we have "tangential" and "exotic" relative torsion invariants $\tau^T(f), \tau^X(f) \in H^{4k}(B;\mathbb{R})$ which measure the extent to which $f$ is not a fiberwise diffeomorphism.

The *tangential relative torsion* measures the difference between the vertical tangent bundles of $E$ and $E'$. It is the push-down of the Chern character of the difference bundle, i.e., it is the relative generalized

Miller-Morita-Mumford class. The *exotic relative torsion* is independent of the vertical tangent bundle. By an argument very similar to the one given in [4], it follows that tangential and exotic relative torsion are unique up to a scalar factor. This implies that exotic relative torsion is proportional to higher relative FR torsion.

What Goette and I proved is that the exotic relative torsion is also proportional to the relative Dwyer-Weiss-Williams torsion when the base and fiber are closed oriented manifolds. This extends to the case of arbitrary base spaces but the definitions become more complicated.

### 4.1. Dwyer-Weiss-Williams smoothing theory

Dwyer-Weiss-Williams smoothing theory works as follows. We take a topological manifold bundle $E \to B$ (with compact topological manifold fiber $M$) together with a linear vertical tangent bundle $V^vE$. This is a vector bundle whose total space is homeomorphic to a neighborhood of the diagonal $\Delta E$ in the fiberwise product $E \times_B E$ (the bundle over $B$ with fiber $M \times M$). The question is: Given the pair $(E, V^vE)$, can we find a smooth bundle $W \to B$ and a homeomorphism $f : W \to E$ which commutes with the projection to $B$ so that $f$ is covered by a nonsingular linear isomorphism of vector bundles $\tilde{f} : T^vW \to V^vE$ so that $\tilde{f}$ is compatible with the exponential maps to $W$ and $E$? We call this a *fiberwise tangential smoothing* of $(E, V^vE)$.

Let $\widetilde{\mathcal{S}}_B^{d/t}(E, V^vE)$ be the space of all fiberwise tangential smoothings of $(E, V^vE)$. We want to know how many components this space has. In their paper [1] Dwyer, Weiss and Williams give a computation of the homotopy type of this space (and in particular of $\pi_0$) in the stable range. Stabilization is given by taking the direct limit with respect to all linear disk bundles $D(\xi)$ over $E$. This gives a space

$$s\widetilde{\mathcal{S}}_B^{d/t}(E, V^vE) = \varinjlim \widetilde{\mathcal{S}}^{d/t}(D(\xi), V^vE \oplus \xi).$$

**Theorem 4.1** (Dwyer-Weiss-Williams [1]). *Assuming that this space is nonempty, we have a homotopy equivalence*

$$s\widetilde{\mathcal{S}}_B^{d/t}(E, V^vE) \simeq \Gamma_B \mathcal{H}^{\%}(E)$$

*where $\Gamma_B\mathcal{H}^{\%}(E)$ is the space of sections of the bundle $\mathcal{H}^{\%}(E)$ over $B$ whose fiber is the zero space $\Omega^\infty(M_+ \wedge \mathcal{H}(*))$ of the homology theory on the fiber of $E$ with coefficients in the stable h-cobordism space $\mathcal{H}(*)$.*

This theorem is not stated in this way in their paper [1]. So, Bruce Williams gave us (Goette and the author) handwritten notes proving this statement. Excerpts were shown in my lecture. This theorem leads to the concept of the *stable smooth structure class* of an exotic smooth structure. The idea is as follows.

First of all, this theorem implies that the set of stable tangential smooth structures on $(E, V^v E)$ forms an abelian group since it is $\pi_0$ of an infinite loop space. However, it would be more accurate to say that it is an affine space which needs a choice of zero to become an additive group. This choice is given by a fixed tangential smoothing $E_0$ of $(E, V^v E)$. Then, any other tangential smoothing $E$ gives an element of this group:

$$\tilde{\theta}(E, E_0) \in \pi_0 \Gamma_B \mathcal{H}^{\%}(E)$$

We call $\tilde{\theta}(E, E_0)$ the *relative stable smooth structure class* of $(E, E_0)$.

### 4.2. Results of Goette-I.

Sebastian Goette and I looked at the special case when both base and fiber are closed manifolds. In this case we have the following results. Details will appear elsewhere.

**Theorem 4.2** (Goette-I). *If the fiber $M$ and base $B$ of the bundle $E \to B$ are closed oriented manifolds then*

$$\pi_0 \Gamma_B \mathcal{H}^{\%}(E) \otimes \mathbb{Q} \cong \bigoplus_{k>0} H_{\dim B - 4k}(E; \mathbb{Q}).$$

We call the image of $\tilde{\theta}(E, E_0)$ in $\bigoplus_{k>0} H_{\dim B-4k}(E;\mathbb{Q})$ the *relative rational stable smooth structure class* of $(E, E_0)$ and denote it by $\theta(E, E_0)$. As a consequence of this calculation we can make the following definition. Again, this is not the same as the definition given in [1] but I claim that it is equivalent in the cases where both are defined.

**Definition 4.3.** Suppose that $E, E_0$ are smooth bundles over $B$ which are tangentially fiberwise homeomorphic and suppose that the fiber $M$ and base $B$ are closed oriented manifolds. Then the degree $4k$ *relative Dwyer-Weiss-Williams torsion* $\tau_{2k}^{DWW}(E, E_0) \in H^{4k}(B; \mathbb{Q})$ is defined to be the Poincaré dual of the image of the relative rational stable smooth structure class $\theta(E, E_0)$ in $H_{\dim B-4k}(B; \mathbb{Q})$.

**Theorem 4.4** (Goette-I). *In the situation above, the relative DWW torsion $\tau_{2k}^{DWW}(E, E_0)$ is a scalar multiple of the relative FR torsion $\tau_{2k}^{FR}(E, E_0)$.*

**Corollary 4.5.** *Suppose that $E \to B$ is a smooth oriented sphere bundle over a closed oriented manifold $B$. Suppose also that $E_0 \to B$ is a linear sphere bundle which is tangentially fiber homeomorphic to $E$. Then the relative DWW torsion $\tau_{2k}^{DWW}(E, E_0) \in H^{4k}(B; \mathbb{Q})$ is proportional to the exotic torsion $\tau_{2k}^{x}(E) \in H^{4k}(B; \mathbb{R})$.*

*Proof.* Let $c_{2k} \in \mathbb{R}$ be the proportionality constant between $\tau_{2k}^{DWW}$ and $\tau_{2k}^{FR}$. Then

$$c_{2k}\tau_{2k}^{DWW}(E, E_0) = \tau_{2k}^{FR}(E, E_0) = \tau_{2k}^{FR}(E) - \tau_{2k}^{FR}(E_0).$$

Since $E$ and $E_0$ are fiber homeomorphic, they have the same topological Pontrjagin classes. Therefore, the difference between their higher FR-torsions is equal to the difference between their exotic torsion invariants. Since $E_0$ is linear, its exotic torsion is zero. Therefore this difference is equal to the exotic torsion of $E$ as claimed. Q.E.D.

## §5. Questions

I will close with two questions, which arise from this correspondence between higher FR torsion and higher Dwyer-Weiss-Williams torsion.

**Question 5.1.** For a smooth oriented $S^{2n-1}$-bundle $p : E \to B$ where $B$ is a smooth closed manifold, does there exist an "absolute rational smooth structure class"

$$\theta(E) = \sum \theta_k(E) \in \bigoplus H^{4k+2n-1}(E; \mathbb{R})$$

so that

$$p_*(\theta_k(E)) = \tau_{2k}^x(E)?$$

The answer to this question would be "Yes" if there were a unique or canonical linear sphere bundle $E_0$ which is tangentially fiber homeomorphic to $E$. Then we could define $\theta(E)$ to be $\theta(E, E_0)$. If $E_0$ does not exist, perhaps we could define $\theta(E)$ to be the average value of $\theta(E, E_0)$ using some canonically defined measure on the set of fiberwise smooth structures on $E \to B$.

By the Gysin sequence

$$\cdots \to H^{4k+2n-1}(E) \xrightarrow{p_*} H^{4k}(B) \xrightarrow{\cup e} H^{4k+2n}(B) \xrightarrow{p^*} H^{4k+2n}(E) \to \cdots$$

this question is almost the same as the question: Is $\tau_{2k}^x(E) \cup e = 0$? where $e \in H^{2n}(B)$ is the Euler class of $E$. This condition is certainly a necessary condition for the existence of $\theta(E)$.

For even dimensional sphere bundles, $\tau_{2k}^x(E) = 0$. So the analogous conjecture would be that $\theta(E)$ exists and is equal to zero. In the relative theory, $\theta(E, E_0) = \tilde{\theta}(E, E_0) \otimes \mathbb{Q}$ is the rational version of the integral obstruction $\tilde{\theta}(E, E_0)$ to fiberwise stable tangential diffeomorphism, i.e., $\tilde{\theta}(E, E_0) = 0$ if and only if $E \times D^N$ is fiberwise diffeomorphic to $E_0 \times D^N$ where $E, E_0$ are tangentially homeomorphic smooth bundles. So, the idea that $\theta(E)$ might be trivial would be expressed as follows.

**Question 5.2.** Are smooth oriented $S^{2n}$-bundles $E \to B$ "stably rationally rigid" in the sense that, for sufficiently large $N$, the smooth bundle $E \times D^N \to B$ is uniquely determined up to finite indeterminacy by the underlying topological bundle of $E$?

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