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A-integrability of geodesic flows and geodesic equivalence

Peter Topalov

§1. Introduction

A common property of the metrics that admit non-trivial geodesic equivalence is that their geodesic flows admit families of pairwise commuting integrals of a special type (see [14, 7, 16, 17, 13]). The metrics studied in [1] and [20] have integrals with analogous properties. We consider this property as a definition of a new class of (pseudo) Riemannian metrics that we call *A-integrable metrics.* We prove that these metrics inherit the main properties of the metrics that admits non-trivial geodesic equivalence (see [16]).

Let M^n be a smooth n-dimensional manifold. By $\Gamma(E)$ we denote the space of the smooth sections of the vector bundle $\pi : E \longrightarrow M^n$. Let $P \in \Gamma(\mathrm{Hom}(TM, TM))$. We say that P is *diagonalizable over* $\mathbb{R}$ *on* M^n if for every point $x \in M^n$ there exists a (real) basis in T_xM^n such that the operator $P(x)$ has diagonal form $P(x) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

Definition 1. *Let g be a (pseudo) Riemannian metric on the manifold M^n and let $A \in \Gamma(\mathrm{Hom}(TM, TM))$ be a self-adjoint operator which is diagonalizable over $\mathbb{R}$ on M^n with respect to g. The metric g is called* A-integrable *if the functions from the one-parameter family*

$$I_c(\xi) \stackrel{\text{def}}{=} \det(A + c\mathbf{1})g((A + c\mathbf{1})^{-1}\xi, \xi) \tag{1}$$

are in involution with respect to the symplectic structure $\omega_g \stackrel{\text{def}}{=} FL_g^*\omega$. Here $FL_g : TM \to T^*M$ is the Legendre transformation corresponding to the metric g and ω is the canonical symplectic structure on T^*M.

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Remark 1. *It seems that the assumption that A is diagonalizable over $\mathbb{R}$ is not essential and the theorems proved in the present paper are likely to be true without this restriction. We refer to* [15] *where the pseudo-Riemannian analogues of the results proved in* [14, 7, 16, 17, 13] *are given. The case of arbitrary A is extremely interesting because in this case we do not have orthogonal separation of the variables and the corresponding systems are not locally of Stäckel form.*

Remark 2. *If the functions I_c are in involution, then they are integrals of the geodesic flow of the metric g. Indeed, it is easily seen that $I_c(\xi) = S_{n-1}(\xi)c^{n-1} + S_{n-2}(\xi)c^{n-2} \cdots + S_0(\xi)$ and $S_{n-1}(\xi) = g(\xi, \xi)$.*

Remark 3. *It follows from the definition that every metric is $\mathbf{1}$-integrable, where $\mathbf{1}$ denotes the identity. Moreover, if g is A-integrable, then for any real constant α the metric g is $(A + \alpha\mathbf{1})$-integrable .*

We assign to any operator A on the manifold M^n a decomposition of the manifold $M^n = S(A) \bigsqcup_{i=1}^n M^i(A)$, where $M^i(A)$ denotes the set of *the stable points of type i*, i.e., the set of the points $x \in M^n$ such that the operator $A(y) \in \mathrm{Hom}(T_yM, T_yM)$ has exactly i distinct eigenvalues for every point y in an open neighborhood of the point x. Denote by $M(A)$ the set of the stable points $\bigsqcup_{i=1}^n M^i(A)$. The closed set $S(M)$ is called *set of the singular points.* It is easily seen that S is nowhere dense.

Definition 2. *The (pseudo) Riemannian metrics g and $\bar{g}$ are called* geodesically equivalent *iff they have the same geodesics (considered as unparametrized curves).*

If g is not a constant multiple of $\bar{g}$, then we say that the metrics are *non-trivially geodesically equivalent.* We say that the (pseudo) Riemannian metric g *admits non-trivial geodesical equivalence* if there exists a (pseudo) Riemannian metric $\bar{g}$ such that g and $\bar{g}$ are non-trivially geodesically equivalent. If, in addition, the metrics g and $\bar{g}$ are simultaneously diagonalizable over $\mathbb{R}$, then we say that the metric g *admits a non-trivial geodesical equivalence of Levi-Civita type (LC-type).*

Theorem 1. *The (pseudo) Riemannian metric g admits a non-trivial geodesical equivalence of LC-type if and only if g is A-integrable, the operator A is non-degenerate on M^n, $A \neq$ constant $\times\mathbf{1}$, and the eigenspaces of A are integrable on $M(A)$. If the metric g is A-integrable and the eigenspaces of A are integrable, then the metrics given by the formula*

$$g_c(\xi, \xi) \stackrel{\text{def}}{=} \frac{1}{\det(A + c\mathbf{1})} g((A + c\mathbf{1})^{-1}\xi, \xi), \tag{2}$$

defined for every real parameter c such that the operator $(A+c\mathbf{1})$ *is non-degenerate on* M^n, *are geodesically equivalent to* g. *Conversely, if there exists a metric* $\bar{g}$ *such that* g *and* $\bar{g}$ *are simultaneously diagonalizable over* $\mathbb{R}$ *and geodesically equivalent, then the metric* g *is* A*-integrable, where the operator* A *is given by the formula*

$$A^i_j(g,\bar{g}) \stackrel{\text{def}}{=} \left|\frac{\det \bar{g}}{\det g}\right|^{\frac{1}{n+1}} \bar{g}^{i\alpha} g_{\alpha j}. \tag{3}$$

By definition, the operator A is *non-degenerate on* M^n iff $\det A \neq 0$ on M^n.

Remark 4. *Suppose that the operator* $A = A(g,\bar{g})$ *is derived from a pair of geodesically equivalent metrics* g *and* $\bar{g}$ *(see formula (3)); then the formula*

$$I_{\alpha,\beta}(g,\bar{g})(\xi) \stackrel{\text{def}}{=} \det(\alpha A + \beta\mathbf{1})g((\alpha A + \beta \mathbf{1})^{-1}\xi, \xi), \tag{4}$$

gives a family of pairwise commuting integrals of the geodesic flow of the metric g.

Remark 5. *Suppose that* g *is a (pseudo) Riemannian metric and let* A *be a self-adjoint operator which is diagonalizable over* $\mathbb{R}$ *with respect to* g. *If the metrics given by formula (2) are geodesically equivalent, then the functions given by formula (1) are pairwise commuting integrals of the geodesic flow of the metric* g. *It is easily checked that* $A(g, g_0) = A$, *where* $g_0 \stackrel{\text{def}}{=} g_c|_{c=0}$ *and the operator* $A(g, g_0)$ *is derived from formula (3).*

The next theorem gives a description of the decomposition of the manifold corresponding to an A-integrable metric.

Theorem 2. *Suppose that the metric* g *is* A*-integrable. If the manifold* M^n *is connected, then* $M^n = S(A) \bigsqcup M^m(A)$, *where* m *is a natural number* $m \leq n$. *The set of singular points* $S(A)$ *is nowhere dense and coincides with the set of points on* M^n *where the number of the distinct eigenvalues of the operator* A *is less than* m. *There exist quadratic forms* $B_1, \dots, B_m$ *on* M^n *such that any form from the family given by formula (1) is a linear combination (with constant coefficients) of the forms* $B_1, \dots, B_m$. *Considered as functions on the tangent bundle* TM^n *the forms* $B_1, \dots, B_m$ *are functionally independent almost everywhere.*

The smooth functions $F_1, \dots, F_k$ given on the smooth manifold V^q are called *functionally independent* on V^q iff the set of the points $x \in V^q$ where the differentials $d_xF_1, \dots, d_xF_k$ are linearly independent is dense in V^q.

Definition 3. *The natural number m that appears in Theorem 2 is called* rank of the A-integrable metric g.

It follows from Theorem 2 that if the operator A of some A-integrable metric g has n different eigenvalues at some point $x_0 \in M^n$, then the geodesic flow of the metric g is completely integrable.

The theorems that follow allow us to produce new A-integrable metrics from a given one.

Theorem 3. *Suppose that the (pseudo) Riemannian metric g is A-integrable; then for every real-analytic function $f(t)$ such that the operator $f(A)$ is non-degenerate and well defined on M^n the metric $g_f(\xi, \xi) \stackrel{\text{def}}{=} g(f(A)\xi, \xi)$ is also A-integrable.*

Theorem 4. *Suppose that the (pseudo) Riemannian metric g is A-integrable; then, provided the operator $(A + c\mathbf{1})$ is non-degenerate on M^n, the metric g_c given by formula (2) is $(\alpha A + \beta \mathbf{1})(A + c\mathbf{1})^{-1}$-integrable, where α and β are arbitrary real parameters.*

Therefore, from an A-integrable metric g we obtain a family (*hierarchy*) of A-integrable metrics

$$\mathbf{S}(g, A) \stackrel{\text{def}}{=} \{g_f | f \text{ satisfies the conditions of Theorem 3}\}. \tag{5}$$

Many classical examples of integrable geodesic flows and mechanical systems lie in hierarchies obtained from a fixed A-integrable metric (see [16] for details). Perhaps the most remarkable fact is that the Liouville integrability of the ellipsoid and the Poisson sphere follow from the Liouville integrability of the standard sphere (all these metrics lie in a fixed hierarchy). Analogous results are true also in the cases of the hyperbolic and the Euclidean plane. Let us describe briefly the corresponding constructions (for details see [16, 13] and [6]).

(a) **The standard sphere and the free motion of a rigid body in Euclidean space.** Let

$$S^n \stackrel{\text{def}}{=} \{\sum_{i=1}^{n+1} x_i^2 = 1\}$$

be the unit sphere embedded in the Euclidean space $\mathbb{R}^{n+1}$ supplied with the Euclidean metric $g_0 \stackrel{\text{def}}{=} dx_1^2 + \cdots + dx_{n+1}^2$. Denote by g the restriction of g_0 to the sphere S^n. Consider a non-degenerate linear transformation C of $\mathbb{R}^{n+1}$. The transformation C induces in a natural way a transformation μ_C of the sphere S^n ($\mu_C(x) \stackrel{\text{def}}{=} \frac{Cx}{||Cx||}$, where $||y||$ is the Euclidean norm of $y \in \mathbb{R}^{n+1}$). It is clear that μ_C preserves the geodesics on the sphere. Therefore, the metrics g and $\bar{g} \stackrel{\text{def}}{=} \mu^* g$ are geodesically equivalent. It follows from Theorem 1 that g is A-integrable where $A \stackrel{\text{def}}{=} A(g, \bar{g})$. Theorem 3 shows that the metrics $g^{(k)}(\xi, \xi) \stackrel{\text{def}}{=} g(A^k \xi, \xi)$ ($k = 1, 2, \dots$) are also A-integrable. It is proved in [16] that $g^{(1)}$ is isometric to the metric of a suitable ellipsoid and the metric $g^{(2)}$ is geodesically equivalent to the metric of a suitable Poisson sphere. Varying the matrix C we obtain all possible ellipsoids and Poisson spheres.

(b) **The hyperbolic plane and the free motion of a rigid body in Minkowski space.** Denote by

$$\mathbf{H}_0^n \stackrel{\text{def}}{=} \{-x_0^2 + x_1^2 + \cdots + x_n^2 = -1\}$$

and

$$\mathbf{S}_1^n \stackrel{\text{def}}{=} \{-x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}$$

the hyperboloids of two sheets and one sheet respectively, embedded in the Minkowski space $\mathbb{R}^{1,n}$, $g_0 \stackrel{\text{def}}{=} -dx_0^2 + dx_1^2 + \cdots + dx_n^2$, and consider the submanifold $M^n \stackrel{\text{def}}{=} \mathbf{H}_0^n \bigsqcup \mathbf{S}_1^n$. Let g be the restriction of the metric g_0 on M^n. Recall that the hyperbolic plane $\mathbf{H}^n$ is isometric to the positive part ($x_0 > 0$) of the hyperboloid of two sheets. As in the previous case any non-degenerate linear transformation C of the plane $\mathbb{R}^{n+1}$ induces a partially defined mapping $\mu_C : M^n \to M^n$ that preserves the geodesics on M^n. It is clear that the metrics g and $\bar{g} \stackrel{\text{def}}{=} \mu_C^* g$ are geodesically equivalent. The metric $\bar{g}$ is not smooth at the points of M^n where the mapping μ_C is not defined. Nevertheless, the operator $A = A(g, \bar{g})$ derived from formula (3) is smooth. As in the previous case we construct the metrics $g^{(k)}$ ($k = 1, 2, \dots$). It is proved in [13] that the metric $g^{(2)}$ (restricted to the hyperbolic plane $\mathbf{H}^n$) is geodesically equivalent to the metric of the analogue of the Poisson sphere corresponding to the free motion of the rigid body in Minkowski space.

(c) **The Euclidean space and the Clebsch case of motion of a rigid body.** Let us consider the projective plane

$$\mathbb{R}P^n \stackrel{\text{def}}{=} \{(x_1 : \cdots : x_{n+1})\}$$

and fix the affine chart $\mathbb{R}^n \cong \{x_{n+1} = 1\} \hookrightarrow \mathbb{R}^{n+1}$. Any non-degenerate linear transformation L of $\mathbb{R}^{n+1}$ gives a projective transformation of $\mathbb{R}P^n$ that acts on the affine chart $\mathbb{R}^n$ as a linear-fractional transformation that we denote by μ_L. It is clear that the partially defined mapping $\mu_L : \mathbb{R}^n - \to \mathbb{R}^n$ preserves the geodesics of the Euclidean metric $g \stackrel{\text{def}}{=} dx_1^2 + \cdots + dx_n^2$ on $\mathbb{R}^n$. It allows us to construct a partially defined metric $\bar{g} \stackrel{\text{def}}{=} \mu_L^* g$ that is geodesically equivalent to the Euclidean metric g. The corresponding operator $A = A(g, \bar{g})$ given by formula (3) is smooth and the metric g is A-integrable. It is proved that the corresponding hierarchy $\mathbf{S}(g, A)$ (given by Theorem 3) contains a metric that is geodesically equivalent to the analogue of the Poisson sphere corresponding to the Clebsch case of motion of the rigid body (see [13]). We consider item $c)$ in more detail in Appendix 1.

The next theorem allows us to construct A-integrable metrics on Cartesian products.

Suppose that the (pseudo) Riemannian metric g_1 given on the manifold M_1 ($\dim M_1 = n_1$) is A_1-integrable and the (pseudo) Riemannian metric g_2 given on the manifold M_2 ($\dim M_2 = n_2$) is A_2-integrable. Denote by M the Cartesian product $M_1 \times M_2$ and let $\pi_i : M \to M_i$ ($i = 1, 2$) be the projections. Let us consider the operators $\tilde{A}_1(\xi) \stackrel{\text{def}}{=} (d_u\pi_1|_{K_2})^{-1}(A_1(d_u\pi_1(\xi)))$, $\xi \in T_uM$, $K_{2u} \stackrel{\text{def}}{=} \ker d_u\pi_2$, and $\tilde{A}_2(\xi) \stackrel{\text{def}}{=} (d_u\pi_2|_{K_1})^{-1}(A_2(d_u\pi_2(\xi)))$, $\xi \in T_uM$, $K_{1u} \stackrel{\text{def}}{=} \ker d_u\pi_1$. Denote by $\sigma_1(A_i), \ldots, \sigma_{n_i}(A_i)$ the elementary symmetric polynomials of the operator A_i ($i = 1, 2$). Let L be a linear operator in the vector space V, $\dim V = n$. The elementary symmetric polynomials of the operator L are defined by the equality $\det(L + c\mathbf{1}) = c^n + \sigma_1(L)c^{n-1} + \cdots + \sigma_n(L)$. It is clear that $\sigma_k(A_i)$ are smooth functions on M_i. Let us consider the functions $\sigma_k^i \stackrel{\text{def}}{=} \pi_i^* \sigma_k(A_i)$. Define the operators $G_1 \stackrel{\text{def}}{=} \tilde{A}_1^{n_2} - \sigma_1^2 \tilde{A}_1^{n_2-1} + \cdots + (-1)^{n_2}\sigma_{n_2}^2 \mathbf{1}_1$ and $G_2 \stackrel{\text{def}}{=} \tilde{A}_2^{n_1} - \sigma_1^1 \tilde{A}_2^{n_1-1} + \cdots + (-1)^{n_1}\sigma_{n_1}^1 \mathbf{1}_2$, where $\mathbf{1}_1$ is the projection on K_{2u} with respect to K_{1u} and $\mathbf{1}_2$ is the projection on K_{1u} with respect to K_{2u}. Define also the forms $\tilde{g}_i(\xi, \eta) \stackrel{\text{def}}{=} g_i(d_u\pi_i(\xi), d_u\pi_i(\eta))$ ($i = 1, 2$), $\xi, \eta \in T_uM$. Let us consider the set $\Lambda_i \stackrel{\text{def}}{=} \{\lambda | \lambda \in \operatorname{Spect} A_i(x), x \in M_i\}$, where $\operatorname{Spect} A$ denotes the spectrum of the linear operator A.

Theorem 5. *Suppose that $\Lambda_1 \cap \Lambda_2 = \emptyset$; then the (pseudo) Riemannian metric*

$$g(\xi, \eta) \stackrel{\text{def}}{=} \tilde{g}_1(G_1\xi, \eta) + \tilde{g}_2(G_2\xi, \eta), \ \xi, \eta \in T_uM, \tag{6}$$

is A-integrable where $A \stackrel{\text{def}}{=} \tilde{A}_1 + \tilde{A}_2$.

§2. Proof of Theorem 1.

We follow the construction of the proof of Theorem 2 in [17]. Assume that the metric g is A-integrable. It follows from our assumption that any two functions from the 1-parameter family

$$I_c(\xi,\xi) \stackrel{\text{def}}{=} \det(A+c\mathbf{1})g((A+c\mathbf{1})^{-1}\xi,\xi) \tag{7}$$

are in involution. Suppose that the point $x_0 \in M^n$ is stable of type m $(0 \le m \le n)$, i.e., $x_0 \in M^m(A)$. Therefore, there is an open neighborhood $U(x_0)$ of the point such that the operator A has exactly $m \le n$ distinct eigenvalues $\phi_1(x) < \phi_2(x) < \cdots < \phi_m(x)$. The corresponding eigenspaces give a natural splitting $TU \cong W_1 \oplus \cdots \oplus W_m$, $\dim W_k = l_k$. Let $X_{k\alpha}$ $(k = 1,\dots,m; \alpha = 1,\dots,l_k)$ be an orthonormal frame on U such that $A(X_{k\alpha}) = \phi_k X_{k\alpha}$, $W_k = \mathrm{Span}(X_{k1},\dots,X_{kl_k})$. Making the Legendre transformation corresponding to the Riemannian metric g we obtain

$$I_c = \Phi(c)\left\{\frac{1}{\phi_1+c}A_1 + \cdots + \frac{1}{\phi_m+c}A_m\right\}, \tag{8}$$

where $\Phi(c) \stackrel{\text{def}}{=} (\phi_1+c)^{l_1}\cdots(\phi_m+c)^{l_m}$, $A_k \stackrel{\text{def}}{=} \sum_{\alpha=1}^{l_k} \epsilon_{k\alpha}V_{k\alpha}^2$, $\epsilon_{k\alpha} = \pm 1$, and $V_{k\alpha} \stackrel{\text{def}}{=} \langle p, X_{k\alpha}\rangle$, $p \in T^*U(x_0)$.

First of all, let us consider the case $m = 1$. We obtain $A = \phi\,\mathbf{1}$. Hence, $I_c = (\phi+c)^{n-1}g$ is a 1-parameter family of first integrals of the geodesic flow of the metric g. This gives $0 \equiv \{g, I_c\} = (n-1)(\phi+c)^{n-2}\{g,\phi\}g$. Hence, $\{g,\phi\} \equiv 0$. This yields $\phi =$ constant and $A =$ constant $\times\mathbf{1}$. This contradicts the assumption that $A \ne$ constant $\times\mathbf{1}$.

We will suppose further that $m \ge 2$. Let us take m distinct real numbers $c_1,\dots,c_m$ such that $c_i \ne -\phi_j$ $(i,j = 1,\dots,m)$ on $U(x_0)$. Note that such constants exist if $U(x_0)$ is sufficiently small. If necessary, we shrink the neighborhood $U(x_0)$. We obtain m integrals $F_k \stackrel{\text{def}}{=} I_{c_k}$ in involution such that

$$\left\{\begin{array}{rcl} \frac{1}{\Phi_1}F_1 & = & \frac{1}{\phi_1+c_1}A_1 + \cdots + \frac{1}{\phi_m+c_1}A_m, \\ \vdots & & \\ \frac{1}{\Phi_m}F_m & = & \frac{1}{\phi_1+c_m}A_1 + \cdots + \frac{1}{\phi_m+c_m}A_m. \end{array}\right.$$

As in section 2 in [17], we prove that

$$A_l = \sum_k (-1)^{l+k} \frac{1}{\Pi_l} \frac{1}{C_k} \left(\frac{\prod_{ij}(\phi_i + c_j)}{\prod_{\substack{i\neq l\\ j\neq k}} (\phi_i + c_j)} \frac{1}{\Phi_k} \right) F_k$$

$$= \sum_k (-1)^{l+k} \frac{1}{\Pi_l} \frac{1}{C_k} \frac{(\phi_l + c_1)\cdots(\phi_l + c_{k-1})(\phi_l + c_{k+1})\cdots(\phi_l + c_m)}{(\phi_1 + c_k)^{l_1-1}(\phi_2 + c_k)^{l_2-1}\cdots(\phi_m + c_k)^{l_m-1}} F_k,$$

where $\Pi_l \stackrel{\text{def}}{=} (\phi_m - \phi_l)\cdots(\phi_{l+1} - \phi_l)(\phi_l - \phi_{l-1})\cdots(\phi_l - \phi_1)$ and $C_k \stackrel{\text{def}}{=} (c_m - c_k)\cdots(c_{k+1} - c_k)(c_k - c_{k-1})\cdots(c_k - c_1)$. Hence,

$$\mathcal{A}_p \stackrel{\text{def}}{=} \frac{A_p}{(\phi_p + c_1)\cdots(\phi_p + c_{m-1})} \Pi_p C_m [(\phi_1 + c_m)^{l_1-1}\cdots(\phi_m + c_m)^{l_m-1}]$$

$$= \sum_k (-1)^{p+k} \frac{C_m}{C_k} \left(\frac{\phi_1 + c_m}{\phi_1 + c_k}\right)^{l_1-1} \cdots \left(\frac{\phi_p + c_m}{\phi_p + c_k}\right)^{l_p} \cdots \left(\frac{\phi_m + c_m}{\phi_m + c_k}\right)^{l_m-1} F_k,$$

and we write this as

$$\mathcal{A}_p = \sum_k a_{pk} F_k. \tag{9}$$

We are going to prove that

1) $\phi_k = \text{constant}$, if $\dim W_k > 1$;
2) $L_{X_{p\alpha}} \phi_q = 0$ $(p \neq q)$, if $\dim W_q = 1$.

It follows from (9) and the Stäckel-Painlevé theorem proved in [17] (see Proposition 1 in [17]) that $L_{X_{q\alpha}} a_{pk} = 0$ $(q \neq p)$. Without loss of generality we can suppose that $l_1 \geq l_2 \geq \cdots \geq l_m$. If $l_1 = 1$, then $a_{qk} = (-1)^{q+k} \frac{C_m}{C_k} \frac{\phi_q + c_m}{\phi_q + c_k}$. Hence, $L_{X_{p1}} \phi_q = 0$ $(p \neq q)$. Suppose that $l_1 > 1$. Denote by s the largest natural number $m \geq s \geq 1$ such that $l_s > 1$. Observing that $\frac{\phi_i + c_k}{\phi_i + c_m} = (c_k - c_m)\left(\frac{1}{c_k - c_m} + \frac{1}{\phi_i + c_m}\right)$ and applying Lemma 3 in [17] to the coefficients a_{pk} $(k = 1, \ldots, s)$ of the first two equations of (9) we obtain $\phi_k = \text{constant}$ $(k = 1, \ldots, s)$. Therefore, $a_{(s+l)1} = \text{constant} \times \left(\frac{\phi_{s+l} + c_m}{\phi_{s+l} + c_1}\right)$ $(l = 1, \ldots, m - s)$. It follows that the functions ϕ_{s+l} $(l = 1, \ldots, m - s)$ satisfy condition 2).

Assume in addition that the eigenspaces of the operator A are integrable on $M(A)$. It follows from (9) and the Stäckel-Painlevé theorem that there is a chart $\{(\bar{x}_1, \ldots, \bar{x}_m)\}$, $\bar{x}_k \stackrel{\text{def}}{=} (x_{k1}, \ldots, x_{kl_k})$, given in a neighborhood of the point x_0, such that $A_k = A_k(\bar{x}_i, \bar{p}_k)$, where $p_{k\alpha}$ are the conjugate impulses. Therefore, $A_k = \frac{1}{\Pi_k} Q_k(\bar{x}_k, \bar{p}_k)$. By definition, $g = A_1 + \cdots + A_m$. Hence,

$$g = \frac{Q_1(\bar{x}_1, \bar{p}_1)}{\Pi_1} + \cdots + \frac{Q_m(\bar{x}_m, \bar{p}_m)}{\Pi_m}. \tag{10}$$

Therefore,

$$g = \Pi_1 G_1(\bar{x}_1, d\bar{x}_1) + \cdots + \Pi_m G_m(\bar{x}_m, d\bar{x}_m), \tag{11}$$

where G_k are quadratic forms. Let us consider the metric $\bar{g} \stackrel{\text{def}}{=} \frac{1}{\det A} g A^{-1}$. We have

$$\bar{g} = \rho^1 \Pi_1 G_1(\bar{x}_1, d\bar{x}_1) + \cdots + \rho^m \Pi_m G_m(\bar{x}_m, d\bar{x}_m), \tag{12}$$

where $\rho^i \stackrel{\text{def}}{=} \frac{1}{\phi_1 \ldots \phi_m} \frac{1}{\phi_i}$. Now, it is easily checked (using the local coordinates) that the metrics g and $\bar{g}$ given by formulae (11) and (12) are geodesically equivalent. The same fact also follows from the Levi-Civita's theorem (see [5, 3, 11, 16]). Note that Levi-Civita proved his theorem when the metrics g and $\bar{g}$ are positive definite. Nevertheless, it is easily seen that his proof works also in the case of simultaneously diagonalizable pseudo-Riemannian metrics. Finally, the first statement of Theorem 1 follows from the fact that the metric $\bar{g}$ is globally defined on M^n and the set of stable points M is everywhere dense in M^n (see [7]). Formula (2) follows from Proposition 1 in [16].

Conversely, suppose that the (pseudo) Riemannian metric g admits non-trivial geodesical equivalence of LC-type, i.e., there exists a metric $\bar{g}$ such that the metrics g and $\bar{g}$ are non-trivially geodesically equivalent and simultaneously diagonalizable over $\mathbb{R}$. Let us consider the operator $A = A(g, \bar{g})$ given by formula (3). A is a self-adjoint operator with respect to the both metrics and $\bar{g}(\xi, \xi) = \frac{1}{\det A} g(A^{-1}\xi, \xi)$. We prove the involutivity of the functions I_c given by formula (1) in Appendix 2. Theorem 1 is proved.

§3. Properties of the A-integrable metrics.

In this section we prove theorems 2, 3, 4 and 5 that give some of the main properties of the A-integrable metrics.

3.1. Proof of Theorem 2.

Denote by $N(x)$ the number of distinct eigenvalues of the operator A at the point $x \in M^n$. Suppose that $m = \max_{x \in M^n} N(x)$. It is clear that if $N(x) = k$, then $N(x) \geq k$ in a neighborhood of the point x. Therefore, if $N(x) = m$, then x is stable point of type m. It is easily seen that the set of stable points is dense in M^n.

Consider the case when the geodesic flow of the metric g is complete. The general case is easily reduced to this one. Suppose that $M^{m_1}(A) \neq \emptyset$ where $m_1 < m$. Let us take $x_0 \in V_0 \subset M^m(A)$ and $x_1 \in V_1 \subset M^{m_1}(A)$ where V_0 and V_1 are open sets. Taking V_0 and V_1 sufficiently small we can find real constants $c_1, \ldots, c_m$ that are not eigenvalues of the operator $(-A_x)$ if x belongs to $V_0 \sqcup V_1$. Let us consider the forms $B_i \stackrel{\text{def}}{=} I_{c_i}$ $(i = 1, \ldots, m)$. It is easily shown that for any real number c there exist constants $a_1, \ldots, a_m$ such that $I_c = \sum_{i=1}^m a_i B_i$ on V_0. Indeed, it follows from formula (8) and Lemma 2 in [17] that there are smooth functions $a_1, \ldots, a_m$ defined in V_0 such that $I_c = \sum_{i=1}^m a_i B_i$. Differentiating the last equality with respect to the Hamiltonian vector field corresponding to the metric g we obtain $0 \equiv \{g, I_c\} = \sum_i \{g, a_i\} B_i$. Hence, $\{g, a_i\} \equiv 0$. This gives $a_i =$ constant.

Let us take a geodesic line $\gamma(t)$ that connects the points x_0 and x_1 ($\gamma(0) = x_0$ and $\gamma(t_0) = x_1$). Without loss of generality we can suppose that the points x_0 and x_1 are not conjugate. It follows from formula (8) and Lemma 2 in [17] that the set Z_{x_0} of points $v \in T_{x_0} M^n$ where the differentials $d_v B_1, \ldots, d_v B_m$ are linearly independent is dense in $T_{x_0} M^n$. Hence, there exists a point $v_0 \in Z_{x_0}$ such that $\exp v_0 \in V_1$. The geodesic flow of the metric g preserves the functions $B_1, \ldots, B_m$. Therefore, the differentials $d_{v_1} B_1, \ldots, d_{v_1} B_m$, where $v_1 \stackrel{\text{def}}{=} \frac{d}{dt}|_{t=1} \exp t v_0$, are also linearly independent. On the other hand, on V_1 we have $B_m = \sum_{j=1}^{m_1} \lambda_j B_j$, $\lambda_j =$ constant. This is a contradiction. Therefore, $m_1 = m$. This proves that all stable points on M^n are of type m. Therefore, $M^n = S(A) \bigsqcup M^m(A)$. It is clear that if $N(x) = m$, then $x \in M^m(A)$. Hence, $S(A) = \{x \in M^n | N(x) < m\}$. Let us prove the last statement of the theorem. Suppose that there is an open set $U \in M^m$ such that for any $x \in U$ the forms $B_1(x), \ldots, B_m(x)$ (defined above) are linearly dependent (as quadratic forms in the tangent space $T_x U$). Taking U sufficiently small we can choose constants $\alpha_1, \ldots, \alpha_m$ such that $B_k = \sum_{j=1}^m \beta_{kj} I_{\alpha_j}$, where β_{kj} are constants and the forms $I_{\alpha_1}(x), \ldots, I_{\alpha_m}(x)$ are linearly independent for any fixed $x \in U$. It is clear that the matrix (β_{kj}) is degenerate. Hence, the differentials $d_v B_1, \ldots, d_v B_m$ are linearly dependent on TU. But the arguments used above show that we can find a point $v_0 \in TU$ where the differentials $d_{v_0} B_1, \ldots, d_{v_0} B_m$ are linearly independent. This contradiction shows that the set K of points $x \in M^n$ where the forms $B_1(x), \ldots, B_m(x)$ form a basis of the linear space generated by the forms $\{I_c(x)|c \in \mathbb{R}\}$ is dense in M^n. Therefore, for any real constant c we have $I_c = \sum_{j=1}^m \mu_j B_j$ where μ_j are functions on K that are locally constants. Let us prove that these functions are constant on K. Suppose that $I_c = \sum_j \mu_j^\alpha B_j$ in some open set $U_\alpha \subset K$ and $I_c = \sum_j \mu_j^\beta B_j$ in some open set $U_\beta \subset K$. Let us take $x_\alpha \in U_\alpha$. It is easily seen that the set of the points $\xi \in T_{x_\alpha}M^n$ where the differentials $d_\xi B_j$ $(j = 1, \dots, m)$ are linearly independent is dense in $T_{x_\alpha}M^n$. Therefore, we can find $\xi_\alpha \in T_{x_\alpha}M^n$ such that $\exp \xi_\alpha = x_\beta \in U_\beta$ and the differentials $d_{\xi_\alpha} B_j$ are linearly independent. Denote by $\zeta^t : TM^n \to TM^n$ the geodesic flow of the metric g and let $\xi_\beta \overset{\text{def}}{=} \frac{d}{dt}|_{t=1} \exp(t\xi_\alpha)$. We have that $d_{\xi_\alpha} I_c = \sum_j \mu_j^\alpha d_{\xi_\alpha} B_j$. On the other hand

$$\begin{aligned} d_{\xi_\alpha} I_c &= d_{\xi_\beta} I_c \circ \zeta_*^1 \\ &= \sum_j \mu_j^\beta d_{\xi_\beta} B_j \circ \zeta_*^1 \\ &= \sum_j \mu_j^\beta d_{\xi_\alpha} B_j. \end{aligned}$$

Hence, $\mu_j^\alpha = \mu_j^\beta \overset{\text{def}}{=} \mu_j$ $(j = 1, \dots, m)$. This gives $I_c = \sum_j \mu_j B_j$ on K. Therefore, $I_c = \sum_j \mu_j B_j$ on M^n. It is clear that the functions $B_1, \dots, B_m$ are functionally independent almost everywhere on TM^n. This proves the theorem.

3.2. Proof of Theorem 3.

Suppose that the metric g is A-integrable. It is sufficient to prove the theorem in a neighborhood of an arbitrary stable point. Suppose that $x_0 \in M^n$ is a stable point of type m and let the smooth functions $\phi_1(x) < \phi_2(x) < \cdots < \phi_m(x)$ be the eigenvalues of the operator $A(x)$ in a neighborhood $U(x_0)$ of the point x_0. As in the proof of Theorem 1 we see that the condition $m = 1$ implies that $A = \text{constant} \times \mathbf{1}$. In this case the statement of Theorem 3 is obvious. Consider the case $m \geq 2$. Let us consider the functions $F_k \overset{\text{def}}{=} I_{c_k}$, where $c_1, \dots, c_m$ are different real constants such that $c_i \neq -\phi_j$ $i, j = 1, \dots, m$ on $U(x_0)$. As in the proof of Theorem 1 we see that equality (9) holds and the functions $\phi_1, \dots, \phi_m$ satisfy conditions 1) and 2) of section 2 (following formula (3)). We have to prove that the 1-parameter family of functions $I_c^f \overset{\text{def}}{=} \det(A + c\mathbf{1}) g_f((A + c\mathbf{1})^{-1}\xi, \xi)$ are in involution. Let us consider the functions $F_k^f \stackrel{\text{def}}{=} I_{c_k}^f$. As in the proof of Theorem 1 we see that

$$\begin{aligned}
A_p^f &\stackrel{\text{def}}{=} \frac{f(\phi_p)^{-1}A_p}{(\phi_p+c_1)\cdots(\phi_p+c_{m-1})}\Pi_p C_m[(\phi_1+c_m)^{l_1-1}\cdots(\phi_m+c_m)^{l_m-1}] \\
&= \sum_k (-1)^{p+k}\frac{C_m}{C_k}\left(\frac{\phi_1+c_m}{\phi_1+c_k}\right)^{l_1-1}\cdots\left(\frac{\phi_p+c_m}{\phi_p+c_k}\right)^{l_p}\cdots\left(\frac{\phi_m+c_m}{\phi_m+c_k}\right)^{l_m-1}F_k^f \\
&= \sum_k a_{pk}F_k^f.
\end{aligned}$$

It is clear that the coefficients a_{pk} satisfy the Stäckel condition (see Proposition 1 in [17]). Moreover, $A_p^f = f(\phi_p)^{-1}A_p$, where the functions $A_1,\dots,A_m$ are in involution. Hence, the functions $A_1^f,\dots,A_m^f$ are in involution. Finally, a variant of the Stäckel-Painlevé theorem shows that the functions $F_1^f,\dots,F_m^f$ are in involution. Note that $g_f = \lim_{c\to\infty} c^{1-n}I_c^f$. Therefore, the functions $F_1^f,\dots,F_m^f$ are integrals of the geodesic flow of the metric g_f. Let us fix an arbitrary real constant α. It follows from the definition of the function I_α^f that $I_\alpha^f = \sum_{j=1}^m \alpha_j F_j^f$, where α_j are smooth functions on $U(x_0)$. The arguments used in the proof of Theorem 2 show that α_j are constants. Therefore, the functions from the family I_c^f are in involution on $M(A)$. Finally, the statement of the theorem follows from the fact that the set of stable points $M(A)$ is everywhere dense in M. Theorem 3 is proved.

3.3. Proof of Theorem 4.

It is sufficient to prove that the functions

$$(13)\qquad I_s(\xi) \stackrel{\text{def}}{=} \det\left(\frac{\alpha A+\beta\mathbf{1}}{A+c\mathbf{1}}+s\mathbf{1}\right)g_c\left(\left(\frac{\alpha A+\beta\mathbf{1}}{A+c\mathbf{1}}+s\mathbf{1}\right)^{-1}\xi,\xi\right)$$

are in involution with respect to the symplectic structure $\omega_{g_c} \stackrel{\text{def}}{=} FL_{g_c}^*\omega$, where FL_{g_c} is the Legendre transformation corresponding to the metric g_c and ω is the canonical symplectic structure on T^*M^n. Applying the Legendre transformation and using matrix representation we obtain

$$\begin{aligned}
I_s &= \frac{\det((c+\alpha)A+(cs+\beta)E)}{\det(A+cE)}((c+\alpha)A+(cs+\beta)E)^{-1}(A+cE)g_c^{-1} \\
&= \det((c+\alpha)A+(s+c\beta)E)((c+\alpha)A+(s+c\beta)E)^{-1}(A+cE)^2g^{-1},
\end{aligned}$$

where E is the unit matrix. It follows from Theorem 3 that the metric $g((A+c\mathbf{1})^{-2}\xi,\xi)$ is A-integrable. This proves Theorem 4.

3.4. Proof of Theorem 5.

Suppose that the conditions of Theorem 5 are satisfied. Let us fix two points $x_0 \in M_1$ and $y_0 \in M_2$ and suppose that x_0 is stable of type $m_1 \le n_1$ and y_0 is stable of type $m_2 \le n_2$. Consider the point $u_0 = (x_0, y_0) \in M$. It is clear that the point u_0 is stable of type $m_1 + m_2$ with respect to the operator $A \overset{\text{def}}{=} \tilde{A}_1 + \tilde{A}_2$. There is an open neighborhood $U(x_0) \subset M_1$ where the operator A_1 has exactly m_1 distinct eigenvalues $\phi_1 < \cdots < \phi_{m_1}$ and an open neighborhood $V(y_0) \subset M_2$ where the operator A_2 has exactly m_2 distinct eigenvalues $\phi_{m_1+1} < \cdots < \phi_{m_1+m_2}$. Denote by $\tilde{\phi}_k$ the functions $\pi_1^*\phi_k$ $(k = 1, \dots, m_1)$ and $\pi_2^*\phi_k$ $(k = m_1+1, \dots, m_1+m_2)$ on $U(x_0) \times V(y_0)$. There exist an orthonormal (with respect to the metric g_1) frame $X_{k\alpha_k}$ $(k = 1, \dots, m_1,\ \alpha_k = 1, \dots, l_k,\ \sum_{k=1}^{m_1} l_k = n_1)$ on $U(x_0)$, such that $A_1(X_{k\alpha_k}) = \phi_k X_{k\alpha_k}$, and an orthonormal (with respect to the metric g_2) frame $X_{k\alpha_k}$ $(k = m_1+1, \dots, m_1+m_2,\ \alpha_k = 1, \dots, l_k,\ \sum_{k=m_1+1}^{m_1+m_2} l_k = n_2)$ on $V(y_0)$, such that $A_2(X_{k\alpha_k}) = \phi_k X_{k\alpha_k}$. Using the natural identification $T_{(x,y)}M = K_2 \oplus K_1 \cong T_x M_1 \oplus T_y M_2$ we obtain a frame $\widetilde{X}_{k\alpha_k}$ $(k = 1, \dots, m_1+m_2,\ \alpha_k = 1, \dots, l_k)$ on $U(x_0) \times V(y_0)$. We have to prove that the one-parameter family of functions $I_c(\xi) \overset{\text{def}}{=} \det(A+c) g((A+c)^{-1}\xi, \xi)$, $\xi \in TM$ are in involution with respect to the symplectic structure ω_g. Making the Legendre transformation corresponding to the metric g we obtain

$$(14) \qquad I_c = \widetilde{\Phi}(c) \left\{ \frac{\widetilde{P}_1}{D_1(\tilde{\phi}_1 + c)} + \cdots + \frac{\widetilde{P}_{m_1+m_2}}{D_{m_1+m_2}(\tilde{\phi}_{m_1+m_2} + c)} \right\},$$

where $\widetilde{\Phi}(c) \overset{\text{def}}{=} (\tilde{\phi}_1 + c)^{l_1} \cdots (\tilde{\phi}_{m_1+m_2} + c)^{l_{m_1+m_2}}$, $\widetilde{P}_k \overset{\text{def}}{=} \sum_{\alpha=1}^{l_k} \epsilon_{k\alpha} \widetilde{P}_{k\alpha}^2$, $\epsilon_{k\alpha} = \pm 1$, $\widetilde{P}_{k\alpha} \overset{\text{def}}{=} \left\langle p, \widetilde{X}_{k\alpha} \right\rangle$, $p \in T^*M$, and

$$\widetilde{D}_k \overset{\text{def}}{=} \begin{cases} (\tilde{\phi}_k - \tilde{\phi}_{m_1+1})^{l_{m_1+1}} \cdots (\tilde{\phi}_k - \tilde{\phi}_{m_1+m_2})^{l_{m_1+m_2}}, & 1 \le k \le m_1, \\ (\tilde{\phi}_k - \tilde{\phi}_1)^{l_1} \cdots (\tilde{\phi}_k - \tilde{\phi}_{m_1})^{l_{m_1}}, & m_1 + 1 \le k \le m_1 + m_2. \end{cases}$$

As in the proof of Theorem 1 we take $m = m_1 + m_2 \ge 2$ real constants $c_1, \dots, c_m$ such that $c_i + \tilde{\phi}_j \ne 0$ $(i, j = 1, \dots, m)$ on $U(x_0) \times V(y_0)$.

Denoting by F_k the functions I_{c_k} we obtain (see formula (9))

$$(15)\quad \widetilde{P}_p \stackrel{\text{def}}{=} \widetilde{P}_p \frac{\widetilde{\Pi}_p}{\widetilde{D}_p} \widetilde{C}_m \frac{(\tilde{\phi}_1 + c_m)^{l_1-1} \cdots (\tilde{\phi}_m + c_m)^{l_m-1}}{(\tilde{\phi}_p + c_1) \cdots (\tilde{\phi}_p + c_{m-1})}$$
$$= \sum_k (-1)^{p+k} \frac{\widetilde{C}_m}{\widetilde{C}_k} \left(\frac{\tilde{\phi}_1 + c_m}{\tilde{\phi}_1 + c_k}\right)^{l_1-1} \cdots \left(\frac{\tilde{\phi}_m + c_m}{\tilde{\phi}_m + c_k}\right)^{l_m-1} \frac{\tilde{\phi}_p + c_m}{\tilde{\phi}_p + c_k} F_k$$
$$= \sum_{k=1}^{m} q_{pk} F_k,$$

where $\widetilde{\Pi}_k \stackrel{\text{def}}{=} (\tilde{\phi}_m - \tilde{\phi}_k) \cdots (\tilde{\phi}_k - \tilde{\phi}_1)$ and $\widetilde{C}_k \stackrel{\text{def}}{=} (c_m - c_k) \cdots (c_k - c_1)$ $(k = 1, \dots, m)$. By assumption, the metric g_1 is A_1-integrable and the metric g_2 is A_2-integrable. Hence, the functions $Q_c(\xi) \stackrel{\text{def}}{=} \det(A_1 + c) g_1((A_1 + c)^{-1}\xi, \xi)$, $\xi \in TM_1$ are in involution with respect to the symplectic structure ω_{g_1} on TM_1 and the functions $R_c(\xi) \stackrel{\text{def}}{=} \det(A_2 + c) g_2((A_2 + c)^{-1}\eta, \eta)$, $\eta \in TM_2$ are in involution with respect to the symplectic structure ω_{g_2} on TM_2. Using the constants $c_1, \dots, c_{m_1}$ and formula (9) we obtain

$$P_s^{(1)} \stackrel{\text{def}}{=} P_s^{(1)} \Pi_s^{(1)} C_{m_1}^{(1)} \frac{(\phi_1 + c_{m_1})^{l_1-1} \cdots (\phi_{m_1} + c_{m_1})^{l_{m_1}-1}}{(\phi_s + c_1) \cdots (\phi_s + c_{m_1-1})}$$
$$= \sum_k (-1)^{s+k} \frac{C_{m_1}^{(1)}}{C_k^{(1)}} \left(\frac{\phi_1 + c_{m_1}}{\phi_1 + c_k}\right)^{l_1-1} \cdots \left(\frac{\phi_{m_1} + c_{m_1}}{\phi_{m_1} + c_k}\right)^{l_{m_1}-1} \frac{\phi_s + c_{m_1}}{\phi_s + c_k} Q_k,$$

which we write as

$$(16)\qquad P_s^{(1)} = \sum_{k=1}^{m_1} a_{sk} Q_k, \; Q_k \stackrel{\text{def}}{=} Q_{c_k}.$$

Here $P_s^{(1)} \stackrel{\text{def}}{=} \sum_{\alpha=1}^{l_s} \epsilon_{s\alpha} P_{s\alpha}^2$, $\epsilon_{s\alpha} = \pm 1$, $P_{s\alpha} \stackrel{\text{def}}{=} \langle p, X_{s\alpha} \rangle$, $p \in T^*M_1$, and $\Pi_s^{(1)} \stackrel{\text{def}}{=} (\phi_{m_1} - \phi_s) \cdots (\phi_s - \phi_1)$, $C_s^{(1)} \stackrel{\text{def}}{=} (c_{m_1} - c_s) \cdots (c_s - c_1)$, $s = 1, \dots, m_1$. Analogously,

$$P_s^{(2)} \stackrel{\text{def}}{=} P_s^{(2)} \Pi_s^{(2)} C_m^{(2)} \frac{(\phi_{m_1+1} + c_m)^{l_{m_1+1}-1} \cdots (\phi_m + c_m)^{l_m-1}}{(\phi_s + c_{m_1+1}) \cdots (\phi_s + c_{m-1})}$$
$$= \sum_k (-1)^{s+k} \frac{C_m^{(2)}}{C_k^{(2)}} \left(\frac{\phi_{m_1+1} + c_m}{\phi_{m_1+1} + c_k}\right)^{l_{m_1+1}-1} \cdots \left(\frac{\phi_m + c_m}{\phi_m + c_k}\right)^{l_m-1} \frac{\phi_s + c_m}{\phi_s + c_k} R_k$$

which we write as

$$(17)\qquad P_s^{(2)} = \sum_{k=m_1+1}^{m} b_{sk} R_k,\ R_k \stackrel{\text{def}}{=} R_{c_k}.$$

Here $P_s^{(2)} \stackrel{\text{def}}{=} \sum_{\alpha=1}^{l_s} \epsilon_{s\alpha} P_{s\alpha}^2$, $\epsilon_{s\alpha} = \pm 1$, $P_{s\alpha} \stackrel{\text{def}}{=} \langle p, X_{s\alpha}\rangle$, $p \in T^*M_2$, and $\Pi_s^{(2)} \stackrel{\text{def}}{=} (\phi_m - \phi_s)\cdots(\phi_s - \phi_{m_1+1})$, $C_s^{(2)} \stackrel{\text{def}}{=} (c_m - c_s)\cdots(c_s - c_{m_1+1})$, $s = m_1 + 1, \ldots, m$. It follows from the Stäckel-Painlevé theorem that the functions $P_s^{(1)}$, $s = 1, \ldots, m_1$ are in involution with respect to the canonical symplectic structure on T^*M_1 and the functions $P_s^{(2)}$, $s = m_1 + 1, \ldots, m$ are in involution with respect to the canonical symplectic structure on T^*M_2. As in the proof of Theorem 1 we see that if $l_k > 1$, then $\phi_k =$ constant, and if $k \neq q$, then $L_{X_{k\alpha}} \phi_q = 0$. It follows from formula (15), (16) and (17) that

$$\widetilde{P}_p = \begin{cases} \widetilde{P}_p^{(1)} B_p & , \quad 1 \le p \le m_1, \\ \widetilde{P}_p^{(2)} B_p & , \quad m_1 + 1 \le p \le m, \end{cases}$$

where B_p are smooth functions on $U(x_0) \times V(y_0)$ such that $L_{\widetilde{X}_{k\alpha}} B_p$, if $k \neq p$, and $\widetilde{P}_p^{(i)} \stackrel{\text{def}}{=} p_i^*(P_p^{(i)}) \in C^\infty(T^*M)$, where $p_i : T^*M \cong (T^*M_1) \times (T^*M_2) \to T^*M_i$ is the natural projection on T^*M_i $(i = 1, 2)$. Therefore, the functions $\widetilde{P}_p$ $(p = 1, \ldots, m)$ are in involution with respect to the canonical symplectic structure on T^*M. Finally, applying a variant of the Stäckel-Painlevé theorem to equation (15) we obtain that the functions F_k are in involution. Theorem 5 is proved.

§4. Appendix 1

In this appendix we apply the theorems proved in the previous sections to the A-integrable metric described in item c) of section 1.

Let us consider the projective transformation of the plane $\mathbb{R}^n$ given by the formula

$$\mu : \begin{cases} x_1 & \mapsto & (\lambda_1 x_1)/x_n \\ & \vdots & \\ x_{n-1} & \mapsto & (\lambda_{n-1} x_{n-1})/x_n \\ x_n & \mapsto & \lambda_n - \lambda_n/x_n \end{cases}$$

where λ_i $(i = 1, \ldots, n)$ are non-zero constants. Consider the Euclidean metric $g \stackrel{\text{def}}{=} dx_1^2 + \cdots + dx_n^2$ in $\mathbb{R}^n$ (see [15], where the pseudo-Riemannian analogue of the present construction is given). The geodesics of the metric g are straight lines. Denote by $\bar g$ the pull-back $\mu^* g$. Note that the metric $\bar g$ is not defined over the hyperplane $Z \stackrel{\text{def}}{=} \{x_n = 0\}$. It is clear that the metrics g and $\bar g$ are geodesically equivalent on $D \stackrel{\text{def}}{=} \mathbb{R}^n \setminus Z$. We have

$$\begin{aligned} \bar g &\stackrel{\text{def}}{=} \mu^*(g) \\ &= \lambda_1^2 \left(\frac{x_n dx_1 - x_1 dx_n}{x_n^2} \right)^2 + \cdots + \lambda_{n-1}^2 \left(\frac{x_n dx_{n-1} - x_{n-1} dx_n}{x_n^2} \right)^2 \\ &\quad + \lambda_n^2 \left(\frac{dx_n}{x_n^2} \right)^2 . \end{aligned} \tag{18}$$

Let us compute the operator $A \stackrel{\text{def}}{=} A(g, \bar g)$. We need the following simple lemmas.

Lemma 1. *Consider the matrix*

$$C \stackrel{\text{def}}{=} \bar a \otimes \bar a + b, \tag{19}$$

where $b \stackrel{\text{def}}{=} \operatorname{diag}(b_1, \ldots, b_n)$, $b_i \neq 0$ $(i = 1, \ldots, n)$, $\bar a \stackrel{\text{def}}{=} (a_1, \ldots, a_n)'$ *(the symbol* $'$ *denotes the transposition of a matrix) and* $\bar a \otimes \bar a$ *stands for the matrix* $\bar a \bar a'$. *Then the inverse matrix* C^{-1} *is given by the formula*

$$C^{-1} = b^{-1} - \frac{1}{D} (b^{-1} \bar a) \otimes (b^{-1} \bar a), \tag{20}$$

where $D \stackrel{\text{def}}{=} \langle b^{-1} \bar a, \bar a \rangle + 1$ *and* $\langle \bar x, \bar y \rangle \stackrel{\text{def}}{=} \sum_{i=1}^n x_i y_i$. *Moreover,* $\det C = (\prod_{k=1}^n b_k) D$.

Proof of Lemma 1. The proof is by direct calculation. We have $(\bar a \bar a' + b)(b^{-1} - (1/D)(b^{-1}\bar a)(b^{-1}\bar a)') = \bar a(\bar a' b^{-1}) + E - \langle \bar a, b^{-1}\bar a \rangle \bar a (b^{-1}\bar a)'/D$ $-(1/D)\bar a (b^{-1}\bar a)' = E$ where E denotes the unit matrix. Lemma 1 is proved.

Lemma 2. *Consider the matrix* $C_0 \stackrel{\text{def}}{=} \bar a \otimes \bar a + \operatorname{diag}(b_1, \ldots, b_{n-1}, 0)$, *where* $b_i \neq 0$ $(i = 1, \ldots, n-1)$. *Then the inverse matrix* C_0^{-1} *is given by the formula* $C_0^{-1} =$

$$\frac{1}{a_n^2} \left\{ \sum_{i=1}^{n-1} \frac{a_n^2}{b_i} E_{ii} + \left(\sum_{i=1}^{n-1} \frac{a_i^2}{b_i} + 1 \right) E_{nn} - \sum_{i=1}^{n-1} \frac{a_n a_i}{b_i} E_{in} - \sum_{i=1}^{n-1} \frac{a_n a_i}{b_i} E_{ni} \right\}$$

where the matrix E_{ij} *has elements* $e_{kl} \stackrel{\text{def}}{=} \delta_{ki} \delta_{lj}$ $(k, l = 1, \ldots, n)$. *We have* $\det C_0 = (\prod_{k=1}^{n-1} b_k) a_n^2$.

Lemma 2 easily follows from Lemma 1 and the formula $C_0^{-1} = \lim_{\alpha\to+0}(C_0+\alpha)^{-1}$.

Denote by $\bar{G}$ the matrix corresponding to the metric $\bar{g}$, i.e. the Gramian of the metric g. Taking $\bar{a} = \bar{x} \stackrel{\text{def}}{=} (x_1,\dots,x_n)$, $b_i = \lambda_n^2/\lambda_i^2$ $(i = 1,\dots,n-1)$, and applying Lemma 2 we see that

$$\bar{G}^{-1} = \frac{x_n^2}{\lambda_n^2}\left(\bar{x}\otimes\bar{x} + \operatorname{diag}\left(\frac{\lambda_n^2}{\lambda_1^2},\dots,\frac{\lambda_n^2}{\lambda_{n-1}^2},0\right)\right). \tag{21}$$

As a corollary we obtain that $\det\bar{G} = (\prod_{i=1}^n \lambda_i^2)/x_n^{2n+2}$. Finally, combining (3) and (21) we obtain (up to multiplication by a constant) that

$$A(g,\bar{g}) = \bar{x}\otimes\bar{x} + \operatorname{diag}\left(\frac{\lambda_n^2}{\lambda_1^2},\dots,\frac{\lambda_n^2}{\lambda_{n-1}^2},0\right). \tag{22}$$

It follows from Theorem 1 formula (22) and Lemma 1 that the quadratic forms

$$\begin{aligned} I_c &= D_c\left(\frac{dx_1^2}{d_1+c}+\cdots+\frac{dx_{n-1}^2}{d_{n-1}+c}+\frac{dx_n^2}{c}\right)\\ &- \left(\frac{x_1dx_1}{d_1+c}+\cdots+\frac{x_{n-1}dx_{n-1}}{d_{n-1}+c}+\frac{x_ndx_n}{c}\right)^2, \end{aligned} \tag{23}$$

where $d_k \stackrel{\text{def}}{=} \lambda_n^2/\lambda_k^2$ $(k=1,\dots,n-1)$, are pairwise commuting integrals of the geodesic flow of the metric g on all half-planes $\{x_n > 0\}$ and $\{x_n < 0\}$. The quadratic forms I_c are smoothly defined on the whole of $\mathbb{R}^n$. Therefore, they are pairwise commuting integrals of the geodesic flow the metric g on $\mathbb{R}^n$.

Suppose that $d_i \neq d_j$ $(i\neq j)$ and $d_i \neq 0$ $(i = 1,\dots,n-1)$. Consider the characteristic polynomial of the matrix $A \stackrel{\text{def}}{=} A(g,\bar{g})$

$$\begin{aligned} \chi_A(c) &\stackrel{\text{def}}{=} \det(A+cE)\\ &= \prod_{k=1}^n (d_k+c)\left\{\frac{x_1^2}{d_1+c}+\cdots+\frac{x_{n-1}^2}{d_{n-1}+c}+\frac{x_n^2}{c}+1\right\}, \end{aligned}$$

where we put $d_n = 0$. It is clear that if $x_1\dots x_n \neq 0$, then the polynomial $\chi_A(c)$ has n different roots. We have proved the next theorem.

Theorem 6. *The metric $g \stackrel{\text{def}}{=} \sum_{i=1}^n dx_i^2$ is A-integrable. If $d_i \neq d_j$ $(i\neq j)$ and $d_i \neq 0$ $(i = 1,\dots,n-1)$, then the rank of the A-integrable metric g is n.*

It follows from Theorem 1 that the one-parameter family of metrics $g_c(g,\bar{g}) \stackrel{\text{def}}{=} \frac{1}{\det(A+c\mathbf{1})} g(A+c\mathbf{1})^{-1}$ are geodesically equivalent to g. We have

$$G_c \stackrel{\text{def}}{=} \frac{1}{\det(A+cE)}(A+cE)^{-1} \tag{24}$$

where G_c denotes the Gramian of the metric g_c. It follows from Lemma 1 that up to a multiplication on a constant

$$\begin{aligned} g_c \quad &= \quad \Bigg\{ D_c \left(\frac{dx_1^2}{d_1+c} + \cdots + \frac{dx_{n-1}^2}{d_{n-1}+c} + \frac{dx_n^2}{c} \right) \\ &- \quad \left(\frac{x_1 dx_1}{d_1+c} + \cdots + \frac{x_{n-1} dx_{n-1}}{d_{n-1}+c} + \frac{x_n dx_n}{c} \right)^2 \Bigg\} / D_c^2, \end{aligned} \tag{25}$$

where

$$D_c \stackrel{\text{def}}{=} \frac{x_1^2}{d_1+c} + \cdots + \frac{x_{n-1}^2}{d_{n-1}+c} + \frac{x_n^2}{c} + 1. \tag{26}$$

Remark 6. *The metric g_c is not defined on the quadric $Z_c \stackrel{\text{def}}{=} \{D_c = 0\}$. Nevertheless, from now on when we say that two metrics are geodesically equivalent we mean they are geodesically equivalent in the domain where both of them are defined.*

The metrics g_c given by formula (25) have the very special property given by the next theorem.

Theorem 7. *Let us assume that $d_i \neq d_j$ $(i \neq j)$ and $d_i \neq 0$ $(i = 1,\ldots,n-1)$. Then the restrictions of the metrics g_c given by formula (25) on every quadric Q_α $(\alpha \neq c)$ from the confocal family $Q_\alpha \stackrel{\text{def}}{=} \{D_\alpha = 0\}$ are geodesically equivalent to the restriction of the standard Euclidean metric g on Q_α.*

As a corollary we obtain the theorem proved in [14] and independently in [12] that the standard ellipsoid admits a non-trivial geodesic equivalence. Theorem 7 shows that the same result is true for the hyperboloids.

Proof of Theorem 7. Suppose that $g = \sum_{i=1}^n dx_i^2$. Let us fix a point $\bar{x}_0 = (x_1^0,\ldots,x_n^0) \in \mathbb{R}^n$ such that $x_1^0 \ldots x_n^0 \neq 0$. It is easily seen that there exist n different real constants $\alpha_1,\ldots,\alpha_n$ such that $Q_{\alpha_1}(\bar{x}_0) = 0,\ldots,Q_{\alpha_n}(\bar{x}_0) = 0$. It follows from formula (23) that

$$I_{\alpha_l}|_{\bar{x}_0} = -(d_{\bar{x}_0} Q_{\alpha_l})^2/4. \tag{27}$$

The forms $d_{\bar{x}_0}Q_{\alpha_l}$ are linearly independent. It follows from Remark 2 and the non-degeneracy of the corresponding Vandermonde determinant that the forms $S_0|_{\bar{x}_0}, \dots, S_{n-1}|_{\bar{x}_0}$ are simultaneously diagonalizable in the frame which is dual to the forms $d_{\bar{x}_0}Q_{\alpha_l}$ $(l = 1, \dots, n)$. The form S_{n-1} is conformally equivalent to the metric g and the form S_0 is conformally equivalent to $\bar{g}$. Hence, the gradients (with respect to the metric g) $\nabla_{\bar{x}_0}Q_{\alpha_1}, \dots, \nabla_{\bar{x}_0}Q_{\alpha_n}$ coincide with the principal directions of the metrics $g|_{\bar{x}_0}$ and $\bar{g}|_{\bar{x}_0}$. Finally, the statement of the theorem follows from Levi-Civita's theorem about the local form of the Riemannian metrics that permit geodesic equivalence (see [5, 16, 17]). Theorem 7 is proved.

Let $(x_1, \dots, x_n)$ be the coordinates in $\mathbb{R}^n$, and $(x_1, \dots, x_n; p_1, \dots, p_n)$ the corresponding coordinates in $T^*\mathbb{R}^n$. Applying Theorem 3 we obtain the next theorem.

Theorem 8. *For every fixed real κ and every fixed integer l, consider the one-parameter family of functions on $T^*\mathbb{R}^n$ given by the formula $I_\alpha^{(l)}(\bar{p}) \stackrel{\text{def}}{=} \langle I_\alpha^{(l)}\bar{p}, \bar{p}\rangle$, where $\bar{p} \stackrel{\text{def}}{=} (p_1, \dots, p_n)$, $\langle \bar{\xi}, \bar{\eta}\rangle \stackrel{\text{def}}{=} \sum_{i=1}^n \xi_i\eta_i$,*

$$\begin{aligned} I_\alpha^{(l)} &\stackrel{\text{def}}{=} \det(A + \alpha E)(A + \alpha E)^{-1}(A + \kappa E)^{-l} \\ &= S_{n-1}^{(l)}(\kappa)\alpha^{n-1} + S_{n-2}^{(l)}(\kappa)\alpha^{n-2} + \dots + S_0^{(l)}(\kappa) \end{aligned}$$

and the matrix A is given by the formula

$$A \stackrel{\text{def}}{=} \bar{x} \otimes \bar{x} + \operatorname{diag}(d_1, \dots, d_{n-1}, 0)\,. \tag{28}$$

These functions are in involution with respect to the canonical symplectic structure on $T^*\mathbb{R}^n$. If $d_i \neq d_j$ $(i \neq j)$ and $d_i \neq 0$ $(i = 1, \dots, n-1)$, then the functions $S_0^{(l)}, \dots, S_{n-1}^{(l)}$ are functionally independent on $T^*\mathbb{R}^n$.

Remark 7. *Taking $l = 0$ we derive the integrals obtained by K. Uhlenbeck in* [19]. *Another way of obtaining these integrals was proposed by J. Moser in* [8] *and K. Kiyohara in* [4]. *The families of functions in involution given by Theorem 8 generalize all these results.*

Remark 8. *The case $l = -1$, $\kappa = 0$, gives a family of pairwise commuting functions $S_0^{(-1)}(0), \dots, S_{n-1}^{(-1)}(0)$ on T^*M^n. The function $S_{n-1}^{(-1)}(0)$ coincides with the Hamiltonian of the metric*

$$g_{hyp} \stackrel{\text{def}}{=} \frac{1}{x_n^2}\sum_{i=1}^n dx_i^2. \tag{29}$$

Therefore, the functions $S_0^{(-1)}(0), \dots, S_{n-1}^{(-1)}(0)$ on T^*M^n are a complete family of integrals of the geodesic flow of the hyperbolic plane.

§5. Appendix 2

Suppose that the (pseudo) Riemannian metrics g and $\bar{g}$, given on the smooth manifold M^n, are geodesically equivalent. In this appendix we prove that the functions given by the formula $I_c(\xi) \overset{\text{def}}{=} \det(A+c\mathbf{1})g((A+c\mathbf{1})^{-1}\xi,\xi)$, $\xi \in TM^n$, where c is real parameter and the operator $A = A(g,\bar{g})$ is given by formula (3), are in involution with respect to the symplectic structure ω_g on TM^n (see [15]).

Let us fix a point $x_0 \in M^n$. It is clear that there exists an open neighborhood $U(x_0)$ of the point x_0 and an interval $(-\epsilon,\epsilon)$, $\epsilon > 0$, such that the operator $(A+c\mathbf{1})|_x$ is invertible if $x \in U(x_0)$ and $c \in (-\epsilon,\epsilon)$. Consider the one-parameter family of geodesically equivalent metrics on $U(x_0)$

$$g_c(g,\bar{g})(\xi,\eta) \overset{\text{def}}{=} \frac{1}{\det(A+c\mathbf{1})} g((A+c\mathbf{1})^{-1}\xi,\eta),\ \xi,\eta \in TM^n, \tag{30}$$

where $c \in (-\epsilon,\epsilon)$ (see Proposition 1 in [16]). Let us fix some $\kappa \in (-\epsilon,\epsilon)$. The metrics g and g_κ are geodesically equivalent on $U(x_0)$. Consider the following subsequence of geodesically equivalent metrics

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
g & \overset{g.e.}{\longleftrightarrow} & g_\kappa \\
\downarrow & & \downarrow \\
g^{(1)} & \overset{g.e.}{\longleftrightarrow} & g_\kappa^{(1)} \\
\downarrow & & \downarrow \\
g^{(2)} & \overset{g.e.}{\longleftrightarrow} & g_\kappa^{(2)} \\
\downarrow & & \downarrow
\end{array}$$

where $g^{(l)} \overset{\text{def}}{=} gA(g,g_\kappa)^l$ and $g_\kappa^{(l)} \overset{\text{def}}{=} g_\kappa A(g,g_\kappa)^l$ (see [10, 16]). The metrics $g^{(2)}$ and $g_\kappa^{(2)}$ are geodesically equivalent. It is easily seen that $A(g,g_\kappa) = A+\kappa\mathbf{1}$. Hence, $g_\kappa^{(2)} \overset{\text{def}}{=} g_\kappa A(g,g_\kappa)^2 = \frac{1}{\det(A+\kappa\mathbf{1})} g(A+\kappa\mathbf{1})$. Let us consider the one-parameter family of integrals $I_\alpha(g_\kappa^{(2)}, g^{(2)})$ of the geodesic flow of the metric $g_\kappa^{(2)}$. A direct computation show that $A(g_\kappa^{(2)}, g^{(2)}) = (A+\kappa\mathbf{1})^{-1}$. Applying the Legendre transformation corresponding to the metric $g_\kappa^{(2)}$ we see that the functions

$${g_\kappa^{(2)}}^{-1} = \det(A+\kappa\mathbf{1})(A+\kappa\mathbf{1})^{-1}g^{-1} \tag{31}$$

and $I_\alpha(g_\kappa^{(2)}, g^{(2)}) \stackrel{\text{def}}{=}$

$$\det((A+\kappa\mathbf{1})^{-1}+\alpha\mathbf{1})((A+\kappa)^{-1}+\alpha\mathbf{1})^{-1}{g_\kappa^{(2)}}^{-1} \tag{32}$$

$$=\det(\alpha A+(\kappa\alpha+1)\mathbf{1})(\alpha A+(\kappa\alpha+1)\mathbf{1})^{-1}g^{-1} \tag{33}$$

are in involution with respect to the canonical symplectic structure on T^*M^n. Note that after applying the Legendre transformation corresponding to the metric g the family of integrals (1) takes the form $I_c(g,\bar{g}) = \det(A+c)(A+c)^{-1}g^{-1}$, where c is an arbitrary real parameter. Hence, the Poisson brackets $\{I_\kappa, I_{\kappa+\frac{1}{\alpha}}\}$ vanish for all $\alpha \neq 0$ and $\kappa \in (-\epsilon, \epsilon)$. Fixing n different real numbers $\kappa_i \in (\epsilon, \epsilon)$, $i = 1, \ldots, n$, $\kappa_i \neq \kappa_j$ $(i \neq j)$, we obtain that $\{I_{\kappa_i}, I_c\} = 0$ for all real values of the parameter c. Finally, recall that $I_\mu = S_{n-1}\mu^{n-1} + S_{n-2}\mu^{n-2} + \cdots + S_0$. The last equality and the non-degeneracy of the corresponding Vandermonde determinant show that the functions S_l $(l = 1, \ldots, n)$ are linear combinations with constant coefficients of the functions I_{κ_i} $(i = 1, \ldots, n)$. Therefore, the integrals given by formula (1) are in involution on $T^*U(x_0)$. The point x_0 was arbitrary. This proves the statement.

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References

[1] S. Benenti, Inertia tensors and Stäckel systems in Euclidean space, Rend. Sem. Mat. Univ. di Torino, **50** (1992), 1–20.

[2] U. Dini, Sopra un problema che si presenta nella theoria generale delle rappresetazioni geografice di una superficie su un'altra, Ann. di Math. ser.2, **3** (1869), 269–293.

[3] L. Eisenhart, Riemannian Geometry, Princeton Univ. Press, 1960.

[4] K. Kiyohara, Two Classes of Riemannian Manifolds Whose Geodesic Flows Are Integrable, Mem. Amer. Math. Soc., **130**, no. 619, Amer. Math. Soc., 1997.

[5] T. Levi-Civita, Sulle trasformazioni delle equazioni dinamiche, Ann. di Mat. serie 2^a, **24** (1896), 255–300.

[6] V. Matveev and P. Topalov, Quantum integrability of Beltrami-Laplace operator as geodesic equivalence, Math. Z., **238** (2001), 833–866.

[7] V. S. Matveev and P. J. Topalov, Trajectory equivalence and corresponding integrals, Regular and Chaotic Dynamics, **2** (1998), 30–45.
[8] J. Moser, Integrable Hamiltonian systems and spectral theory, Lezioni Fermiante, Academia Nazionale Dei Lincei, Pisa, 1981.
[9] P. Painlevé, Sur les intégrale quadratiques des équations de la Dinamique, Compt. Rend., **124** (1897), 221–224.
[10] N. S. Sinyukov, An invariant transformation of Riemannian manifolds with common geodesics, Soviet Math. Docl., **137** (1961), 1312–1314.
[11] N. S. Sinyukov, Geodesic mappings of Riemannian spaces, Nauka, Moscow, 1979.
[12] S. Tabachnikov, Projectively equivalent metrics, exact transverse line field and the geodesic flow on the ellipsoid, Comment. Math. Helv., **74** (1999), 306–321.
[13] P. Topalov, Families of metrics geodesically equivalent to the analogues of the Poisson sphere, J. Math. Phys., **41** (2000), 7510–7520.
[14] P. J. Topalov and V. S. Matveev, Geodesic equivalence via integrability, Geom. Dedicata, **96** (2003), 91–115.
[15] P. Topalov, Geodesic hierarchies and involutivity, J. Math. Phys., **42** (2001), 3898–3914.
[16] P. Topalov, Hierarchy of integrable geodesic flows, Publ. Mat.,**44** (2000), 257–276.
[17] P. Topalov, Integrability criterion of geodesical equivalence. Hierarchies, Acta Appl. Math., **59** (1999), 271–298.
[18] P. Topalov, Tensor invariants of natural mechanical systems on compact surfaces, and the corresponding integrals, Sb. Mat., **188** (1997), 307–326.
[19] K. Uhlenbeck, Minimal 2-spheres and tori in S^k, preprint, 1975.
[20] K. Yamato, A class of Riemannian manifolds with integrable geodesic flows, J. Math. Soc. Japan, **47** (1995), 719–733.

Department of Mathematics
Northeastern University
Boston, MA 02115
USA

E-mail address:
`p.topalov@neu.edu`