Advanced Studies in Pure Mathematics 51, 2008
Surveys on Geometry and Integrable Systems
pp. 259–283

A generalized Weierstrass representation for a submanifold S in $\mathbb{E}^n$ arising from the submanifold Dirac operator

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§1. Introduction

Using the submanifold quantum mechanical scheme [dC, JK], the restricted Dirac operator for a k dimensional spin (k-spin) submanifold S immersed in Euclidean space $\mathbb{E}^n$ $(0 < k < n)$ was defined [BJ, Ma1-10]. We call it the submanifold Dirac operator. The zero modes of the Dirac operator express the local properties of the submanifold, such as the Frenet-Serret and generalized Weierstrass formulae. We shall give a survey of this method from the point of view of quantum physics.

As motivation, we recall three facts.

(1) Let us consider an element Q of a ring of operators $\mathbf{P}$ defined over a Riemannian manifold M. The concept of the adjoint of Q is very subtle, as we shall explain briefly, following the book of Björk (see [Remark 1.2.16 in Bj]). Assume that M is Riemannian. For smooth functions f_1 and f_2 whose support is compact, we consider the following integral as a bilinear form of f_1 and f_2:

$$\int_M d\,\mathrm{vol}\ (f_1 Q f_2). \tag{1-1}$$

What is the natural adjoint of Q? One might regard the action on f_1 obtained by integration by parts as defining the adjoint. However the measure here depends on the local coordinates.

Received January 3, 2001
Revised May 5, 2003 and April 30, 2008
2000 *Mathematics Subject Classification.* Primary 53C42.

(2) In a quantum mechanical problem, we sometimes encounter the situation that an eigenfunction ψ of a differential operator P,

$$P\psi = E\psi, \tag{1-2}$$

belongs to a representation space of a group G. Suppose that P is decomposed as $P = P_1 + P_2$. Let us consider the kernel of P_2, KerP_2, in a certain function space. If an element $\psi_1 \in \mathrm{Ker}P_2$ satisfies

$$P_1\psi_1 = E\psi_1, \tag{1-3}$$

we may also obtain a representation of the group G.

(3) In quantum mechanics over a manifold M, there are typically two pairings on functions: 1) the global pairing $< \ , \ >$ induced from the L^2-norm, such as (1-1), and 2) the pointwise pairing $\cdot$ which is connected with the probability density. (The pointwise pairing can be regarded as $< \circ, \delta(p) \times >$ in terms of the Dirac distribution δ at p in M.)

Even though the concept of adjoint operator is subtle, we can sometimes define a natural adjoint operator if the measure is fixed for some reason, e.g. Haar measure. We note that the ordinary Lebesgue measure in Euclidean space $\mathbb{E}^n$ is Haar measure for the translation group. Quantum mechanics in $\mathbb{E}^n$ is based on this measure and the concept of adjoint operator plays an essential role. In such a case, by fixing a measure and L^2-type pairing $< \ , \ >: \Omega^* \times \Omega \to \mathbb{C}$, for an operator $Q \in \mathbf{P}$ whose domain is a function space Ω, we can define a right-adjoint operator, $Ad(Q)$, with the domain Ω^*, by

$$< f, Qg >=< f Ad(Q), g >, \quad \text{for } (f,g) \in \Omega^* \times \Omega. \tag{1-4}$$

Assume that there is a linear isomorphism $\varphi : \Omega \to \Omega^*$. Then we can define the left-adjoint operator Q^* by $Q^* f = \varphi^{-1}(\varphi(f) Ad(Q))$. The triple $(\Omega^* \times \Omega, < \ , \ >, \varphi)$ becomes a pre-Hilbert space $\mathcal{H}$ by taking the inner product $(\ , \ )_\varphi : \Omega \times \Omega \to \mathbb{C}$ as $(f,g)_\varphi =< \varphi(f), g >$. (Then $(P^* f, g)_\varphi = (f, Pg)_\varphi$.)

Suppose that $Q \in \mathbf{P}$ is *self-adjoint*, i.e. $Q^* = Q$. Then we have the following properties:

(1) The kernel of Q is isomorphic to that of $Ad(Q)$:

$$(\mathrm{Ker}(Q))^* = \varphi(\mathrm{Ker}(Q)) = \mathrm{Ker}(Ad(Q)). \tag{1-5}$$

(2) $((\mathrm{Ker}Q)^* \times \mathrm{Ker}Q, < \ , \ >, \varphi)$ becomes a pre-Hilbert space.

The projection π from $\Omega^* \times \Omega$ to $(\mathrm{Ker}Q)^* \times \mathrm{Ker}Q$ commutes with φ, i.e.

$$(1\text{-}6) \qquad \varphi\pi|_\Omega = \pi|_{\Omega^*}\varphi, \quad (\varphi(\pi|_\Omega f) = \pi|_{\Omega^*}\varphi(f) = \varphi(f)Ad(\pi|_\Omega)).$$

For π satisfying (1-6), we will say that π *is compatible with the inner product.* In fact (1-6) means that $\pi_\Omega{}^* = \pi_\Omega$ due to the relation $\pi|_\Omega{}^* f = \varphi^{-1}(\varphi(f)Ad(\pi|_\Omega)) = \pi|_\Omega f$.

Next assume that P_2 is not necessarily self-adjoint on the pre-Hilbert space $\mathcal{H} = (\Omega^* \times \Omega, < \ , \ >, \varphi)$. For certain P_2, we may be able to construct a transformation η_{sa} of the pre-Hilbert space and its operators, and find a pre-Hilbert space $\mathcal{H}'$ satisfying the following conditions:

(1) There exists an isomorphism $\eta_{\mathrm{sa}} : \Omega^* \times \Omega \to \tilde{\Omega}^* \times \tilde{\Omega}$.

(2) By defining a pairing $< \circ, \times >_{P_2} = < \eta_{\mathrm{sa}}|_{\Omega^*}\circ, \eta_{\mathrm{sa}}|_\Omega \times >$, and $\tilde{\varphi} = \eta_{\mathrm{sa}}|_{\Omega^*}\varphi\eta_{\mathrm{sa}}^{-1}|_{\tilde{\Omega}}$, we obtain $\mathcal{H}' = (\tilde{\Omega}^* \times \tilde{\Omega}, (\ ,\ ), \tilde{\varphi})$.

(3) Every operator P is transformed as $\eta_{\mathrm{sa}}|_{\Omega^*} P \eta_{\mathrm{sa}}^{-1}|_{\tilde{\Omega}}$.

(4) P_2 is self-adjoint.

We call η_{sa} a *self-adjointization*: $\eta_{\mathrm{sa}} : \mathcal{H} \to \mathcal{H}'$.

As mentioned above, $\mathrm{Ker}(P_2) \subset \tilde{\Omega}$ also becomes a pre-Hilbert space, denoted by $\mathcal{H}''$. Letting the projection $\mathcal{H}' \to \mathcal{H}''$ be denoted by π_{P_2}, we have a sequence,

$$(1\text{-}7) \qquad \mathcal{H} \xrightarrow{\eta_{\mathrm{sa}}} \mathcal{H}' \xrightarrow{\pi_{P_2}} \mathcal{H}''.$$

This sequence is the key to submanifold quantum mechanics. Instead of considering $P\psi = E\psi$ in $\mathcal{H}$, we search for a solution of $\eta_{\mathrm{sa}}(P)\psi_1 = E\psi_1$.

Let us explain the main idea. For a smooth k-submanifold S embedded in $\mathbb{E}^n$ $(0 < k < n)$, we can find a natural adjoint operator for a differential operator defined over S by using the induced metric of S from $\mathbb{E}^n$. For the Schrödinger equation in $\mathbb{E}^n$ with the L^2-type Hilbert space $\mathcal{H}$,

$$(1\text{-}8) \qquad -\Delta\psi = E\psi,$$

we regard the Laplace operator Δ as a Casimir operator for the translation group. By considering Δ over a tubular neighborhood of S, Δ includes the normal differential operator $\partial_\perp$. We regard $\partial_\perp$ as the above P_2. As $\partial_\perp$ is not self-adjoint in general, we use the above sequence. In the self-adjointization $\eta_{\rm sa}$, we obtain an extra potential in the differential equation. By considering the kernel of the normal component of the differential $\partial_\perp$ and restricting the domain of $\eta_{\rm sa}(\Delta)$ to S, we define a differential operator

$$\Delta_{S\hookrightarrow\mathbb{E}^n} = \eta_{\rm sa}(\Delta)|_{\mathrm{Ker}\partial_\perp}|_S. \tag{1-9}$$

Then it turns out that

$$\Delta_{S\hookrightarrow\mathbb{E}^n} = \Delta_S + U(\kappa_i), \tag{1-10}$$

where Δ_S is the Beltrami-Laplace operator on S and $U(\kappa_i)$ is an invariant functional of the principal curvature functions κ_i of S in $\mathbb{E}^n$.

Due to the self-adjointness of $\partial_\perp$ in $\mathcal{H}''$, we may naturally consider the Hilbert space $\mathcal{H}''$ for $\Delta_{S\hookrightarrow\mathbb{E}^n}$. The pointwise product is valid in $\mathrm{Ker}(\partial_\perp)|_S$. Thus we can consider the submanifold Schrödinger equation,

$$-\Delta_{S\hookrightarrow\mathbb{E}^n}\psi = E\psi,$$

as a quantum mechanical problem and a representation of the translation group.

For the case of a smooth surface S embedded in $\mathbb{E}^3$, where K and H denote the Gauss and mean curvature functions, we obtain

$$\Delta_{S\hookrightarrow\mathbb{E}^3} = \Delta_S + H^2 - K. \tag{1-11}$$

It is expected that $\Delta_{S\hookrightarrow\mathbb{E}^n}$ and its zero mode exhibit extrinsic properties, e.g., umbilical points, of the submanifold.

Submanifold quantum mechanics was started by Jensen and Koppe about thirty years ago, and rediscovered by da Costa [JK, dC]. In this paper we give an exposition of the Dirac operator version of the above quantum system from our point of view [Ma1-10].

We recall the fact that the solutions $\{\Psi\}$ of the Dirac equation

$$\not{D}_{\mathbb{E}^n}\Psi = 0$$

give a local representation of the spin group. By letting $\cdot$ denote pointwise pairing, $(\varphi_{pt}(\{\Psi\}) \times \{\Psi\}, \cdot, \varphi_{pt})$ for a certain map φ_{pt} becomes a pre-Hilbert space and for an appropriate γ-matrix and solution Ψ, $\varphi_{pt}(\Psi)\gamma\Psi$ exhibits a section of the SO(n) principal bundle $\mathrm{SO}_{\mathbb{E}^n}(T\mathbb{E}^n)$ over $\mathbb{E}^n$.

Thus we apply the submanifold quantum mechanical scheme to the Dirac operator over a k-spin submanifold S immersed in $\mathbb{E}^n$. We obtain a representation of $\mathrm{SO}_{\mathbb{E}^n}(T\mathbb{E}^n)|_S$, which is the generalized Weierstrass representation. Our main theorem is Theorem 3.15.

The organization of this article is as follows. Section 2 gives preliminaries on the geometrical setting [E] and conventions regarding Clifford modules [BGV, Tas]. In §3, we give a construction algorithm for the submanifold Dirac equation and investigate its properties. Section 4 presents an example.

I thank the organizers of the conference, especially Prof. M. A. Guest and Prof. Y. Ohnita for giving me a chance to talk there. It is acknowledged that Prof. M. A. Guest gave me helpful suggestions to write this article. I am grateful to Prof. B. G. Konopelchenko, Prof. U. Pinkall and Prof. F. Pedit for their interest in this work, Prof. T. Kori and Dr. Y. Homma for inviting me to their seminar and for crucial discussions. I thank Prof. S. Saito, Prof. K. Tamano and H. Mitsuhashi for critical discussions and encouragement.

§2. Preliminaries

2.1 Conventions [Mal]. For a fiber bundle A over a differential manifold M and an open set $U \subset M$, let $\Gamma(U, A)$ denote the set of smooth sections of the fiber bundle A over U. For a point p in M, let $\Gamma(p, A)$ denote the stalk at p.

We use the Einstein summation convention and let $\mathbb{C}$ ($\mathbb{R}$) denote the complex (real) field. For brevity, we use the notation $\partial_{u^\mu} = \partial/\partial u^\mu$. For a real number $x \in [n, n+1)$ and an integer n, [x] denotes n. Let $\mathbb{C}_M$ ($\mathbb{R}_M$) denote a complex (real) line bundle over a manifold M.

In order to define the submanifold Dirac operator, let us consider a smooth spin k-submanifold S immersed in Euclidean space $\mathbb{E}^n$. As mentioned in §3, the Dirac operator can be constructed locally. We shall assume that S satisfies several properties:

2.2 Assumptions/Notation.

(1) Let S be diffeomorphic to $\mathbb{R}^k$ with coordinates $(s^1, \ldots, s^k)$.

(2) Let T_S be a tubular neighborhood of S with projection map $\pi_{T_S} : T_S \to S$.

(3) Let T_S have the natural bundle structure $T_S \approx \mathbb{R}^k \times \mathbb{R}^{n-k}$ with coordinates $(s^1, \ldots, s^k, q^{k+1}, \ldots, q^n) = (u^1, \ldots, u^n)$. (Let $\alpha, \beta, \ldots$ run from 1 to k and $\dot{\alpha}, \dot{\beta}, \ldots$ run from $k+1$ to n. Let $\mu, \nu, \ldots$ run from 1

to n.)

(4) For a fixed $q \in \mathbb{R}^{n-k}$, let $S_q = u^{-1}(\mathbb{R}^k \times \{q\})$.

(5) Let $\mathfrak{g}_{T_S}$, $\mathfrak{g}_{S_q}$ and $\mathfrak{g}_S$ denote the Riemannian metrics of T_S, S_q and S induced from that of $\mathbb{E}^n$, respectively.

(6) Let $g_{T_S\ i,j} = \mathfrak{g}_{T_S}(\partial_{x^i}, \partial_{x^j})$, $g_{T_S\ \mu,\nu} = \mathfrak{g}_{T_S}(\partial_{u^\mu}, \partial_{u^\nu})$, $g_{T_S} = \det g_{T_S\ \mu,\nu}$ and similarly for T_S, S and $\mathbb{E}^n$.

(7) Let $(dq^{\dot\alpha})_{\dot\alpha=k+1,\dots,n}$ be an orthonormal basis, $\mathfrak{g}_{T_S}(\partial_{q^{\dot\alpha}}, \partial_{q^{\dot\beta}}) = \delta_{\dot\alpha,\dot\beta}$, $\mathfrak{g}_{T_S}(\partial_{q^{\dot\alpha}}, \partial_{s^\alpha}) = 0$ for $\alpha = 1, \dots, k$ and $\dot\alpha = k+1, \dots, n$. In other words,

$$\mathfrak{g}_{T_S} = \mathfrak{g}_{S_q} \oplus \delta_{\dot\alpha\dot\beta} dq^{\dot\alpha} \otimes dq^{\dot\beta}. \tag{2-1}$$

(8) At a point p in T_S, let an orthonormal frame of the cotangent space T^*T_S be denoted by $d\xi = (d\xi^\mu) = (d\zeta^\alpha, dq^{\dot\alpha})$.

We call a parametrization q satisfying (1)-(8) a *canonical parametrization.*

2.3 Notation. For a point p of S, let the *Weingarten map* be denoted by $-\gamma_{\dot\beta} : T_pS \to T_p\mathbb{E}^n$; for bases $\mathbf{e}_\alpha$ of TS and $\tilde{\mathbf{e}}_{\dot\beta} \in TS^\perp$ ($T_p\mathbb{E}^n = T_pS \oplus T_pS^\perp$),

$$\gamma_{\dot\beta}(\mathbf{e}_\alpha) = \partial_\alpha \tilde{\mathbf{e}}_{\dot\beta} = \gamma^{\dot\alpha}_{\ \dot\beta\alpha} \tilde{\mathbf{e}}_{\dot\alpha} + \gamma^{\beta}_{\ \dot\beta\alpha} \mathbf{e}_\beta. \tag{2-2}$$

In general it is not obvious whether a submanifold in $\mathbb{E}^n$ has a canonical parametrization q. In particular property (7) requires some arguments on the Weingarten map but we can use the following Proposition [Ma10].

2.4 Proposition. For a basis $\mathbf{e}_\alpha$ of TS, there is an orthonormal frame $\mathbf{e}_{\dot\alpha} = \delta_{\dot\alpha\dot\beta} dq^{\dot\beta} \in TS^\perp$ satisfying

$$\partial_\alpha \mathbf{e}_{\dot\beta} = \gamma^{\beta}_{\ \dot\beta\alpha} \mathbf{e}_\beta. \tag{2-3}$$

In terms of the above properties, we have the moving frame and the metric as follows.

2.5 Lemma. *For the moving frame $\mathbf{e}_{\dot\alpha} = \delta_{\dot\alpha\dot\beta} dq^{\dot\beta}$ in Proposition 2.4, the moving frame $\mathbf{E}_\mu = E^i{}_\mu \partial_i$, $(E^i{}_\mu = \partial_\mu x^i)$ in S_q is expressed by*

$$E^i{}_\alpha = e^i{}_\alpha + q^{\dot\alpha}\gamma^\beta{}_{\dot\alpha\alpha} e^i{}_\beta, \quad E^i{}_{\dot\alpha} = e^i{}_{\dot\alpha}. \tag{2-4}$$

2.6 Corollary. *Let $g_{T_S} = \det_{n\times n}(g_{T_S,\mu.\nu})$, $g_{S_q} = \det_{k\times k}(g_{S_q,\alpha.\beta})$ and $\mathfrak{g}_S = \mathfrak{g}_{S_q}|_{q=0}$.*

(1) $g_{T_S} = g_{S_q}$.

(2) $\mathfrak{g}_{S_q} = \mathfrak{g}_S + \mathfrak{g}_S^{(1)} q^{\dot\alpha} + \mathfrak{g}_S^{(2)}(q^{\dot\alpha})^2$, and locally

(2-5)
$$g_{TS_q}(\partial_\alpha, \partial_\beta) = g_{S\alpha\beta} + [\gamma^\gamma{}_{\dot\alpha\alpha} g_{S\gamma\beta} + g_{S\alpha\gamma}\gamma^\gamma{}_{\dot\alpha\beta}]q^{\dot\alpha} + [\gamma^\delta{}_{\dot\alpha\alpha} g_{S\delta\gamma}\gamma^\gamma{}_{\dot\beta\beta}]q^{\dot\alpha}q^{\dot\beta}.$$

(3) When we factorize g_{S_q} as $g_{S_q} = g_S \cdot \rho_{S_q}$, the factor ρ_{S_q} is given by

(2-6)
$$\begin{aligned}\rho_{S_q} =& 1 + 2\mathrm{tr}_{k\times k}(\gamma^\alpha{}_{\dot\alpha\beta})q^{\dot\alpha} \\ &+ \left[2\mathrm{tr}_{k\times k}(\gamma^\alpha{}_{\dot\alpha\beta})\mathrm{tr}_{k\times k}(\gamma^\alpha{}_{\dot\beta\beta}) - \mathrm{tr}_{k\times k}(\gamma^\delta{}_{\dot\alpha\beta}\gamma^\alpha{}_{\dot\beta\delta})\right] q^{\dot\alpha}q^{\dot\beta} + \mathcal{O}(q^{\dot\alpha}q^{\dot\beta}q^{\dot\gamma})\end{aligned}$$

where $\mathcal{O}$ is the Landau symbol.

2.7 Clifford algebras and spinor representations [BGV, Tas].

In this section we recall some properties of the Clifford algebra and related notation.

(1) Let $\mathrm{CLIFF}(\mathbb{R}^n)$ denote the Clifford algebra for the vector space $\mathbb{R}^n$ and let $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n) = \mathrm{CLIFF}(\mathbb{R}^n)\otimes\mathbb{C}$. Let $\mathrm{CLIFF}^{\mathbb{C}^{\mathrm{even}}}(\mathbb{R}^n)$ denote the subalgebra of $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n)$ generated by elements of even degree in $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n)$. Further let $\mathrm{SPIN}(\mathbb{R}^n)$ denote the spin group for $\mathbb{R}^n$.

(2) For the exterior algebra $\wedge\mathbb{R}^n = \oplus_{j=0}^n \wedge^j \mathbb{R}^n$, there is a vector space isomorphism (symbol map) $\mathrm{CLIFF}(\mathbb{R}^n) \to \wedge\mathbb{R}^n$. Let its inverse be denoted by γ (γ-matrix).

(3) For n even, let $\mathrm{Cliff}(\mathbb{R}^n)$ denote a $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n)$-module whose endomorphism ring is isomorphic to $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n)$, which is a $2^{[n/2]}$ dimensional $\mathbb{C}$-vector space. For n odd, let $\mathrm{Cliff}(\mathbb{R}^n)$ denote a $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n)$-module whose endomorphism ring is isomorphic to $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n)$ as a

$2^{[n/2]}$ dimensional $\mathbb{C}$-vector space and whose elements are invariant for the action of $\gamma(e_1)\gamma(e_2)\dots\gamma(e_n)$. Here $e_1, e_2, \dots, e_n$ are an orthonormal basis of $\mathbb{R}^n$.

2.8 Conventions. (1) We recall the fact $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^{n+2}) \approx \mathrm{END}(\mathbb{C}^2)\otimes \mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n)$, where $\mathrm{END}(\mathbb{C}^2)$ is the ring of endomorphisms of $\mathbb{C}^2$; $\mathrm{END}(\mathbb{C}^2)$ can be generated by the Pauli matrices

(2-8)
$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(2) For even n, since $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n) \approx \mathrm{END}(\mathbb{C}^{[n/2]})$, we use the conventions

(2-9)
$$\begin{aligned}
\gamma(e_1) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \\
\gamma(e_2) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2, \\
\gamma(e_3) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3, \\
\gamma(e_4) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_0 \\
\gamma(e_5) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_0 \\
&\cdots \\
\gamma(e_{n-1}) &= \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0, \\
\gamma(e_n) &= \sigma_2 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0.
\end{aligned}$$

(3) For odd n, since $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n) \approx \mathrm{END}(\mathbb{C}^{(n-1)/2}) \oplus \mathrm{END}(\mathbb{C}^{(n-1)/2})$, we use the conventions

(2-10)
$$\begin{aligned}
\gamma(e_1) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \mathfrak{e}, \\
\gamma(e_2) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1, \\
\gamma(e_3) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes 1, \\
&\cdots \\
\gamma(e_{n-2}) &= \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes 1, \\
\gamma(e_{n-1}) &= \sigma_2 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes 1. \\
\gamma(e_n) &= \sigma_3 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes 1,
\end{aligned}$$

where $\mathfrak{e}$ is the generator of $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}) = \mathbb{R}[\mathfrak{e}]/(1-\mathfrak{e}^2)$.

(4) Let $b_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $b_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $\mathrm{Cliff}(\mathbb{R}^n)$ is spanned by

(2-11)
$$\Xi_\epsilon = b_{\epsilon_1} \otimes b_{\epsilon_2} \otimes \cdots \otimes b_{\epsilon_{[n/2]-1}} \otimes b_{\epsilon_{[n/2]}},$$

where $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{[n/2]})$ and $\epsilon_a = \pm$ $(a = 1, \dots, [n/2])$. Similarly, $\mathrm{Cliff}(\mathbb{R}^n)^*$ is spanned by

$$\overline{\Xi}_\epsilon = \overline{b}_{\epsilon_1} \otimes \overline{b}_{\epsilon_2} \otimes \cdots \otimes \overline{b}_{\epsilon_{[n/2]-1}} \otimes \overline{b}_{\epsilon_{[n/2]}}, \tag{2-12}$$

where $\overline{b}_+ = (1,0)$ and $\overline{b}_- = (0,1)$. Renumbering, let $\epsilon^{[c]}$ $(c = 1, \dots, 2^{[n/2]})$ denotes the ϵ's. The isomorphism $\varphi : \mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{Cliff}(\mathbb{R}^n)^*$ is given by

$$\varphi\Big(\sum_{c=1}^{2^{[n/2]}} a_c \Xi_{\epsilon[c]}\Big) = \sum_{c=1}^{2^{[n/2]}} \overline{a}_c \overline{\Xi}_{\epsilon[c]}, \tag{2-13}$$

for $a_c \in \mathbb{C}$ and its complex conjugate $\overline{a}_c$.

(5) Defining

$$b_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, b_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, b_3 = \begin{pmatrix}1\\0\end{pmatrix}$$

and $\overline{b}_1 = \frac{1}{\sqrt{2}}(1,1)$, $\overline{b}_2 = \frac{1}{\sqrt{2}}(1,-i)$, $\overline{b}_3 = (1,0)$, we have

$$\overline{b}_a \sigma_b b_a = \delta_{a,b}, \quad (a, b = 1,2,3), \text{ (not summed over } a). \tag{2-14}$$

(6) For $j = 1, \dots, [n/2]$, define

$$\Psi^{(2j-1)} = b_1 \otimes b_1 \otimes \cdots \otimes b_1 \otimes b_3 \otimes b_1 \otimes \cdots \otimes b_1,$$

$$\Psi^{(2j)} = b_1 \otimes b_1 \otimes \cdots \otimes b_1 \otimes b_2 \otimes b_1 \otimes \cdots \otimes b_1. \tag{2-15}$$

For odd n, we define,

$$\Psi^{(n)} = b_3 \otimes b_1 \otimes \cdots \otimes b_1 \otimes b_1 \otimes \cdots \otimes b_1, \tag{2-16}$$

and assume that $\mathfrak{e}\Psi^{(i)} = \Psi^{(i)}$. We define $\overline{\Psi}^{(k)}$ as in (2-13). Then $\Psi^{(a)}$ is an element of $\mathrm{Cliff}(\mathbb{R}^n)$ $(a = 1, \dots, n)$ satisfying

$$\overline{\Psi}^{(a)}_{\{dx\}} \gamma(e^b) \Psi^{(a)}_{\{dx\}} = \delta^a_b, \quad \text{(not summed over } a). \tag{2-17}$$

As we wish to consider a spin principal subbundle over S induced from a spin principal bundle over $\mathbb{E}^n$, we recall the following facts:

2.9 Lemma. *(1) For $k < n$,* $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^k)$ *is a natural vector subspace of* $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n)$ *(from the generators in 2.8), and* $\mathrm{CLIFF}^{\mathbb{C}^{\mathrm{even}}}(\mathbb{R}^k)$ *is a subring of* $\mathrm{CLIFF}^{\mathbb{C}^{\mathrm{even}}}(\mathbb{R}^n)$. *(2) For $k < n$,* $\mathrm{SPIN}(\mathbb{R}^k)$ *is a natural subgroup of* $\mathrm{SPIN}(\mathbb{R}^n)$.

Proof. It will suffice to comment on the cases $k = 2l$ and $n = 2l+2$. As in 2.8, we have a natural inclusion defined on the basis elements e_i of $\mathbb{R}^k$ and E_i of $\mathbb{R}^n$ by $\tau_{k,n} : \gamma(e_i) \mapsto \gamma(E_i) := \sigma_1 \otimes \gamma(e_i)$, $i = 1, 2, \cdots, k$; then we define $\tau_{k,n}(\gamma(e_i)\gamma(e_j)) := \tau_{k,n}(\gamma(e_i))\tau_{k,n}(\gamma(e_j))$ and so on. On the other hand, for $c^i \in \mathrm{CLIFF}(\mathbb{R}^k)$, we have a ring homomorphism $\iota_{k,n} : \mathrm{CLIFF}(\mathbb{R}^k) \hookrightarrow \mathrm{CLIFF}(\mathbb{R}^n)$ by $C_i = \sigma_0 \otimes c_i$. For $1 \le i, j \le k$, $\gamma(E_i)\gamma(E_j) = \iota_{k,n}(\gamma(e_i)\gamma(e_j))$. In $\mathrm{CLIFF}^{\mathrm{even}}(\mathbb{R}^n)$, the images of τ and ι agree. Thus (1) is proved. Accordingly $\exp(\tau_{k,n}(\gamma(e_j)\gamma(e_i)))$ can be regarded as elements of $\mathrm{SPIN}(\mathbb{R}^n)$ and (2) is proved. ■

§3. Construction of the submanifold Dirac operator

An algorithm to construct the submanifold Dirac operator is presented in the following six steps.

Step 1: Consider the Dirac equation $\not{D}_{\mathbb{E}^n}\Psi_{\mathbb{E}^n} = 0$ in a Euclidean space $\mathbb{E}^n$ and embed the k-spin submanifold S into $\mathbb{E}^n$ $(0 < k < n)$

Here we explain our notation for the Dirac equation $\not{D}_{\mathbb{E}^n}\Psi_{\mathbb{E}^n} = 0$.

3.1 Clifford modules. (1) Let ${\mathrm{CLIFF}^{\mathbb{C}}}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ denote the Clifford bundle over $\mathbb{E}^n$ which has the $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^n)$ structure associated with the cotangent bundle $T^*\mathbb{E}^n$ and the set of differential forms $\Omega(\mathbb{E}^n) = \sum_{a=1}^n \Omega^a(\mathbb{E}^n)$.

(2) Let $\mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ denote the Clifford module over $\mathbb{E}^n$ associated with the cotangent space $T^*\mathbb{E}^n$, which is modelled on $\mathrm{Cliff}(\mathbb{R}^n)$.

(3) Let φ_{pt} denote the natural bijection of ${\mathrm{CLIFF}^{\mathbb{C}}}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$-modules

$$\varphi_{pt} : \Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)) \to \Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)^*), \tag{3-1}$$

for p in $\mathbb{E}^n$.

(4) For an orthonormal frame $\mathbf{e}^i \in T^*\mathbb{E}^n$ at $p \in \mathbb{E}^n$, let $\gamma_{\{\mathbf{e}\}}$ denote the γ-matrix as the map from $\Omega(\mathbb{E}^n)$ to ${\mathrm{CLIFF}^{\mathbb{C}}}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ as in (2-7),

$$\gamma_{\{\mathbf{e}\}} : \mathbf{e}^i \mapsto \gamma_{\{\mathbf{e}\}}(\mathbf{e}^i) \in {\mathrm{CLIFF}^{\mathbb{C}}}_{\mathbb{E}^n}(T^*\mathbb{E}^n)|_p. \tag{3-2}$$

For later convenience, we also employ the following notation for the one-form $du^\mu = E^\mu_i \mathbf{e}^i \in \Gamma(p, \Omega^1(\mathbb{E}^n))$:

$$\gamma_{\{\mathbf{e}\}}(du^\mu) = E^\mu_i \gamma_{\{\mathbf{e}\}}(\mathbf{e}^i). \tag{3-3}$$

For a Cartesian coordinate system $(x^i) = (x^1, \dots x^n)$ in $\mathbb{E}^n$ we fix $\mathbf{e}^i = dx^i$ and in T_S let $\mathbf{e}^\mu = d\xi^\mu$ in 2.2 (8).

(5) Let $\mathrm{SPIN}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ denote a spin principal bundle over $\mathbb{E}^n$. Let the natural bundle map from $\mathrm{SPIN}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ to the $\mathrm{SO}(n)$-principal bundle $\mathrm{SO}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ be denoted by $\tau_{\mathbb{E}^n}$; for $p \in \mathbb{E}^n$, the orthonormal frame $\mathbf{e} \in T^*_p\mathbb{E}^n$ and $\mathrm{e}^\Omega \in \mathrm{SPIN}_p(T^*\mathbb{E}^n)$, the action of $\mathrm{SO}(n)$ is defined by

$$\tau_{\mathbb{E}^n}(\mathrm{e}^\Omega)\mathrm{e}^i = \gamma^{-1}_{\{\mathbf{e}\}}(\mathrm{e}^\Omega \gamma_{\{\mathbf{e}\}}(\mathrm{e}^i)\mathrm{e}^{-\Omega}). \tag{3-4}$$

(6) The Dirac operator $\not{D}_{\mathbb{E}^n}$ is a map

$$\not{D}_{\mathbb{E}^n} : \Gamma(U, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)) \to \Gamma(U, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)). \tag{3-5}$$

3.2 Proposition. *For a point $p \in \mathbb{E}^n$ and a pair $(\overline{\Psi}_{\{dx\}}, \Psi_{\{dx\}})$ in $\Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)^*) \times \Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$, the following holds:*

(1) $\delta_{ij}\overline{\Psi}_{\{dx\}}(x)\gamma_{\{dx\}}(dx^i)\Psi_{\{dx\}}(x)dx^j$ is an element of $\Gamma(p, \Omega^1(T^*\mathbb{E}^n))$.

(2) $\overline{\Psi}_{\{dx\}}(x)\Psi_{\{dx\}}(x)$ is an element of $\Gamma(p, \mathbb{C}_{\mathbb{E}^n})$.

Using the conventions (2-11), we have:

3.3 Proposition. *For a point p in $\mathbb{E}^n$, $\{\Psi^{[a]}_{\{dx\}} = \Xi_{\epsilon^{[a]}}\}_{a=1,\dots,2^{[n/2]}}$ satisfy the Dirac equation*

$$\not{D}_{x,\{dx\}}\Psi_{\mathbb{E}^n}(x) = 0,$$

and are bases of $\Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ as a $\Gamma(p, \mathbb{C}_{\mathbb{E}^n})$-vector space. Thus they are also bases of the fiber $\mathrm{Cliff}_p(T^*\mathbb{E}^n)$ of $\mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ at p.

Using (2-17), we have the following important proposition.

3.4 Proposition. *There exist $\Psi^{(i)}_{\{dx\}}$ $(i = 1, \dots, n)$ and $\overline{\Psi}^{(i)}_{\{dx\}} = \varphi_{pt}(\Psi^{(i)}_{\{dx\}})$ satisfying $\sum_{j,k=1}^n \overline{\Psi}^{(i)}_{\{dx\}}\gamma_{\{dx\}}(dx^j)\Psi^{(i)}_{\{dx\}}dx^k = dx^i$.*

As mentioned in Proposition 3.2 (2), $\mathcal{W}_p = \Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)^*) \times \Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ has a pointwise-pairing $\cdot$ for any point $p \in \mathbb{E}^n$, so

$\mathcal{H}_p = (\mathcal{W}_p, \cdot, \varphi_{pt})$ becomes a pre-Hilbert space, and by Proposition 3.2 (1), $\mathcal{W}_p$ gives the stalk $\Gamma(p, \mathrm{SO}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$.

Step 2: Express the Dirac operator using the canonical parametrization in the tubular neighborhood T_S of S.

Let $\Gamma_{vc}(T_S, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ denote the set of sections of the Clifford module $\mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ whose support is in T_S. We go on to express the stalk at $p \in T_S$ by $\Gamma_{vc}(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$. Let $\Psi_{\{d\xi\}}$ denote a germ of $\Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ for a point $p \in T_S$. Recalling (3-3), $dx^i = E^i{}_\mu du^\mu$ and decomposing $E^i{}_\mu = G_\mu{}^\nu \Lambda^i{}_\nu$ as $dx^i = \Lambda^i{}_\mu d\xi^\mu$, we can find $\mathrm{e}^\Omega \in \Gamma(p, \mathrm{SPIN}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ satisfying the following relations, called a *gauge transformation*:

$$\Psi_{\{d\xi\}}(u) = \mathrm{e}^{-\Omega}\Psi_{\{dx\}}(x), \quad \overline{\Psi}_{\{d\xi\}}(u) = \overline{\Psi}_{\{dx\}}(x)\mathrm{e}^{\Omega},$$

$$\begin{aligned} \mathrm{e}^{-\Omega}\gamma_{\{dx\}}(dx^i)\mathrm{e}^{\Omega} &= \gamma_{\{d\xi\}}(\Lambda^i{}_\mu d\xi^\mu) \\ &= \Lambda^i{}_\rho G_\mu{}^\rho G_\nu{}^\mu \gamma_{\{d\xi\}}(d\xi^\nu) \\ &= E^i{}_\mu \gamma_{\{d\xi\}}(du^\mu). \end{aligned} \tag{3-6}$$

As $\overline{\Psi}_{\{d\xi\}} = \varphi_{pt}(\Psi_{\{d\xi\}})$, φ_{pt} does not depend upon the orthonormal frame. In terms of these expressions, we have (see Proposition 3.2)

$$\begin{aligned} &\delta_{ij}\overline{\Psi}_{\{dx\}}(x)\gamma_{\{dx\}}(dx^i)\Psi_{\{dx\}}(x)dx^j \\ &\qquad = g_{T_S,\mu\nu}\overline{\Psi}_{\{d\xi\}}(u)\gamma_{\{d\xi\}}(du^\mu)\Psi_{\{d\xi\}}(u)du^\nu, \end{aligned}$$

$$\overline{\Psi}_{\{dx\}}(x)\Psi_{\{dx\}}(x) = \overline{\Psi}_{\{d\xi\}}(u)\Psi_{\{d\xi\}}(u). \tag{3-7}$$

Further, the representation of the Dirac operator $\not{D}_{x,\{dx\}}$ is transformed to $\not{D}_{u,\{d\xi\}}$ by means of the gauge transformation:

$$\begin{aligned} \not{D}_{u,\{d\xi\}} &= \mathrm{e}^{-\Omega}\,\not{D}_{x,\{dx\}}\mathrm{e}^{\Omega} = \gamma_{\{d\xi\}}(du^\mu)\mathrm{e}^{-\Omega}\partial_\mu \mathrm{e}^{\Omega} \\ &= \gamma_{\{d\xi\}}(du^\mu)(\partial_\mu + \partial_\mu\Omega). \end{aligned} \tag{3-8}$$

Then we have the following lemma:

3.5 Lemma. *We can regard $\not{D}_{u,\{d\xi\}}$ as a representation of a map $\not{D}_{T_S}$,*

$$\not{D}_{T_S} : \Gamma_{vc}(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)) \to \Gamma_{vc}(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)),$$

for a point p in T_S.

By Lemma 3.5 and the fact that the zero-section is in $\mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$, it is not difficult to prove that the solution space of $\not{D}_{u,\{d\xi\}}\Psi = 0$ in $\Gamma(p, \mathbb{C}^{2^{[n/2]}}_{\mathbb{E}^n})$ belongs to $\Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$. Since $\not{D}_{u,\{d\xi\}}$ is a $2^{[n/2]} \times 2^{[n/2]}$-matrix type first order differential operator of rank $2^{[n/2]}$, the solution space gives an orthonormal frame in $\Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$. In fact, by Proposition 3.2, we can take $\mathrm{e}^{-\Omega}\Psi^{[a]}_{\{dx\}}$.

From Propositions 3.2-3.5 and (3-6)-(3-8), we have a key proposition:

3.6 Proposition. *(1) For a point $p \in T_S$, there exists some $(\Psi^{[a]}_{\{d\xi\}})_{a=1,\dots,2^{[n/2]}}$ in $\Gamma(p, \mathbb{C}^{2^{[n/2]}}_{\mathbb{E}^n})$ forming an orthonormal frame in $\Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ and satisfying the Dirac equation*

$$\not{D}_{u,\{d\xi\}}\Psi_{\{d\xi\}}(u) = 0, \tag{3-9}$$

i.e.,

$$\overline{\Psi}^{[a]}_{\{d\xi\}}\Psi^{[b]}_{\{d\xi\}} = \delta_{a,b}, \tag{3-10}$$

for $\overline{\Psi}^{[a]}_{\{d\xi\}} = \varphi_{pt}(\Psi^{[a]}_{\{d\xi\}})$.

(2) For $\Psi^{[a]}_{\{dx\}} \in \Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ as in (1) and $\Psi^{[a]}_{\{dx\}} \in \Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ as in Proposition 3.3, there exists a $(2^{[n/2]} \times 2^{[n/2]})$-matrix $\mathrm{e}^{\Omega} \in \Gamma(p, \mathrm{SPIN}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ satisfying

$$\Psi^{[a]}_{\{dx\}} = \mathrm{e}^{\Omega}\Psi^{[a]}_{\{d\xi\}}, \quad (a = 1, \dots, 2^{[n/2]}).$$

(3) For $\mathrm{e}^{\Omega} \in \Gamma(p, \mathrm{SPIN}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ in (2), by letting $\Psi^{(i)}_{\{d\xi\}} = \mathrm{e}^{-\Omega}\Psi^{(i)}_{\{dx\}}$ for $(i = 1, \dots, n)$ in Proposition 3.4, we have the relation

$$g_{T_S,\mu\nu}\overline{\Psi}^{(i)}_{\{d\xi\}}(u)\gamma_{\{d\xi\}}(du^\mu)\Psi^{(i)}_{\{d\xi\}}(u)du^\nu = dx^i \quad (i = 1, \dots, n). \tag{3-11}$$

Proof. It suffices to establish the well-definedness of (3-11). We should check the dependence on the orthonormal frame $\Psi^{[a]}_{\{d\xi\}}$. Let us take another one $\Psi'^{[a]}_{\{d\xi\}}$. Defining $\mathrm{e}^{\Omega'}$ by $\Psi^{[a]}_{\{dx\}} = \mathrm{e}^{\Omega'}\Psi'^{[a]}_{\{d\xi\}}$, we have $\Psi^{[a]}_{\{d\xi\}} = \mathrm{e}^{-\Omega}\mathrm{e}^{\Omega'}\Psi'^{[a]}_{\{d\xi\}}$. When one rewrites (3-11) in terms of the frame

$\Psi'^{[a]}_{\{d\xi\}}$, there appears $\mathrm{e}^{-\Omega'}\mathrm{e}^{\Omega}\gamma_{\{d\xi\}}(du^\mu)\mathrm{e}^{-\Omega}\mathrm{e}^{\Omega'}$. However as both are solutions of the Dirac equation (3-9), the gauge transformation for $\mathrm{e}^{-\Omega}\mathrm{e}^{\Omega'}$ must leave the Dirac operator invariant. Hence the du^μ component gives

$$\mathrm{e}^{-\Omega'}\mathrm{e}^{\Omega}\gamma_{\{d\xi\}}(du^\mu)\mathrm{e}^{-\Omega}\mathrm{e}^{\Omega'} = \gamma_{\{d\xi\}}(du^\mu).$$

Thus (3-11) does not depend on the choice of orthonormal frame. ■

In order to investigate the domain of the new Dirac operator, we apply Lemma 2.9 to the fiber bundles. Let $\mathrm{CLIFF}^{\mathbb{C}}{}_S(T^*S)$ (or $\mathrm{Cliff}_S(T^*S)$) be restricted to S as $\mathrm{CLIFF}^{\mathbb{C}}{}_{\mathbb{E}^n}(T^*\mathbb{E}^n)|_S$ (or $\mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)|_S$) as a vector bundle over S. As the fiber of $\mathrm{SPIN}_p(T^*S)$ is a subgroup of $\mathrm{SPIN}_p(T^*\mathbb{E}^n)$, we also have $\mathrm{SPIN}_S(T^*\mathbb{E}^n) = \mathrm{SPIN}_{\mathbb{E}^n}(T^*\mathbb{E}^n)|_S$. The reader should note the difference between $\Gamma(p, \mathrm{Cliff}_S(T^*\mathbb{E}^n))$ and $\Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$; the former is $\{\psi(s)\}$ and the latter is $\{\psi(u)\}$.

3.7 Definition. *(1) The inclusion modelled by Lemma 2.9 is denoted by* $\tau_{S,\mathbb{E}^n} : \mathrm{CLIFF}^{\mathbb{C}}{}_S(T^*S) \to \mathrm{CLIFF}^{\mathbb{C}}{}_S(T^*\mathbb{E}^n)$, $\gamma_{S,\{d\zeta\}}(d\zeta) \mapsto \gamma_{\{d\xi\}}(d\zeta))$. *This induces* $\mathrm{CLIFF}^{\mathbb{C}\,\mathrm{even}}_S(T^*S) \to \mathrm{CLIFF}^{\mathbb{C}\,\mathrm{even}}_S(T^*\mathbb{E}^n)$ *(a ring homomorphism). Here* $\gamma_{S,\{d\zeta\}}$ *denotes the* γ*-matrix over* S *associated with* T^*S *and its orthonormal frame* $\{d\zeta\}$*. (2) The corresponding inclusion of groups is* $\iota_{S,\mathbb{E}^n} : \mathrm{SPIN}_S(T^*S) \to \mathrm{SPIN}_S(T^*\mathbb{E}^n)$.

Step 3: To construct a self-adjointization of the normal differential operator, we define a pairing in T_S

As q is a natural coordinate of T_S, let us introduce the Haar measure with respect to the local affine transformation along the normal direction,

$$\mathrm{e}^{b^{\dot\alpha}\partial_{q^{\dot\alpha}}}\ g_S^{1/2} d^k s d^{n-k} q = g_S^{1/2} d^k s d^{n-k} q.$$

We call this measure the *normal affine invariant measure.* We have a generic normal differential operator $\partial_\perp = b^{\dot\alpha}\partial_{q^{\dot\alpha}}$ for generic real constants $b^{\dot\alpha}$. As mentioned in the Introduction, we apply the submanifold quantum mechanics scheme to this operator.

3.8 Definition. *For a point* $p \in S$ *and* $(\overline{\Psi}_{1,\{d\xi\}}, \Psi_{2,\{d\xi\}}) \in$

$$\Gamma_{vc}(\pi_{T_S}^{-1}(p), \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)^*) \times \Gamma_{vc}(\pi_{T_S}^{-1}(p), \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)),$$

we introduce the L^2-type pairing $<\,|\,>$ as the fiber integral $< \Psi_{1,\{dx\}} | \not{D}_{u,\{d\xi\}} \Psi_{2,\{d\xi\}} >=$

$$\int_{\pi_{T_S}^{-1}(p)} g_{T_S}^{1/2} d^{n-k} q \overline{\Psi}_{1,\{d\xi\}}(u)\, \not{D}_{u,\{d\xi\}} \Psi_{2,\{d\xi\}}(u) \in \Gamma(p, \mathbb{C}_S),$$

and a transformation η_{sa} whose measure is the normal affine invariant measure $< \Psi_{1,\{d\xi\}} | \not{D}_{u,\{d\xi\}} \Psi_{2,\{d\xi\}} >= (\Phi_{1,\{d\xi\}} | \not{\mathbb{D}}_{u,\{d\xi\}} \Phi_{2,\{dx\}}) =$

$$\int_{\pi_{T_S}^{-1}(p)} g_S^{1/2} d^{n-k} q \overline{\Phi}_{1,\{dx\}}(u)\, \not{\mathbb{D}}_{u,\{d\xi\}}(u) \Phi_{2,\{dx\}}(u) \in \Gamma(p, \mathbb{C}_S),$$

where $\overline{\Phi}_{1,\{d\xi\}} = \eta_{\mathrm{sa}}(\overline{\Psi}_{1,\{d\xi\}}) = {\rho_{S_q}}^{1/4} \overline{\Psi}_{1,\{d\xi\}}$, $\Phi_{2,\{d\xi\}} = \eta_{\mathrm{sa}}(\Psi_{2,\{d\xi\}}) = {\rho_{S_q}}^{1/4} \Psi_{2,\{d\xi\}}$,

$$\not{\mathbb{D}}_{u,\{d\xi\}} = \eta_{\mathrm{sa}}(\not{D}_{u,\{d\xi\}}) = {\rho_{S_q}}^{1/4}\, \not{D}_{u,\{d\xi\}} {\rho_{S_q}}^{-1/4}. \tag{3-12}$$

Further let $\tilde{\varphi}_{pt}$ denote $\eta_{\mathrm{sa}} \varphi_{pt} \eta_{\mathrm{sa}}^{-1}$.

We note that η_{sa} can be defined for more general differential operators but for simplicity, we only define it for the Dirac operator. Further as the metric $\mathfrak{g}_{T_S}$ is not singular, η_{sa} gives a diffeomorphism.

3.9 Lemma. *For points $p \in S_q$ and $p' \in S$, the triples*

$$\mathcal{H}'_p = (\eta_{\mathrm{sa}}(\Gamma_{vc}(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)^*) \times \Gamma_{vc}(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)), \cdot, \tilde{\varphi}_{pt}),$$

and

$$\mathcal{H}' = (\eta_{\mathrm{sa}}(\Gamma_{vc}(\pi_{T_S}^{-1} p', \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)^*) \times \Gamma_{vc}(\pi_{T_S}^{-1} p', \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)), (|), \tilde{\varphi}_{pt}),$$

become pre-Hilbert spaces. In $\mathcal{H}'$, The operator $\partial_\perp$ is skew-Hermitian, i.e., $\partial_\perp^* = -\partial_\perp$ for each $b_{\dot{\alpha}}$.

Step 4: Decompose $\not{\mathbb{D}}_{T_S}$ into its normal and tangential parts

Using the orthonormal frame $(d\zeta^1, \dots, d\zeta^k, dq^{k+1}, \dots, dq^n)$, let us decompose $\not{\mathbb{D}}_{T_S}$ into

$$\not{\mathbb{D}}_{u,\{d\xi\}} = \not{\mathbb{D}}^{\parallel}_{u,\{d\xi\}} + \not{\mathbb{D}}^{\perp}_{u,\{d\xi\}}, \tag{3-13}$$

where $\not{\mathbb{D}}^{\perp}_{u,\{d\xi\}} = \gamma_{\{d\xi\}}(dq^{\dot{\alpha}}) \partial_{q^{\dot{\alpha}}}$.

3.10 Lemma. *(1)* $\not{\mathbb{D}}^{\parallel}_{u,\{d\xi\}}$ *does not contain the vertical differential operator* $\partial_{q^{\dot\alpha}}$.

(2) ${\not{\mathbb{D}}^{\perp}_{u,\{d\xi\}}}^{*} = -\not{\mathbb{D}}^{\perp}_{u,\{d\xi\}}$ *in* $\mathcal{H}'$.

(3) For a point $p \in T_S$, $\not{\mathbb{D}}^{\perp}_{u,\{d\xi\}}$ *is a homomorphism of Clifford modules,*

$$\not{\mathbb{D}}^{\perp}_{u,\{d\xi\}} : \eta_{\rm sa}(\Gamma_{vc}(p, \mathrm{Cliff}(T^*T_S))) \to \eta_{\rm sa}(\Gamma_{vc}(p, \mathrm{Cliff}(T^*T_S))).$$

For $p \in T_S$, let us define a projection $\pi : \eta_{\rm sa}((\Gamma_{vc}(p, \mathrm{Cliff}(T^*T_S)^*) \times \Gamma_{vc}(p, \mathrm{Cliff}(T^*T_S)))) \to (\mathrm{Ker}_p(Ad(\partial_\perp)) \times \mathrm{Ker}_p(\partial_\perp))$, where $Ad(P)$ denotes the right-adjoint of P. We note that $\mathrm{Ker}_p(\partial_\perp)(\subset \Gamma_{vc}(p, \mathrm{Cliff}(T^*T_S))$ is given as the intersection of the $\mathrm{Ker}_p(\partial_{\dot\alpha})$ for all $\dot\alpha = k+1, \dots, n$.

Using (2-6) and the fact that $\rho_{S_q} = 1$, we obtain the following relations.

3.11 Lemma. *(1)* $\tilde\varphi_{pt}|_S = \varphi_{pt}|_S$ *and* $\eta_{\rm sa}(\Psi_{\{d\xi\}})|_S = \Psi_{\{d\xi\}}|_S$ *for* $p \in S$ *and* $\Psi_{\{d\xi\}} \in \Gamma_{vc}(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$.

(2) For a point $p \in T_S$, $\tilde\varphi_{pt}|_{\mathrm{Ker}_p(\partial_\perp)}$ *is a bijection:*

$$(\mathrm{Ker}_p(\partial_\perp))^* = \tilde\varphi_{pt}\,(\mathrm{Ker}_p(\partial_\perp)) \approx \mathrm{Ker}_p(Ad(\partial_\perp)).$$

(3) $\pi^* = \pi$, $\pi\tilde\varphi_{pt} = \tilde\varphi_{pt}\pi$ *and* $\pi\varphi_{pt} = \varphi_{pt}\pi$ *on* S.

(4) For a point $p \in T_S$, *the triple* $\mathcal{H}''_p = ((\mathrm{Ker}_p(\partial_\perp))^* \times \mathrm{Ker}_p(\partial_\perp), \cdot, \tilde\varphi_{pt})$ *becomes a pre-Hilbert space. Let* $\mathcal{H}_p(S \hookrightarrow \mathbb{E}^n) = \mathcal{H}''_p$ *for* $p \in S$.

(5) $\mathrm{Ker}_p(\partial_\perp) \subset \mathrm{Ker}_p(\not{\mathbb{D}}^{\perp}_{u,\{d\xi\}})$.

Step 5: Define the submanifold Dirac operator $\not{D}_{S\hookrightarrow\mathbb{E}^n}$ for $S \hookrightarrow \mathbb{E}^n$ by restricting to $\mathrm{Ker}_S(\partial_\perp)$

Let us define $\not{D}_{S\hookrightarrow\mathbb{E}^n}$ by restricting to $\coprod_{p\in S} \mathrm{Ker}_p(\partial_\perp)$, i.e.,

$$\not{D}_{S\hookrightarrow\mathbb{E}^n} = \not{\mathbb{D}}_{T_S}|_{\coprod_{p\in S}\mathrm{Ker}_p(\partial_\perp)}, \tag{3-14}$$

with the local expression $\not{D}_{s,\{d\xi\}} = \not{\mathbb{D}}_{u,\{d\xi\}}|_{q=0,\partial_q=0} = \not{\mathbb{D}}^{\parallel}_{u,\{d\xi\}}|_{q=0}$ whose explicit form is given by the following proposition.

3.12 Proposition. *Abbreviating $\tau_{S,\mathbb{E}^n}(\gamma_{S,d\zeta}(ds^\alpha))$ by its image $\gamma_{\{d\xi\}}(ds^\alpha)$, the explicit form of $\not{D}_{s,\{d\xi\}}$ is*

$$\not{D}_{s,\{d\xi\}} = \tau_{S,\mathbb{E}^n}(\not{\mathbf{D}}_{S,s,\{d\zeta\}}) + \gamma_{\{d\xi\}}(dq^{\dot\alpha})\mathrm{tr}_{k\times k}(\gamma^{\alpha}{}_{\dot\alpha\beta}), \tag{3-15}$$

where $\not{\mathbf{D}}_{S,s,\{d\zeta\}}$ is a proper (or intrinsic) Dirac operator of S,

$$\not{\mathbf{D}}_{S,s,\{d\zeta\}} = \gamma_{\{d\xi\}}(ds^\alpha)(\partial_{s^\alpha} + \partial_{s^\alpha}\Omega|_{q=0}). \tag{3-16}$$

Here we fix the coordinate s and the orthonormal frame $\{d\zeta\}$.

Proof. [BJ, Ma1-10, MT] From (2-6), ${\rho_{S_q}}^{1/4} = 1+\frac{1}{2}\mathrm{tr}_{k\times k}(\gamma^{\alpha}{}_{\dot\alpha\beta})q^{\dot\alpha}+$ $\mathcal{O}(q^{\dot\alpha}q^{\dot\beta})$. Hence,

$${\rho_{S_q}}^{1/4}\partial_{\dot\alpha}{\rho_{S_q}}^{-1/4} = \partial_{\dot\alpha} - \frac{1}{2}\mathrm{tr}_{k\times k}(\gamma^{\alpha}{}_{\dot\alpha\beta}) + \mathcal{O}(q^{\dot\alpha}),$$

$${\rho_{S_q}}^{1/4}\partial_{\alpha}{\rho_{S_q}}^{-1/4} = \partial_{\alpha} + \mathcal{O}(q^{\dot\alpha}).$$

The second term in (3-15) is thus obtained. As $\tau_{S,\mathbb{E}^n}$ induces the group inclusion $\iota_{S,\mathbb{E}^n}$, this completes the proof. ■

3.13 Proposition. *For a point S, the stalks $\Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ and $\Gamma_{vc}(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ are equivalent.*

As we work at a point from now on, we identify $\Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ and $\Gamma_{vc}(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$. We are considering the kernel of $\not{\mathbb{D}}^{\perp}_{u,\{d\xi\}}$ at p as the domain of $\not{D}_{S\hookrightarrow\mathbb{E}^n}$. For $\psi_0(s) \in \Gamma(p, \mathrm{Cliff}_S(T^*\mathbb{E}^n))$, we assume that $\Phi(u) \in \Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ satisfies $\not{\mathbb{D}}^{\perp}_{u,\{d\xi\}}\Phi = 0$ with the boundary condition $\Phi(s,0) = \psi_0(s)$ at S. We regard $\Gamma(p, \mathrm{Cliff}_S(T^*\mathbb{E}^n))$ as a subset of $\mathrm{Ker}_p(\partial_\perp)$. Further we can evaluate $\Phi(u) \in \mathrm{Ker}_p(\partial_\perp)$ around the point p in S as

$$\Phi(u) = \psi_0(s)+\sum\psi_{\dot\alpha}(s)q^{\dot\alpha}+\sum\psi_{\dot\alpha\dot\beta}(s)q^{\dot\alpha}q^{\dot\beta}+\dots, \quad \psi_{\dot\alpha}(s) = 0 \tag{3-17}$$

where $\psi_0(s) \in \Gamma(p, \mathrm{Cliff}_S(T^*\mathbb{E}^n))$, $\psi_{\dot\beta}(s), \psi_{\dot\gamma\dot\delta}(s) \in \Gamma(p, \mathbb{C}^{2^{[n/2]}}_{\mathbb{E}^n})$ and so on. Accordingly we can identify $\mathrm{Ker}_p(\partial_\perp)|_{q=0}$ with $\Gamma(p, \mathrm{Cliff}_S(T^*\mathbb{E}^n))$.

Noting that $\not{D}_{s,\{d\xi\}}$ is expressed in coordinate-free fashion and does not include $\partial_{q^{\dot\alpha}}$, we see that Definition 3.7, Lemmas 2.9, 3.11 and Proposition 3.12 have the following consequence:

3.14 Proposition. *The Dirac operator* $\not{D}_{S\hookrightarrow\mathbb{E}^n}$ *is an endomorphism of* $\mathrm{Cliff}_S(T^*\mathbb{E}^n)$ *at a point* p *in* S, *more precisely*

$$\not{D}_{S\hookrightarrow\mathbb{E}^n} : \Gamma(p, \mathrm{Cliff}_S(T^*\mathbb{E}^n)) \to \Gamma(p, \mathrm{Cliff}_S(T^*\mathbb{E}^n)). \tag{3-18}$$

Step 6: Consider the solution of the submanifold Dirac equation $\not{D}_{S\hookrightarrow\mathbb{E}^n}\psi = 0$

Let us consider the solution of $\not{D}_{S\hookrightarrow\mathbb{E}^n}\psi = 0$ at a point $p \in S$. For non-vanishing $\psi \in \Gamma(p, \mathbb{C}_{\mathbb{E}^n}^{2^{[n/2]}}))$ such that

$$\not{D}_{s,\{d\xi\}}\psi(s) = 0, \tag{3-19}$$

$\psi(s)$ can be regarded as an element of $\Gamma(S, \mathrm{Cliff}_S(T^*\mathbb{E}^n))$. By solving a boundary problem for $\Phi(u) \in \Gamma(S, \mathbb{C}_{\mathbb{E}^n}^{2^{[n/2]}})$,

$$\not{\mathbb{D}}^{\perp}_{u,\{d\xi\}}\Phi(u) = 0, \tag{3-20}$$

with boundary condition

$$\Phi|_S = \psi, \text{ on } S, \tag{3-21}$$

we have $\Phi(u)$ satisfying $\not{\mathbb{D}}_{u,\{d\xi\}}\Phi(u) = 0$ and $\not{\mathbf{D}}_{u,\{d\xi\}}\eta_{\mathrm{sa}}^{-1}(\Phi(u)) = 0$ at $p \in S$.

Noting that $\not{\mathbb{D}}_{u,\{d\xi\}}|_{q=0} = \not{D}_{s,\{d\xi\}} + \not{\mathbb{D}}^{\perp}_{u,\{d\xi\}}$, and $\not{D}_{s,\{d\xi\}}$ does not include the parameter q, we can apply the separation of variables method to this system. Thus it can be expected that there exists an orthonormal frame $(\eta_{\mathrm{sa}}^{-1}\Phi^{[a]}_{\{d\xi\}})_{a=1,\dots,2^{[n/2]}}$ in $\Gamma(p, \mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ such that each $\Phi^{[a]}_{\{d\xi\}}$ belongs to $\mathrm{Ker}_p(\partial_\perp)$ and satisfies the Dirac equation (3-20); each $(\eta_{\mathrm{sa}}^{-1}\Phi^{[a]}_{\{d\xi\}})$ is a solution of the Dirac equation (3-9). Noting Lemma 3.11 (1), we regard $(\Phi^{[a]}_{\{d\xi\}}|_{q=0})_{a=1,\dots,2^{[n/2]}}$ as an orthonormal frame of $\Gamma(p, \mathrm{Cliff}_S(T^*\mathbb{E}^n))$ and solutions of the submanifold Dirac equation (3-19).

Conversely, as the Dirac operator $\not{D}_{s,\{d\xi\}}$ is also a $2^{[n/2]} \times 2^{[n/2]}$-matrix type first order differential operator, let us assume that we can find an orthonormal frame of $\Gamma(p, \mathrm{Cliff}_S(\mathbb{E}^n))$ belonging to the solution space of the submanifold Dirac equation (3-19). Let it be denoted by $\psi^{[a]}_{\{d\xi\}}$ $(a = 1, \dots, 2^{[n/2]})$, i.e., $\varphi_{pt}(\psi^{[a]}_{\{d\xi\}})\psi^{[b]}_{\{d\xi\}} = \delta_{a,b}$. For each $\psi^{[a]}_{\{d\xi\}}(s)$, we find an element $\tilde{\Psi}^{[a]}_{\{d\xi\}}(u)$ in the solution space of $\not{\mathbb{D}}_{u,\{d\xi\}}\eta_{\mathrm{sa}}(\Psi(u)) =$

0 or $\not{\mathbf{D}}_{u,\{d\xi\}}\Psi(u) = 0$ with the boundary condition $\tilde{\Psi}^{[a]}_{\{d\xi\}}(u)|_{q=0} = \psi^{[a]}_{\{d\xi\}}(s)$ at p.

By Proposition 3.6 (2), we have an element of $\mathrm{e}^{\Omega} \in \Gamma(p, \mathrm{SPIN}_{\mathbb{E}^n}(\mathbb{E}^n))$ such that

$$\Psi^{[a]}_{\{dx\}} = \mathrm{e}^{\Omega}\tilde{\Psi}^{[a]}_{\{d\xi\}}, \quad (a = 1, \dots, 2^{[n/2]}).$$

Using this element, we define $\Psi^{(i)}_{\{d\xi\}} = \mathrm{e}^{-\Omega}\Psi^{(i)}_{\{dx\}}$, $(i = 1, \dots, n)$. Then Proposition 3.6 (3) gives the relation,

$$g_{T_S,\mu\nu}\overline{\Psi}^{(i)}_{\{d\xi\}}\gamma_{\{d\xi\}}(du^{\mu})\Psi^{(i)}_{\{d\xi\}}du^{\nu} = dx^i, \quad (i = 1, \dots, n). \tag{3-22}$$

Therefore by defining $\psi^{(i)}_{\{d\xi\}}(s) = \Psi^{(i)}_{\{d\xi\}}(u)|_{q=0}$ and extracting the ds^{α} component, we have
(3-23)

$$g_{S,\alpha\beta}\overline{\psi}^{(i)}_{\{d\xi\}}\gamma_{\{d\xi\}}(ds^{\alpha})\mathrm{e}^{\Omega_i}\psi^{(i)}_{\{d\xi\}} = \frac{\partial x^i}{\partial s^{\alpha}}, \quad (i = 1, \dots, n, \alpha = 1, \dots, k).$$

This is a representation of the tangent space of S, which we regard as the generalized Weierstrass representation. We note that $\psi^{(i)}_{\{d\xi\}}$ is also a solution of the submanifold Dirac equation (3-19).

As mentioned above, our theory is locally constructed but we can extend it to the case of a general smooth spin k-submanifold immersed in $\mathbb{E}^n$. Since $\mathbb{E}^n$ is spin, we can define $\mathrm{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$, $\mathrm{SPIN}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ and their global sections. By restricting it to S, we have $\mathrm{Cliff}_S(T^*\mathbb{E}^n)$ and a $\mathrm{SPIN}_S(T^*\mathbb{E}^n)$ structure over S. On the other hand, S has its own $\mathrm{Cliff}_S(T^*S)$ and $\mathrm{SPIN}_S(T^*S)$ structure. By Proposition 3.12, S has the submanifold Dirac operator $\not{D}_{S\hookrightarrow\mathbb{E}^n}$.

3.15 Theorem. *A smooth spin k-submanifold S immersed in $\mathbb{E}^n$ has a Clifford module $\mathrm{Cliff}_S(T^*\mathbb{E}^n)$, constructed locally in terms of the solution space of the submanifold Dirac equation $\not{D}_{S\hookrightarrow\mathbb{E}^n}\psi = 0$ at each chart. The solutions give the data at each point as follows:*

Let $p \in S$ be expressed in terms of an affine coordinate (x^i). For $\{\psi\}$ in $\Gamma(p, \mathbb{C}_S^{2^{[n/2]}})$ satisfying $\not{D}_{s,\{d\xi\}}\psi = 0$, there exists an orthonormal frame $\{\psi^{[a]}_{\{d\xi\}}\}_{a=1,\dots,2^{[n/2]}}$ $(\subset \{\psi\})$ of $\Gamma(p, \mathrm{Cliff}_S(T^*\mathbb{E}^n))$ as a $\Gamma(p, \mathbb{C}_{\mathbb{E}^n})$-vector space. For these bases, there exists some $\mathrm{e}^{\Omega} \in \Gamma(p, \mathrm{Spin}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ which can be transformed to $\{\Psi^{[a]}_{\{dx\}}|_S\}_{a=1,\dots,2^{[n/2]}}$ by $\psi^{[a]}_{\{d\xi\}} = \mathrm{e}^{-\Omega}\Psi^{[a]}_{\{dx\}}|_S$,

$(a = 1, \dots, 2^{[n/2]})$. *We define* $\psi^{(i)} = \mathrm{e}^{-\Omega}\Psi^{(i)}_{\{dx\}}|_S$ *and* $\overline{\psi}^{(i)} = \overline{\Psi}^{(i)}_{\{dx\}}\mathrm{e}^{\Omega}|_S$ $(i = 1, \dots, n)$. *Then the following relation holds:*

$$g_{S,\alpha,\beta}\overline{\psi}^{(i)}[\tau_{S,\mathbb{E}^n}(\gamma_{S,\{d\xi\}}(ds^\beta))]\psi^{(i)} = \partial_{s^\alpha}x^i \tag{3-24}$$

$(i = 1, \dots, n,\ s = 1, \dots, k)$.

This is the generalized Weierstrass representation for a spin submanifold immersed in $\mathbb{E}^n$.

Even though we fix the coordinate system, for a point p in S, $\left(\partial x^i/\partial s^\alpha\right)_{\alpha=1,\dots,k, i=1,\dots,n}$ gives the data of an embedding $T^*_pS \approx \mathbb{R}^k$ in $T^*_p\mathbb{E}^n \approx \mathbb{R}^n$ up to action of $\mathrm{GL}(\mathbb{R}^k)$ and $\mathrm{GL}(\mathbb{R}^{n-k})$. In other words, the submanifold Dirac operator $\not{D}_{S\hookrightarrow\mathbb{E}^n}$ exhibits the data of a Grassmannian bundle over S.

3.16 Remark. The generalized Weierstrass representation for a conformal surface in $\mathbb{E}^3$ was discovered by Kenmotsu [Ke] as a generalization of the Weierstrass representation for a minimal surface in $\mathbb{E}^3$. A Dirac type representation was found by Konopelchenko in 1995 [Ko1] and the author showed that the submanifold Dirac equation can be identified with this representation [Ma8]. He computed the submanifold Dirac operators for the generalized Weierstrass representation for a conformal surface in $\mathbb{E}^4$; this was also discovered by Konopelchenko [KL, Ko1-2] and Pinkall and Pedit [PP]. Recently there has been much work on the relations between submanifolds and Clifford bundles [Bo, Ko1-2, KL, KT, Fr, Tai1-2, Tr]. However, as far as the author is aware, the transformation η_{sa} has not appeared before.

Over the past decade, the author has developed the relation between Dirac operators and submanifolds, especially for curves immersed in $\mathbb{E}^n$, where the submanifold Dirac equation is identified with the Frenet-Serret equation [Ma1-7]. After the present conference, he was lead to the generalized Weierstrass representation of Theorem 3.15.

Although not mentioned in this article so far, physical considerations suggest that the Dirac operator has the following properties:

(1) The index of the Dirac operator is related to the topological index for curves [Ma2, 5].

(2) The operator determinants are associated with energy functionals, such as the Euler-Bernoulli functional and the Willmore functional [Ma6, 10].

(3) Deformations preserving all eigenvalues of the submanifold Dirac operators give rise to soliton equations, such as the MKdV equation [Ma2, 6, 7, MT], complex MKdV equation, nonlinear Schrödinger equation [Ma1, 3], and modified Novikov-Veselov equations [Ko1, Ko2, KL, KT, Tai1].

As Dirac operators are generally related to characteristic classes of fiber bundles, it is expected that the submanifold Dirac operator might be related to the Thom class and/or a generalization of the Riemann-Roch Theorem [PP].

§4. Dirac operator on a conformal surface in $\mathbb{E}^4$

As an example, we will consider the case of a conformal surface immersed in $\mathbb{E}^4$. In [Ma10], we gave an explicit local form (4-1) of the submanifold Dirac operator in this case and conjectured that the submanifold Dirac operator represents the surface. The conjecture was proved by Konopelchenko [KL, Ko2] and Pedit and Pinkall [PP], but we will sketch another proof by means of the submanifold Dirac operator method.

First we will give the properties of the Dirac operators for a conformal surface S in $\mathbb{E}^n$ [P]. We take the metric $g_{S\alpha\beta} = \rho\delta_{\alpha\beta}$, and the orthonormal frame $\{d\zeta\}$ given by $d\zeta^\alpha = \rho^{-1/2}ds^\alpha$. Take a complex parametrization of S, $dz = ds^1 \pm i ds^2$. We introduce another transformation η_{conf} as follows. For a point $p \in S$ and $\psi_{\{d\xi\}} \in \Gamma(p, \mathrm{Cliff}_S(T^*\mathbb{E}^n))$, let $\varphi_{\{d\xi\}} = \eta_{\mathrm{conf}}(\psi_{\{d\xi\}}) = \rho^{1/2}\psi_{\{d\xi\}}$, $\overline{\varphi}_{\{d\xi\}} = \eta_{\mathrm{conf}}(\overline{\psi}_{\{d\xi\}}) = \varphi_{pt}(\psi_{\{d\xi\}})$. Then we have the following properties.

4.1 Lemma. *(1) The Dirac operator of S is given by*

$$\not{D}_{s,\{d\zeta\}} = \rho^{-1}\sigma^\alpha\partial_\alpha\rho^{1/2},$$

by letting $\gamma_{S,\{d\zeta\}}(d\zeta^a) = \sigma^a$.

(2) For $(\overline{\varphi}_{\{d\xi\}}, \varphi_{\{d\xi\}})$ in $\eta_{\mathrm{conf}}\mathcal{H}_p(S \hookrightarrow \mathbb{E}^n)$, $\varphi_{\{d\xi\}}\tau_{S,\mathbb{E}^n}[\gamma_{\{d\xi\}}(d\xi^\alpha)]\varphi_{\{d\xi\}}ds^\alpha$ is independent of choice of local parametrization of S.

Proof. (1) is obvious [P, Ma8-9]. Setting $\gamma_{\{d\xi\}}(ds^\alpha) = \rho^{-1/2}\gamma_{\{d\xi\}}(d\xi^\alpha)$, the relation in Proposition 3.2 (1) becomes

$$g_{S\alpha\beta}\overline{\psi}_{\{d\xi\}}\tau_{S,\mathbb{E}^n}[\gamma_{\{d\xi\}}(ds^\alpha)]\psi_{\{d\xi\}}ds^\alpha = \overline{\varphi}_{\{d\xi\}}\tau_{S,\mathbb{E}^n}[\gamma_{\{d\xi\}}(d\xi^\alpha)]\varphi_{\{d\xi\}}ds^\alpha,$$

and thus (2) is proved ■

4.2 Proposition. *(1) For the case of a conformal surface in $\mathbb{E}^4$, the submanifold Dirac operator is given by*

$$\not{D}_{s,\{d\xi\}} = 2\begin{pmatrix} & & \overline{p}_c & \partial \\ & & \overline{\partial} & -p_c \\ p_c & \partial & & \\ \overline{\partial} & -\overline{p}_c & & \end{pmatrix}, \tag{4-1}$$

where $\partial = (\partial_{s^1} - i\partial_{s^2})/2$, $\overline{\partial} = (\partial_{s^1} + i\partial_{s^2})/2$ and p_c is

$$p_c = -\frac{1}{2}\rho^{1/2}\mathrm{tr}_{2\times 2}(\gamma^\alpha_{3\beta} + i\gamma^\alpha_{4\beta}).$$

(2) Let $\mathbb{E}^4 \approx \mathbb{C}\times\mathbb{C} \in (Z^1, Z^2) = (x^1 + ix^2, x^3 + ix^4)$. For the affine coordinate (Z^1, Z^2) of the surface, the relations

$$dZ^1 = fmdz - gnd\overline{z}, \quad dZ^2 = f\overline{n}dz + g\overline{m}d\overline{z}, \quad d\overline{Z}^1 = \overline{dZ^1}, \quad d\overline{Z}^2 = \overline{dZ^2},$$

hold if $\varphi_1 = \begin{pmatrix} f \\ g \\ 0 \\ 0 \end{pmatrix}$ $\varphi_2 = \begin{pmatrix} 0 \\ 0 \\ m \\ n \end{pmatrix}$ *are solutions of*

$$\not{D}_{s,\{d\xi\}}\varphi_a = 0,$$

and $(|f|^2 + |g|^2)(|m|^2 + |n|^2) = \rho^{1/2}$.

Direct computation leads to the next lemma.

4.3 Lemma. *For the four dimensional Euclidean space $\mathbb{E}^4$, we can fix the representation of the γ-matrices of $\{dx^i\}$ system as $\gamma_{\{dx\}}(dx^i) = \sigma^1 \otimes \sigma^i$ for $i = 1, 2, 3$ and $\gamma_{\{dx\}}(dx^4) = \sigma^2 \otimes 1$. By letting*

$$\Psi_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \Psi_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \Psi_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

$\overline{\Psi}_1 = (0, 1, 0, 1)$, $\overline{\Psi}_2 = (1, 0, 1, 0)$, $\overline{\Psi}_3 = (0, -1, 1, 0)$, $\overline{\Psi}_4 = (1, 0, 0, -1)$, we have the following relations:

$$\sum_i \overline{\Psi}_1\gamma_{\{dx\}}(dx^i)\Psi_1 dx^i = 2dZ_1, \quad \sum_i \overline{\Psi}_2\gamma_{\{dx\}}(dx^i)\Psi_2 dx^i = 2d\overline{Z}_1,$$

$$\sum_i \overline{\Psi}_3\gamma_{\{dx\}}(dx^i)\Psi_3 dx^i = 2dZ_2, \quad \sum_i \overline{\Psi}_4\gamma_{\{dx\}}(dx^i)\Psi_4 dx^i = 2d\overline{Z}_2.$$

Proof of Proposition 4.2. (1) follows from Theorem 3.15. We consider (2). For the φ_a's, their independent partner solutions are given by $\varphi_3 = \begin{pmatrix} -\overline{g} \\ \overline{f} \\ 0 \\ 0 \end{pmatrix}$ $\varphi_4 = \begin{pmatrix} 0 \\ 0 \\ -\overline{n} \\ \overline{m} \end{pmatrix}$. Since we have fixed $\gamma_{S,\{d\xi\}}(d\zeta^\beta) = \sigma^\beta$, the convention in 2.8 gives $\tau_{S,\mathbb{E}^2}(\gamma_{S,\{d\xi\}}(d\zeta^\beta)) = \sigma^1 \otimes \sigma^\beta$. We define $\tilde{\varphi}_1 = \varphi_1 + \varphi_2$, $\tilde{\varphi}_2 = \varphi_3 + \varphi_4$, $\tilde{\varphi}_3 = \varphi_1 + \varphi_4$, and $\tilde{\varphi}_4 = \varphi_3 + \varphi_2$. Let us assume that $\tilde{\varphi}_a = \rho^{1/2}\mathrm{e}^{\Omega}\Psi_a|_{q=0}$ $(a = 1, 2, 3, 4)$ as in Lemma 4.3. Then we find the spin matrix as

$$\rho^{1/2}\mathrm{e}^{\Omega} = \begin{pmatrix} f & -\overline{g} & 0 & 0 \\ g & \overline{f} & 0 & 0 \\ 0 & 0 & m & -\overline{n} \\ 0 & 0 & n & \overline{m} \end{pmatrix}.$$

We have these dual bases, $\overline{\tilde{\varphi}}_a = \overline{\Psi}_a \mathrm{e}^{-\Omega}|_{q^{\beta}=0}$, and obtain the relation,

$$2dZ_1 = \overline{\tilde{\varphi}}_1\sigma_1 \otimes \sigma^\alpha \tilde{\varphi}_1 ds^\alpha, \quad 2d\overline{Z}_1 = \overline{\tilde{\varphi}}_2\sigma_1 \otimes \sigma^\alpha \tilde{\varphi}_2 ds^\alpha,$$

$$2dZ_2 = \overline{\tilde{\varphi}}_3\sigma_1 \otimes \sigma^\alpha \tilde{\varphi}_3 ds^\alpha, \quad 2d\overline{Z}_2 = \overline{\tilde{\varphi}}_4\sigma_1 \otimes \sigma^\alpha \tilde{\varphi}_4 ds^\alpha.$$

Explicit computation gives (2). ■

Update added, April 2008: After writing this article, we realised that the operator η_{sa} is closely related to the half-density form in Theorem 18.1.34 in [H] and the quasi-regular representation or L^2 induced representation as an infinite-dimensional representation of a non-compact subgroup [Kob]. These are alluded to in [Ma11, Ma12].

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