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Integral representations of q-analogues of the Barnes multiple zeta functions

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Abstract.

Integral representations of q-analogues of the Barnes multiple zeta functions are studied. The integral representation provides a meromorphic continuation of the q-analogue to the whole plane and describes its poles and special values at non-positive integers. Moreover, for any weight, employing the integral representation, we show that the q-analogue converges to the Barnes multiple zeta function when $q \uparrow 1$ for all complex numbers.

§1. Introduction.

In 1904, E. Barnes introduced his multiple zeta functions with a weight $\boldsymbol{\omega} := (\omega_1, \ldots, \omega_r) \in \mathbb{C}^r$ by the following multiple series ([1]):

$$\zeta_r(s, z, \boldsymbol{\omega}) := \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geq 0}^r} (\boldsymbol{n} \cdot \boldsymbol{\omega} + z)^{-s} \qquad (\mathrm{Re}(s) > r),$$

where $\boldsymbol{n} \cdot \boldsymbol{\omega} = \sum_{j=1}^{r} n_j \omega_j$ for $\boldsymbol{n} = (n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r$. It is known that the function $\zeta_r(s, z, \boldsymbol{\omega})$ can be meromorphically continued to the whole plane $\mathbb{C}$ via the contour integral representation

$$(1) \quad \zeta_r(s, z, \boldsymbol{\omega}) = -\frac{\Gamma(1-s)}{2\pi\sqrt{-1}} I_r(s, z, \boldsymbol{\omega}; a) + \frac{1}{\Gamma(s)} \int_a^{\infty} t^{s-1} G_r(t, z, \boldsymbol{\omega}) \frac{dt}{t},$$

where $I_r(s, z, \boldsymbol{\omega}; a)$ is an entire function defined by

$$(2) \qquad I_r(s, z, \boldsymbol{\omega}; a) := \int_{C(\varepsilon, a)} (-t)^{s-1} G_r(t, z, \boldsymbol{\omega}) \frac{dt}{t}.$$

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Here $C(\varepsilon, a)$ is a contour for $0 < a \le \infty$ and $0 < \varepsilon < \min\{a, b(\boldsymbol{\omega})\}$ with $b(\boldsymbol{\omega}) := \min_{1\le j\le r}\{|2\pi/\omega_j|\}$ along the real axis from a to ε, counterclockwise around the circle of radius ε with the center at the origin, and then along the real axis from ε to a (see [9]), and

$$(3)\quad G_r(t,z,\boldsymbol{\omega}) := \frac{te^{(\omega_1+\cdots+\omega_r-z)t}}{\prod_{j=1}^{r}(e^{\omega_j t}-1)} = \sum_{k=1}^{r-1}(-1)^k{}_rA_{-k}(z,\boldsymbol{\omega})t^{-k} + \sum_{n=0}^{\infty}(-1)^n{}_rB_n(z,\boldsymbol{\omega})\frac{t^n}{n!}.$$

Note that the series expression (3) is valid for $|t| < b(\boldsymbol{\omega})$. These coefficients ${}_rA_{-k}(z,\boldsymbol{\omega})$ and ${}_rB_n(z,\boldsymbol{\omega})$ are called the Barnes multiple Bernoulli polynomials with the weight $\boldsymbol{\omega}$ ([1], see also [5]). We also put ${}_rA_0(z,\boldsymbol{\omega}) := {}_rB_0(z,\boldsymbol{\omega})$. From the expression (1), one can see that $\zeta_r(s,z,\boldsymbol{\omega})$ has simple poles at $s = 1, 2, \ldots, r$ with residues

$$(4)\qquad \operatorname*{Res}_{s=n}\zeta_r(s,z,\boldsymbol{\omega}) = \frac{(-1)^{n-1}}{(n-1)!}{}_rA_{-(n-1)}(z,\boldsymbol{\omega}) \qquad (1 \le n \le r)$$

and

$$(5)\qquad \zeta_r(1-m,z,\boldsymbol{\omega}) = -\frac{{}_rB_m(z,\boldsymbol{\omega})}{m} \qquad (m\in\mathbb{N}).$$

The main purpose of this paper is, as a generalization of the previous work in [9], to obtain an integral representation of the q-analogue $\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})$ of the Barnes multiple zeta function $\zeta_r(s,z,\boldsymbol{\omega})$ defined by the following Dirichlet type q-series ([10]):

$$(6)\qquad \zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega}) := q^{z(s-\nu-r+1)}\sum_{\boldsymbol{n}\in\mathbb{Z}_{\ge 0}^r}\frac{\prod_{j=1}^{r}q^{n_j\omega_j(s-\nu-j+1)}}{[\boldsymbol{n}\cdot\boldsymbol{\omega}+z]_q^s}.$$

The series converges absolutely for $\mathrm{Re}(s) > \nu+r-1$. Here $0<q<1$ and $[x]_q := (1-q^x)/(1-q)$ for $x \in \mathbb{C}$. We always denote by ν a positive integer and assume $\omega_j > 0$ (to ensure that $\delta_j := 2\pi\sqrt{-1}/(\omega_j\log q) \in \sqrt{-1}\mathbb{R}$) for $1\le j\le r$. Note that the factor $q^{z(s-\nu-r+1)}$ is normalization so that $\zeta_{q,1}^{(\nu)}(s,z,1)$ coincides with the q-analogue of the Hurwitz zeta function studied in [3, 4, 9]. In [10], we show a meromorphic continuation of $\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})$ to the whole plane $\mathbb{C}$ by the binomial theorem and calculate the special values at non-positive integers (see Remark 4.5). Moreover, for the special weight $\boldsymbol{\omega} = \mathbf{1}_r := (1,\ldots,1)$, using the Euler-Maclaurin summation formula, we prove that $\lim_{q\uparrow 1}\zeta_{q,r}^{(\nu)}(s,z,\mathbf{1}_r) = \zeta_r(s,z,\mathbf{1}_r)$ for any $s \in \mathbb{C}$ except for the points $s = 1, 2, \ldots, \nu + r - 1$. Note that the points $s = 1, 2, \ldots, \nu + r - 1$ are the poles of $\zeta_{q,r}^{(\nu)}(s, z, \mathbf{1}_r)$ on the real axis. For a general weight $\boldsymbol{\omega}$, however, it is hard to see the classical limit $q \uparrow 1$ of $\zeta_{q,r}^{(\nu)}(s, z, \boldsymbol{\omega})$ since we can not apply the Euler-Maclaurin summation formula. The integral representation also gives a meromorphic continuation of $\zeta_{q,r}^{(\nu)}(s, z, \boldsymbol{\omega})$ to the entire plane $\mathbb{C}$ and allows us to describe the poles and special values at non-negative integers as (4) and (5). Furthermore, we can obtain the following theorem by employing the integral representation of $\zeta_{q,r}^{(\nu)}(s, z, \boldsymbol{\omega})$. Notice that this theorem gives a part of the answer of Conjecture 4.2 in [10].

Theorem 1.1. *For $s \in \mathbb{C}$, $s \neq 1, 2, \ldots, \nu + r - 1$, we have*

$$\lim_{q \uparrow 1} \zeta_{q,r}^{(\nu)}(s, z, \boldsymbol{\omega}) = \zeta_r(s, z, \boldsymbol{\omega}).$$

We remark here that this kind of limit theorems are obtained for the other types of q-zeta functions (cf. [6, 7, 8]), which are not of the form of the Dirichlet type q-series (actually, they need some extra term). For Dirichlet type q-analogues of the multiple zeta values, see [2, 11].

The paper is organized as follows. In Section 2, we define functions $F_{q,r,j}^{(\nu)}(t, z, \boldsymbol{\omega})$ for $0 \leq j \leq r + 1$ and study their analytic properties. In particular, for $1 \leq j \leq r$, we give another expression of $F_{q,r,j}^{(\nu)}(t, z, \boldsymbol{\omega})$ by using the Poisson summation formula (Proposition 2.2). In Section 3, we introduce q-analogues ${}_rA_{-k}^{(\nu)}(z, \boldsymbol{\omega}; q)$ of ${}_rA_{-k}(z, \boldsymbol{\omega})$ and ${}_rB_n^{(\nu)}(z, \boldsymbol{\omega}; q)$ of ${}_rB_n(z, \boldsymbol{\omega})$ respectively by the generating function $G_{q,r}^{(\nu)}(t, z, \boldsymbol{\omega})$, which is defined via the functions $F_{q,r,j}^{(\nu)}(t, z, \boldsymbol{\omega})$. In fact, using a certain relation among $F_{q,r,j}^{(\nu)}(t, z, \boldsymbol{\omega})$'s (Lemma 2.4), we show that $G_{q,r}^{(\nu)}(t, z, \boldsymbol{\omega})$ essentially gives a q-analogue of $G_r(t, z, \boldsymbol{\omega})$ (Theorem 3.1). In Section 4, we first express $\zeta_{q,r}^{(\nu)}(s, z, \boldsymbol{\omega})$ as the Mellin transform of $F_{q,r,+}^{(\nu)}(t, z, \boldsymbol{\omega}) := F_{q,r,r+1}^{(\nu)}(t, z, \boldsymbol{\omega})$ (Proposition 4.1), and then establish a contour integral representation of $\zeta_{q,r}^{(\nu)}(s, z, \boldsymbol{\omega})$ (Theorem 4.3). As an application of this integral representation, we give the proof of Theorem 1.1.

Throughout the present paper, we denote by $\mathbb{Z}_P$ the set of all integers satisfying the condition P.

§2. Functions $F_{q,r,j}^{(\nu)}(t, z, \boldsymbol{\omega})$.

Let $0 \leq j \leq r + 1$. We study functions $F_{q,r,j}^{(\nu)}(t, z, \boldsymbol{\omega})$ defined by

$$(7)\quad F^{(\nu)}_{q,r,j}(t,z,\boldsymbol{\omega}) := (tq^{-z})^{\nu+r-1} \times \sum_{\boldsymbol{n}\in\mathbb{D}_j} \Big(\prod_{h=1}^{r} q^{-n_h\omega_h(\nu+h-1)}\Big) \exp\big(-tq^{-(\boldsymbol{n}\cdot\boldsymbol{\omega}+z)}[\boldsymbol{n}\cdot\boldsymbol{\omega}+z]_q\big),$$

where

$$\mathbb{D}_j := \left\{ \boldsymbol{n} = (n_1,\ldots,n_r) \in \mathbb{Z}^r \;\middle|\; \begin{array}{l} n_k \geq 0\ (1 \leq k \leq j-1), \\ n_j \in \mathbb{Z}, \\ n_k < 0\ (j+1 \leq k \leq r) \end{array} \right\}.$$

In this paper, for simplicity, we assume $z > 0$ (it is easy to follow the subsequent discussion for general setting. For details, see [9]). Here we understand $\mathbb{D}_0 = \mathbb{Z}^r_{<0}$ (resp. $\mathbb{D}_{r+1} = \mathbb{Z}^r_{\geq 0}$) and write $F^{(\nu)}_{q,r,-} := F^{(\nu)}_{q,r,0}$ (resp. $F^{(\nu)}_{q,r,+} := F^{(\nu)}_{q,r,r+1}$). We first study analytic properties of $F^{(\nu)}_{q,r,\pm}$.

Lemma 2.1. *(i) $F^{(\nu)}_{q,r,-}(t,z,\boldsymbol{\omega})$ is entire as a function of t.*
(ii) $F^{(\nu)}_{q,r,+}(t,z,\boldsymbol{\omega})$ is holomorphic for $\mathrm{Re}(t) > 0$. *Moreover, if* $\mathrm{Re}(\alpha) > \frac{1}{2}r(r+2\nu-1)-\nu$, *$t^{\alpha}F^{(\nu)}_{q,r,+}(t,z,\boldsymbol{\omega})$ is integrable on $[0,\infty)$.*

Proof. Using the relation

$$q^{-(\boldsymbol{n}\cdot\boldsymbol{\omega}+z)}[\boldsymbol{n}\cdot\boldsymbol{\omega}+z]_q = q^{-z}[z]_q + \sum_{h=1}^{r} q^{-(n_h\omega_h+\cdots+n_r\omega_r+z)}[n_h\omega_h]_q,$$

we have

$$(8)\quad F^{(\nu)}_{q,r,j}(t,z,\boldsymbol{\omega}) = (tq^{-z})^{\nu+r-1}\exp\big(-tq^{-z}[z]_q\big) \times \sum_{\boldsymbol{n}\in\mathbb{D}_j}\prod_{h=1}^{r}\Big(q^{-n_h\omega_h(\nu+h-1)}\exp\big(-tq^{-(n_h\omega_h+\cdots+n_r\omega_r+z)}[n_h\omega_h]_q\big)\Big).$$

Let $j=0$. Then, since the exponential factors in the series in (8) are bounded for $\boldsymbol{n}\in\mathbb{Z}^r_{<0}$, $F^{(\nu)}_{q,r,-}(t,z,\boldsymbol{\omega})$ converges absolutely for all $t\in\mathbb{C}$, whence defines an entire function. Suppose next $j=r+1$. Then the series in (8) is bounded by $\prod_{h=1}^{r} S^{(\nu)}_{q,r,h}(\mathrm{Re}(t),\boldsymbol{\omega})$, where

$$S^{(\nu)}_{q,r,h}(t,\boldsymbol{\omega}) := \sum_{n\geq 0} q^{-n\omega_h(\nu+h-1)}\exp\big(-tq^{-n\omega_h}[n\omega_h]_q\big)$$

because $q^{-(n_{h+1}\omega_{h+1}+\cdots+n_r\omega_r+z)} > 1$ for any $\boldsymbol{n}\in\mathbb{Z}^r_{\geq 0}$. This shows that $F^{(\nu)}_{q,r,+}(t,z,\boldsymbol{\omega})$ converges absolutely for $\mathrm{Re}(t) > 0$ since the series

$S^{(\nu)}_{q,r,h}(t,\boldsymbol{\omega})$ does for $t>0$. Hence $F^{(\nu)}_{q,r,+}(t,z,\boldsymbol{\omega})$ is holomorphic for $\mathrm{Re}(t)>0$. Moreover, by the same argument as the one in Lemma 2.2 in [9], one can show

$$(9)\qquad S^{(\nu)}_{q,r,h}(t,\boldsymbol{\omega}) \le 1 + \left((\nu+h-1)e^{-1}\right)^{\nu+h-1}\frac{t^{-(\nu+h-1)}}{1-e^{-t}} \qquad (t>0).$$

Notice that the following equation is valid for $a>0$ and $\mathrm{Re}(\alpha)>\frac{1}{2}r(r+2\nu-1)-\nu$

$$(10)\qquad \int_0^\infty t^{\alpha}\cdot t^{\nu+r-1}e^{-at}\prod_{h=1}^{r}\frac{t^{-(\nu+h-1)}}{1-e^{-t}}dt$$
$$= \Gamma\Big(\alpha-\frac{1}{2}r(r+2\nu-3)+\nu\Big)\zeta_r\Big(\alpha-\frac{1}{2}r(r+2\nu-3)+\nu,a,\mathbf{1}_r\Big).$$

Therefore we obtain the rest of assertion in (ii) by (9) and (10) with $a=q^{-z}[z]_q>0$. This shows the claims. Q.E.D.

For $1\le j\le r$, the Poisson summation formula asserts the following

Proposition 2.2. *Let* $1\le j\le r$. *Then* $F^{(\nu)}_{q,r,j}(t,z,\boldsymbol{\omega})$ *is holomorphic for* $\mathrm{Re}(t)>0$ *with* $-\pi/2<\arg(t)<\pi/2$ *via the expression*

$$(11)\qquad F^{(\nu)}_{q,r,j}(t,z,\boldsymbol{\omega}) = -\frac{(-1)^{r-j}(1-q)^{\nu+j-1}}{\omega_j\log q}(tq^{-z})^{r-j}e^{\frac{1}{1-q}t}$$
$$\times\sum_{m\in\mathbb{Z}}\Big(\frac{1-q}{t}\Big)^{m\delta_j}\frac{\Gamma(\nu+j-1+m\delta_j)}{\prod_{h\ne j}\left(1-q^{\omega_h(j-h+m\delta_j)}\right)}e^{2\pi\sqrt{-1}mz/\omega_j},$$

where $\delta_j := 2\pi\sqrt{-1}/(\omega_j\log q)\in\sqrt{-1}\mathbb{R}$.

Proof. From the definition, it can be expressed as

$$F^{(\nu)}_{q,r,j}(t,z,\boldsymbol{\omega}) = (tq^{-z})^{\nu+r-1}e^{\frac{1}{1-q}t}$$
$$\times\sum_{\check{\boldsymbol{n}}(j)\in\check{\mathbb{D}}_j}\prod_{h\ne j}q^{-n_h\omega_h(\nu+h-1)}\sum_{n_j\in\mathbb{Z}}f^{(\nu)}_{q,r,j}(n_j),$$

where

$$\check{\mathbb{D}}_j := \left\{\check{\boldsymbol{n}}(j) := (n_1,\dots,\check{n}_j,\dots,n_r)\in\mathbb{Z}^{r-1}\ \middle|\ \begin{array}{l} n_k\ge 0\ (1\le k\le j-1),\\ n_k<0\ (j+1\le k\le r)\end{array}\right\}$$

(here $\check{n}_j$ means that n_j is omitted) and

$$f^{(\nu)}_{q,r,j}(x) := q^{-x\omega_j(\nu+j-1)} \exp\Big(-\frac{t}{1-q}q^{-z}\cdot q^{-x\omega_j}\prod_{h\neq j} q^{-n_h\omega_h}\Big).$$

Note that, for a fixed $\check{\boldsymbol{n}}(j) \in \check{\mathbb{D}}_j$, the series $\sum_{n_j\in\mathbb{Z}} f^{(\nu)}_{q,r,j}(n_j)$ converges absolutely for $\mathrm{Re}(t) > 0$. Then, since the Fourier transform $\tilde{f}^{(\nu)}_{q,r,j}(\xi)$ of $f^{(\nu)}_{q,r,j}(x)$ is given by

$$\begin{aligned}\tilde{f}^{(\nu)}_{q,r,j}(\xi) &= \int_{-\infty}^{\infty} f^{(\nu)}_{q,r,j}(x)e^{-2\pi\sqrt{-1}x\xi}dx\\ &= -\frac{(1-q)^{\nu+j-1}}{\omega_j\log q}(tq^{-z})^{-(\nu+j-1)}\prod_{h\neq j} q^{n_h\omega_h(\nu+j-1+\xi\delta_j)}\\ &\quad\times\Big(\frac{1-q}{t}\Big)^{\xi\delta_j}\Gamma(\nu+j-1+\xi\delta_j)e^{2\pi\sqrt{-1}\xi z/\omega_j},\end{aligned}$$

the Poisson summation formula $\sum_{n\in\mathbb{Z}} f^{(\nu)}_{q,r,j}(n) = \sum_{m\in\mathbb{Z}} \tilde{f}^{(\nu)}_{q,r,j}(m)$ yields the desired formula (11). Remark that, for $j+1 \leq h \leq r$, it can be calculated as

$$\sum_{n_h<0} q^{n_h\omega_h(j-h+m\delta_j)} = \frac{q^{\omega_h(h-j-m\delta_j)}}{1-q^{\omega_h(h-j-m\delta_j)}} = \frac{-1}{1-q^{\omega_h(j-h+m\delta_j)}}.$$

Hence we have the factor $(-1)^{r-j}$ in (11). Now, it is easy to see that the series in (11) converges absolutely for $\mathrm{Re}(t) > 0$. In fact, by the Stirling formula, we have

$$(12)\quad \big|\Gamma(\nu+j-1+m\delta_j)\big| \sim \frac{(2\pi)^{\nu+j-1}|m|^{\nu+j-\frac{3}{2}}}{|\omega_j\log q|^{\nu+j-\frac{3}{2}}}e^{-\frac{\pi^2|m|}{\omega_j|\log q|}} \quad (|m|\to\infty)$$

and $\big|t^{-m\delta_j}\big| < \exp\big(\frac{\pi^2|m|}{\omega_j|\log q|}\big)$. Therefore $F^{(\nu)}_{q,r,j}(t,z,\boldsymbol{\omega})$ is holomorphic for $\mathrm{Re}(t) > 0$. This completes the proof of proposition. Q.E.D.

Remark 2.3. From the expression (11) and using the relation $q^{\omega_j\delta_j} = 1$, $F^{(\nu)}_{q,r,j}(t,z,\boldsymbol{\omega})$ satisfies the following functional equation for each $1 \leq j \leq r$:

$$F^{(\nu)}_{q,r,j}(q^{\omega_j}t,z,\boldsymbol{\omega}) = q^{\omega_j(r-j)}e^{-t[\omega_j]_q}F^{(\nu)}_{q,r,j}(t,z,\boldsymbol{\omega}).$$

Let us denote by $F^{(\nu)}_{q,r,j(0)}(t,z,\boldsymbol{\omega})$ the term for $m=0$ in (11);

$$\text{(13)}\quad F^{(\nu)}_{q,r,j(0)}(t,z,\boldsymbol{\omega}) := \frac{(-1)^{r+1-j}(1-q)^{\nu+j-1}}{\omega_j \log q} \frac{(\nu+j-2)!(tq^{-z})^{r-j}}{\prod\limits_{h\neq j}\left(1-q^{\omega_h(j-h)}\right)} e^{\frac{1}{1-q}t}.$$

Then $F^{(\nu)}_{q,r,j(0)}(t,z,\boldsymbol{\omega})$ is clearly entire as a function of t. Moreover, we put $F^{(\nu)}_{q,r,j(\neq 0)} := F^{(\nu)}_{q,r,j} - F^{(\nu)}_{q,r,j(0)}$ for the other terms in (11).

The next lemma is crucial in the subsequent discussion.

Lemma 2.4. *For* $\mathrm{Re}(t)>0$, *we have*

$$\text{(14)}\qquad \sum_{j=0}^{r+1}(-1)^{r+1-j}F^{(\nu)}_{q,r,j}(t,z,\boldsymbol{\omega}) \equiv 0.$$

Proof. Write $F^{(\nu)}_{q,r,j}(t,z,\boldsymbol{\omega}) = \sum_{\boldsymbol{n}\in\mathbb{D}_j} h(\boldsymbol{n})$. For $0\le j\le r+1$, we define the partial series $\widetilde{F}^{(\nu)}_{q,r,j}$ of $F^{(\nu)}_{q,r,j}$ by $\widetilde{F}^{(\nu)}_{q,r,j}(t,z,\boldsymbol{\omega}) := \sum_{\boldsymbol{n}\in\widetilde{\mathbb{D}}_j} h(\boldsymbol{n})$, where

$$\widetilde{\mathbb{D}}_j := \left\{ \boldsymbol{n}=(n_1,\dots,n_r)\in\mathbb{Z}^r \;\middle|\; \begin{array}{l} n_k\ge 0\ (1\le k\le j), \\ n_k<0\ (j+1\le k\le r) \end{array} \right\} \subseteq \mathbb{D}_j.$$

Then it holds that $F^{(\nu)}_{q,r,0} = \widetilde{F}^{(\nu)}_{q,r,0}$, $F^{(\nu)}_{q,r,j} = \widetilde{F}^{(\nu)}_{q,r,j} + \widetilde{F}^{(\nu)}_{q,r,j-1}$ for $1\le j\le r$ and $F^{(\nu)}_{q,r,r+1} = \widetilde{F}^{(\nu)}_{q,r,r}$. Now, the relation (14) immediately follows from these equations. Q.E.D.

§3. Function $G^{(\nu)}_{q,r}(t,z,\boldsymbol{\omega})$.

Let

$$\text{(15)}\quad G^{(\nu)}_{q,r}(t,z,\boldsymbol{\omega}) := \sum_{j=1}^{r}(-1)^{r-j}F^{(\nu)}_{q,r,j(0)}(t,z,\boldsymbol{\omega}) + (-1)^r F^{(\nu)}_{q,r,-}(t,z,\boldsymbol{\omega}).$$

It follows from Lemma 2.1 (*i*) and the expression (13) that $G^{(\nu)}_{q,r}(t,z,\boldsymbol{\omega})$ has an infinite radius of convergence at $t=0$ when $0<q<1$ and is entire. Then we define ${}_rA^{(\nu)}_{-k}(z,\boldsymbol{\omega};q)$ and ${}_rB^{(\nu)}_{n}(z,\boldsymbol{\omega};q)$ as the coefficients of the Taylor expansion of $G_{q,r}^{(\nu)}(t,z,\boldsymbol{\omega})$ at $t=0$:

$$
\begin{aligned}
(16)\quad & G_{q,r}^{(\nu)}(t,z,\boldsymbol{\omega}) \\
&= t^{\nu+r-2}\Big\{\sum_{k=1}^{\nu+r-2}(-1)^k{}_rA_{-k}^{(\nu)}(z,\boldsymbol{\omega};q)t^{-k}+\sum_{n=0}^{\infty}(-1)^n{}_rB_n^{(\nu)}(z,\boldsymbol{\omega};q)\frac{t^n}{n!}\Big\}.
\end{aligned}
$$

We also put ${}_rA_0^{(\nu)}(z,\boldsymbol{\omega};q) := {}_rB_0^{(\nu)}(z,\boldsymbol{\omega};q)$. The following theorem asserts that ${}_rA_{-k}^{(\nu)}(z,\boldsymbol{\omega};q)$ and ${}_rB_n^{(\nu)}(z,\boldsymbol{\omega};q)$ are q-analogues of the Barnes multiple Bernoulli polynomials.

Theorem 3.1. *For $0 < t < b(\boldsymbol{\omega})$, we have*

$$
\lim_{q\uparrow 1} G_{q,r}^{(\nu)}(t,z,\boldsymbol{\omega}) = t^{\nu+r-2}G_r(t,z,\boldsymbol{\omega}). \tag{17}
$$

In particular, it holds that

$$
\lim_{q\uparrow 1} {}_rA_{-k}^{(\nu)}(z,\boldsymbol{\omega};q) = \begin{cases} {}_rA_{-k}(z,\boldsymbol{\omega}) & \textit{for } 0\le k\le r-1, \\ 0 & \textit{for } r\le k\le \nu+r-2, \end{cases} \tag{18}
$$

$$
\lim_{q\uparrow 1} {}_rB_n^{(\nu)}(z,\boldsymbol{\omega};q) = {}_rB_n(z,\boldsymbol{\omega}) \quad \textit{for } n\ge 0. \tag{19}
$$

Proof. The assertions (18) and (19) follow immediately from (3), (16) and (17). Hence it suffices to show the formula (17). For $t>0$, we have $\lim_{q\uparrow 1} F_{q,r,+}^{(\nu)}(t,z,\boldsymbol{\omega}) = t^{\nu+r-2}G_r(t,z,\boldsymbol{\omega})$ because $F_{q,r,+}^{(\nu)}(t,z,\boldsymbol{\omega})$ converges absolutely for $\mathrm{Re}(t)>0$. On the other hand, from the relation (14), we have

$$
\begin{aligned}
F_{q,r,+}^{(\nu)}(t,z,\boldsymbol{\omega}) &= -\sum_{j=1}^{r}(-1)^{r+1-j}F_{q,r,j}^{(\nu)}(t,z,\boldsymbol{\omega}) - (-1)^{r+1}F_{q,r,-}^{(\nu)}(t,z,\boldsymbol{\omega}) \\
(20)\quad &= G_{q,r}^{(\nu)}(t,z,\boldsymbol{\omega}) + \sum_{j=1}^{r}(-1)^{r-j}F_{q,r,j(\neq 0)}^{(\nu)}(t,z,\boldsymbol{\omega}).
\end{aligned}
$$

Therefore it is enough to show that for all $1\le j\le r$

$$
\lim_{q\uparrow 1} F_{q,r,j(\neq 0)}^{(\nu)}(t,z,\boldsymbol{\omega}) = 0 \qquad (0<t<b(\boldsymbol{\omega})). \tag{21}
$$

Put $\mu_j := \#\{1\le h\le r \,|\, \omega_h/\omega_j\in\mathbb{Z},\ h\neq j\}$. Then notice that if $m\neq 0$, we have

$$
\prod_{h\neq j}\big(1-q^{\omega_h(j-h+m\delta_j)}\big) = O\big((1-q)^{\mu_j}\big) \qquad (q\uparrow 1).
$$

Hence, using the formula $1/\log q = -1/(1-q) + O(1)$ as $q \uparrow 1$, we have from (12)

$$\frac{(1-q)^{\nu+j-1}}{\log q} e^{\frac{1}{1-q}t} \frac{\left|\Gamma(\nu+j-1+m\delta_j)\right|}{\left|\prod_{h\neq j}\left(1-q^{\omega_h(j-h+m\delta_j)}\right)\right|}$$

$$= \frac{(1-q)^{\nu+j-1}}{\log q} \cdot O\Big(\frac{e^{-\frac{1}{4}\frac{\pi^2|m|}{\omega_j|\log q|}}}{(1-q)^{\mu_j}(\log q)^{\nu+j-\frac{1}{2}}}\Big) \exp\Big(\frac{1}{1-q}t - \frac{3}{4}\frac{\pi^2|m|}{\omega_j|\log q|}\Big)$$

$$= O\Big(\frac{e^{-\frac{1}{4}\frac{\pi^2|m|}{\omega_j|\log q|}}}{(1-q)^{\mu_j+\frac{1}{2}}}\Big) \exp\Big(-\big(\frac{3\pi^2|m|}{4\omega_j} - t\big)\frac{1}{1-q} + O(1)\Big) \to 0 \qquad (q \uparrow 1)$$

because $0 < t < b(\boldsymbol{\omega}) \leq \frac{2\pi}{\omega_j} \leq \frac{2\pi}{\omega_j}\frac{3\pi|m|}{8} = \frac{3\pi^2|m|}{4\omega_j}$. This shows that each summand of $F^{(\nu)}_{q,r,j(\neq 0)}(t,z,\boldsymbol{\omega})$ vanishes as $q \uparrow 1$, whence the claim (21) follows. This completes the proof of the theorem. Q.E.D.

One can obtain the following explicit expressions of ${}_rA^{(\nu)}_{-k}(z,\boldsymbol{\omega};q)$ and ${}_rB^{(\nu)}_n(z,\boldsymbol{\omega};q)$.

Proposition 3.2. *We have for* $0 \leq k \leq \nu + r - 2$

$${}_rA^{(\nu)}_{-k}(z,\boldsymbol{\omega};q) = \frac{(q-1)^{1+k}}{\log q} \times \sum_{j=\max\{k-\nu+2,1\}}^{r} \frac{q^{z(j-r)}}{\omega_j \prod_{h\neq j}\left(1-q^{\omega_h(j-h)}\right)} \frac{(\nu+j-2)!}{(-k+\nu+j-2)!}$$

and for $n \geq 0$

$${}_rB^{(\nu)}_n(z,\boldsymbol{\omega};q) = (q-1)^{1-n}\left\{\sum_{\ell=1}^{n}(-1)^\ell\binom{n}{\ell}\frac{\ell q^{z(-\ell-\nu-r+2)}}{\prod_{j=1}^{r}\left(1-q^{\omega_j(-\ell-\nu-j+2)}\right)} + \frac{1}{\log q}\sum_{j=1}^{r}\binom{n+\nu+j-2}{\nu+j-2}^{-1}\frac{q^{z(j-r)}}{\omega_j\prod_{h\neq j}\left(1-q^{\omega_h(j-h)}\right)}\right\}.$$

Proof. These formulas are directly derived from (15) by calculating the Taylor expansions of the exponential functions at $t = 0$. Q.E.D.

§4. Main results.

Now we are ready to study an integral representation of $\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})$. We first show that $\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})$ can be expressed as the Mellin transform of $F_{q,r,+}^{(\nu)}(t,z,\boldsymbol{\omega})$.

Proposition 4.1. *For* $\mathrm{Re}(s) > \frac{1}{2}r(r+2\nu+1)$, *we have*

$$\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega}) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-(\nu+r-1)}F_{q,r,+}^{(\nu)}(t,z,\boldsymbol{\omega})\frac{dt}{t}. \tag{22}$$

Proof. From Lemma 2.1 (ii), the integral

$$\int_0^\infty t^{s-(\nu+r-1)}F_{q,r,+}^{(\nu)}(t,z,\boldsymbol{\omega})\frac{dt}{t}$$

converges absolutely for $\mathrm{Re}(s) > \frac{1}{2}r(r+2\nu+1)$. Then, from the integral expression of the gamma function $\Gamma(s)$, one can easily obtain the formula (22) by changing the variable $tq^{-(\boldsymbol{n}\cdot\boldsymbol{\omega}+z)}[\boldsymbol{n}\cdot\boldsymbol{\omega}+z]_q \mapsto t$. Q.E.D.

To establish our main result, we introduce the function $\varphi_{q,r,j}^{(\nu)}(s;a,m)$ for $0<a<\infty$, $m\in\mathbb{Z}\backslash\{0\}$ and $1\le j\le r$ by the following integral:

$$\varphi_{q,r,j}^{(\nu)}(s;a,m) := \int_0^a t^{s-\nu-j-m\delta_j}e^{\frac{1}{1-q}t}dt.$$

Since the integral converges absolutely for $\mathrm{Re}(s) > \nu+j-1$, it defines a holomorphic function on the region. Further, we have the following

Lemma 4.2. *The function* $\varphi_{q,r,j}^{(\nu)}(s;a,m)$ *can be meromorphically continued to the whole plane* $\mathbb{C}$. *It has simple poles at* $s=n+m\delta_j$ *for* $n\in\mathbb{Z}_{\le\nu+j-1}$ *with*

$$\operatorname*{Res}_{s=n+m\delta_j}\varphi_{q,r,j}^{(\nu)}(s;a,m) = \frac{1}{(\nu+j-1-n)!(1-q)^{\nu+j-1-n}}. \tag{23}$$

These exhaust all poles of $\varphi_{q,r,j}^{(\nu)}(s;a,m)$.

Proof. This is obtained by integration by parts. Precisely, see Proposition 2.5 in [9]. Q.E.D.

Moreover, we put

$$\widetilde{\varphi}_{q,r,j}^{(\nu)}(s;a,m) := \frac{(1-q)^{m\delta_j}\Gamma(\nu+j-1+m\delta_j)q^{z(j-r+m\delta_j)}}{\prod_{h\neq j}\left(1-q^{\omega_h(j-h+m\delta_j)}\right)}\varphi_{q,r,j}^{(\nu)}(s;a,m).$$

The following theorem is our main result, which gives a generalization of Theorem 3.6 in [9].

Theorem 4.3. *(i) For $0 < a < \infty$ and $0 < \varepsilon < \min\{a, b(\boldsymbol{\omega})\}$, we have*

$$(24)\quad \zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega}) = \frac{(-1)^{\nu+r-1}\Gamma(1-s)}{2\pi\sqrt{-1}} I_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega};a) - \frac{1}{\Gamma(s)}\sum_{j=1}^{r}\frac{(1-q)^{\nu+j-1}}{\omega_j\log q}\sum_{m_j\in\mathbb{Z}\backslash\{0\}}\widetilde{\varphi}_{q,r,j}^{(\nu)}(s;a,m_j) + \frac{1}{\Gamma(s)}\int_a^{\infty} t^{s-(\nu+r-1)}F_{q,r,+}^{(\nu)}(t,z,\boldsymbol{\omega})\frac{dt}{t},$$

where

$$I_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega};a) := \int_{C(\varepsilon,a)}(-t)^{s-(\nu+r-1)}G_{q,r}^{(\nu)}(t,z,\boldsymbol{\omega})\frac{dt}{t}$$

and $C(\varepsilon,a)$ is the same contour as the one in (1). This provides a meromorphic continuation of $\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})$ to the entire plane $\mathbb{C}$.
(ii) The poles of $\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})$ are all simple and are located at $s = 1,2,\ldots,\nu+r-1$ and $s = \nu+j-1-\ell+m_j\delta_j$ for $1 \le j \le r$, $\ell \in \mathbb{Z}_{\ge 0}$ and $m_j \in \mathbb{Z}\backslash\{0\}$. For $n \in \mathbb{Z}_{\le \nu+r-1}$, $m \in \mathbb{Z}$ and $\omega \in \{\omega_1,\ldots,\omega_r\}$ with $\delta := 2\pi\sqrt{-1}/(\omega\log q) \in \sqrt{-1}\mathbb{R}$, we have

$$(25)\quad \operatorname*{Res}_{s=n+m\delta}\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega}) = -\frac{(1-q)^{n+m\delta}}{\log q} \times \sum_{j=\max\{n-\nu+1,1\}}^{r}\binom{\nu+j-2+m\delta}{\nu+j-1-n}\frac{d_j q^{z(j-r+m\delta)}}{\omega_j\prod_{h\neq j}\left(1-q^{\omega_h(j-h+m\delta)}\right)},$$

where $d_j := \#\{m_j \in \mathbb{Z}\backslash\{0\} \mid m_j\delta_j = m\delta\}$.
(iii) For a positive integer m, we have

$$(26)\quad \zeta_{q,r}^{(\nu)}(1-m,z,\boldsymbol{\omega}) = -\frac{{}_rB_m^{(\nu)}(z,\boldsymbol{\omega};q)}{m}.$$

Proof. Suppose $\mathrm{Re}(s) > \frac{1}{2}r(r+2\nu+1)$. Then, from Proposition 4.1, (20) and (11), it holds that

$$\begin{aligned}
&\Gamma(s)\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})\\
&= \int_0^a t^{s-(\nu+r-1)} G_{q,r}^{(\nu)}(t,z,\boldsymbol{\omega})\frac{dt}{t} + \int_a^\infty t^{s-(\nu+r-1)} F_{q,r,+}^{(\nu)}(t,z,\boldsymbol{\omega})\frac{dt}{t}\\
&\quad + \int_0^a t^{s-(\nu+r-1)} \sum_{j=1}^{r}(-1)^{r-j} F_{q,r,j(\neq 0)}^{(\nu)}(t,z,\boldsymbol{\omega})\frac{dt}{t}\\
&= \int_0^a t^{s-(\nu+r-1)} G_{q,r}^{(\nu)}(t,z,\boldsymbol{\omega})\frac{dt}{t} + \int_a^\infty t^{s-(\nu+r-1)} F_{q,r,+}^{(\nu)}(t,z,\boldsymbol{\omega})\frac{dt}{t}\\
&\quad - \sum_{j=1}^{r}\frac{(1-q)^{\nu+j-1}}{\omega_j \log q}\sum_{m\in\mathbb{Z}\setminus\{0\}} \widetilde{\varphi}_{q,r,j}^{(\nu)}(s;a,m).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(27)\qquad &\int_0^a t^{s-(\nu+r-1)} G_{q,r}^{(\nu)}(t,z,\boldsymbol{\omega})\frac{dt}{t}\\
&\qquad = \frac{(-1)^{\nu+r-1}\Gamma(s)\Gamma(1-s)}{2\pi\sqrt{-1}} I_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega};a).
\end{aligned}$$

Actually, since the integral $I_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega};a)$ converges absolutely and uniformly with respect to s, it defines an entire function in s. Further, by the Cauchy integral theorem, it does not depend on ε. Then, taking the limit $\varepsilon \to 0$ and using the relation $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, we have (27). Note that the integral on the path $|t| = \varepsilon$ in $I_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega};a)$ vanishes as $\varepsilon \to 0$ since $\mathrm{Re}(s) > \frac{1}{2}r(r+2\nu+1) > \nu+r-1$. Hence we obtain the desired formula (24). Since the last integral on the right hand side of (24) clearly defines an entire function, from Lemma 4.2, (24) provides a meromorphic continuation of $\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})$ to the entire plane $\mathbb{C}$. Further, since $I_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega};a) = 0$ for $s \in \mathbb{Z}_{\geq \nu+r}$ by the residue theorem, one can see from (24) that $\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})$ has simple poles at $s = 1,2,\ldots,\nu+r-1$ with residues

$$\begin{aligned}
\operatorname*{Res}_{s=n} \zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega}) &= -\Big(\operatorname*{Res}_{s=n} \Gamma(1-s)\Big)\, {}_rA_{-(n-1)}^{(\nu)}(z,\boldsymbol{\omega};q)\\
(28)\qquad &= \frac{(-1)^{n-1}}{(n-1)!}\, {}_rA_{-(n-1)}^{(\nu)}(z,\boldsymbol{\omega};q) \qquad (1 \leq n \leq \nu+r-1).
\end{aligned}$$

Hence, from Proposition 3.2, we have (25) for $m = 0$. Moreover, from Lemma 4.2, $\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})$ has also simple poles at $s = \nu+j-1-\ell+m_j\delta_j$

for $1 \leq j \leq r$, $\ell \in \mathbb{Z}_{\geq 0}$ and $m_j \in \mathbb{Z}\backslash\{0\}$. Retaining the notation in the statement (ii) above, we have

$$\operatorname*{Res}_{s=n+m\delta} \zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega}) = -\frac{1}{\Gamma(n+m\delta)} \times \sum_{j=\max\{n-\nu+1,1\}}^{r} \sum_{\substack{m_j\in\mathbb{Z}\backslash\{0\} \\ m_j\delta_j=m\delta}} \frac{(1-q)^{\nu+j-1}}{\omega_j \log q} \Big(\operatorname*{Res}_{s=n+m_j\delta_j} \widetilde{\varphi}_{q,r,j}^{(\nu)}(s;a,m_j) \Big).$$

Therefore, by the formula (23), we have (25) for $m \neq 0$. From (27) again, it follows that

$$\begin{aligned} \zeta_{q,r}^{(\nu)}(1-m,z,\boldsymbol{\omega}) &= \frac{(-1)^{\nu+r-1}\Gamma(m)}{2\pi\sqrt{-1}} I_{q,r}^{(\nu)}(1-m,z,\boldsymbol{\omega};a) \\ &= -\frac{{}_rB_m^{(\nu)}(z,\boldsymbol{\omega};q)}{m}. \end{aligned}$$

This completes the proof of the theorem. Q.E.D.

Remark 4.4. From (4), (18) and (28), we have

$$\lim_{q\uparrow 1} \operatorname*{Res}_{s=n} \zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega}) = \begin{cases} \operatorname*{Res}_{s=n} \zeta_r(s,z,\boldsymbol{\omega}) & \text{for } n=1,2,\ldots,r, \\ 0 & \text{for } n=r+1,\ldots,\nu+r-1. \end{cases}$$

We finally give the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose $0 < a < b(\boldsymbol{\omega})$. Compare the integral expression (24) with (1). Then, from Theorem 3.1, it is sufficient to show that $\lim_{q\uparrow 1} \widetilde{\varphi}_{q,r,j}^{(\nu)}(s;a,m_j) = 0$ for all $1 \leq j \leq r$ and $m_j \in \mathbb{Z}\backslash\{0\}$. Indeed, using the mean-value theorem, one can show the formula by the same way as the proof of (21) (more precisely, see Corollary 3.8 in [9]). Hence we obtain the desired claim. Q.E.D.

Remark 4.5. Using the binomial theorem, we obtain the following series expression of $\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})$ (see [10], also [3, 4]):

$$\text{(29)} \quad \zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega}) = (1-q)^s \sum_{\ell=0}^{\infty} \binom{s+\ell-1}{\ell} \frac{q^{z(s-\nu-r+1+\ell)}}{\prod_{j=1}^{r} \left(1-q^{\omega_j(s-\nu-j+1+\ell)}\right)}.$$

This also gives a meromorphic continuation of $\zeta_{q,r}^{(\nu)}(s,z,\boldsymbol{\omega})$ to the whole plane $\mathbb{C}$. One can obtain the same facts (25) and (26) from the expression (29), however, it seems to be difficult to show Theorem 1.1.

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