The universality of L-functions attached to Maass forms

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Abstract.

   We establish the universality theorem on the strip $\{s \in \mathbb{C} \mid 1/2 < \operatorname{Re} s < 1\}$ for automorphic L-functions attached to Maass forms for $SL(2, \mathbb{Z})$, without the assumption of the Ramanujan conjecture. From this theorem, some results concerning the value distribution of the derivatives of those L-functions are obtained.

§1. Introduction

Let $\zeta(s)$ be the Riemann zeta-function. In 1914 Bohr and Courant [3] showed that the set $\{\zeta(s_0 + i\tau) \mid \tau \in \mathbb{R}\}$ is dense in $\mathbb{C}$ for any fixed $s_0 \in D$. Here and henceforth D denotes the region $\{s \in \mathbb{C} \mid \frac{1}{2} < \operatorname{Re} < 1\}$. Extending this result, in 1972 Voronin [22] showed that if n is a positive integer, then the set $\{(\zeta(s_0 + i\tau), \zeta'(s_0 + i\tau), \ldots, \zeta^{(n-1)}(s_0 + i\tau)) \in \mathbb{C}^n \mid \tau \in \mathbb{R}\}$ is dense in $\mathbb{C}^n$ for any fixed $s_0 \in D$. Furthermore, in 1975 Voronin [23] [6] obtained a remarkable result, which is called the *universality theorem* for $\zeta(s)$. His result above in 1972 is a corollary of this theorem (see [6, p. 252, Theorem 2]). A modern form of the universality theorem for $\zeta(s)$ is the following (see [8, p. 225, Theorem 5.2]):

   Let K be a compact subset of D with connected complement. Let $h(s)$ be a non-vanishing continuous function on K which is holomorphic in the interior (if any) of K. Then for any $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \, m\left(\{\tau \in [0, T] \mid \max_{s \in K} |\zeta(s + i\tau) - h(s)| < \varepsilon\}\right) > 0,$$

where m denotes the Lebesgue measure on $\mathbb{R}$.

After Voronin's discovery, in 1981 Bagchi [1] (see also [8]) gave another proof of this theorem, using probability-theoretic ideas. Also, Bagchi [2], Gonek [5] and Voronin independently showed the universality theorem for the Dirichlet L-function $L(s, \chi)$ with a character χ mod q; in fact, the *joint* universality theorem for $L(s, \chi)$'s, which is a stronger result, was shown.

Let f be a holomorphic normalized Hecke eigen cusp form of integral weight for $SL(2, \mathbb{Z})$. Laurinčikas and Matsumoto [9] succeeded in proving (without any assumption) the universality theorem for the automorphic L-function associated with f. Their proof is influenced by Bagchi's thesis [1] and accomplished by introducing an idea, which is called "the positive density method" in [13]. This theorem was generalized in [11] to the case of holomorphic normalized newforms of integral weight with respect to Hecke congruence subgroups of $SL(2, \mathbb{Z})$. See also [10].

As is well known, the Ramanujan conjecture for those automorphic forms was proved by Deligne. This conjecture is used in their proofs of those universality theorems. Generalizing their theorems further, Steuding [20] established the universality theorem for Dirichlet series in the Selberg class, under certain assumptions including the Ramanujan conjecture. This result is very extensive.

We note, however, that it is conjectured by Linnik and Ibragimov that all Dirichlet series, except for trivial exceptions, would have the universality property. See [13, p. 65]. In addition, it should be noted that so far automorphic cuspidal representations of $GL(n)$ ($n \geq 2$) for which the Ramanujan conjecture has been proved are few and that this conjecture is difficult to prove. For a study towards this conjecture, see e.g. [12]. Hence, it would be desirable to seek proofs of the universality theorems above without using the Ramanujan conjecture and to study the universality theorem for a Dirichlet series whose coefficients are not certain to possess a bound like the Ramanujan conjecture.

Let $\Gamma = PSL(2, \mathbb{Z})$ and $\mathbb{H}$ be the upper half-plane $\{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\}$. Throughout this paper, we fix a simultaneous $L^2 (\Gamma \backslash \mathbb{H})$-eigenfunction φ of the operators Δ, T_n ($n = 1, 2, \dots$) and R which is not a constant function (see Section 2). This is a typical example of $GL(2)/\mathbb{Q}$-automorphic forms for which the Ramanujan conjecture is still open. The universality of the automorphic L-function $L(s, \varphi)$ attached to φ was discussed by the author [17]. He proved that $L(s, \varphi)$ has the universality property on the strip $1/2 < \mathrm{Re}\, s < 25/32$, by using Kim-Sarnak's bound (2.2). It was also shown that if the Ramanujan conjecture for φ is true then $L(s, \varphi)$ has the universality property on D.

The purpose of the present paper is to establish the universality theorem for the L-function $L(s, \varphi)$ on the strip D, irrespective of the truth of the Ramanujan conjecture. Our main result is the following:

Theorem 1.1. *Let K be a compact subset of $D = \{s \in \mathbb{C} \mid \frac{1}{2} < \operatorname{Re} s < 1\}$ with connected complement. Let $h(s)$ be a non-vanishing continuous function on K which is holomorphic in the interior (if any) of K. Then for any $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} m \left(\left\{ \tau \in [0, T] \mid \max_{s \in K} |L(s + i\tau, \varphi) - h(s)| < \varepsilon \right\} \right) > 0,$$

where m denotes the Lebesgue measure on $\mathbb{R}$.

We remark that our way of proving Theorem 1.1 gives a proof of the above universality theorem of [9], without using the Ramanujan conjecture (shown by Deligne). An important ingredient of our proof is the asymptotic formula (4.2) below. Theorem 1.1 suggests that, in general, such a bound as the Ramanujan conjecture would not be necessary for the universality property. (See also [17].) We can apply our argument to automorphic L-functions for Hecke congruence subgroups of $SL(2, \mathbb{Z})$ as in [11] and [10], and presumably to a general Dirichlet series with suitable assumptions milder than those of the paper [20] mentioned above.

Although Theorem 1.1 above is an improvement on the main result of [17], we note that a certain argument in [17], which is inspired by the papers [23] and [5] (rather than Bagchi's thesis [1] and his paper [2]) and not employed in the present paper, is essential in [18].

Theorem 1.1 yields the universality theorem for the derivatives $L^{(r)}(s, \varphi)$ of $L(s, \varphi)$, as in [17]. Further, the following corollaries can be deduced. Let $\mathbb{N}$ denote the set of all positive integers.

Corollary 1.2. *Let $n \in \mathbb{N}$. Let $(s_1, \ldots, s_n) \in D^n$ such that $s_j \neq s_k$ if $j \neq k$. Then the set $\{(L(s_1 + i\tau, \varphi), \ldots, L(s_n + i\tau, \varphi)) \in \mathbb{C}^n \mid \tau \in \mathbb{R}\}$ is dense in $\mathbb{C}^n$. More precisely, it holds that for any $(a_1, \ldots, a_n) \in \mathbb{C}^n$ and any $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} m \left(\{ \tau \in [0, T] \mid |L(s_j + i\tau, \varphi) - a_j| < \varepsilon \right.$$
$$\left. \text{for all } j = 1, \ldots, n \} \right) > 0.$$

Corollary 1.3. *Let $s_0 \in D$ and $n \in \mathbb{N}$. Then the set*

$$\left\{ \left(L(s_0 + i\tau, \varphi), L'(s_0 + i\tau, \varphi), \ldots, L^{(n-1)}(s_0 + i\tau, \varphi) \right) \in \mathbb{C}^n \mid \tau \in \mathbb{R} \right\}$$

*is dense in $\mathbb{C}^n$. More precisely, it holds that for any $(a_1, \ldots, a_n) \in \mathbb{C}^n$
and any $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \, m \left(\{ \tau \in [0, T] \mid |L^{(j-1)}(s_0 + i\tau, \varphi) - a_j| < \varepsilon \right.$$

$$\left. \text{for all } j = 1, \ldots, n \} \right) > 0.$$

Corollary 1.4. *Let C be a region whose closure is a compact subset
of D. Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{C} - \{0\}$. Then*

$$\liminf_{T \to \infty} \frac{1}{T} \, m \left(\{ \tau \in [0, T] \mid \text{there exists a number } z_j \in C + i\tau \text{ such that} \right.$$

$$L(z_j, \varphi) = a_j, \text{ for every } j = 1, \ldots, n \}) > 0,$$

where $C + i\tau$ means the set $\{ w + i\tau \mid w \in C \}$ in D.

Corollary 1.5. *Let C be as in Corollary 1.4 and let $n \in \mathbb{N}$. Let
$a_1 \in \mathbb{C} - \{0\}$ and $a_2, \ldots, a_n \in \mathbb{C}$. Then*

$$\liminf_{T \to \infty} \frac{1}{T} \, m \left(\{ \tau \in [0, T] \mid \text{there exists a number } z_j \in C + i\tau \text{ such that} \right.$$

$$L^{(j-1)}(z_j, \varphi) = a_j, \text{ for every } j = 1, \ldots, n \}) > 0,$$

where $C + i\tau$ is as in Corollary 1.4.

§2.   Notation and preliminaries

Let $\Gamma = PSL(2, \mathbb{Z})$ and $\mathbb{H} = \{ z = x + iy \in \mathbb{C} \mid x \in \mathbb{R}, y > 0 \}$, as
before. Let $L^2(\Gamma \backslash \mathbb{H})$ denote the Hilbert space

$$\{ f : \mathbb{H} \to \mathbb{C} \mid f(\gamma z) = f(z) \text{ for } z \in \mathbb{H} \text{ and } \gamma \in \Gamma,$$

$$\int_{\Gamma \backslash \mathbb{H}} |f(z)|^2 \, y^{-2} dx dy < \infty \}$$

with the inner product $\langle f_1, f_2 \rangle = \int_{\Gamma \backslash \mathbb{H}} f_1(z) \overline{f_2(z)} y^{-2} dx dy$. An eigen-
function in $L^2(\Gamma \backslash \mathbb{H})$ of the Laplacian $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ is called a
Maass form (of weight 0) for Γ. The n-th Hecke operator T_n on $L^2(\Gamma \backslash \mathbb{H})$
is defined by

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n, a>0 \\ 0 \le b < d}} f \left(\frac{az + b}{d} \right).$$

Further, we define the operator R by $Rf(z) = f(-\bar{z})$. Then, Δ, R and
T_n's form a commutative family of symmetric operators on $L^2(\Gamma \backslash \mathbb{H})$.

As mentioned in Section 1, throughout this paper let us fix a simultaneous $L^2(\Gamma\backslash\mathbb{H})$-eigenfunction φ of Δ, R and all T_n's which is not a constant function. It is a cusp form. Several symbols and constants (e.g. the implied constants by the symbols $\ll$ and O) in the following will depend on φ. Let $\lambda(n)(\in \mathbb{R})$ denote the n-th Hecke eigenvalue of φ, i.e. $T_n\varphi(z) = \lambda(n)\varphi(z)$. Rankin-Selberg's theory yields

$$(2.1) \qquad \sum_{n \leq x} |\lambda(n)|^2 \sim Ax, \qquad x \to \infty,$$

where A is some positive constant. For a prime p we write

$$\lambda(p) = \alpha(p) + \beta(p) \quad \text{with } \alpha(p)\beta(p) = 1,$$

and throughout we assume $|\alpha(p)| \geq |\beta(p)|$. Then the Ramanujan conjecture for φ asserts that $|\alpha(p)| = |\beta(p)| = 1$ for all primes p. This is still open. Towards the conjecture Kim and Sarnak [7] have recently established the bound

$$(2.2) \qquad |\alpha(p)| \leq p^{\frac{7}{64}}$$

for every prime p.

The automorphic L-function $L(s,\varphi)$ associated with φ is defined by

$$L(s,\varphi) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.$$

According to (2.1), this series converges absolutely for $\operatorname{Re} s > 1$. By a certain multiplicative property of T_n's, we have the Euler product $L(s,\varphi) = \prod_{p\,:\,\text{prime}} L_p(s,\varphi)$ with

$$L_p(s,\varphi) := \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} = \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta(p)}{p^s}\right)^{-1}.$$

The function $L(s,\varphi)$ has analytic continuation to the whole s-plane. Further, it satisfies a certain functional equation (see e.g. [4, Proposition 1.9.1]), whose critical line is $\operatorname{Re} s = \frac{1}{2}$.

The following notation also will be used. Let $\mathbb{P}$ denote the set of all primes. Let p stand for a prime number, and p_n the n-th prime. Let S be the torus $\{s \in \mathbb{C} \mid |s| = 1\}$. As usual, for $s \in \mathbb{C}$ we write $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$.

§3. A limit result

Let $\mathcal{H}(D)$ denote the space of analytic functions on D, and we equip $\mathcal{H}(D)$ with the following topology of uniform convergence on compacta. Let us fix a sequence $\{K_n \mid n \in \mathbb{N}\}$ of compact subsets of D such that $K_n \subset K_{n+1}$ for every $n \in \mathbb{N}$ and such that if K is a compact set in D then $K \subset K_n$ for some n. For $f, g \in \mathcal{H}(D)$, let

$$\rho(f,g) := \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f,g)}{1 + \rho_n(f,g)},$$

where $\rho_n(f,g) := \sup_{s \in K_n} |f(s) - g(s)|$. Then ρ is a metric on $\mathcal{H}(D)$.

For a topological space X, we denote by $\mathcal{B}(X)$ the class of Borel sets of X. Let Ω denote the infinite-dimensional torus $\prod_p S_p = \prod_{n=1}^{\infty} S_{p_n}$, where $S_p = S$ for each prime p. With the product topology and point-wise multiplication, Ω is a compact Abelian topological group. Denote by m_Ω the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. For a prime p, let ω_p denote the projection of $\omega \in \Omega$ to the coordinate space S_p. Note that m_Ω is the product of probability Haar measures on coordinate spaces S_p. Hence $\{\omega_p \mid p \in \mathbb{P}\}, \omega \in \Omega$, is a sequence of independent random variables on Ω.

In this paper, we define $\log(1 + z) := -\sum_{n=1}^{\infty} \frac{(-z)^n}{n}$ for $z \in \mathbb{C}$ with $|z| < 1$. For a prime $p \in \mathbb{P}$ and a number $a_p \in S$, we define the function $f_p(s, a_p) = f_p(s, a_p, \varphi)$ on the region $\operatorname{Re} s > \frac{1}{2}$ by

(3.1)

$$f_p(s, a_p) := -\log\left(1 - \frac{\alpha(p)a_p}{p^s}\right) - \log\left(1 - \frac{\beta(p)a_p}{p^s}\right)$$

$$= \sum_{n=1}^{\infty} \frac{\alpha(p)^n a_p^n}{np^{ns}} + \sum_{n=1}^{\infty} \frac{\beta(p)^n a_p^n}{np^{ns}} = \frac{\lambda(p)a_p}{p^s} + \sum_{n=2}^{\infty} \frac{(\alpha(p)^n + \beta(p)^n) a_p^n}{np^{ns}}.$$

Note that

(3.2) $$e^{\sum_{p \leq x} f_p(s, a_p)} = \prod_{p \leq x} \left(1 - \frac{\alpha(p)a_p}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)a_p}{p^s}\right)^{-1}$$

for $x \geq 2$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > \frac{1}{2}$.

For $\omega \in \Omega$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > \frac{1}{2}$, let us define

$$L(s, \varphi, \omega) := \prod_p \left(1 - \frac{\alpha(p)\omega_p}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega_p}{p^s}\right)^{-1}.$$

As is shown essentially in the proof of Proposition 4.6 below, the series $\sum_p f_p(s, \omega_p)$ converges in $\mathcal{H}(D)$ for almost all $\omega \in \Omega$. If ω is an element in Ω such that $\sum_p f_p(s, \omega_p)$ is convergent for $\operatorname{Re} s > \frac{1}{2}$, then by (3.2), the product $L(s, \varphi, \omega)$ also is convergent for $\operatorname{Re} s > \frac{1}{2}$ and we have

$$(3.3) \qquad e^{\sum_p f_p(s, \omega_p)} = L(s, \varphi, \omega).$$

Therefore, $L(s, \varphi, \omega)$ is an $\mathcal{H}(D)$-valued random element on Ω.

Let $\widetilde{P}_\varphi$ denote the distribution of the random element $L(s, \varphi, \omega)$, i.e.

$$\widetilde{P}_\varphi(A) := m_\Omega\left(\{\omega \in \Omega \mid L(s, \varphi, \omega) \in A\}\right), \qquad A \in \mathcal{B}\left(\mathcal{H}(D)\right).$$

For $T > 0$, we define on $\mathcal{H}(D)$ the probability measure

$$(3.4)$$
$$P_{\varphi,T}(A) := \frac{1}{T} m\left(\{\tau \in [0, T] \mid L(s + i\tau, \varphi) \in A\}\right), \qquad A \in \mathcal{B}\left(\mathcal{H}(D)\right).$$

Then, the next result is proved in [17].

Proposition 3.1. *The measure $P_{\varphi,T}$ converges weakly to $\widetilde{P}_\varphi$ as $T \to \infty$.*

§4. A denseness result

The aim of this section is to prove Proposition 4.6 below. The theory of Hardy's space was used in [2] and implicitly in [9] to obtain similar results to Proposition 4.6. In the present paper, more fundamentally we will use an usual L^2-space, as in [5].

The next result is essentially shown in the proof of [2, Proposition 4.3].

Lemma 4.1. *Let H be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let $\{u_n \mid n \in \mathbb{N}\}$ be a sequence in H satisfying the following two conditions:*

  (i) $\sum_{n=1}^{\infty} \|u_n\|^2 < \infty$,
  (ii) $\sum_{n=1}^{\infty} |\langle u_n, u \rangle| = \infty$ *for any non-zero element $u \in H$.*

Let any $m \in \mathbb{N}$ be fixed. Then the set $\{\sum_{n=m}^{m'} c_n u_n \mid m' \in \mathbb{N}, m' \geq m, c_n \in S$ for $m \leq n \leq m'\}$ is dense in H.

Let U be a bounded, simply connected region in D which satisfies $\overline{U} \subset D$, where $\overline{U}$ denotes the closure of U. As usual, $L^2(U)$ denotes

the set of all $\mathbb{C}$-valued measurable functions on U which are square-integrable with respect to the Lebesgue measure. Then $L^2(U)$ is a complex Hilbert space with the inner product given by

$$\langle g_1(s), g_2(s) \rangle_U := \int_U g_1(s)\overline{g_2(s)}\, d\sigma dt, \qquad s = \sigma + it \ (\sigma, t \in \mathbb{R}).$$

and the norm given by

$$\|g(s)\|_U := \sqrt{\langle g(s), g(s) \rangle_U} = \left(\int_U |g(s)|^2 \, d\sigma dt \right)^{\frac{1}{2}}.$$

If $g(s)$ is a holomorphic function on D, then it is bounded on U and hence $g(s)$ (precisely, the function given by the restriction of the domain D of $g(s)$ to U) belongs to $L^2(U)$. Noting this, we define $H(U)$ to be the closure in $L^2(U)$ of the set

$$\{g(s) \mid g(s) \text{ is a holomorphic function on } D\} \ (\subset L^2(U)\,).$$

Then $H(U)$ is a closed subspace of the complex Hilbert space $L^2(U)$, so that $H(U)$ is also a complex Hilbert space.

To prove Proposition 4.4 below, we shall apply Lemma 4.1 with $H = H(U)$ and $u_n = \lambda(p_n)p_n^{-s}$ ($s \in U$), where p_n denotes the n-th prime. The condition (i) of Lemma 4.1 is then satisfied by (2.1).

Let $g(s) \in H(U)$. For each prime p we have

$$\langle \lambda(p)p^{-s}, g(s) \rangle_U = \int_U \lambda(p)p^{-s}\overline{g(s)}d\sigma dt = \lambda(p)\triangle_g(\log p),$$

where

$$\triangle_g(z) = \triangle_{g,U}(z) := \int_U e^{-sz}\overline{g(s)}d\sigma dt, \qquad z \in \mathbb{C}.$$

The function $\triangle_g(z)$ is entire. By the Cauchy-Schwarz inequality,

$$(4.1) \qquad |\triangle_g(z)| \leq e^{A|z|} \left(\int_U \left|\overline{g(s)}\right|^2 d\sigma dt \int_U 1 \, d\sigma dt \right)^{\frac{1}{2}},$$

where $A := \sup_{s \in U} |s| < \infty$. Hence $\triangle_g(z)$ is of exponential type.

Lemma 4.2. *Let U be as above. Let $g(s)$ be an element in $H(U)$ such that* $\limsup_{x \to \infty,\, x \in \mathbb{R}} \frac{\log|\triangle_g(x)|}{x} > -1$. *Then* $\sum_p |\lambda(p)\triangle_g(\log p)| = \infty$.

This lemma is an analogue of [2, Lemma 4.8] and [9, Section 4], and will be proved in the next section. An ingredient of the proof is the asymptotic formula

$$(4.2) \qquad \sum_{p \leq x} |\lambda(p)|^4 \sim \frac{2x}{\log x}, \qquad x \to \infty.$$

This formula is derived from [16, Theorem 2] and partial summation.

Lemma 4.3. *If $g(s)$ is a non-zero element of $H(U)$, then the function $\triangle_g(z)$ is not identically zero.*

Proof. This is proved in [17]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Now we check the condition (ii) of Lemma 4.1 with $H = H(U)$ and $u_n = \lambda(p_n)p_n^{-s}$. Let $g(s)$ be a non-zero element of $H(U)$. Then, from Lemma 4.3 and [2, Lemma 4.5] we have

$$\limsup_{x \to \infty,\, x \in \mathbb{R}} \frac{\log |\triangle_g(x)|}{x} > -1.$$

Hence Lemma 4.2 gives $\sum_{n=1}^{\infty} |\langle \lambda(p_n)p_n^{-s}, g(s) \rangle_U| = \infty$.

Thus, by Lemma 4.1 we obtain

Proposition 4.4. *Let U be a bounded, simply connected region in D which satisfies $\overline{U} \subset D$. Let any $y > 0$ be fixed. Then the set*

$$\left\{ \sum_{y \leq p \leq \nu} \frac{\lambda(p)c_p}{p^s} \,\middle|\, \nu \geq y,\ c_p \in S \text{ for } y \leq p \leq \nu \right\}$$

is dense in $H(U)$. That is, for any $g(s) \in H(U)$ and any $\varepsilon > 0$, there exist $\nu \geq y$ and $c_p \in S$, for each prime p with $y \leq p \leq \nu$, such that

$$\int_U \left| g(s) - \sum_{y \leq p \leq \nu} \frac{\lambda(p)c_p}{p^s} \right|^2 d\sigma dt < \varepsilon.$$

The next lemma is a generalization of [21, p. 303, Lemma] and proved in [14].

Lemma 4.5. *Let U be a bounded region in $\mathbb{C}$. Let K be a compact set in $\mathbb{C}$ such that $K \subset U$. Let $B > 0$. Suppose that $h(s)$ is a holomorphic function on U. If $h(s)$ satisfies*

$$\int_U |h(s)|^2 d\sigma dt \leq B,$$

then

$$\max_{s \in K} |h(s)| \le b(U, K) \sqrt{B},$$

where $b(U, K)$ is a certain positive constant depending only on U and K.

Finally we can prove the next result, in which let $f_p(s, a_p)$ be as in (3.1).

Proposition 4.6. *The set of all series $\sum_p f_p(s, a_p)$, $a_p \in S$, which converge in $\mathcal{H}(D)$ is dense in $\mathcal{H}(D)$.*

Proof. It suffices to prove that for any $g(s) \in \mathcal{H}(D)$, any compact subset K of D and any $\varepsilon' > 0$, there exists a series $\sum_p f_p(s, a_p)$, $a_p \in S$, which converges in $\mathcal{H}(D)$ and which satisfies $\sup_{s \in K} \left| g(s) - \sum_p f_p(s, a_p) \right| < \varepsilon'$.

Let $g(s) \in \mathcal{H}(D)$ and K be a compact subset of D. Let $\varepsilon > 0$ be arbitrary. Set $\sigma_1 := \min\{\operatorname{Re} s \,|\, s \in K\} > \frac{1}{2}$. We take a bounded, simply connected region U in D which satisfies $K \subset U$ and $\overline{U} \subset D$.

As in [8, p. 183] and [17], it follows from Kolmogorov's theorem ([8, p. 5, Theorem 2.11]) and (2.1) that the series $\sum_p \frac{\lambda(p)\omega_p}{p^s} = \sum_{n=1}^{\infty} \frac{\lambda(p_n)\omega_{p_n}}{p_n^s}$ converges on the region $\operatorname{Re} s > \frac{1}{2}$ for almost all $\omega \in \Omega$. In view of this, let us take a sequence $\{b_p \in S \,|\, p \in \mathbb{P}\}$ such that $\sum_p \frac{\lambda(p)b_p}{p^s}$ converges on the region $\operatorname{Re} s > \frac{1}{2}$. We note that it converges uniformly on compacta in that region, by a property of Dirichlet series. By (2.1), the series

$$\sum_p \frac{(\alpha(p)^2 + \beta(p)^2)\, b_p^2}{2p^{2s}} = \sum_p \frac{(\lambda(p)^2 - 2)\, b_p^2}{2p^{2s}}$$

converges absolutely and uniformly on compacta in the region $\operatorname{Re} s > \frac{1}{2}$. Using (2.1) and (4.2), we have

$$\sum_{p \le x} |\lambda(p)|^3 \le \left(\sum_{p \le x} |\lambda(p)|^2 \right)^{1/2} \left(\sum_{p \le x} |\lambda(p)|^4 \right)^{1/2} \ll x.$$

By this, (2.1) and (4.2), the series

$$\sum_p \frac{(\alpha(p)^3 + \beta(p)^3)\, b_p^3}{3p^{3s}} \quad \text{and} \quad \sum_p \frac{(\alpha(p)^4 + \beta(p)^4)\, b_p^4}{4p^{4s}}$$

converge absolutely and uniformly on compacta in the region $\operatorname{Re} s > \frac{1}{2}$. Since it is known (see (2.2)) that $|\alpha(p)| \leq p^\theta$ with some absolute positive constant $\theta < 3/10$ for all primes p, we have

$$\left| \sum_p \sum_{n=5}^\infty \frac{(\alpha(p)^n + \beta(p)^n)\, b_p^n}{np^{ns}} \right| \ll \sum_p \sum_{n=5}^\infty \frac{1}{p^{n(\sigma-\theta)}} \ll \sum_p \frac{1}{p^{5(\sigma-\theta)}} < \infty$$

uniformly for $\operatorname{Re} s > \frac{1}{2}$. Consequently, the series $\sum_p f_p(s, b_p)$ converges uniformly on compacta in the region $\operatorname{Re} s > \frac{1}{2}$, so that $\sum_p f_p(s, b_p)$ converges in $\mathcal{H}(D)$.

Since the series $\sum_p \frac{\lambda(p) b_p}{p^s}$ and $\sum_p \sum_{n=2}^4 \left| \frac{\alpha(p)^n + \beta(p)^n}{np^{ns}} \right|$ converge uniformly on compacta in the region $\operatorname{Re} s > \frac{1}{2}$, we may take a large real number y satisfying

$$\frac{y^{1-5(\sigma_1-3/10)}}{5(\sigma_1 - 3/10) - 1} < \varepsilon, \quad \sup_{s \in K} \left(\sum_{p \geq y} \sum_{n=2}^4 \left| \frac{\alpha(p)^n + \beta(p)^n}{np^{ns}} \right| \right) < \varepsilon$$

and $\sup_{s \in K} \left| \sum_{p \geq y_1} \frac{\lambda(p) b_p}{p^s} \right| < \varepsilon$ for all $y_1 > y$.

By Proposition 4.4, there exist a number $\nu \geq y$ and $c_p \in S$, for each prime p with $y \leq p \leq \nu$, such that

$$\int_U \left| \left(g(s) - \sum_{p<y} f_p(s,1) \right) - \sum_{y \leq p \leq \nu} \frac{\lambda(p) c_p}{p^s} \right|^2 d\sigma dt < \varepsilon^2.$$

This and Lemma 4.5 yield

$$(4.3) \qquad \max_{s \in K} \left| g(s) - \sum_{p<y} f_p(s,1) - \sum_{y \leq p \leq \nu} \frac{\lambda(p) c_p}{p^s} \right| \ll_{U,K} \varepsilon.$$

Now for each prime p we set a_p to be 1 if $p < y$, c_p if $y \leq p \leq \nu$, and b_p if $p > \nu$. Noting that $\sum_{p \geq y} f_p(s, a_p)$ converges for $\operatorname{Re} s > \frac{1}{2}$ and using the definition of y, we have, uniformly for $s \in K$,

$$\left| \sum_{p \geq y} f_p(s, a_p) - \sum_{p \geq y} \frac{\lambda(p) a_p}{p^s} \right|$$

$$= \left| \sum_{p \geq y} \sum_{n=2}^4 \frac{(\alpha(p)^n + \beta(p)^n)\, a_p^n}{np^{ns}} + \sum_{p \geq y} \sum_{n=5}^\infty \frac{(\alpha(p)^n + \beta(p)^n)\, a_p^n}{np^{ns}} \right|$$

$$\leq \varepsilon + \sum_{p \geq y} \sum_{n=5}^{\infty} \frac{|\alpha(p)^n + \beta(p)^n|}{np^{n\sigma_1}} \ll \varepsilon + \sum_{p \geq y} \frac{1}{p^{5(\sigma_1 - 3/10)}}$$

$$\ll \varepsilon + \sum_{n \geq y, n \in \mathbb{N}} \frac{1}{n^{5(\sigma_1 - 3/10)}} \ll \varepsilon + \frac{y^{1-5(\sigma_1 - 3/10)}}{5(\sigma_1 - 3/10) - 1} \ll \varepsilon.$$

Therefore using (4.3) and the definition of y, we conclude that uniformly for $s \in K$

$$\left| g(s) - \sum_p f_p(s, a_p) \right| = \left| \left(g(s) - \sum_{p<y} f_p(s, a_p) - \sum_{y \leq p \leq \nu} \frac{\lambda(p)a_p}{p^s} \right) \right.$$

$$\left. - \sum_{p>\nu} \frac{\lambda(p)a_p}{p^s} - \left(\sum_{p \geq y} f_p(s, a_p) - \sum_{p \geq y} \frac{\lambda(p)a_p}{p^s} \right) \right|$$

$$\ll_{U,K} \varepsilon + \left| \sum_{p>\nu} \frac{\lambda(p)b_p}{p^s} \right| + \varepsilon \ll \varepsilon.$$

This completes the proof.   □

In this proof, the bound $|\alpha(p)| \leq p^\theta$ with an absolute positive constant $\theta < 3/10$ is used. If instead a much better bound as in (2.2) is used, then we will have a shorter proof (see [17]).

§5.   Proof of Lemma 4.2

In our proof of Lemma 4.2, the following result is used. This is [2, Lemma 4.6] and a variant of the Bernstein theorem.

Lemma 5.1. *Let $h(z)$ be an entire function of exponential type. Let $\{r_m \mid m \in \mathbb{N}\}$ be a sequence of complex numbers and α, β, γ be positive real numbers such that*

(a) $\displaystyle \limsup_{y \to \infty, \, y \in \mathbb{R}} \frac{\log |h(\pm iy)|}{y} \leq \alpha,$

(b) $|r_m - r_n| \geq \gamma |m - n|, \qquad m, n \geq 1,$

(c) $\displaystyle \lim_{m \to \infty} \frac{r_m}{m} = \beta,$

(d) $\alpha\beta < \pi.$

Then

$$\limsup_{m \to \infty} \frac{\log |h(r_m)|}{|r_m|} = \limsup_{x \to \infty, \, x \in \mathbb{R}} \frac{\log |h(x)|}{|x|}.$$

Proof of Lemma 4.2. Let $\alpha = \alpha(U) := 10 + \sup_{s \in U} |s|$. By (4.1) we have

(5.1) $$\limsup_{y \to \infty, \, y \in \mathbb{R}} \frac{\log |\triangle_g(\pm iy)|}{y} \leq \alpha.$$

Fix a positive number β satisfying

(5.2) $$\beta \alpha < \pi.$$

Let $\mathcal{M} = \mathcal{M}(\beta, g)$ denote the set of positive integers m such that there exists a number $r \in \left(\left(m - \frac{1}{4} \right) \beta, \left(m + \frac{1}{4} \right) \beta \right]$ with $|\triangle_g(r)| \leq e^{-r}$.

Now let us suppose

(5.3) $$\sum_{m \notin \mathcal{M}} \frac{1}{m} < \infty.$$

Write $\mathcal{M} = \{ c_n \mid n \in \mathbb{N} \}$ $(\subset \mathbb{N})$ with $c_1 < c_2 < \ldots$. From (5.3) we see that

$$\lim_{m \to \infty} \frac{c_m}{m} = 1.$$

By the definition of $\mathcal{M}$, there exists a number r_m for each $m \in \mathbb{N}$ such that $(c_m - 1/4)\beta < r_m \leq (c_m + 1/4)\beta$ and $|\triangle_g(r_m)| \leq e^{-r_m}$. Then

(5.4) $$\lim_{m \to \infty} \frac{r_m}{m} = \beta \quad \text{and} \quad \limsup_{m \to \infty} \frac{\log |\triangle_g(r_m)|}{r_m} \leq -1.$$

If $m > n$, then

(5.5) $$r_m - r_n \geq (c_m - c_n)\beta - \frac{\beta}{2} \geq (m - n)\beta - \frac{\beta}{2} \geq \frac{\beta}{2}(m - n).$$

Therefore Lemma 5.1, together with (5.1), (5.2), (5.4) and (5.5), yields

$$\limsup_{x \to \infty, \, x \in \mathbb{R}} \frac{\log |\triangle_g(x)|}{x} \leq -1,$$

This contradicts the assumption on $g(s)$, so that (5.3) must be false. Thus we have

(5.6) $$\sum_{m \notin \mathcal{M}} \frac{1}{m} = \infty.$$

Fix a number μ with $0 < \mu < \sqrt{\frac{2}{3}}$. Define a set of primes $\mathcal{P}_\mu = \mathcal{P}_{\mu, \varphi}$ by

$$\mathcal{P}_\mu := \{ p : \text{prime} \mid \mu \leq |\lambda(p)| \leq 2 \}.$$

Set $a = a_{m,\beta} := \exp\left(\left(m - \frac{1}{4}\right)\beta\right)$ and $b = b_{m,\beta} := \exp\left(\left(m + \frac{1}{4}\right)\beta\right)$ for $m \in \mathbb{N}$. Then we obtain, for a large real number x,

$$\sum_{p \leq e^{(x+1/4)\beta}} |\lambda(p)\triangle_g(\log p)| \gg_\mu \sum_{\substack{p \in \mathcal{P}_\mu \\ p \leq e^{(x+1/4)\beta}}} |\triangle_g(\log p)|$$

$$(5.7) \qquad \geq \sum_{\substack{m \notin \mathcal{M} \\ m \leq x}} \sum_{\substack{p \in \mathcal{P}_\mu \\ a < p \leq b}} |\triangle_g(\log p)| \geq \sum_{\substack{m \notin \mathcal{M} \\ m \leq x}} \sum_{\substack{p \in \mathcal{P}_\mu \\ a < p \leq b}} \frac{1}{p}.$$

Fix a positive number δ with $\delta < \beta/2$. For a number $x > 0$, put $\pi(x) := \#\{p \leq x\}$ as usual and define $\pi_\mu(x) = \pi_{\mu,\varphi}(x)$ by

$$\pi_\mu(x) := \#\{p \leq x \mid p \in \mathcal{P}_\mu\}.$$

Then, as is shown below, we have

$$(5.8) \qquad \pi_\mu(u) - \pi_\mu(a) \gg_\mu \pi(u) - \pi(a)$$

uniformly for $u \in \mathbb{R}$ with $a(1 + \delta) \leq u \leq b$, if m is sufficiently large.

Now we prove the inequality (5.8). Consider the polynomial $Q_\mu(x) := -x^4 + (4 + \mu^2)x^2 - 4\mu^2 = -(x - 2)(x + 2)(x - \mu)(x + \mu)$. Set the interval $I_\mu := [-2, -\mu] \cup [\mu, 2]$. Note that $Q_\mu(x) < 0$ if $x \notin I_\mu$. Hence we get

$$\sum_{a < p \leq u} Q_\mu\left(\lambda(p)\right) \leq \sum_{\substack{a < p \leq u \\ \lambda(p) \in I_\mu}} Q_\mu\left(\lambda(p)\right)$$

$$(5.9) \qquad \leq M_\mu \sum_{\substack{a < p \leq u \\ \lambda(p) \in I_\mu}} 1 = M_\mu\left(\pi_\mu(u) - \pi_\mu(a)\right),$$

where $M_\mu := \max\{Q_\mu(x) \mid x \in I_\mu\} > 0$.

Let m be sufficiently large. Recalling $0 < \mu < \sqrt{2/3}$, let us take a small positive real number ε such that $0 < \varepsilon < \delta/100$ and $2 - 3\mu^2 - \varepsilon(5 + \mu^2)\left(4e^{\beta/2} + 4\right)\delta^{-1} > 0$. From (4.2) and the prime number theorem $\pi(x) \sim \frac{x}{\log x}$, we obtain

$$(5.10) \qquad \sum_{a < p \leq u} \lambda(p)^4 = \sum_{p \leq u} \lambda(p)^4 - \sum_{p \leq a} \lambda(p)^4$$

$$\leq \pi(u)(2 + \varepsilon) - \pi(a)(2 - \varepsilon) \leq \left(2 + \varepsilon \frac{4e^{\beta/2} + 4}{\delta}\right)(\pi(u) - \pi(a)).$$

Here we have used the relation

$$(5.11) \qquad \pi(u) + \pi(a) \leq \frac{4e^{\beta/2} + 4}{\delta}(\pi(u) - \pi(a))$$

for any u with $a(1 + \delta) \le u \le b$ (see [9, p. 353]). It is known (see e.g. [15, Remark 5.2.2] and [17]) that

$$\sum_{p \le x} |\lambda(p)|^2 \sim \frac{x}{\log x}, \qquad x \to \infty.$$

This, the prime number theorem and (5.11) imply that

$$\sum_{a < p \le u} \lambda(p)^2 = \sum_{p \le u} \lambda(p)^2 - \sum_{p \le a} \lambda(p)^2 \ge \pi(u)(1 - \varepsilon) - \pi(a)(1 + \varepsilon)$$

$$(5.12) \qquad \ge \left(1 - \varepsilon \frac{4e^{\beta/2} + 4}{\delta}\right)(\pi(u) - \pi(a)).$$

Thus, from (5.12) and (5.10) it follows that for all large m,

$$\sum_{a < p \le u} Q_\mu(\lambda(p)) = \sum_{a < p \le u} \left(-\lambda(p)^4 + \left(4 + \mu^2\right)\lambda(p)^2 - 4\mu^2\right)$$

$$\ge (\pi(u) - \pi(a))\left(2 - 3\mu^2 - \varepsilon\left(5 + \mu^2\right)\left(4e^{\beta/2} + 4\right)\delta^{-1}\right).$$

This and (5.9) complete the proof of (5.8).

Using partial summation and (5.8), we obtain for all large m

$$(5.13) \qquad \sum_{p \in \mathcal{P}_\mu, a < p \le b} \frac{1}{p} \gg_\mu \sum_{a(1+\delta) < p \le b} \frac{1}{p}.$$

From the prime number theorem $\pi(x) = \int_2^x \frac{du}{\log u} + O\left(xe^{-c\sqrt{\log x}}\right)$ with some constant $c > 0$ and partial summation, we find that for all large m

$$(5.14) \qquad \sum_{a(1+\delta) < p \le b} \frac{1}{p} \gg \left(\frac{1}{2} - \frac{\log(1 + \delta)}{\beta}\right)\frac{1}{m} > 0,$$

since $\delta < e^{\beta/2} - 1$.

Combining (5.7), (5.13) and (5.14), we have

$$(5.15) \qquad \sum_{p \le e^{(x+1/4)\beta}} |\lambda(p)\triangle_g(\log p)| \gg_{\mu,\beta,\delta} \sum_{\substack{m \notin \mathcal{M} \\ m \le x}} \frac{1}{m},$$

which goes to ∞ as $x \to \infty$, by (5.6). This completes the proof of the lemma. $\qquad\square$

§6.   Proof of Theorem 1.1

We shall deduce Theorem 1.1 from Propositions 3.1 and 4.6 in the same way as in [8, Section 6.5]. The next result is obtained in [8, p. 23, Theorem 7.10].

Lemma 6.1. *Let $\{X_m\}$ be a sequence of independent $\mathcal{H}(D)$-valued random elements and suppose that $\sum_{m=1}^{\infty} X_m$ converges in $\mathcal{H}(D)$ almost surely. Let S_{X_m} be the support of X_m. Then the support of $\sum_{m=1}^{\infty} X_m$ is the closure of the set of all $f \in \mathcal{H}(D)$ which may be written as a convergent sum $f = \sum_{m=1}^{\infty} f_m$ with $f_m \in S_{X_m}$.*

Using this lemma, Proposition 4.6 and (3.3), we can prove the next result about the measure $\widetilde{P}_\varphi$ defined in Section 3, by the same argument as in the proof of [8, p. 230, Lemma 5.5].

Lemma 6.2. *Let $\mathcal{H}_0(D)$ be the set $\{h(s) \in \mathcal{H}(D) | h(s) \text{ has no zeros}\}$. Then the support of the measure $\widetilde{P}_\varphi$ contains $\mathcal{H}_0(D)$.*

In fact, it can be proved that the support of the measure $\widetilde{P}_\varphi$ is the set

$$\{h(s) \in \mathcal{H}(D) \mid h(s) \equiv 0 \quad \text{or} \quad h(s) \text{ has no zeros}\},$$

as in [8, p. 230, Lemma 5.5]. However, Lemma 6.2 is sufficient for our purpose.

Proof of Theorem 1.1. First we consider the case that $h(s)$ has non-vanishing analytic continuation to $\mathcal{H}(D)$. Let $G = G_{h,\varepsilon}$ be the set of functions $g \in \mathcal{H}(D)$ such that $\max_{s \in K} |g(s) - h(s)| < \varepsilon$. Then G is open in $\mathcal{H}(D)$, and by Lemma 6.2 $h(s)$ is in the support of $\widetilde{P}_\varphi$. Hence $\widetilde{P}_\varphi(G) > 0$. From this, (3.4), Proposition 3.1 and a property of *weak convergence* we conclude that

$$\liminf_{T \to \infty} \frac{1}{T} m\left(\left\{\tau \in [0, T] \mid \max_{s \in K} |L(s + i\tau, \varphi) - h(s)| < \varepsilon\right\}\right)$$
$$= \liminf_{T \to \infty} P_{\varphi, T}(G) \geq \widetilde{P}_\varphi(G) > 0,$$

which completes the proof of the present case.

Now we consider the general case. Let $h(s)$ be as in the theorem and $\varepsilon > 0$ be an arbitrary small number. Since $h(s)$ has no zeros on K, by Mergelyan's theorem [19, Theorem 20.5] there exists a polynomial $q(s)$ such that $q(s) \neq 0$ on K and

(6.1) $\displaystyle \max_{s \in K} |h(s) - q(s)| < \varepsilon/4.$

Noting that $q(s)$ has only finitely many zeros and using the condition of K, we can find a simply connected region U such that $K \subset U$ and $q(s) \neq 0$ on U (see [19, Theorem 13.11, (b), (d)]). Then there exists a holomorphic function $\log q(s)$ (see [19, Theorem 13.11, (h)]). Using Mergelyan's theorem again, we find another polynomial $r(s)$ such that

$$\max_{s \in K} \left| q(s) - e^{r(s)} \right| = \max_{s \in K} |q(s)| \left| e^{r(s) - \log q(s)} - 1 \right| < \varepsilon/4.$$

This and (6.1) give

$$(6.2) \qquad \max_{s \in K} \left| h(s) - e^{r(s)} \right| < \varepsilon/2.$$

Since $e^{r(s)} \neq 0$ for all $s \in D$, we can use the result of the former case already proved, which yields

$$\liminf_{T \to \infty} \frac{1}{T} \, m \left(\{\tau \in [0, T] \mid \max_{s \in K} \left| L(s + i\tau, \varphi) - e^{r(s)} \right| < \varepsilon/2 \} \right) > 0.$$

This together with (6.2) completes the proof. $\qquad\qquad \square$

References

[1] B. Bagchi, The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series, Ph. D. thesis, Indian Statistical Institute, Calcutta, 1981.

[2] B. Bagchi, A joint universality theorem for Dirichlet L-functions, Math. Z., **181** (1982), 319–334.

[3] H. Bohr and R. Courant, Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannsche Zetafunktion, J. Reine Angew. Math., **144** (1914), 249–274.

[4] D. Bump, Automorphic Forms and Representations, Cambridge Univ. Press, 1998.

[5] S. M. Gonek, Analytic properties of zeta and L-functions, Ph. D. thesis, Univ. of Michigan, 1979.

[6] A. A. Karatsuba and S. M. Voronin, The Riemann Zeta-Function, Walter de Gruyter, 1992.

[7] H. Kim and P. Sarnak, Refined estimates towards the Ramanujan and Selberg conjectures, Appendix 2 of Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2, J. Amer. Math. Soc., **16** (2003), 139–183.

[8] A. Laurinčikas, Limit Theorems for the Riemann Zeta-Function, Kluwer, 1996.

[9] A. Laurinčikas and K. Matsumoto, The universality of zeta-functions attached to certain cusp forms, Acta Arith., **98** (2001), 345–359.

[10] A. Laurinčikas and K. Matsumoto, The joint universality of twisted automorphic L-functions, J. Math. Soc. Japan, **56** (2004), 923–939.

[11] A. Laurinčikas, K. Matsumoto and J. Steuding, The universality of L-functions associated with newforms, Izv. Ross. Akad. Nauk Ser. Mat., **67** (2003), 83–98; Izv. Math., **67** (2003), 77–90.

[12] W. Luo, Z. Rudnick and P. Sarnak, On the generalized Ramanujan conjectures for $GL(n)$, Proc. Symposia in Pure Math., **66**, Part 2, 301–310, Amer. Math. Soc., 1999.

[13] K. Matsumoto, Probabilistic value-distribution theory of zeta-functions, Sugaku Expositions, **17** (2004), 51–71.

[14] H. Mishou and H. Nagoshi, Functional distribution of $L(s, \chi_d)$ with real characters and denseness of quadratic class numbers, to appear in Trans. Amer. Math. Soc.

[15] C. J. Moreno, Explicit formulas in the theory of automorphic forms, Lecture Notes in Math., **626**, Springer-Verlag, 1977, pp. 73–216.

[16] M. Ram Murty, On the estimation of eigenvalues of Hecke operators, Rocky Mountain J. Math., **15** (1985), 521–533.

[17] H. Nagoshi, On the universality for L-functions attached to Maass forms, Analysis, **25** (2005), 1–22.

[18] H. Nagoshi, A joint universality theorem for automorphic L-functions, in preparation.

[19] W. Rudin, Real and Complex Analysis, Third Edition, McGraw-Hill, 1987.

[20] J. Steuding, On the universality for functions in the Selberg class, In: Proceedings of the session in analytic number theory and Diophantine equations, Bonn, 2002, (eds. D. R. Heath-Brown and B. Z. Moroz), Bonner Mathematische Schriften, **360**, 2003.

[21] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, Second Edition, Oxford Univ. Press, 1986.

[22] S. M. Voronin, On the distribution of nonzero values of the Riemann zeta-function, Trudy Mat. Inst. Steklov., **128** (1972), 131–150; Proc. Steklov Inst. Math., **128** (1972), 153–175.

[23] S. M. Voronin, Theorem on the universality of the Riemann zeta-function, Izv. Akad. Nauk. SSSR Ser. Mat., **39** (1975), 475–486; Math. USSR-Izv., **9** (1975), 443–453.

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