Advanced Studies in Pure Mathematics 49, 2007
Probability and Number Theory — Kanazawa 2005
pp. 219–244

On Q-multiplicative functions having a positive upper-meanvalue

Jean-Loup Mauclaire

Abstract.

A classical approach to study properties of Q–multiplicative functions $f(n)$ is to associate to the mean $\frac{1}{x}\sum_{0\le n\le x} f(n)$ the product $\prod_{0\le j\le k}\frac{1}{q_j}\sum_{0\le a\le q_j-1} f(aQ_j)$. We discuss its validity in the case of non-negative Q–multiplicative functions $f(n)$ with a positive upper meanvalue, defined via a Cantor numeration system.

§1. Introduction and notations

1.1. Numeration systems and associated additive functions

Let N be the set of non-negative integers, and $Q = (Q_k)_{k\ge 0}$, $Q_0 = 1$, be an increasing sequence of positive integers. Using the greedy algorithm to every element n of N, one can associate a representation

$$n = \sum_{k=0}^{+\infty} \varepsilon_k(n) Q_k,$$

which is unique if for every K,

$$\sum_{k=0}^{K-1} \varepsilon_k(n) Q_k < Q_K.$$

Such a condition provides a numeration scale and in this case, we can define on N a complex-valued arithmetic function $f(n)$ by $f(0.Q_k) = 1$

Received May 15, 2006.
Revised January 23, 2007.
2000 *Mathematics Subject Classification.* Primary 11A25; Secondary 11N64, 11N56.
Key words and phrases. mean-value, Q-multiplicative functions.

and $f(n) = \prod_{k\geq 0} f(\varepsilon_k(n)Q_k)$, and it will be called a Q-multiplicative function.

Simple examples of numeration scales are the q-adic scale, where $Q_k = q^k$, q integer, $q \geq 2$, and its generalization, the Cantor scale $Q_{k+1} = q_k Q_k$, $Q_0 = 1$, $q_k \geq 2$, $k \geq 0$.

A classical approach to study properties of Q-multiplicative functions $f(n)$ is to associate to the mean $\frac{1}{x}\sum_{0\leq n<x} f(n)$ the product

$$\prod_{0\leq j\leq k} \frac{1}{q_j} \sum_{0\leq a\leq q_j-1} f(aQ_j)),$$

and in fact, this correspondence essentially explains a natural underlying probabilistic structure.

Now, although the q-adic scale and its generalization, the Cantor scale, seem very similar, basic differences may exist between them. More precisely, if a Cantor system is such that there exists some uniform bound B of the q_k, there is practically no differences, and this is due essentially to this uniformity condition. Otherwise, if we allow the q_k to be unbounded, the situation is not so simple. An example was given in [4], where the case of the mean-value of unimodular Q-multiplicative functions is considered.

§2. Results

In the simple case of non-negative Q-multiplicative functions, the existence of some essential difference can be shown. In fact, we have the following result:

Theorem 1. *1) For a given Cantor scale with uniformly bounded q_k and for any non-negative q-multiplicative function f, the condition*

$$\limsup_{x\to+\infty} \frac{1}{x} \sum_{0\leq n<x} f(n) \text{ exists and is positive}$$

is equivalent to the condition

$$\limsup_{k\to+\infty} \prod_{0\leq j\leq k} \frac{1}{q_j} \sum_{0\leq a\leq q_j-1} f(aQ_j)) \text{ exists and is positive.}$$

2) There exist Cantor scales (Q) with not uniformly bounded q_k and non-negative Q-multiplicative functions f such that the condition

$$\limsup_{x\to+\infty} \frac{1}{x} \sum_{0\leq n<x} f(n) \text{ exists and is positive}$$

will not imply the condition

$$\limsup_{k\to+\infty} \prod_{0\le j\le k} \frac{1}{q_j} \sum_{0\le a\le q_j-1} f(aQ_j)) \text{ exists and is positive,}$$

and non-negative Q-multiplicative functions f such that the condition

$$\limsup_{k\to+\infty} \prod_{0\le j\le k} \frac{1}{q_j} \sum_{0\le a\le q_j-1} f(aQ_j)) \text{ exists and is positive}$$

will not imply the condition

$$\limsup_{x\to+\infty} \frac{1}{x} \sum_{0\le n<x} f(n) \text{ exists and is positive.}$$

In this article, we shall consider the case of non-negative Q- multiplicative functions with a positive upper meanvalue defined via an unbounded Cantor system.

Given an arbitrary arithmetical function f, we set

$$S_N(f) = \sum_{0\le n<N} f(n),$$

$$\varpi_k(f) = q_k^{-1} \sum_{0\le n\le q_k-1} f(aQ_k),$$

$$\prod\nolimits_{k-}(f) = \prod_{0\le r\le k-1} \varpi_k(f).$$

For our convenience, the result of a summation (resp. a product) on an empty set will be 0 (resp.1).

Now, for a given f of non-negative Q-multiplicative function, we define a sequence of arithmetical functions $f_{k-}(x)$ on Z_Q (resp. $f^*_{k-}(x)$) by $f_{k-}(x) = \prod_{0\le j<k} f(a_jQ_r)$ (resp. $f^*_{k-}(x) = \prod_{0\le j<k} f(a_jQ_j).\varpi_j(f)^{-1}$), where x being written in base Q as $x = \sum_{j=0}^{+\infty} a_jQ_k$. For simplicity, we shall also use the notations $f_j(x) = f(aQ_j)$ and $f^*(aQ_j) = f(aQ_j).\varpi_j(f)^{-1}$.

We denote by Z_Q the compact group$Z_Q = \lim_{k\to+\infty} Z/Q_kZ$ equipped with the natural Haar measure μ, and we shall identify it with the compact space $\prod_k Z/q_kZ$ equipped with the measure $\mu = \otimes_k \mu_{q_k}$, where μ_{q_k} is the uniform measure on Z/q_kZ. An element a of Z_Q can be written as $a = (a_0, a_1, ...), 0 \le a_k \le q_k - 1, 0 \le k$, and an integer is an element of Z_Q which has only a finite number of digits different from zero. For

$a = (a_0, a_1, ...)$ in Z_Q, we denote by $x_{k_-}(a)$ the sequence of random variables defined by $x_{k_-}(a) = \{a_j\}_{0 \le j \le k-1}$, and by $x_{k_+}(a)$ the sequence of random variables defined by $x_{k_+}(a) = \{a_j\}_{k \le j}$. We shall use also the notation x_k for an integer $x_k = \sum_{j=0}^{k-1} a_j Q_k$ when $x = \sum_{j=0}^{+\infty} a_j Q_k$.

We have the following result:

Theorem 2. *Let (Q) be an unbounded Cantor system, and $f(n)$ be a non-negative Q-multiplicative function such that*

$$\limsup_{x \to +\infty} \frac{1}{x} \sum_{0 \le n \le x-1} f(n)$$

exists and is positive. Then, there are two possibilities:

1) $\sum_{1 \le k} q_k^{-1} \sum_{0 \le a \le q_k - 1} \left(1 - f(aQ_k)^{1/2} \varpi_k (f^{1/2})^{-1}\right)^2$ is bounded, and in this case, for any r, $0 \le r \le 1$, we have μ-almost surely

$$\frac{1}{x_k} \sum_{0 \le n \le x_k - 1} f(n)^r = \left(\prod_{0 \le j \le k} \frac{1}{q_j} \sum_{0 \le a \le q_j - 1} f(aQ_j)^r \right) . (1 + o(1)),$$

as $x_k \to x$.

2) $\sum_{1 \le k} q_k^{-1} \sum_{0 \le a \le q_k - 1} \left(1 - f(aQ_k)^{1/2} \varpi_k (f^{1/2})^{-1}\right)^2$ is not bounded, and in this case, for any r, $0 < r < 1$, we have

$$\frac{1}{x} \sum_{0 \le n \le x-1} f(n)^r = o(1), \qquad \text{as } x \to +\infty.$$

§3. Proof of the results

3.1. Proof of Theorem 1

1) We begin with a proof of assertion 1).

Proof. Assume that $S = \limsup_{x \to +\infty} x^{-1} \sum_{0 \le n < x} f(n)$ exists and is positive.

Let x_i be a sequence such that

$$\frac{1}{2} S \le x_i^{-1} \sum_{0 \le n < x_i} f(n).$$

A fortiori, if $\kappa(x_i)$ denotes the maximal index k for which $a_k(x_i)$ is different from zero, then we have

$$\frac{1}{2} S \le x_i^{-1} \sum_{0 \le n < Q_{\kappa(x_i)+1}} f(n),$$

and so

$$\left(\frac{Q_{\kappa(x_i)+1}}{x_i}\right)^{-1} \times \left(\frac{1}{2}S\right) \leq \left(\frac{1}{Q_{\kappa(x_i)+1}} \sum_{0\leq n<Q_{\kappa(x_i)+1}} f(n)\right).$$

Since $\left(\frac{Q_{\kappa(x_i)+1}}{x_i}\right)^{-1} \geq \frac{1}{\max(q_k)}$ and $\max(q_k)$ is bounded, this gives us that there is some $S' \geq \frac{1}{2.\max(q_k)}S$, hence > 0, such that

$$0 < S' \leq \limsup_{k\to+\infty} \frac{1}{Q_k} \sum_{0\leq n\leq Q_k-1} f(n) < +\infty.$$

Conversely, if there exists some positive S'' such that

$$\limsup_{k\to+\infty} \frac{1}{Q_k} \sum_{0\leq n\leq Q_k-1} f(n) = S'' < +\infty,$$

then by using the same notations as above, we remark that, since

$$\sum_{0\leq n\leq Q_{\kappa(x)}} f(n) \leq \sum_{0\leq n\leq x} f(n) \leq \sum_{0\leq n<Q_{\kappa(x)+1}} f(n),$$

we have

$$(x^{-1}Q_{\kappa(x)}) \left(Q_{\kappa(x)}^{-1} \sum_{0\leq n<Q_{\kappa(x)}} f(n)\right) \leq x^{-1} \sum_{0\leq n\leq x} f(n)$$

and

$$x^{-1} \sum_{0\leq n\leq x} f(n) \leq (x^{-1}Q_{\kappa(x)+1}) \left(Q_{\kappa(x)+1}^{-1} \sum_{0\leq n<Q_{\kappa(x)+1}} f(n)\right).$$

Hence we get that

$$0 < \frac{1}{\max(q_k)} S'' \leq \limsup_{x\to+\infty} x^{-1} \sum_{0\leq n\leq x} f(n),$$

for $(x^{-1}Q_{\kappa(x)}) \geq \frac{1}{\max(q_k)} > 0$, and

$$\limsup_{x\to+\infty} x^{-1} \sum_{0\leq n\leq x} f(n) \leq \max(q_k)S'' < +\infty,$$

since

$$(x^{-1}Q_{\kappa(x)+1}) \leq \max(q_k) < +\infty,$$

and so

$$\limsup_{x \to +\infty} x^{-1} \sum_{0 \le n \le x} f(n)$$

exists and its value is positive.

Q.E.D.

2) We prove now assertion 2).

Proof. We consider the following (with indexation shifted for convenience of notations) Q-system, satisfying $\limsup(q_k) = +\infty$:

$$q_k = k, \ k \ge 2,$$

and the Q-multiplicative function f defined by

$$\begin{aligned} &f(aQ_k) = 1 \ \text{ if } k \ne 2^r \text{ and } 0 \le a \le q_k - 2, \\ &f((q_k - 1)Q_k) = 0 \ \text{ if } k \ne 2^r, \\ &f(Q_{2^r}) = 2^r - 1, \\ &f(aQ_{2^r}) = 0 \ \text{ if } 2 \le a \le 2^r - 1. \end{aligned}$$

We have

$$\begin{aligned} &\prod_{2 \le j \le k} \frac{1}{q_j} \sum_{0 \le a \le q_j - 1} f(aQ_j)) \\ &= \left(\prod_{2 \le j \le k, j \ne 2^r} \frac{1}{j}(j-1) \right) \left(\prod_{2 \le j \le k, j = 2^r} \frac{1}{2^r}(1 + (2^r - 1)) \right) \end{aligned}$$

and

$$\begin{aligned} \prod_{2 \le j \le k, j \ne 2^r} \frac{1}{j}(j-1) &= \left(\prod_{2 \le j \le k} \frac{1}{j}(j-1) \right) \left(\prod_{2 \le j \le k, j = 2^r} \frac{1}{2^r}(2^r - 1) \right)^{-1} \\ &= ((k-1)!/k!) \left(\prod_{2 \le j \le k, j = 2^r} \frac{1}{2^r}(2^r - 1) \right)^{-1} \\ &= \frac{1}{k} \prod_{2 \le j \le k, j = 2^r} (1 - \frac{1}{2^r})^{-1}, \end{aligned}$$

and so, since $\prod_{2 \le r}(1 - \frac{1}{2^r})^{-1}$ is convergent, we have

$$\lim_{k \to +\infty} \prod_{2 \le j \le k} \frac{1}{q_j} \sum_{0 \le a \le q_j - 1} f(aQ_j)) = 0.$$

Now, for $x = 2Q_{2^k} - 1$, we have

$$\begin{aligned} \frac{1}{x+1} \sum_{0 \le n \le x} f(n) &= \frac{1}{2Q_{2^k}} \sum_{0 \le n \le 2Q_{2^k}-1} f(n) \\ &= \left(\frac{1}{2} \left(f(0.Q_{2^k} + f(1.Q_{2^k})\right) \right) \times \left(\prod_{r=2}^{2^k-1} \frac{1}{q_r} \sum_{a=0}^{q_r-1} f(aQ_r) \right) \\ &= \left(\frac{1}{2} 2^k \right) \times \left(\frac{1}{2^k - 1} \prod_{2 \le r \le k-1} (1 - \frac{1}{2^r})^{-1} \right) \\ &\ge \frac{1}{2} \\ &> 0. \end{aligned}$$

As a consequence, the condition

$$0 < \limsup_{x \to +\infty} \frac{1}{x} \sum_{0 \le n < x} f(n) < +\infty$$

will not imply

$$0 < S' = \limsup_{k \to +\infty} \prod_{2 \le j \le k} \frac{1}{q_j} \sum_{0 \le a \le q_j - 1} f(aQ_j)) < +\infty$$

for some S'.

In a similar way, it is possible, using the same kind of approach as above, to provide an example of Q-multiplicative function such that the condition

$$\limsup_{k \to +\infty} \prod_{0 \le j \le k} \frac{1}{q_j} \sum_{0 \le a \le q_j - 1} f(aQ_j)) < +\infty$$

will not imply the condition

$$\limsup_{x \to +\infty} \frac{1}{x} \sum_{0 \le n < x} f(n) < +\infty.$$

It is sufficient to consider the following (again with indexation shifted for convenience of notations) Q-system, satisfying $\limsup(q_k) = +\infty$:

$$q_k = k, \ \ k \ge 2,$$

and the Q-multiplicative function f defined by

$$f(aQ_k) = 1 \ \text{ if } k \ne 2^r,$$

$$f(Q_{2^r}) = 2^r - 1,$$
$$f(aQ_{2^r}) = 0 \ \text{ if } 2 \leq a \leq 2^r - 1.$$

We have

$$\prod_{2\leq j\leq k} \frac{1}{q_j} \sum_{0\leq a\leq q_j-1} f(aQ_j))$$
$$= \left(\prod_{2\leq j\leq k, j\neq 2^r} \frac{1}{j} \sum_{0\leq a\leq j-1} 1 \right) \left(\prod_{2\leq j\leq k, j=2^r} \frac{1}{2^r}(1 + (2^r - 1)) \right) = 1.$$

Now, for $x = 2Q_{2^k} - 1$, we have

$$\frac{1}{x+1} \sum_{0\leq n\leq x} f(n) = \frac{1}{2Q_{2^k}} \Sigma_{0\leq n\leq 2Q_{2^k}-1} f(n)$$
$$= \left(\frac{1}{2} \left(f(0.Q_{2^k} + f(1.Q_{2^k})\right) \right) \times \left(\prod_{r=2}^{2^k-1} \frac{1}{q_r} \sum_{a=0}^{q_r-1} f(aQ_r) \right)$$
$$= \left(\frac{1}{2} 2^k \right)$$
$$= 2^{k-1}.$$

Q.E.D.

3.2. Proof of theorem 2

3.2.1. Method of proof

The method is as follows:

i) We associate to f a Radon measure ν_f on Z_Q.

ii) We prove that ν_f is absolutely continuous with respect to μ if

$$\sum_{1\leq k} q_k^{-1} \sum_{0\leq a\leq q_k-1} \left(1 - f(aQ_k)^{1/2} \varpi_k (f^{1/2})^{-1}\right)^2$$

is bounded, and orthogonal to μ if

$$\sum_{1\leq k} q_k^{-1} \sum_{0\leq a\leq q_k-1} \left(1 - f(aQ_k)^{1/2} \varpi_k (f^{1/2})^{-1}\right)^2$$

is not bounded.

Remark that this dichotomy leaves no other eventuality.

iii) We prove part 1) of Theorem 2 in the case $r = 1$ as a simple consequence of the absolute continuity of ν_f.

iv) We show that to f^r, $0 < r < 1$, one can associate a Radon measure which is absolutely continuous with respect to μ. As a consequence, with iii), this gives the proof of part 1) of Theorem 2.

v) We prove directly part 2) of Theorem 2.

3.2.2.

We denote by $(a, k(a))$ an arithmetical progression $\{a + Q_{k(a)}n\}_{n\in N}$, where a is in N, $k(a)$ is a positive integer such that $Q_{k(a)} > a$. Let $I_{a,k(a)}$ be its characteristic function. Remark that $I_{a,k(a)}$ is the restriction to N of the characteristic function, still denoted $I_{a,k(a)}$, of the open subset $O_{(a,k(a))}$ of Z_Q defined by $O_{(a,k(a))} = \left(x_{k(a)_-}(a), \prod_{k\geq k(a)} Z/q_k Z\right)$, and that this function is continuous, which implies that

$$\lim \frac{1}{x} \sum_{0\leq n<x} I_{a,k(a)}(n) = \mu(O_{(a,k(a))}).$$

i) Radon measure associated to f.

Let $f(n)$ be a nonnegative Q-multiplicative function with a positive bounded upper mean-value $\overline{M}(f)$. Since $\overline{M}(f)$ exists, the series $\sum_{n\in N} f(n)x^n$ converges for $|x| < 1$ and can be written as

$$\sum_{n\in N} f(n)x^n = \lim_{k\to+\infty} \sum_{0\leq n\leq Q_k-1} f(n)x^n = \prod_{0\leq k}(\sum_{0\leq b\leq q_k-1} f(bQ_k)x^{bQ_k}).$$

Moreover, since $f(n)$ is non-negative for all n in N, as a consequence of a theorem of Hardy and Littlewood ([1], theorem 4), we get that there exists some $L > 0$ and a sequence $(x_k)_{k\in N}$ such that $\lim_{k\to+\infty} x_k = 1$ and $\lim_{k\to+\infty}(1 - x_k)^{-1} \sum_{n\in N} f(n)x_k^n = L$.

In fact if not, then,

$$\lim_{x\to 1_-} (1 - x)^{-1} \sum_{n\in N} f(n)x^n = 0,$$

which implies that the mean value of $f(n)$ is equal to zero, a contradiction with our hypothesis that $f(n)$ has a positive bounded upper mean-value $\overline{M}(f)$.

Now, we remark that

$$\sum_{n\in N} f(n)I_{a,k(a)}(n)x^n = \sum_{n\in N, n\equiv a (\mathrm{mod}\, Q_{k(a)})} f(n)x^n$$

and, since the function $f_{k(a)}(n)$ defined by $f_{k(a)}(n) = f(Q_{k(a)}n)$ can be regarded as a Q-multiplicative function for the Cantor system defined by $q'_k = q_{k+k(a)}$, $k \geq 0$, we get that

$$\sum_{n\in N} f(n)I_{a,k(a)}(n)x^n = \sum_{m\in N} f(a+Q_{k(a)}m)x^{a+Q_{k(a)}m}$$
$$= f(a)x^a \sum_{m\in N} f(Q_{k(a)}m)x^{Q_{k(a)}m}$$
$$= f(a)x^a \prod_{k\geq k(a)} \Big(\sum_{0\leq b\leq q_k-1} f(bQ_k)x^{bQ_k}\Big)$$
$$= f(a)x^a \left(\left(\prod_{0\leq k\leq k(a)-1} \Big(\sum_{0\leq b\leq q_k-1} f(bQ_k)x^{bQ_k}\Big)\right)^{-1} \times \left(\prod_{0\leq k} \Big(\sum_{0\leq b\leq q_k-1} f(bQ_k)x^{bQ_k}\Big)\right)\right)$$
$$= \left(f(a)x^a \left(\prod_{0\leq k\leq k(a)-1} \Big(\sum_{0\leq b\leq q_k-1} f(bQ_k)x^{bQ_k}\Big)\right)^{-1}\right) \times \left(\sum_{n\in N} f(n)x^n \right).$$

Since $f(n)$ is non-negative and $f(0.Q_k) = 1$, the function $F_{a,k(a)}(x)$ defined by

$$F_{a,k(a)}(x) = \left(f(a)x^a \left(\prod_{0\leq k\leq k(a)-1} \Big(\sum_{0\leq b\leq q_k-1} f(bQ_k)x^{bQ_k}\Big)\right)^{-1}\right)$$

is analytic on a neighborhood of 1, and as a consequence of the relation

$$\sum_{n\in N} f(n)x_k^n \backsim (1-x_k)L \quad \text{as } k \to +\infty,$$

we get that

$$\sum_{n\in N} f(n)I_{a,k(a)}(n)x_k^n \backsim (1-x_k)LF_{a,k(a)}(1) \quad \text{as } k \to +\infty,$$

i.e.

$$\lim_{k\to+\infty} (1-x_k)^{-1} \sum_{n\in N} f(n)I_{a,k(a)}(n)x_k^n$$
$$= Lf(a) \left(\prod_{0\leq k\leq k(a)-1} \Big(\sum_{0\leq b\leq q_k-1} f(bQ_k)\right)^{-1} \quad \text{as } k \to +\infty.$$

And so, we shall define $\nu_f(I_{a,k(a)})$ by

$$\nu_f(I_{a,k(a)}) = f(a) \left(\prod_{0\leq k\leq k(a)-1} (\sum_{0\leq b\leq q-1} f(bq^k)) \right)^{-1} .$$

Now, we check that ν_f is a Radon measure. (For the definition, properties of the Radon measures, see [3], ch2, p.57 et seq.). To do that, we consider the set $\mathcal{A}$ of complex-valued continuous functions defined on Z_Q by

$$\mathcal{A} = \left\{ h = \sum_{l_a\in L} l_a.I_{a,k(a)},\ L \text{ finite, } l_a \text{ complex numbers} \right\}.$$

This is an algebra of step functions, and by the Stone-Weierstrass theorem ([2], p. 101, note 1.a), $\mathcal{A}$ is dense with respect to the uniform topology in the set of the complex-valued continuous functions defined on Z_Q. If h is in $\mathcal{A}$, we define $\nu_f(h)$ by $\nu_f(h) = \sum_{l_a\in L} l_a.\nu_f(I_{a,k(a)})$. It is a simple remark that we have

$$\nu_f(h) = L^{-1} \lim_{k\to+\infty} (1-x_k)^{-1} \sum_{n\in N} f(n)h(n)x_k^n.$$

Since $\nu_f(1) = 1$, for a given $\varepsilon > 0$, if h and $h'$are in $\mathcal{A}$ and satisfy $\sup_{t\in Z_Q} |h'(t) - h(t)| \leq \varepsilon$, then we have $|\nu_f(h' - h)| \leq \varepsilon$, since $|\nu_f(h' - h)| \leq \nu_f(1). \sup_{t\in Z_Q} |h'(t) - h(t)| \leq 1.\varepsilon$, and so ν_f defines a continuous linear form on the set of the complex-valued continuous functions defined on Z_Q. By Riesz representation theorem ([2], p. 129, (11.37)), this gives us that ν_f is a positive Radon measure on Z_Q.

ii) Characterization of the absolute continuity (resp. orthogonality) of ν_f with respect to μ.

For K in N, we have

$$\begin{aligned} &1 - f_{K-}(t)^{1/2}\textstyle\prod_{K-}(f^{1/2})^{-1} \\ &= \sum_{1\leq k\leq K} \left(f_{(k-1)-}(t)^{1/2}\textstyle\prod_{(k-1)-}(f^{1/2})^{-1} - f_{k-}(t)^{1/2}\prod_{k-}(f^{1/2})^{-1} \right) \\ &= \sum_{1\leq k\leq K} \left(f_{(k-1)-}(t)^{1/2}\textstyle\prod_{(k-1)-}(f^{1/2})^{-1} \right) \left(1 - f_{k-1}(t)^{1/2}\varpi_{k-1}(f^{1/2})^{-1} \right). \end{aligned}$$

We remark that

$$\int \left(1 - f_{k-1}(t)^{1/2}\varpi_{k-1}(f^{1/2})^{-1} \right) d\mu(t) = 0,$$

$$\int \left(1 - f_k(t)^{1/2}\varpi_k(f^{1/2})^{-1}\right)\left(1 - f_l(t)^{1/2}\varpi_l(f^{1/2})^{-1}\right) d\mu(t)$$
$$\begin{cases} = 0 \quad \text{if } k \neq l, \ \text{and} \\ = q_k^{-1} \displaystyle\sum_{0 \leq a \leq q_k+1-1} \left(1 - f(aQ_k)^{1/2}\varpi_k(f^{1/2})^{-1}\right)^2 \quad \text{if } k = l. \end{cases}$$

As a consequence of these orthogonality relations, we get that

$$\begin{aligned} &\int \left(1 - f_{K-}(t)^{1/2}\prod\nolimits_{K-}(f^{1/2})^{-1}\right)^2 d\mu(t) \\ &\quad = \sum_{1 \leq k \leq K} \int \left(f_{(k-1)-}(t)^{1/2}\prod\nolimits_{(k-1)-}(f^{1/2})^{-1}\right)^2 d\mu(t) \\ &\qquad \times \int \left(1 - f_{k-1}(t)^{1/2}\varpi_{k-1}(f^{1/2})^{-1}\right)^2 d\mu(t). \end{aligned}$$

Now, since we have

$$\begin{aligned} &\int \left(f_{(k-1)-}(t)^{1/2}\prod\nolimits_{(k-1)-}(f^{1/2})^{-1}\right)^2 d\mu(t) \\ &\quad = \prod\nolimits_{(k-1)-}(f) \times \prod\nolimits_{(k-1)-}(f^{1/2})^{-2}, \end{aligned}$$

we obtain that

$$\begin{aligned} &\int \left(1 - f_{K-}(t)^{1/2}\prod\nolimits_{K-}(f^{1/2})^{-1}\right)^2 d\mu(t) \\ &\quad = \prod\nolimits_{K-}(f) \times \prod\nolimits_{K-}(f^{1/2})^{-2} - 1 \\ &\quad = \sum_{1 \leq k \leq K-1} \left(\prod\nolimits_{(k-1)-}(f) \times \prod\nolimits_{(k-1)-}(f^{1/2})^{-2}\right) \\ &\qquad \times \left(q_k^{-1} \sum_{0 \leq a \leq q_k-1} \left(1 - f(aQ_k)^{1/2}\varpi_k(f^{1/2})^{-1}\right)^2\right), \end{aligned}$$

and if we are in the situation such that $\lim_{k\to+\infty}\left(\prod_{k-}(f) \times \prod_{k-}(f^{1/2})^{-2}\right)$ exists and is > 0, we get that the series

$$\sum_{1 \leq k} q_k^{-1} \sum_{0 \leq a \leq q_k-1} \left(1 - f(aQ_k)^{1/2}\varpi_k(f^{1/2})^{-1}\right)^2$$

is convergent.

Assuming that we are in the case where

$$\lim_{K\to+\infty} \prod\nolimits_{K-}(f)^{-1} \times \prod\nolimits_{K-}(f^{1/2})^2 = 0,$$

we consider the equality

$$\begin{aligned}&\prod_{K-}(f) \times \prod_{K-}(f^{1/2})^{-2} - 1 \\ &= \sum_{1\le k\le K-1} \left(\prod_{(k-1)-}(f) \times \prod_{(k-1)-}(f^{1/2})^{-2}\right) \\ &\qquad \times \left(q_k^{-1} \sum_{0\le a\le q_k+1-1} \left(1 - f(aQ_k)^{1/2}\varpi_k(f^{1/2})^{-1}\right)^2 \right).\end{aligned}$$

We multiply each member of this equality by $\prod_{K-}(f)^{-1}\times\prod_{K-}(f^{1/2})^2$, and we get that

$$\begin{aligned}&1 - \prod_{K-}(f)^{-1} \times \prod_{K-}(f^{1/2})^2 \\ &= \sum_{1\le k\le K-1} A(K,k) \times \left(q_k^{-1} \sum_{0\le a\le q_k-1} \left(1 - f(aQ_k)^{1/2}\varpi_k(f^{1/2})^{-1}\right)^2 \right),\end{aligned}$$

where $A(K,k)$ is defined by

$$A(K,k) = \prod_{(k-1)-}(f) \times \prod_{(k-1)-}(f^{1/2})^{-2} \times \prod_{K-}(f)^{-1} \times \prod_{K-}(f^{1/2})^2.$$

Now, we remark that if

$$\lim_{K\to+\infty} \prod_{K-}(f)^{-1} \times \prod_{K-}(f^{1/2})^2 = 0,$$

then, for a fixed k, we have

$$\lim_{K\to+\infty} A(K,k) = 0.$$

Since we have

$$\lim_{K\to+\infty} (1 - \prod_{K-}(f)^{-1} \times \prod_{K-}(f^{1/2})^2) = 1,$$

we get that the series of general term $q_k^{-1}\sum_{0\le a\le q_k-1}\left(1 - f(aQ_k)^{1/2}\varpi_k\right.$ $\left.(f^{1/2})^{-1}\right)^2$ is not convergent, i.e.

$$\limsup_{K\to+\infty} \sum_{1\le k\le K} q_k^{-1} \sum_{0\le a\le q_k-1} \left(1 - f(aQ_k)^{1/2}\varpi_k(f^{1/2})^{-1}\right)^2 = +\infty.$$

This proves that the measure ν_f is continuous with respect to μ (resp. orthogonal to μ) if and only if the series of general term q_k^{-1} $\sum_{0\le a\le q_k-1}\left(1 - f(aQ_k)^{1/2}\varpi_k(f^{1/2})^{-1}\right)^2$ is convergent (resp. divergent).

iii) Part 1) of Theorem 2 in the case $r = 1$ is a simple consequence of the absolute continuity of ν_f.

Proof. We shall apply to the present situation the method of proof given in [4].

1) First we prove

Lemma 1. *There exists a subset F_∞ of $\mathbf{Z}_Q$ such that $\mu(F_\infty) = 1$ and for every $x = (a_0(x), a_1(x), ...)$ in F_∞, we have*

$$\lim_{\substack{k\to+\infty \\ a_k(x)\neq 0}} \frac{1}{a_k(x)} \sum_{0\leq a<a_k(x)} (1 - f^*(aQ_k)) = 0.$$

2) This is a consequence of the following result:

Lemma 2. *There exists a subset F_∞ of $\mathbf{Z}_Q$ such that $\mu(F_\infty) = 1$ and for every $x = (a_0(x), a_1(x), ...)$ in F_∞, we have*

$$\lim_{\substack{k\to+\infty \\ a_k(x)\neq 0}} \frac{1}{a_k(x)} \sum_{0\leq a<a_k(x)} \left(1 - f^*(aQ_k)^{1/2}\right)^2 = 0.$$

Proof. 2) $\Rightarrow$ 1).

We have

$$\left(1 - f^*(aQ_k)^{1/2}\right)^2 = 2.(1 - f^*(aQ_k)^{1/2}) - (1 - f^*(aQ_k)),$$

which gives us that

$$(1 - f^*(aQ_k)) = 2.(1 - f^*(aQ_k)^{1/2}) - \left(1 - f^*(aQ_k)^{1/2}\right)^2.$$

As a consequence, we get that

$$\sum_{0\leq a<a_k(x)} (1 - f^*(aQ_k))$$
$$= \sum_{0\leq a<a_k(x)} 2.(1 - f^*(aQ_k)^{1/2}) - \sum_{0\leq a<a_k(x)} \left(1 - f^*(aQ_k)^{1/2}\right)^2,$$

which gives that

$$\left|\sum_{0\leq a<a_k(x)} (1 - f^*(aQ_k))\right|$$
$$\leq 2\left|\sum_{0\leq a<a_k(x)} \left(1 - f^*(aQ_k)^{1/2}\right)\right| + \sum_{0\leq a<a_k(x)} \left(1 - f^*(aQ_k)^{1/2}\right)^2.$$

By the Cauchy inequality, we have

$$\left|\sum_{0\le a<a_k(x)} \left(1-f^*(aQ_k)^{1/2}\right)\right| \le a_k(x)^{1/2}.$$

$$\left(\sum_{0\le a<a_k(x)} \left(1-f^*(aQ_k)^{1/2}\right)^2\right)^{1/2},$$

and so we get that

$$\left|\frac{1}{a_k(x)}\sum_{0\le a<a_k(x)} (1-f^*(aQ_k))\right|$$
$$\le 2.\left(\frac{1}{a_k(x)}\sum_{0\le a<a_k(x)} \left(1-f^*(aQ_k)^{1/2}\right)^2\right)^{1/2}$$
$$+\frac{1}{a_k(x)}\sum_{0\le a<a_k(x)} \left(1-f^*(aQ_k)^{1/2}\right)^2.$$

Hence we have

$$\lim_{\substack{k\to+\infty\\ a_k(x)\neq 0}} \frac{1}{a_k(x)}\sum_{0\le a<a_k(x)} (1-f^*(aQ_k)) = 0.$$

Q.E.D.

3) We prove that there exists a subset F_∞ of $\mathbf{Z}_Q$ such that $\mu(F_\infty)=1$ and for every $x=(a_0(x),a_1(x),...)$ in F_∞, we have

$$\lim_{\substack{k\to+\infty\\ a_k(x)\neq 0}} \frac{1}{a_k(x)}\sum_{0\le a<a_k(x)} \left(1-f(aQ_k)^{1/2}\varpi_k(f^{1/2})^{-1}\right)^2 = 0.$$

Proof. Since the series

$$\sum_{1\le k} q_k^{-1} \sum_{0\le a\le q_k-1} \left(1-f(aQ_k)^{1/2}\varpi_k(f^{1/2})^{-1}\right)^2$$

is convergent, let σ_k be defined by $\sigma_k=\frac{1}{q_k}\sum_{a=0}^{q_k-1}\left(1-f(aQ_k)^{1/2}\varpi_k\right.$ $\left.(f^{1/2})^{-1}\right)^2$. For x in $\mathbf{Z}_Q$, we write $x=(a_0(x),a_1(x),...)$, $0\le a_k(x)\le$

$q_k - 1, 0 \leq k$ and we remark that, on the sequence of the $a_k(x)$ different from 0, one has

$$\frac{1}{a_k(x)} \sum_{0 \leq a < a_k(x)} \left(1 - f(aQ_k)^{1/2} \varpi_k (f^{1/2})^{-1}\right)^2$$
$$\leq \frac{1}{a_k(x)} \sum_{0 \leq a < q_k} \left(1 - f(aQ_k)^{1/2} \varpi_k (f^{1/2})^{-1}\right)^2$$
$$\leq \frac{q_k}{a_k(x)} \frac{1}{q_k} \sum_{0 \leq a < q_k} \left(1 - f(aQ_k)^{1/2} \varpi_k (f^{1/2})^{-1}\right)^2.$$

Since $\sum_k \sigma_k < +\infty$, there exists an increasing positive function h tending to infinity as k tends to infinity such that $\sum_k \sigma_k h(k) < +\infty$ and $\prod_{k=0}^{+\infty}(1 - \sigma_k h(k)) > 0$. We consider the set $F(h)$ of points x in $\mathbf{Z}_Q$ such that for all k, the inequality

$$[q_k \sigma_k h(k)] \leq a_k(x) \leq q_k - 1$$

holds, where $[\cdot]$ denotes the integer part function. This set $F(h)$ is closed, and its measure $\mu(F(h))$ is equal to

$$\prod_{k=0}^{+\infty} \frac{1}{q_k}(q_k - [q_k \sigma_k h(k)]),$$

and we have

$$\mu F(h) \geq \prod_{k=0}^{+\infty} \frac{1}{q_k}(q_k - q_k \sigma_k h(k)).$$

Now, we remark that this last product can be written as $\prod_{k=0}^{+\infty}(1 - \sigma_k h(k))$ and so, $\mu F(h) \neq 0$. For an x in $F(h)$, we consider the condition $[q_k \sigma_k h(k)] \leq a_k(x) \leq q_k - 1$, for $a_k(x) \neq 0$. If $[q_k \sigma_k h(k)]$ is not 0, then we have

$$\frac{q_k}{a_k(x)} \sigma_k \leq \frac{q_k}{[q_k \sigma_k h(k)]} \sigma_k$$
$$\leq \frac{q_k \sigma_k h(k)}{[q_k \sigma_k h(k)]} \cdot \frac{q_k}{q_k \sigma_k h(k)} \sigma_k \leq \frac{q_k \sigma_k h(k)}{[q_k \sigma_k h(k)]} \frac{1}{h(k)} \leq \frac{2}{h(k)}$$

and in this case, we get $\lim_{k \to +\infty} \frac{q_k}{a_k(x)} \sigma_k = 0$. Now the remaining case is that $[q_k \sigma_k h(k)] = 0$. We have $0 \leq q_k \sigma_k h(k) < 1$, i.e. $q_k \sigma_k < 1/h(k)$. Hence

$$\frac{q_k}{a_k(x)} \sigma_k \leq \frac{q_k}{1} \sigma_k \leq q_k \sigma_k \leq \frac{1}{h(k)} = o(1), \quad k \to +\infty.$$

To obtain the result, we remark that the sequence of functions h_r indexed by positive integers r and defined by $h_r(n) = h(n)$ if $n > r$ and $h(n)r^{-1}$ otherwise, satisfies the same requirements as h. Now, the sequence of closed sets $F(h_r)$ is increasing with r and $\lim_{r\to+\infty} \mu(F(h_r)) = 1$. This gives that F_∞, the union of the $F(h_r)$, is a measurable set of measure 1. Now, if x belongs to F_∞, it belongs to some $F(h_r)$ and as a consequence, along the sequence k such that $a_k(x) \neq 0$, we have

$$\begin{aligned}
&\frac{1}{a_k(x)} \sum_{0\le a<a_k(x)} \left(1 - f(aQ_k)^{1/2}\varpi_k(f^{1/2})^{-1}\right)^2 \\
&\le \frac{q_k}{a_k(x)}\sigma_k \\
&\le q_k\sigma_k \\
&\le \frac{2}{h_r(k)} = o(1), \quad k \to +\infty.
\end{aligned}$$

Q.E.D.

4) We shall need the following result:

Lemma 3. *There exists a subset E_∞ of $\mathbf{Z}_Q$ such that $\mu(E_\infty) = 1$ and for every $x = (a_0(x), a_1(x), ...)$ in E_∞ and $\varepsilon > 0$, there exists a positive integer $K(x)$ such that for $s \ge r \ge K(x)$, and we have*

$$\left|\left(\prod_{s\ge r\ge K(x)} f(aQ_j)\varpi_j(f)^{-1}\right) - 1\right| \le \varepsilon.$$

Proof. We consider the sequence of real-valued functions $f^*_{(k+1)-}$ defined on Z_Q by $x \longmapsto f^*_{(k+1)-}(x) = \prod_{0\le j\le k} f(a_j(x)Q_j)\varpi_j(f)^{-1}$, $x = (a_0(x), a_1(x), ...)$. Kakutani's Theorem ([5], p. 109) gives us that $f^*_{(k+1)-}(x)$ converges $\mu - a.s.$ and in $L^1(Z_Q, d\mu)$. Hence we get that $f^*_\infty(x) = \prod_{0\le j} f(a_j(x)Q_j)\varpi_j(f)^{-1}$ exists μ-a.s. and is in $L^1(Z_Q, d\mu)$. Now, as a consequence of Jessen's Theorem [5, p.108],

$$\lim_{k\to+\infty} \int f^*_\infty(x) \bigotimes_{0\le j\le k} d\mu_j(x) = \int f^*_\infty d\mu = 1 \quad \mu\text{-a.s.},$$

i.e.

$$\lim_{k\to+\infty} \prod_{k\le j} f(a_j(x)Q_j)\varpi_j(f)^{-1} = 1 \quad \mu\text{-a.s.},$$

and as a consequence, by Cauchy's criterion, we get our result.

Q.E.D.

5) End of the proof

We consider the intersection of the sets E_∞ and F_∞. We shall prove that, for every ξ in $E_\infty \cap F_\infty$ which is not an integer, we have

$$\frac{1}{x_k(\xi)} \sum_{n<x_k(\xi)} f(n) = \Big(\prod_{0\le j\le k} \frac{1}{q_j} \sum_{0\le a\le q_j-1} f(aQ_j)).(1+o(1)), \quad \text{as } k \to +\infty.$$

Let $\xi = (a_0, a_1, a_2, ...)$ be an element of $E_\infty \cap F_\infty$ and abbreviate $x_k(\xi)$ by x_k. We have:

$$S_{x_k}(f) = \left(\sum_{0\le a<a_k} f(aQ_k) \right) \left(\prod_{r=0}^{k-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right) + (f(a_kQ_k)) S_{x_{k-1}}(f)$$

and by iteration

$$S_{x_k}(f) = \sum_{j=0}^{k} \left(\prod_{j+1\le r\le k} f(a_rQ_r) \right) \left(\sum_{0\le a<a_j(\xi)} f(aQ_j) \right) \left(\prod_{r=0}^{j-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right)$$

$$= \sum_{j=0}^{k} \left(\prod_{j+1\le r\le k} f(a_rQ_r) \right) \left(\sum_{0\le a<a_j(\xi)} f(aQ_j) \right) \left(\prod_{r=0}^{j-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right).$$

We remark now that this equality can be written as

$$S_{x_k}(f) \left(\prod_{r=0}^{k} q_r^{-1} \sum_{a=0}^{q_r-1} f(aQ_r) \right)^{-1}$$

$$= \sum_{j=0}^{k} \left[\left(\prod_{j+1\le r\le k} f^*(a_rQ_r) \right) \left(\sum_{0\le a<a_j(\xi)} f^*(aQ_j) \right) \left(\prod_{r=0}^{j-1} \sum_{a=0}^{q_r-1} f^*(aQ_r) \right) \right].$$

Since

$$\sum_{a=0}^{q_r-1} f^*(aQ_r) = q_r,$$

we have

$$\left(\prod_{r=0}^{j-1} \sum_{a=0}^{q_r-1} f^*(aQ_r) \right) = \left(\prod_{r=0}^{j-1} q_r \right) = Q_j.$$

The choice of ξ in F_∞ implies that

$$\sum_{0\le a<a_j(\xi)} f^*(aQ_r) = a_j(\xi)(1+\varepsilon_j),$$

with $\varepsilon_j = o(1)$ as j tends to infinity. The choice of ξ in E_∞ implies that

$$\prod_{j+1\le r\le k} f^*(a_r Q_r) = 1 + \varepsilon'_j,$$

with $\varepsilon'_j = o(1)$ as j tends to infinity.

This gives us that

$$S_{x_k}(f)\left(\prod_{r=0}^{k} q_r^{-1}\sum_{a=0}^{q_r-1} f(aQ_r)\right)^{-1} = \sum_{j=0}^{k} a_j(\xi)Q_j(1+\varepsilon_j)(1+\varepsilon'_j),$$

as $j \to +\infty$,

and so, since

$$\sum_{j=0}^{k} a_j(\xi)Q_j = x_k,$$

we remark that we have

$$\lim_{k\to+\infty}(\sum_{j=0}^{k} a_j(\xi)Q_j)^{-1}(\sum_{j=0}^{k} a_j(\xi)Q_j(1+\varepsilon_j)(1+\varepsilon'_j))$$

$$= \lim_{k\to+\infty}(x_k)^{-1}(\sum_{j=0}^{k} a_j(\xi)Q_j(1+\varepsilon_j)(1+\varepsilon'_j))$$

$$= 1$$

and as a consequence, we obtain that

$$S_{x_k}(f)x_k^{-1} = \left(\prod_{r=0}^{k} q_r^{-1}\sum_{a=0}^{q_r-1} f(aQ_r)\right)(1+o(1)), \quad \text{as } k \to +\infty.$$

Q.E.D.

iv) To f^r, $0 < r < 1$, one can associate a Radon measure absolutely continuous with respect to μ.

By 3) above, this will give the end of the proof of part 1) of Theorem 2.

We consider the sequence of real-valued functions f_k^* defined on Z_Q by $x \longmapsto f_{k-}^*(x) = \prod_{0\le j<k} f(a_j(x)Q_j)\varpi_j(f)^{-1}$, $x = (a_0(x), a_1(x), \ldots)$. Kakutani's Theorem ([5], p. 109) gives us that $f_{k-}^*(x)$ converges $\mu - a.s.$ and in $L^1(Z_Q, d\mu)$. As a consequence, we get that. $\left(f_{k-}^*(x)\right)^r$ converges

$\mu - a.s.$ and in $L^{1/r}(Z_Q, d\mu)$. This implies that

$$\lim_{K\to+\infty}(\prod_{r=0}^{K-1}(1/q_r).\sum_{a=0}^{q_r-1} f^r(aQ_r))(\prod_{r=0}^{K-1}(1/q_r).\sum_{a=0}^{q_r-1} f(aQ_r))^{-r}$$

exists, and the value is less or equal to 1, but is not zero.

Hence we get that the sequence of functions

$$\left((\prod_{r=0}^{k-1}(1/q_r).\sum_{a=0}^{q_r-1} f^r(aQ_r))((\prod_{r=0}^{k-1}(1/q_r).\sum_{a=0}^{q_r-1} f(aQ_r))^{-r}\right)^{-1}(f^*_{k-}(x))^r$$

converges μ-a.s. and in $L^{1/r}(Z_Q, d\mu)$, i.e.

$$(f_{k-}(x))^r\left((\prod_{r=0}^{k-1}(1/q_r).\sum_{a=0}^{q_r-1} f^r(aQ_r))\right)^{-1}$$

converges μ-a.s. and in $L^{1/r}(Z_Q, d\mu)$.

As a consequence, since $L^1(Z_Q, d\mu) \supset L^{1/r}(Z_Q, d\mu)$, this product defines a measure absolutely continuous with respect to μ.

Q.E.D.

v) We prove directly part 2) of Theorem 2.

1) Assume that $\lim_{k\to+\infty}\int(f^*_{k-})^{1/2}d\mu = 0$. Then, we have

$$\lim_{x\to+\infty}\frac{1}{x}S_x(f^{1/2}) = 0.$$

Proof. If $x = \sum_{k=0}^{K} a_k Q_k$ and K denotes the maximal index k for which $a_k(x)$ is different from zero, we have

$$a_K Q_K \leq x \leq (a_K + 1)Q_K,$$

and so,

$$((a_K + 1)Q_K)^{-1} \leq x^{-1}.$$

But

$$((a_K + 1)Q_K)^{-1} = \left((a_K Q_K) \times ((a_K + 1)Q_K)^{-1}\right) \times (a_K Q_K)^{-1},$$

and since

$$(a_K Q_K) \times ((a_K + 1)Q_K)^{-1} = (a_K) \times (a_K + 1)^{-1}$$

and $a_K \geq 1$, we get that

$$(a_K) \times (a_K+1)^{-1} \geq 1/2.$$

This implies that

$$((a_K+1)Q_K)^{-1}$$
$$= \left((a_K Q_K) \times ((a_K+1)Q_K)^{-1}\right) \times (a_K Q_K)^{-1} \geq (1/2) \times (a_K Q_K)^{-1},$$

and as a consequence, since

$$((a_K+1)Q_K)^{-1} \leq x^{-1},$$

we get that

$$(1/2) \times (a_K Q_K)^{-1} \leq x^{-1}.$$

Similarly, since we have $x^{-1} \leq (a_K Q_K)^{-1}$, we get that $x^{-1} \leq 2 \times ((a_K+1)Q_K)^{-1}$.

Now, if $g(n)$ is any non-negative Q-multiplicative function, from the inequality

$$a_K Q_K \leq x \leq (a_K+1)Q_K,$$

we obtain that

$$S_{a_K Q_K}(g) \leq S_x(g) \leq S_{(a_K+1)Q_K}(g)$$

i.e.

$$x^{-1} S_{a_K Q_K}(g) \leq x^{-1} S_x(g) \leq x^{-1} S_{(a_K+1)Q_K}(g)$$

and so, using the above inequalities, we get that

$$(1/2) \times \left((a_K Q_K)^{-1} S_{a_K Q_K}(g)\right) \leq x^{-1} S_{a_K Q_K}(g) \leq x^{-1} S_x(g),$$

i.e.,

$$(1/2) \times \left((a_K Q_K)^{-1} S_{a_K Q_K}(g)\right) \leq x^{-1} S_x(g)$$

and similarly,

$$x^{-1} S_x(g) \leq x^{-1} S_{(a_K+1)Q_K}(g) \leq 2 \times \left(((a_K+1)Q_K)^{-1} S_{(a_K+1)Q_K}(g)\right),$$

i.e.,

$$x^{-1} S_x(g) \leq 2 \times \left(((a_K+1)Q_K)^{-1} S_{(a_K+1)Q_K}(g)\right).$$

Replacing g by f, since $\limsup_{x\to+\infty} \frac{1}{x} S_x(f) = L > 0$, we have, if K is large enough,

$$S_{a_K Q_K}(f) \leq 2L a_K Q_K,$$

$$S_{(a_K+1)Q_K}(f) \leq 2L(a_K+1)Q_K.$$

Now, replacing g by $f^{1/2}$, we have

$$x^{-1}S_x(f^{1/2}) \leq 2 \times \left(((a_K+1)Q_K)^{-1}S_{(a_K+1)Q_K}(f^{1/2})\right)$$

with

$$S_{(a_K+1)Q_K}(f^{1/2}) = \left(\sum_{0\leq a\leq a_K} f^{1/2}(aQ_K)\right)\left(\prod_{r=0}^{K-1}\sum_{a=0}^{q_r-1} f^{1/2}(aQ_r)\right),$$

and by Cauchy's inequality, we get that

$$S_{(a_K+1)Q_K}(f^{1/2}) \leq \left((a_K+1)\,(\sum_{0\leq a\leq a_K} f(aQ_K))\right)^{1/2}\left(\prod_{r=0}^{K-1}\sum_{a=0}^{q_r-1} f^{1/2}(aQ_r)\right).$$

This gives us that

$$x^{-1}S_x(f^{1/2}) \leq 2\times((a_K+1)Q_K)^{-1} \times \left((a_K+1)\,(\sum_{0\leq a\leq a_K} f(aQ_K))\right)^{1/2}\left(\prod_{r=0}^{K-1}\sum_{a=0}^{q_r-1} f^{1/2}(aQ_r)\right),$$

and we write the right member of this inequality as

$$2\times\left(((a_K+1)Q_K)^{-1}(\sum_{0\leq a\leq a_K} f(aQ_K))\right)^{1/2}\left(\prod_{r=0}^{K-1}\sum_{a=0}^{q_r-1} f(aQ_r)\right) \times\left\{\left((Q_K)^{-1/2}\right)\times\left((\prod_{r=0}^{K-1}\sum_{a=0}^{q_r-1} f^{1/2}(aQ_r))((\prod_{r=0}^{K-1}\sum_{a=0}^{q_r-1} f(aQ_r))^{-1/2}\right)\right\},$$

i.e.,

$$2\times\left[((a_K+1)Q_K)^{-1}S_{(a_K+1)Q_K}(f)\right]^{1/2}\times \left[(\prod_{r=0}^{K-1}(1/q_r).\sum_{a=0}^{q_r-1} f^{1/2}(aQ_r))(\prod_{r=0}^{K-1}(1/q_r).\sum_{a=0}^{q_r-1} f(aQ_r))^{-1/2}\right],$$

and so we have

$$x^{-1}S_x(f^{1/2})$$

$$\leq 2 \times \left[((a_K+1)Q_K)^{-1} S_{(a_K+1)Q_K}(f)\right]^{1/2}$$
$$\times \left[\left(\prod_{r=0}^{K-1} (1/q_r). \sum_{a=0}^{q_r-1} f^{1/2}(aQ_r)\right)\left(\prod_{r=0}^{K-1} (1/q_r). \sum_{a=0}^{q_r-1} f(aQ_r)\right)^{-1/2} \right].$$

Since

$$((a_K+1)Q_K)^{-1} S_{(a_K+1)Q_K}(f) \leq 2L$$

and

$$\left(\prod_{r=0}^{K-1} (1/q_r). \sum_{a=0}^{q_r-1} f^{1/2}(aQ_r)\right)\left(\prod_{r=0}^{K-1} (1/q_r). \sum_{a=0}^{q_r-1} f(aQ_r)\right)^{-1/2}$$
$$= o(1), \qquad \text{as } K \to +\infty,$$

we get that $\lim_{x\to+\infty} x^{-1} S_x(f^{1/2}) = 0$.

Q.E.D.

2) For any r in $]0,1[$, we have $\lim_{x\to+\infty} \frac{1}{x} \sum_{0\leq n\leq x} f(n)^r = 0$.

Proof. Since

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{0\leq n\leq x} f(n)^{1/2} = 0,$$

we get that

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{0\leq n\leq x,\ f(n)^{1/2}\geq 1} f(n)^{1/2} = 0,$$

i.e.

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{0\leq n\leq x,\ f(n)\geq 1} f(n)^{1/2} = 0,$$

and as a consequence

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{0\leq n\leq x,\ f(n)\geq 1} 1 = 0,$$

which implies that

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{0\leq n\leq x,\ f(n)\leq 1} 1 = 1.$$

If r is in $]0,1[$, we have

$$\sum_{0\leq n\leq x} f(n)^r = \sum_{0\leq n\leq x,\ f(n)\geq 1} f(n)^r + \sum_{0\leq n\leq x,\ f(n)\leq 1} f(n)^r.$$

Using Hölder's inequality, we get that

$$\sum_{0\le n\le x,\ f(n)\ge 1} f(n)^r \le \Big(\sum_{0\le n\le x,\ f(n)\ge 1} 1\Big)^{1-r}.\Big(\sum_{0\le n\le x,\ f(n)\ge 1} f(n)\Big)^r.$$

Since

$$\limsup_{x\to+\infty} x^{-1}\sum_{0\le n\le x} f(n) = L,$$

we get that

$$\limsup_{x\to+\infty} x^{-1}\sum_{0\le n\le x,\ f(n)\ge 1} f(n) \le L,$$

and since

$$\lim_{x\to+\infty} x^{-1}\sum_{0\le n\le x,\ f(n)\ge 1} 1 = 0,$$

we obtain that

$$\begin{aligned}
&\limsup_{x\to+\infty} x^{-1}\sum_{0\le n\le x,\ f(n)\ge 1} f(n)^r\\
&\le \Big(\limsup_{x\to+\infty} x^{-1}\sum_{0\le n\le x,\ f(n)\ge 1} 1\Big)^{1-r}.\Big(\limsup_{x\to+\infty} x^{-1}\sum_{0\le n\le x,\ f(n)\ge 1} f(n)\Big)^r\\
&\le \Big(\limsup_{x\to+\infty} x^{-1}\sum_{0\le n\le x,\ f(n)\ge 1} 1\Big)^{1-r}.L^r\\
&= 0.
\end{aligned}$$

Now, we remark that as above, we have

$$\begin{aligned}
&x^{-1}\sum_{0\le n\le x,\ f(n)\le 1} f(n)^r\\
&\quad\le \Big(x^{-1}\sum_{0\le n\le x,\ f(n)\le 1} 1\Big)^{1-r}.\Big(x^{-1}\sum_{0\le n\le x,\ f(n)\le 1} f(n)\Big)^r
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\limsup_{x\to+\infty} x^{-1}\sum_{0\le n\le x,\ f(n)\le 1} f(n)^r\\
&\le \Big(\limsup_{x\to+\infty} x^{-1}\sum_{0\le n\le x,\ f(n)\le 1} 1\Big)^{1-r}.\Big(\limsup_{x\to+\infty} x^{-1}\sum_{0\le n\le x,\ f(n)\le 1} f(n)\Big)^r\\
&\le 1.\Big(\limsup_{x\to+\infty} x^{-1}\sum_{0\le n\le x,\ f(n)\le 1} f(n)\Big)^r.
\end{aligned}$$

But if $0 \leq f(n) \leq 1$, then the inequality $0 \leq f(n) \leq f(n)^{1/2}$ holds, and as a consequence, we get that

$$\sum_{0 \leq n \leq x,\ f(n) \leq 1} f(n) \leq \sum_{0 \leq n \leq x,\ f(n) \leq 1} f(n)^{1/2}$$

and a fortiori,

$$\sum_{0 \leq n \leq x,\ f(n) \leq 1} f(n) \leq \sum_{0 \leq n \leq x} f(n)^{1/2}.$$

Now, since

$$\lim_{x \to +\infty} x^{-1} \sum_{0 \leq n \leq x} f(n)^{1/2} = 0,$$

we get that

$$\lim_{x \to +\infty} x^{-1} \sum_{0 \leq n \leq x,\ f(n) \leq 1} f(n) = 0$$

and so, we have

$$\limsup_{x \to +\infty} x^{-1} \sum_{0 \leq n \leq x,\ f(n) \geq 1} f(n)^r = 0.$$

This proves that for any r in $]0,1[$, we have

$$\lim_{x \to +\infty} x^{-1} \sum_{0 \leq n \leq x} f(n)^r = 0.$$

Q.E.D.

References

[1] G. H. Hardy and J. E. Littlewood, Tauberian theorems concerning power series and Dirichlet series whose coefficients are positive, Proc. London Math. Soc. (2), **13**, 174–191.

[2] E. Hewitt and K. Ross, Abstract harmonic analysis. I, 1963, Springer-Verlag.

[3] P. Malliavin, Intégration et probabilité, Analyse de Fourier et Analyse spectrale, Masson, Paris, 1982.

[4] J. L. Mauclaire, An almost-sure estimate for the mean of generalized Q-multiplicative functions of modulus 1, J. Théor. Nombres Bordeaux, **12** (2000), 1–12.

[5] A. Tortrat, Calcul des probabilités, Masson, 1971, Paris.

Jean-Loup Mauclaire
THEORIE DES NOMBRES
Institut de mathématiques, (UMR 75867 du CNRS)
Université Pierre et Marie Curie
175 rue du chevaleret, Plateau 7D
F-75013 Paris
France
E-mail address: `mauclai@ccr.jussieu.fr`