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On the Backlund equivalent for the Lindelöf hypothesis

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Abstract.

Backlund showed that the Lindelöf hypothesis (LH) for the Riemann zeta-function is equivalent to some regularity of the distribution of zeros to the right from the critical line. We generalize the Backlund equivalent, by showing that LH is equivalent to the same type regularity of the distribution of *any* fixed complex value (not only zero). This generalized Backlund equivalent also can be applied for the Lerch zeta-function and, in our opinion, supports the idea that the Lindelöf hypothesis also is reasonable for zeta functions without the Euler product (usually having zeros off the critical line). Further we show that this generalized Backlund equivalent for LH can be formulated for zeta functions of the Selberg class and for the Selberg zeta-function, for which the Riemann hypothesis is true.

§1. Introduction

As usual, let $s = \sigma + it$ be a complex variable. Denote by $\mathbb{P}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of prime, real and complex numbers accordingly. Denote by $\{\lambda\}$ the fractional part of a real number λ. We write $f(x) = O(g(x))$ and $f(x) \ll g(x)$, resp., when $\limsup_{x\to\infty} \frac{|f(x)|}{g(x)}$ is bounded, $f(x) \gg g(x)$, when $\liminf_{x\to\infty} \frac{|f(x)|}{g(x)} \neq 0$, and $f(x) = o(g(x))$ if this limit equals 0. Further, $f(x) \asymp g(x)$ denotes that the estimate $g(x) \ll |f(x)| \ll g(x)$ holds and $f(x) \sim g(x)$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

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The Riemann zeta-function for $\sigma > 1$ is given by the following Dirichlet series or Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The Lindelöf hypothesis states, that for any $\varepsilon > 0$,

$$\zeta(1/2 + it) \ll_{\varepsilon} t^{\varepsilon}.$$

Backlund [1] proved, that the Lindelöf hypothesis is equivalent to the following statement: *for any $\sigma' > 1/2$ the number of zeros of $\zeta(s)$ in the region $\sigma > \sigma'$, $T \leq t \leq T+1$ is $o(\log T)$* (see also Titchmarsh [19], §13.5). Here and further the number of zeros (roots) are always counted with multiplicities. Backlund's theorem immediately shows, that the Lindelöf hypothesis follows from the celebrated Riemann hypothesis, which states that the Riemann zeta-function has no zeros to the right of the critical line $\sigma = 1/2$.

The Riemann zeta-function belongs to the family of Lerch zeta-functions, which for $0 < \lambda, \alpha \leq 1$, $\sigma > 1$ are given by

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+\alpha)^s}.$$

These functions can be continued analytically to the whole complex plane, may be, except the point $s = 1$ (see [12]). Note, that

$$\begin{aligned} L(1,1,s) &= \zeta(s), \quad L\left(1, \frac{1}{2}, s\right) = (2^s - 1)\zeta(s), \\ L\left(\frac{1}{2}, 1, s\right) &= (1 - 2^{1-s})\zeta(s) \quad \text{and} \quad L\left(\frac{1}{2}, \frac{1}{2}, s\right) = 2^s L(s, \chi), \end{aligned}$$

where $L(s, \chi)$ is the Dirichlet L-function with the character $\chi \bmod 4$, $\chi(3) = -1$. Lerch zeta-functions have growing properties similar to the Riemann zeta-function. For λ and α in $(\delta, 1-\delta) \bigcup 1$ with $0 < \delta < 1/2$, we have ([5])

$$L(\lambda, \alpha, 1/2 + it) \ll_{\delta, \varepsilon} t^{32/205 + \varepsilon}.$$

This bound coincides with the best known bound for $\zeta(1/2 + it) = L(1, 1, 1/2 + it)$ obtained by Huxley [8].

In [6] we with Steuding proved that the Backlund equivalent also is valid for $L(\lambda, \alpha, s)$ with fixed parameters λ and α. However the zero distribution to the right from the critical line of $L(\lambda, \alpha, s)$, say if α is a transcendental number, is very different from that of $\zeta(s)$. Denote by $N(\sigma', T; \alpha, \lambda)$ the number of zeros of $L(\lambda, \alpha, s)$ in a region $\sigma > \sigma'$, $0 \leq t \leq T$. Then it is known (Titchmarsh [19]) that for the Riemann zeta-function

$$N(\sigma, T) := N(\sigma, T; 1, 1) = o(T),$$

if $\sigma > 1/2$ and for the Lerch zeta-function

$$N(\sigma, T; \alpha, \lambda) \asymp T,$$

if $1/2 < \sigma < 1 + 0.6\alpha$ and α is a transcendental number ([12], §8.4).

Here we prove a variant of the Backlund equivalent, which connects the Lindelöf hypothesis with a general value distribution, where the zero value, possibly, is only the exceptional case. For $a \in \mathbb{C}$ denote by $N_a(\sigma', T, \alpha, \lambda)$ the number of roots of $L(\lambda, \alpha, s) - a$ in a region $\sigma > \sigma'$, $0 \leq t \leq T$.

Theorem 1. *Let $a \in \mathbb{C}$. Let $0 < \delta < 1/2$ and let I_λ and I_α be compact sets contained in $(\delta, 1-\delta) \bigcup 1$. Then for any $\varepsilon > 0$,*

$$L(\lambda, \alpha, \frac{1}{2} + it) \ll_{\varepsilon,\delta} t^\varepsilon$$

uniformly in $\lambda \in I_\lambda$, $\alpha \in I_\alpha$ if and only if for every $\sigma > \frac{1}{2}$,

$$N_a(\sigma, T+1; \lambda, \alpha) - N_a(\sigma, T; \lambda, \alpha) = o(\log T)$$

holds uniformly in $\lambda \in I_\lambda$, $\alpha \in I_\alpha$.

The theorem will be proved in the next section. If $a \neq 0$ and $1/2 < \sigma < 1$, then Bohr and Jessen [2] proved that for the Riemann zeta-function

$$N_a(\sigma, T) \sim c(\sigma)T$$

with $c(\sigma) > 0$. For the error term see Matsumoto [13].

The similar situation, where $a = 0$ is the exceptional value, appears in universality theorems. As an example we give the universality theorem for the Lerch zeta-function. For simplicity we state it only for special values of α.

Universality theorem. *Let α be a transcendental number and $0 < \lambda \leq 1$ or $\alpha = \lambda = 1$. Let K be a compact subset of the strip $1/2 < \sigma < 1$ with connected complement. Suppose that $g(s)$ is a continuous function* *on K which is analytic in the interior of K and, if $\alpha = \lambda = 1$, moreover suppose that $g(s) \neq 0$ on K. Then for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \max_{|s|\le r} \left| L\left(\lambda, \alpha, s + \frac{3}{4} + i\tau\right) - g(s) \right| < \varepsilon \right\} > 0.$$

For $\alpha = \lambda = 1$ this is a famous Voronin's universality theorem, the proof for this case can be found in Laurinčikas [10] and for the case that α is a transcendental number and $0 < \lambda \le 1$ the proof can be found in [12].

By universality and Roushé's theorems, in the same way as in ([12], §8.4, proof of Theorem 4.7), it is easy to derive that for $a \neq 0$, $1/2 < \sigma < 1$ and α, λ satisfying conditions of Universality theorem,

$$N_a(\sigma, T, \alpha, \lambda) \gg T. \tag{1}$$

Considering the second moment of $|L(\lambda, \alpha, s) - a|$, similarly as in ([12], §8.4, proof of Theorem 4.10), one can obtain that the upper bound in formula (1) is $\ll T$. Thus we see that the distribution of values $a \neq 0$ for the Riemann zeta-function is similar to that for Lerch zeta-functions.

According to the Linnik-Ibragimov conjecture all zeta-functions satisfying some natural conditions should have the universality property. For almost all known zeta-functions it is already proved. In all proved cases in the universality theorem for zeta-functions with Euler type product (e.g., Riemann, Dedekind zeta functions, Dirichlet L-functions, some automorphic L-functions) the value $a = 0$ is exceptional and for zeta-functions without Euler type product (e.g., Hurwitz, Lerch, Estermann zeta-functions) there is no such exceptional value. All the facts mentioned here concerning universality can be found in interesting surveys written by Laurinčikas [11], Matsumoto [14].

By the above we hope that, similarly to universality theorems, the analog of the Lindelöf hypothesis should be valid for a wide class of zeta-functions.

We will derive Theorem 1 from

Proposition 2. *Let poles of the family of meromorphic functions*

$$\{f(b, q, s) : b \in B \subset \mathbb{R}^n, q \in \mathbb{R}^m\}$$

be contained in a compact subset of $\mathbb{C}$. Let $R : \mathbb{R}^n \to \mathbb{R}$ and $P : \mathbb{R} \to \mathbb{R}$ be non decreasing in each positive variable functions and $R(q) \ge 2$, $P(t) \ge 2$. Suppose that $|f(b,q,s)| \ge c > 0$, $f'_s(b,q,s)/f(b,q,s) = o(\log(R(q)P(t)))$ on some strip $\sigma_0 - \omega \le \sigma \le \sigma_0 + \omega$ $(\omega > 0)$ and $|f(b,q,s)| > 0$ for $\sigma \ge \sigma_0 + \omega$, uniformly in $b \in B$ and q. Let for *some fixed numbers $D > 0$, $\sigma_1 < \sigma_0$ we have $f(b,q,s) \ll (R(q)P(t))^D$ uniformly for $\sigma \geq \sigma_0 - 4(\sigma_0 - \sigma_1)$, $b \in B$ and q. Then we have that if for $\sigma \geq \sigma_1$*

$$f(b,q,\sigma+it) \ll_\varepsilon (R(q)P(t))^\varepsilon \tag{2}$$

uniformly in $b \in B$ and q then

$$N(\sigma, T+1; b, q) - N(\sigma, T; b, q) = o(\log(R(q)P(T))) \tag{3}$$

holds for every $\sigma > \sigma_1$ uniformly in $b \in B$ and q. Here $N(\sigma', T; b, q)$ denotes the number of zeros of $f(b,q,s)$ in the region $\sigma > \sigma'$, $0 \leq t \leq T$.

From the other side, if (3) is true, then (2) is true for $\sigma = \sigma_1$ uniformly in $b \in B$ and q.

This proposition will be proved in the next section. As we can see from Proposition 2, the above type equivalent for the Lindelöf hypothesis should work for many of zeta-functions. We will show it for the Selberg class and later for the Selberg zeta-function.

The Selberg class S consists of Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

satisfying the following hypotheses.

1. Analyticity: $(s-1)^m F(s)$ is an entire function of finite order for some nonnegative integer m.

2. Ramanujan Hypothesis: $a(n) \ll_\varepsilon n^\varepsilon$ for any fixed $\varepsilon > 0$.

3. Functional equation: for $1 \leq j \leq k$, there are positive real numbers Q, λ_j, and there are complex numbers μ_j, ω with $\Re \mu_j \geq 0$ and $|\omega| = 1$, such that

$$\Lambda_F(s) = \omega \overline{\Lambda_F(1-\overline{s})}, \tag{4}$$

where

$$\Lambda_F(s) = F(s) Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j).$$

4. Euler product: $a(1) = 1$, and

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \tag{5}$$

where $b_n = 0$ unless n is a positive power of a prime and $b(n) \ll n^\theta$ for some $\theta < 1/2$.

Note that Perelli [15] defined a class of "general L-functions" similar to S and among other things proved the Backlund equivalent for his class (see Theorm 7 in [15]).

We denote by $S^\sharp$ the larger class of functions $F(s)$ which are not identically vanishing and satisfies 1.-3. above.

Theorem 3. *Let a be a complex number and $F(s) \in S^\sharp$. Then for any $\varepsilon > 0$,*

$$F(\frac{1}{2} + it) \ll_\varepsilon t^\varepsilon \tag{6}$$

if and only if for any $\bar{\sigma} > 1/2$ the number of roots of $F(s) - a$ in the region $\sigma > \bar{\sigma}$, $T \le t \le T + 1$ is $o(\log T)$.

The theorem will be proved in the next section. Examples of functions from the Selberg class are the Riemann zeta-function, Dirichlet L-functions attached to primitive characters, Dedekind zeta-functions, normalized L-functions associated with holomorphic newforms, the Rankin-Selberg L-function of any two holomorphic newforms. Selberg has conjectured the Riemann Hypothesis for this class, that is, that all of the non-trivial zeros of any element of S have real part equal to $1/2$. By Theorem 3 this hypothesis implies the Lindelöf hypothesis (6) for S. The universality property of functions from the Selberg class was considered by Steuding [17].

In view of the work of Conrey and Ghosh [3] the Lindelöf hypothesis for S can be formulated in the aspect of parameters Q and μ_j appearing in the functional equation. Suppose that $F \in S$ is entire. Suppose further that $\lambda_j = 1/2$ for each j and that the Euler product condition (5) is changed to the stronger condition:

$$F(s) = \prod_p \prod_{j=1}^{k} (1 - \alpha_{p,j} p^{-s})^{-1}, \tag{7}$$

where for all p and j, either $|\alpha_{p,j}| = 1$ or $\alpha_{p,j} = 0$. If F satisfies the Riemann hypothesis and (7), then Conrey and Ghosh [3] proved that, for any $\varepsilon > 0$, there exists a constant $c = c(\varepsilon, k)$ such that

$$|F(\frac{1}{2} + it)| \le c \left(Q(1 + |t|)^{\frac{k}{2}} \prod_{j=1}^{k} (1 + |\mu_j|) \right)^{\varepsilon}.$$

Note that the last statement means the same if the term $(1+|t|)^{\frac{k}{2}}$ is replaced by $(1+|t|)$. We formulate the generalized Backlund equivalent for this case.

Theorem 4. *Let $a \neq 1$ be a complex number. Let $F(s) \in S$, $\lambda_j = 1/2$ for each j and Euler product condition (7) is valid. Then for any $\varepsilon > 0$,*

$$F(\frac{1}{2}+it) \ll_{\varepsilon,k} \left(Q(1+|t|) \prod_{j=1}^{k} (1+|\mu_j|) \right)^{\varepsilon}, \quad for \quad t \to \infty,$$

uniformly in Q and μ_j, if and only if for any $\bar{\sigma} > 1/2$ the number of roots of $F(s) - a$ in the region $\sigma > \bar{\sigma}$, $T \leq t \leq T+1$ is

$$o\left(\log\left(Q(1+|T|) \prod_{j=1}^{k} (1+|\mu_j|) \right) \right), \quad for \quad T \to \infty,$$

uniformly in Q and μ_j.

Next we will consider the simplest Selberg zeta-function attached to a compact Riemann surface F of genus $g \geq 2$. F can be represented as a quotient space $\Gamma \setminus H$, where $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a strictly hyperbolic Fuchsian group and H is the upper half-plane. The Γ conjugacy class determined by $P \in \Gamma$ will be denoted by $\{P\}$ and its norm by $N\{P\}$. By P_0 will be denoted the primitive element of Γ. Then the Selberg zeta-function for $\sigma > 1$ is given by

$$Z(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k})$$

(see Hejhal [7] for details). It is an entire function with a functional equation

$$Z(s) = Z(1-s) \exp\left(4\pi(g-1) \int_0^{s-\frac{1}{2}} v \tan(\pi v) dv \right),$$

and for this function the Riemann hypothesis is true, i.e. its nontrivial zeros are located at the critical line $\sigma = 1/2$ (Hejhal [7], §2.4). In the same chapter of [7] we find that $Z(s) = 1 + o(1)$ for $\sigma \to \infty$,

$$|Z(s)| \leq \exp(\frac{6}{\pi}(g-1)t + O(1))$$

for $\sigma \geq -1$, $t \geq 0$, and

$$\exp\left(4\pi(g-1)\int_0^{s-\frac{1}{2}} v\tan(\pi v)dv\right) = \exp\left(i2\pi(g-1)(s-\frac{1}{2})^2 + O(1) + O((\sigma-\frac{1}{2})^2 e^{-2\pi t}) + O((\sigma-\frac{1}{2})te^{-2\pi t})\right)$$

for $t \geq 1$. By this and the functional equation we have that for $\sigma_2 \leq -1$, $\sigma \geq \sigma_2$ and $t \geq 1$

$$Z(s) = O(\exp(2\pi(g-1)(\frac{1}{2}-\sigma_2)t)).$$

The logarithmic derivative $Z'/Z(s)$ is bounded for $\sigma \geq 2$ (Hejhal [7], §2.3). Choosing sufficiently large σ_0 and $\sigma_1 = 1/2$, $P(t) = \exp(t)$, by Proposition 2, we obtain

Theorem 5. *For any* $\varepsilon > 0$

$$Z(\frac{1}{2}+it) \ll_\varepsilon \exp(\varepsilon t),$$

as $t \to \infty$.

From the above we see that $|Z(s)\exp(i\varepsilon s)|$ is bounded by a constant on the upper part of lines $\sigma = 1/2$, $\sigma = \sigma_0$, and on the horizontal segment joining points $\sigma = 1/2$, $t = 1$ and $\sigma = \sigma_0$, $t = 1$. Thus by the Phragmén-Lindelöf theorem (Titchmarsh [18], §5.6.4) we obtain that $Z(\sigma+it) \ll \exp(\varepsilon|t|)$ for $1/2 \leq \sigma \leq \sigma_0$, $t \geq 1$. In view of $\overline{Z(s)} = Z(\bar{s})$ the same bound is valid in the halfplane $\sigma \geq 1/2$. Again, by Proposition 2 we derive

Theorem 6. *Let a be a complex number. For any $\bar{\sigma} > 1/2$ the number of roots of $Z(s) - a$ in the region $\sigma \geq \bar{\sigma}$, $T \leq t \leq T+1$ is $o(T)$.*

§2. Proofs of Proposition 2 and Theorems 1, 3, 4

Proof of Proposition 2. First we assume the truth of Lindelöf's hypothesis (2). Therefore, we make use of

Lemma 7 (Jensen's formula). *Let $f(s)$ be analytic for $|s| < R$. Suppose that $f(0)$ is not zero, and let $r_1, r_2, \ldots$ be the moduli of the zeros of $f(s)$ in the circle $|s| < R$, arranged as a non-decreasing sequence. Then, if $r_n \leq r < r_{n+1}$,*

$$\log\frac{r^n|f(0)|}{r_1\cdot\ldots\cdot r_n} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(r\exp(i\phi))|d\phi.$$

For a proof see Titchmarsh [18], §3.61.

Applying Jensen's theorem to $f(b,q,s)$ and to the circle with center σ_0+it and radius $\sigma_0-\sigma_1-\frac{\delta}{4}$, we obtain

$$\sum_{|\rho-\sigma_0-it|<\sigma_0-\sigma_1-\frac{\delta}{4}} \log\frac{\sigma_0-\sigma_1-\frac{\delta}{4}}{|\rho-\sigma_0-it|}$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(b,q,\sigma_0+it+\left(\sigma_0-\sigma_1-\frac{\delta}{4}\right)\exp(i\phi)\right)\right| \mathrm{d}\phi$$
$$-\log|f(b,q,\sigma_0+it)|.$$

On the Lindelöf hypothesis (2) the right hand side is $o(\log(R(q)P(t)))$. Further, if there are m zeros in the concentric circle of radius $\sigma_0-\sigma_1-\frac{\delta}{2}$, the left hand side is bounded below by

$$m\log\frac{\sigma_0-\sigma_1-\frac{\delta}{4}}{\sigma_0-\sigma_1-\frac{\delta}{2}} = m\log(1+O(\delta)).$$

Therefore, the number of zeros in the circle of radius $\sigma_0-\sigma_1-\frac{\delta}{2}$ is $o(\log(R(q)P(t)))$, and the result with $\sigma=\sigma_1+\delta$ follows by superposing a finite number (not depending on t, b, and q) of such circles.

Now we have to prove the converse. Therefore we quote (Titchmarsh [19], §3.9)

Lemma 8 (Landau). *If $f(s)$ is regular, and*

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M$$

in $\{s : |s-s_0|\le r\}$ with $M>1$, then

$$\left|\frac{f'(s)}{f(s)}-\sum_{\rho}\frac{1}{s-\rho}\right| < C\frac{M}{r}$$

for $|s-s_0|\le\frac{r}{4}$, where C is some constant and ρ runs through the zeros of $f(s)$ such that $|\rho-s_0|\le\frac{r}{2}$.

We apply this lemma with $s_0=\sigma_0+iT$, where T is sufficiently large, and some $r=2(\sigma_0-\sigma_1-2\delta)$. Then we may choose $M=c\log(R(q)P(T))$ and obtain

$$\frac{f'}{f}(b,q,s) = \sum_{|\rho-s_0|\le r}\frac{1}{s-\rho}+O\left(\log(R(q)P(T))\right)$$

for $|s-s_0| \le r/2$.

Let C_1 be the circle with the center s_0 and a radius $\sigma_0-\sigma_1-\delta$. By the bound $f(b,q,s) \ll (R(q)P(t))^D$ and Jensen's formula, in the same way as in the first part of the proof, we obtain that $N(\sigma, t+r; b, q) - N(\sigma, t-r; b, q) \ll \log(R(q)P(t))$ for fixed r. For s and ρ such that $|s-s_0| \le \sigma_0-\sigma_1-2\delta$ and $\sigma_0-\sigma_1-\delta \le |\rho-s_0| \le 2(\sigma_0-\sigma_1-2\delta)$ the bound $|s-\rho| \ge \delta$ is valid. By this we have, that

$$\Psi(s) := \frac{f'}{f}(a,q,s) - \sum_{\rho\in C_1} \frac{1}{s-\rho} = O\left(\frac{\log(R(q)P(t))}{\delta}\right),$$

for $|s-s_0| \le \sigma_0-\sigma_1-2\delta$. Let C_3 be the concentric circle of radius $\sigma_0-\sigma_1-3\delta$, and C be the concentric circle of radius ω. Then $\Psi(s) = o(\log R(q)P(T))$ for s in C, since each term is $O(1)$ and by the hypothesis the number of terms is $o(\log R(q)P(T))$. Now we will use

Lemma 9 (Hadamard's three circle theorem). *Let $f(s)$ be an analytic function, regular for $r_1 \le |s| \le r_3$. Let $r_1 < r_2 < r_3$, and let M_1, M_2, M_3 be the maxima of $|f(s)|$ on the three circles $|s| = r_1, r_2, r_3$ respectively. Then*

$$\log\frac{r_3}{r_1}\log M_2 \le \log\frac{r_3}{r_2}\log M_1 + \log\frac{r_2}{r_1}\log M_3.$$

For a proof see once more [18], §5.3.

Hadamard's three circle theorem yields for $s \in C_3$

$$\Psi(s) = (o(\log(R(q)P(T))))^{\kappa}\left(O\left(\frac{\log(R(q)P(T))}{\delta}\right)\right)^{\iota},$$

where $\kappa+\iota = 1, 0<\iota<1$ and κ, ι depending on δ only. Hence we have $\Psi(s) = o(\log(R(q)P(t)))$ for any given δ in C_3. Since $o(\log(R(q)P(T)))$ zeros lie inside C_1, we get

$$\begin{aligned}\int_{\sigma_1+3\delta}^{\sigma_0} \Psi(s)\mathrm{d}\sigma &= \log f(b,q,\sigma_0+it) - \log f(b,q,\sigma_1+3\delta+it) \\ &\quad - \sum_{\rho\in C_1}\left(\log(\sigma_0+it-\rho) - \log(\sigma_1+3\delta+it-\rho)\right) \\ &= O(1) - \log f(b,q,\sigma_1+3\delta+it) + o(\log T) \\ &\quad + \sum_{\rho\in C_1} \log(\sigma_1+3\delta+it-\rho).\end{aligned}$$

Now setting $t = T$ the left-hand side is $o(\log(R(q)P(T)))$. Taking the real parts we obtain

$$\log|f(b,q,\sigma_1+3\delta+it)| = o(\log(R(q)P(T))) + \sum_{\rho\in C_1} \log|\sigma_1+3\delta+iT-\rho|\,.$$

Since $|\sigma_1+3\delta+it-\rho| < const$ in C_1, it follows that

$$\log|f(b,q,\sigma_1+3\delta+it)| = o(\log(R(q)P(T))),$$

which proves (2). •

Proof of Theorem 1. We apply Proposition 2 with $f(b,q,s) = L(\lambda,\alpha,s)-a$, $b=(\lambda,\alpha)$, $B = I_\lambda \times I_\alpha$, $R(q)=1$, $P(t)=|t|+1$, $\omega=1/2$, and $\sigma_1 = 1/2$. Now we will show that there exists $\sigma_0 > 1$ such that, for $\sigma_0 - 1/2 \le \sigma \le \sigma + 1/2$, $\alpha \in I_\alpha$, $\lambda \in I_\lambda$,

$$|L(\lambda,\alpha,s) - a| \ge c(\sigma_0, a) > 0$$

and

$$\left|\frac{(L(\lambda,\alpha,s)-a)'_s}{L(\lambda,\alpha,s)-a}\right| \le C(\sigma_0,a).$$

This follows by the following inequalities: for $a \ne 1$ and for all sufficiently large σ,

$$|L(\lambda,\alpha,s)-a| \ge |1-a| - \sum_{m=2}^{\infty}\frac{1}{m^\sigma} > 0;$$

for $a=1$, $\alpha\in(\delta,1-\delta)$ and all sufficiently large σ,

$$|L(\lambda,\alpha,s)-1| \ge \left|\frac{1}{(1-\delta)^\sigma} - 1\right| - \sum_{m=2}^{\infty}\frac{1}{m^\sigma} > 0$$

and for $\alpha = 1$, $a = 1$,

$$|L(\lambda,\alpha,s)-1| \ge \frac{1}{2^\sigma} - \sum_{m=3}^{\infty}\frac{1}{m^\sigma} > 0;$$

for $\sigma \ge 2.5$,

$$|L'_s(\lambda,\alpha,s)| \le \delta^{-\sigma} + \sum_{n=1}^{\infty}\frac{\log n}{n^{2.5}} \le c(\delta).$$

We also see, that if we choose σ_0 satisfying above conditions, then $|L(\lambda,\alpha,s)-a|>0$ for $\sigma\geq\sigma_0$, $\alpha\in I_\alpha$, and $\lambda\in I_\lambda$.

From [4] we have that, for $\sigma\geq 1/2$, $\alpha\in I_\alpha$, and $\lambda\in I_\lambda$,

$$L(\lambda,\alpha,s)\ll_\delta t^{\frac{1}{2}}.$$

By the functional equation ([12])

$$\begin{aligned} L(\lambda,\alpha,1-s) &= (2\pi)^{-s}\Gamma(s)\Big(\exp\Big(2\pi i(\frac{s}{4}-\alpha\lambda)\Big)\,L(-\alpha,\lambda,s) \\ &\quad+\exp\Big(-2\pi i(\frac{s}{4}+\alpha(1-\{\lambda\}))\Big)\,L(\alpha,1-\{\lambda\},s)\Big) \end{aligned}$$

and by the growing properties of Euler gamma function $\Gamma(s)$ we have, that for any σ' there exists a constant $A=A(\sigma')$, such that for $\sigma\geq\sigma'$, $\alpha\in I_\alpha$, and $\lambda\in I_\lambda$,

$$L(\lambda,\alpha,s)\ll_{\delta,\sigma'} t^A.$$

Now Theorem 1 follows from Proposition 2. •

Proof of Theorem 3 is analogous to the previous proof.

Proof of Theorem 4. We will check conditions of Proposition 2 for a function $f(b,q,s)=F(s)-a$ (independent on the parameter b), with $P(t)=1+|t|$, $q=(Q,\mu_1,\dots,\mu_k)$, $R(q)=Q\prod_{j=1}^k\mu_j$, $\sigma_1=1/2$; constants σ_0 and ω will be chosen later. By the Euler product condition (7) we have for $\sigma>1$, that

$$|F(s)|\leq\zeta^k(\sigma)=\sum_{n=1}^{\infty}\frac{d_k}{n^\sigma}, \tag{8}$$

where $d_k(n)$ is the number of ways of expressing n as a product of k factors. Thus we can find sufficiently large σ_1 such that for $\sigma>\sigma_1$

$$|F(s)-a|\geq|1-a|-\sum_{m=2}^{\infty}\frac{d_k}{m^\sigma}>c(\sigma_1,k)>0$$

and $|F'(s)|\leq C(\sigma_1,k)$. By this we can choose σ_0 such that $|F(s)-a|\geq c(\sigma_0,k)>0$,

$$\frac{(F(s)-a)'_s}{F(s)-a}\leq C(\sigma_0,k)$$

on the strip $\sigma_0-1\leq\sigma\leq\sigma_0+1$ and $|F(s)-a|>0$ for $\sigma\geq\sigma_0+1$, uniformly in Q and μ_j.

Next we will show that for any $\sigma' < -\sigma_0$ there exists a constant $A = A(\sigma', k)$, such that for $\sigma \geq \sigma'$

$$F(s) \ll_{\sigma',k} (Q(1+|t|)(1+\mu))^A \tag{9}$$

uniformly in Q and μ_j. By the functional equation $\Gamma(s+1) = s\Gamma(s)$ we have that

$$\begin{aligned}\left|\frac{\Gamma\left(\frac{1-(\sigma'+\{\sigma'\}-1.5+it)}{2}+\overline{\mu}\right)}{\Gamma\left(\frac{\sigma'+\{\sigma'\}-1.5+it}{2}+\mu\right)}\right| &\leq (2-\sigma')(1+|t|+|\mu|)^{2-\sigma'} \\ &\leq (2-\sigma')\left((1+|t|)(1+|\mu|)\right)^{2-\sigma'}.\end{aligned}$$

Then (9) follows by the bound (8), functional equation (4) and the generalized Phragmen-Lindelöf Theorem ([16]). Now Proposition 2 yields Theorem 4. •

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