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Dynamical systems of Lagrangian and Hamiltonian mechanical systems

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Dedicated to the memory of Professor Makoto Matsumoto

Abstract.

In Part I of this paper the dynamical systems of the Lagrangian mechanical system $\Sigma_L = (M, L(x, y),\ F_e(x, y))$ are defined and investigated. In Theorem 3.1 we prove the existence of a canonical dynamical system on the phase space whose integral curves are given by the Lagrange equations of Σ_L. The particular case of Finslerian mechanical systems is considered. The geometry of Σ_L on TM is also described. Part I is a survey of the author's papers [18] [22] [23].

In the Part II for the first time the same problems for the Hamiltonian mechanical systems $\Sigma_H = (M, H(x, p),\ F_e(x, p))$ are studied. In Theorem 10.1, we prove the existence of a canonical dynamical system ξ on the momenta space, whose integral curves are given by the Hamilton equations of Σ_H. As a particular case the Cartan mechanical systems are examined.

Introduction

The geometric study of dynamical systems is an important chapter of contemporary mathematics due to its applications in Mechanics, Theoretical Physics, Control Systems, Economy or Biology. If M is a differentiable manifold that correspond to the configurations space, a dynamical system can be locally given by a system of ordinary differential equations of the form $\dot{x}^i = f^i(t, x)$, which are called the equations of evolution. Globally, a dynamical system is a vector field X on the manifold $M \times R$ whose integral curves $c(t)$ are given by the equations $x \circ c(t) = \dot{c}(t)$.

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The theory of dynamical systems deals with the integration of such systems determining the general solution that correspond to some initial conditions and with the qualitative problems of these solutions concerning especially the stability.

For instance, such kind of theory can be developed for the Riemannian mechanical systems $\Sigma_{\mathcal{R}} = (M, g(x), F_e(x))$ which have as evolution equations the well known Lagrange equations, even though these equations are of second order. However, as we will prove in the present paper, it is preferable to study these dynamical systems on the phase space TM. Following this idea, we can consider the more general Riemannian mechanical systems $\Sigma_{\mathcal{R}}$ whose external forces $F_e(x, \dot{x})$ depend on points $x \in M$ and on velocities $\dot{x}$ such that $(x, \dot{x}) \in TM$.

Consequently, we must define the dynamical systems on the phase space TM by a second order ordinary differential equations: $\dfrac{d^2x^i}{dt^2} + 2G^i(x, \dot{x}) = 0$, which are invariant with respect to changes of local coordinates on TM. But in this case G^i are the local coefficients of a vector field S on the phase space M, given by $S = y^i \dfrac{\partial}{\partial x^i} - 2G^i(x, y) \dfrac{\partial}{\partial y^i}$. Thus, the geometry of the dynamical system is the geometry of S on TM. S is called a semispray.

In the case of Lagrange space $L^n = (M, L(x, y))$, a canonical semispray S is determined. Its evolution curves are given by the Euler - Lagrange equations of L^n. It is interesting to remark that the Euler - Lagrange equations are selfadjoint, [8, 17, 26, 27].

A similar theory can be done for the dynamical systems on the momenta space T^*M, considering the Hamilton spaces $H^n = (M, H(x, y))$.

In the present paper, in Part I, we define and investigate the notion of Lagrangian mechanical system $\Sigma_L = (M, L(x, y), F_e(x, y))$, where $L^n = (M, L(x, y))$ is a Lagrange space [16, 17] and $F_e(x, y) = F^i(x, y) \dfrac{\partial}{\partial y^i}$ are the given external forces. The evolution equation Σ_L are given by Postulate 1 from section 3 and they are expressed by the Lagrange equations (3.3). In Theorem 3.1 one proves that on $\widetilde{TM}$ there exists a canonical semispray S_L (which is determined only on Σ_L), whose integral curves are given by the evolution equations of Σ_L. However the system of evolution curves is not selfadjoint.

Now the geometry of Σ_L is reduced to the geometry of the pair (L^n, S_L). We develop this geometry determining for Σ_L the canonical nonlinear connection, the canonical N-metrical connection and the almost Hermitian model on TM. We deduce the remarkable formulas

(4.5), (4.6) for the $h-$ electromagnetic tensor $\mathcal{F}_{ij}$ and the Maxwell equations of Σ_L. The particular case of Finslerian mechanical systems is presented, also.

Part II deals with the dual (via Legendre transformation) of the previous theory, introducing for the first time the notion of Hamiltonian mechanical systems. These are defined by a set $\Sigma_H = (M, H(x,y), F_e(x,p))$ where $H^n = (M, H(x,p))$ is a Hamilton space and $F_e(x,p) = F_i(x,p)\dot{\partial}^i$ are the given external forces. A good example is obtained by considering $F_i(x,p) = a(x,p)p_i$.

For more details from the Geometry of Hamilton spaces we refer to the book [19] of R. Miron, D. Hrimiuc, H. Shimada and S. Sabău.

Postulate 2 introduces the evolution equations of Σ_H as being the Hamilton equations (10.3). But, these equations being $\mathcal{L}-$dual of Lagrange equations (3.3"), which are not selfadjoint, are not selfadjoint, also.

The main result is contained in Theorem 10.1 of Part II, in which one proves the existence of the canonical dynamical system ξ of the Hamilton mechanical systems. Thus the geometry of the Hamiltonian mechanical system Σ_H is the geometry of pair (H^n, ξ).

The particular case of Cartan mechanical systems Σ_C is studied, as well.

Part I. The dynamical systems of the Lagrangian mechanical systems

The dynamical system of a Lagrange mechanical system can not be correctly defined without geometrical frameworks of the phases manifold TM, the manifold M being the configuration space of the considered mechanical system.

Because of this, at the beginning of the present paper we briefly present in section 1 some elements of the differential geometry of the manifold TM. The Lagrangian mechanical systems Σ_L, their evolution equations and the associated dynamical systems will be studied in sections 2, 3. A special attention is paid to the cases of Finslerian or Riemannian mechanical systems, since for some special external forces, the evolution curves are given by the Lorentz equations. The content of this part is a survey of the author's papers [18, 20, 21, 22, 23, 24].

§1. The geometry of phase space

Let M be a C^∞ real n dimensional manifold called the space of configurations. The local coordinate of the points $x \in M$ are denoted by x^i, $(i = 1, ..., n)$. Let (TM, π, M) be the tangent bundle of the manifold M. The $2n$ dimensional manifold TM is called the phases space of M. A point $u \in TM$, with $\pi(u) = x$, will be denoted by (x, y) and its local coordinate will be (x^i, y^i), $(i = 1, ..., n)$. The coordinate (y^i) can be thought as a tangent vector (or a velocity vector) $y^i = \dfrac{dx^i}{dt}$ at the point $x \in M$.

A change of local coordinates on TM, $(x, y) \to (\tilde{x}, \tilde{y})$ is given by

$$(1.1) \qquad \begin{cases} \tilde{x}^i = \tilde{x}^i(x^j), \\ \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j} y^j, \end{cases} \quad \operatorname{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n.$$

The tangent space T_uTM has the natural basis $\left(\left.\dfrac{\partial}{\partial x^i}\right|_u, \left.\dfrac{\partial}{\partial y^i}\right|_u\right)$. With respect to a change of coordinates (1.1), the natural basis changes as follows:

$$(1.2) \qquad \frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{y}^j}{\partial x^i}\frac{\partial}{\partial \tilde{y}^j}; \quad \frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{y}^j}.$$

Remark that $\frac{\partial}{\partial y^i}$ generates locally an n dimensional distribution V called the vertical distribution on TM. Obviously, V is an integrable distribution.

By means of (1.1) and (1.2), on TM there exists a globally defined vector field

$$\mathbb{C} = y^i \frac{\partial}{\partial y^i}, \tag{1.3}$$

and $\mathbb{C}$ vanishes nowhere on the manifold $\widetilde{TM} = TM \setminus \{0\}$.

This is called the **Liouville vector field**.

On the manifold TM there exists a tangent structure J defined by

$$J = \frac{\partial}{\partial y^i} \otimes dx^i. \tag{1.4}$$

J is an integrable structure [9, 16, 20, 21].

A semispray is a vector field $S \in \chi(TM)$ which has the property

$$JS = \mathbb{C}. \tag{1.5}$$

Proposition 1.1. *A vector field $S \in \chi(TM)$ is a semispray if and only if there exists the functions $G^i(x, y)$ such that*

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}, \tag{1.6}$$

and with respect to (1.1) we have

$$2\tilde{G}^i = \frac{\partial \tilde{x}^i}{\partial x^j} 2G^j - \frac{\partial \tilde{y}^i}{\partial x^j} y^j, \tag{1.7}$$

where G^i are called the coefficients of the semispray S.

The integral curves of S are given by

$$\frac{dx^i}{dt} = y^i, \ \frac{dy^i}{dt} + 2G^i(x, y) = 0. \tag{1.8}$$

We will say that S is a dynamical system on the phase manifold TM and the equations (1.8) are the evolution equations of dynamical system S.

If M is a paracompact manifold, then on TM there exists dynamical systems S.

Now the geometry of a pair (TM, S) can be constructed. First of all we begin with the notion of nonlinear connection.

A nonlinear connection N on TM is a distribution N on TM supplementary of the vertical distribution V:

$$T_uTM = N_u \oplus V_u, \quad u \in TM. \tag{1.9}$$

A local adapted basis to N is (δ_i), $i = 1, ..., n$, where

$$\delta_i = \partial_i - N_i^j(x,y)\dot{\partial}_j, \tag{1.10}$$

and $N_j^i(x,y)$ are the coefficients of N. Here $\partial_i = \dfrac{\partial}{\partial x^i}$, $\delta_i = \dfrac{\delta}{\delta x^i}$, $\dot{\partial}_i = \dfrac{\partial}{\partial y^i}$ being the usual notations.

With respect to (1.1) the coefficients $N_j^i(x,y)$ transform as follows:

$$\widetilde{N}_k^j \frac{\partial \tilde{x}^k}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^k} N_i^k - \frac{\partial \tilde{y}^j}{\partial x^i}. \tag{1.11}$$

The nonlinear connection N can be defined by the functions $N_j^i(x,y)$ which verify (1.11). We have:

Theorem 1.1. *If S is a semispray with the coefficients $G^i(x,y)$, then the functions*

$$N_j^i(x,y) = \frac{\partial G^i}{\partial y^j}, \tag{1.12}$$

are the coefficients of a nonlinear connection N on TM.

Remarking that $\delta_i, \dot{\partial}_i$ is an adapted basis to N and V, then its dual basis is $(dx^i, \delta y^i)$, where

$$\delta y^i = dy^i + N_j^i dx^j. \tag{1.10'}$$

Therefore the autoparallel curves of N are given by

$$\frac{dx^i}{dt} = y^i, \ \frac{\delta y^i}{dt} = 0. \tag{1.13}$$

The Lie brackets $[\delta_i, \delta_j]$ can be expressed by

$$[\delta_i, \delta_j] = R^k{}_{ij}\dot{\partial}_k, \tag{1.14}$$

where $R^k{}_{ij}$ is the following $d-$ tensor field

$$R^k{}_{ij} = \delta_j N_i^k - \delta_i N_j^k. \tag{1.15}$$

Here "*d*" means "distinguished", [18, 20, 21].

The condition ${R^k}_{ij} = 0$ characterizes the **integrability of the non-linear connection** N.

Now we can introduce the notion of N linear connection, like in [18].

A linear connection D on TM is called an N linear connection if D preserves by parallelism the distribution N and V and the tangent structure J is absolutely parallel by D.

In adapted basis $(\delta_i, \dot{\partial}_i)$ an $N-$ linear connection D has two types of coefficients $D\Gamma(N) = (L^i_{jk}(x,y), C^i_{jk}(x,y))$.

With respect to (1.1) these coefficients transform as follows:

$$\begin{aligned} \tilde{L}^k_{ij} &= \frac{\partial \tilde{x}^k}{\partial x^l} L^l_{pq} \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} - \frac{\partial^2 \tilde{x}^k}{\partial x^p \partial x^q} \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j}, \\ \tilde{C}^k_{ij} &= \frac{\partial \tilde{x}^k}{\partial x^l} C^l_{pq} \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j}. \end{aligned} \tag{1.16}$$

These equations characterize the coefficients of an $N-$ linear connection D.

Denoting the operator of $h-$ covariant derivation by "$_|$" and the operator of $v-$ covariant derivation by "$|$", for the $d-$ tensor field $g_{ij}(x,y)$ we have:

$$\begin{cases} g_{ij|k} = \delta_k g_{ij} - L^s_{ik} g_{sj} - L^s_{jk} g_{is}, \\ g_{ij}|_k = \dot{\partial}_k g_{ij} - C^s_{ik} g_{sj} - C^s_{jk} g_{is}. \end{cases} \tag{1.17}$$

Other details can be found in the books [2, 18, 19, 20].

§2. Lagrange Spaces

In the last thirty years many geometrical models in Mechanics, Physics, Control theory, Biology were based on the notion of Lagrangian or Hamiltonian, concepts studied by the author in [16, 17, 20]. Nowadays these geometrical theories are considerable developed and used in various fields.

We start with the following definitions.

A differentiable Lagrangian on the configurations manifold M is a scalar function

$$L : (x,y) \in TM \to L(x,y) \in R$$

of $C^\infty-$class on $\widetilde{TM}$ and continuous on the null section.

The $d-$ tensor field

$$g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L, \tag{2.1}$$

is a covariant and symmetric tensor, called the fundamental or metric tensor.

The Lagrangian $L(x,y)$ is "regular" if

$$\text{rank}(g_{ij}) = n \text{ on } \widetilde{TM}. \tag{2.2}$$

If $L(x,y)$ is a regular Lagrangian we can consider the contravariant tensor $g^{ij}(x,y)$ of $g_{ij}(x,y)$.

Definition 2.1. *A Lagrange space is a pair* $L^n = (M, L(x,y))$, *where* $L(x,y)$ *is a regular Lagrangian and its fundamental tensor* g_{ij} *has constant signature on* $\widetilde{TM}$.

If M is a paracompact manifold, then there exist Lagrange spaces $L^n = (M, L(x,y))$.

Example.

Consider the Lagrangian used in electrodynamics:

$$L(x,y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i + \mathcal{U}(x), \tag{2.3}$$

where $m \neq 0$, c, e are the well-known physical constants $\gamma_{ij}(x)$ being the gravitational potentials, $A_i(x)$ is a covector field, the functions $A_i(x)$ are the electromagnetic potentials and $\mathcal{U}(x)$ is a potential function.

It is not difficult to prove that $L(x,y)$ is a scalar function on $\widetilde{TM}$ with respect to (1.1) and that $L^n = (M, L(x,y))$ is a Lagrange space. Its fundamental tensor field is given by $g_{ij} = mc\gamma_{ij}(x)$.

In order to study the geometry of Lagrange spaces based only on the principles of Analytical Mechanics, we present briefly the variational problem for the differentiable Lagrangian $L(x,y)$.

Let $c : t \in [0,1] \to (x^i(t)) \in \mathcal{U} \subset M$ be a parametrized curve having the image in a domain of a chart $\mathcal{U}$ on M. Its extension to $\pi^{-1}(\mathcal{U}) \subset \widetilde{TM}$ is $c^* : t \in [0,1] \to \left(x(t), \dfrac{dx}{dt}\right) \in \pi^{-1}(\mathcal{U})$.

The integral of action of the Lagrangian $L(x,y)$ along the curve c is given by the functional

$$I(c) = \int_0^1 L(x, \dot{x})dt. \tag{2.4}$$

Consider the curves $c_\varepsilon : t \in [0,1] \to (x^i(t) + \varepsilon v^i(t)) \in M$, which have the same end point as c and v^i is a vector field on $\mathcal{U}$, ε is a real number. The integral of action $I(c_\varepsilon)$ is:

$$I(c_\varepsilon) = \int_0^1 L\left(x + \varepsilon v, \frac{dx}{dt} + \varepsilon\frac{dv}{dt}\right) dt. \tag{2.4'}$$

A necessary condition for $I(c)$ to be an extremal value of $I(c_\varepsilon)$ is

$$\left.\frac{dI(c_\varepsilon)}{dt}\right|_{\varepsilon=0} = 0. \tag{2.5}$$

Taking into account the previous considerations, equation (2.5) leads us to

$$\int_0^1 \left(\frac{d}{dt}\dot{\partial}_i L - \partial_i L\right) v^i dt = 0, \tag{2.6}$$

where $v^i(t)$ is arbitrary. Therefore, from (2.6) follows:

Theorem 2.1. *In order for the functional $I(c)$ to be an extremal value of the functionals $I(c_\varepsilon)$ it is necessary for $c(t)$ to be a solution of the Euler- Lagrange equations:*

$$E_i(L) \stackrel{def}{=} \frac{d}{dt}\dot{\partial}_i L - \partial_i L = 0, \; y^i = \frac{dx^i}{dt}. \tag{2.7}$$

The curves $c(t)$ which verify (2.7) are called the extremal curves of $L(x,y)$.

A first property is obtained for **the energy** of the Lagrangian L:

$$\mathcal{E}_L = y^i \frac{\partial L}{\partial y^i} - L. \tag{2.8}$$

Theorem 2.2. *The energy $\mathcal{E}_L$ is constant along every extremal curve $c(t)$ of $L(x,y)$.*

Remarking that $E_i(L)$ is a $d-$ covector field we can apply the Euler-Lagrange equations to determine a canonical semispray of a Lagrange space L^n.

Theorem 2.3. *For a Lagrange space $L^n = (M, L(x,y))$ the Euler-Lagrange equations $E_i(L) = 0$ determine the semispray*

$$\overset{\circ}{S} = y^i \partial_i - 2 \overset{\circ}{G}{}^i (x,y) \dot{\partial}_i, \tag{2.9}$$

where

$$2 \overset{\circ}{G}{}^i = \frac{1}{2} g^{is} \left(y^k \partial_k \dot{\partial}_s L - \partial_s L\right). \tag{2.10}$$

Proof. The equations $E_i(L) = 0$ are equivalent to the equations $g^{ij}E_j(L) = 0$, which are

$$\frac{d^2x^i}{dt^2} + 2\overset{\circ}{G}{}^i\left(x, \frac{dx}{dt}\right) = 0, \tag{2.11}$$

where $\overset{\circ}{G}{}^i$ are given by (2.10). We can prove that with respect to (1.1) $\overset{\circ}{G}{}^i$ have the rule of transformation (1.11). So $\overset{\circ}{G}{}^i$ are the coefficients of a semispray $\overset{\circ}{S}$. Q.E.D.

$\overset{\circ}{S}$ is determined only by the Lagrange space L^n. So it is called the canonical semispray.

Therefore we may say that *the geometry of Lagrange space L^n is the geometry of the dynamical system determined by the canonical semispray $\overset{\circ}{S}$ on $\widetilde{TM}$.*

So far we have the following.

1° The canonical nonlinear connection $\overset{\circ}{N}$ of L^n has the coefficients

$$\overset{\circ}{N}{}^i_j = \dot{\partial}_j \overset{\circ}{G}{}^i. \tag{2.12}$$

2° The canonical metrical connection $\overset{\circ}{D}$ of L^n, has the coefficients given by the generalized Christoffel symbols

$$\begin{aligned} \overset{\circ}{L}{}^i_{jk} &= \frac{1}{2}g^{is}\left(\overset{\circ}{\delta}_j\, g_{sk} + \overset{\circ}{\delta}_k\, g_{js} - \overset{\circ}{\delta}_s\, g_{jk}\right), \\ \overset{\circ}{C}{}^i_{jk} &= \frac{1}{2}g^{is}\left(\dot{\partial}_j g_{sk} + \dot{\partial}_k g_{js} - \dot{\partial}_s g_{jk}\right), \end{aligned} \tag{2.13}$$

where $\overset{\circ}{\delta}_i = \partial_i - \overset{\circ}{N}{}^j_i\, \dot{\partial}_j$.

Remarks. 1° The integral curves of $\overset{\circ}{S}$ are

$$\frac{dx^i}{dt} = y^i, \quad \frac{dy^i}{dt} + 2\overset{\circ}{G}{}^i(x, y) = 0. \tag{2.14}$$

2° The geometrical object fields $\overset{\circ}{S}, \overset{\circ}{N}, \overset{\circ}{\Gamma}(N)$ can be calculated without difficulties for the Lagrange space L^n of the electrodynamics.

§3. The Lagrangian Mechanical systems

The notion of Lagrangian mechanical system can be introduced as a natural extension of the classical one, considering the regular Lagrangians $L(x,y)$ and the external forces $F_e(x,\dot{x})$ defined on the phase space.

Definition 3.1. *A Lagrangian mechanical system is a triple*

$$\Sigma_L = (M, L(x,y), F_e(x,y)), \tag{3.1}$$

where $L^n = (M, L(x,y))$ *is a Lagrange space and* $F_e(x,y)$ *is a given vertical vector field:*

$$F_e(x,y) = F^i(x,y)\dot{\partial}_i. \tag{3.2}$$

F_e are called the external forces, $F^i(x,y)$, $(i = 1,...,n)$ determine a $d-$ vector field on the manifold TM.

The fundamental tensor $g_{ij}(x,y)$ of L^n is called the fundamental tensor, or the metric tensor of Σ_L.

Taking into account the variational problem of the integral action of $L(x,y)$ we introduce the evolution equations of Σ_L by:

Postulate 1. *The evolution equations of the Lagrangian mechanical system* Σ_L *are the following Lagrange equations:*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = F_i(x,y), \; y^i = \frac{dx^i}{dt}, \tag{3.3}$$

where

$$F_i(x,y) = g_{ij}(x,y)F^j(x,y). \tag{3.3'}$$

If $L(x,y) = \mathcal{E}(x,y) = \gamma_{ij}(x)y^i y^j$ is the kinetic energy of a Riemannian space $\mathcal{R}^n = (M, \gamma_{ij}(x))$ and $\dot{\partial}_j F^i = 0$, then (3.3) are the classical Lagrange equations of a Riemannian mechanical system $\Sigma_{\mathcal{R}} = (M, \mathcal{E}, F_e(x))$ [25, 26, 27].

For us it is useful to remark:

Proposition 3.1. *The Lagrange equations (3.3) are equivalent to the equations:*

$$\frac{d^2x^i}{dt^2} + 2\overset{\circ}{G}{}^i\left(x, \frac{dx}{dt}\right) = \frac{1}{2}F^i\left(x, \frac{dx}{dt}\right), \tag{3.3''}$$

where

$$2\overset{\circ}{G}{}^{i} = \frac{1}{2}g^{is}(y^k\dot{\partial}_s\partial_k L - \partial_s L), \tag{3.3'''}$$

are the coefficients of the canonical semispray $\overset{\circ}{S}$ of the Lagrange space L^n.

Remarking that the functions

$$G^i(x,y) = \overset{\circ}{G}{}^{i}(x,y) - \frac{1}{4}F^i(x,y), \tag{3.4}$$

are the coefficients of a semispray, we can prove, without difficulties, the following theorem:

Theorem 3.1. *The following properties hold:*
1° *S_L given by*

$$S_L = y^i\partial_i - 2\left(\overset{\circ}{G}{}^{i} - \frac{1}{4}F^i\right)\dot{\partial}_i, \tag{3.5}$$

is a semispray on $\widetilde{TM}$.

2° *S_L is a dynamical system on $\widetilde{TM}$ depending only on the Lagrangian mechanical system Σ_L.*

3° *The integral curves of S_L are the evolution curves of Σ_L given by (3.3).*

Clearly, by means of (3.4) we can write

$$S_L = y^i\partial_i - 2G^i\dot{\partial}_i = \overset{\circ}{S} + \frac{1}{2}F^i\dot{\partial}_i, \tag{3.5'}$$

which shows, directly, that S_L has the properties expressed in the previous theorem.

Looking at the energy ε_L of the Lagrangian $L(x,y)$, given by (2.8), and using the Lagrange equations (3.3), we obtain

Theorem 3.2. *The variation of the energy $\mathcal{E}_L$ along the evolution curves of Σ_L is given by:*

$$\frac{d\varepsilon_L}{dt} = F_i\left(x, \frac{dx}{dt}\right)\frac{dx^i}{dt}. \tag{3.6}$$

Therefore we can say that the geometry of the Lagrangian mechanical system Σ_L is the geometry of the pair (L^n, S_L), where S_L is canonical semispray (or its dynamical system).

Hence, the canonical nonlinear connection N of Σ_L has the coefficients

(3.7) $$N^i_j = \overset{\circ}{N}{}^i_j - \frac{1}{4}\dot{\partial}_j F^i,$$

where $\overset{\circ}{N}$ ($\overset{\circ}{N}{}^i_j$) is the canonical nonlinear connection of the Lagrange space L^n.

The adapted basis to the distribution N and V are $(\delta_i, \dot{\partial}_i)$, where

(3.8) $$\delta_i = \overset{\circ}{\delta}_i + \frac{1}{4}\delta_i F^j \dot{\partial}_j,$$

with the dual basis $(dx^i, \delta y^i)$, where δy^i is given by

(3.8') $$\delta y^i = \overset{\circ}{\delta} y^i - \frac{1}{4}\partial_j F^i dx^j.$$

The Berwald connection $B\Gamma(N) = (B^i_{jk}, 0)$, $B^i_{jk} = \dot{\partial}_k N^i_j$ has the coefficients

(3.9) $$B^i_{jk} = \overset{\circ}{B}{}^i_{jk} - \frac{1}{4}\dot{\partial}_j \dot{\partial}_k F^i.$$

We have $B^i_{jk} = B^i_{kj}$.

The tensor of integrability of the canonical nonlinear connection N is as follows:

(3.10)
$$R^i_{jk} = \overset{\circ}{R}{}^i_{jk} + \frac{1}{4}[(\dot{\partial}_k F^i)_{\|j} - (\dot{\partial}_j F^i)_{\|k}] - \frac{1}{16}[\dot{\partial}_k F^s \cdot \dot{\partial}_s \dot{\partial}_j F^i - \dot{\partial}_j F^s \cdot \dot{\partial}_s \dot{\partial}_k F^i],$$

where "$_{\|}$" is the $h-$ covariant derivative with respect to $B\Gamma(N)$.

§4. The canonical metrical connection of Σ_L

The canonical $N-$ metrical connection of Σ_L, with the coefficients $C\Gamma(N) = (L^i_{jk}, C^i_{jk})$ is uniquely determined by the conditions

1° N is the canonical nonlinear connection (3.7);

2° $g_{ij|k} = 0$,

3° $g_{ij}|_k = 0$,

4° $T^i_{jk} = L^i_{jk} - L^i_{kj} = 0$,

5° $S^i_{jk} = C^i_{jk} - C^i_{kj} = 0$.

$C\Gamma(N)$ has the coefficients L^i_{jk} and C^i_{jk} given by the generalized Christoffel symbols

$$(4.1)\quad \begin{aligned} L^i_{jk} &= \frac{1}{2}g^{is}(\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}), \\ C^i_{jk} &= \frac{1}{2}g^{is}(\dot{\partial}_j g_{sk} + \dot{\partial}_k g_{js} - \dot{\partial}_s g_{jk}). \end{aligned}$$

We can prove without difficulties:

Theorem 4.1. *The canonical metrical connection $C\Gamma(N)$ has the coefficients*

$$(4.2)\quad L^i_{jk} = \overset{\circ}{L}{}^i_{jk} + \frac{1}{2}(\overset{\circ}{C}{}^i_{ks}\, \dot{\partial}_j F^s + \overset{\circ}{C}{}^i_{js}\, \dot{\partial}_k F^s - g^{ir}\, \overset{\circ}{C}_{jks}\, \dot{\partial}_r F^s),\ C^i_{jk} = \overset{\circ}{C}{}^i_{jk},$$

where $C\Gamma(\overset{\circ}{N})$ is $\overset{\circ}{N}$ canonical metrical connection of Lagrange space L^n.

Consider the $h-$ and $v-$ deflection tensors(see [16, 17] for details) of $C\Gamma(N)$, defined by $D^i_j = y^i_{|j}$, $d^i_j = y^i|_j$, as well as the $h-$ and $v-$ electromagnetic tensors

$$(4.3)\quad \mathcal{F}_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), f_{ij} = \frac{1}{2}(d_{ij} - d_{ji}),$$

where $D_{ij} = g_{ir}D^r_j$, $d_{ij} = g_{ir}d^r_j$.

Let us consider the helicoidal tensor of Σ_L:

$$(4.4)\quad F_{ij} = \frac{1}{2}(\dot{\partial}_j F_i - \dot{\partial}_i F_j).$$

From (4.2), (4.3) and (4.4) it follows $f_{ij} = 0$, and therefore we obtain the following Theorem.

Theorem 4.2. *Between the $h-$electromagnetic tensors $\mathcal{F}_{ij}$, $\overset{\circ}{\mathcal{F}}_{ij}$ of Σ_L and L^n and the helicoidal tensor F_{ij} of Σ_L the following relation holds:*

$$(4.5)\quad \mathcal{F}_{ij} = \overset{\circ}{\mathcal{F}}_{ij} + \frac{1}{4}F_{ij}.$$

Also we can prove:

Theorem 4.3. *The following generalized Maxwell equations hold good:*

$$(4.6)\quad \begin{aligned} &\mathcal{F}_{ij|k} + \mathcal{F}_{jk|i} + \mathcal{F}_{ki|j} = \frac{1}{2}\underset{ijk}{\sigma}(y^s R_{sijk} - R_{ijk}), \\ &\mathcal{F}_{ij}|_k + \mathcal{F}_{jk}|_i + \mathcal{F}_{ki}|_j = 0, \end{aligned}$$

where $\underset{ijk}{\sigma}$ *is the cyclic sum symbol.*

§5. The almost Hermitian model of Σ_L

The $N-$ lift, [15, 16, 17], denoted by $\mathbb{G}$ of the fundamental tensor g_{ij} of the Lagrangian mechanical system Σ_L together with the almost complex structure $\mathbb{F}$ determined by the canonical nonlinear connection N define an almost Hermitian structure $(\mathbb{G}, \mathbb{F})$ on the phases space $\widetilde{TM}$. The almost Hermitian manifold $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$ is the almost Hermitian model of the system Σ_L. Applying the well-known methods ([17]), we can study the Einstein equations of Σ_L.

The $N-$ lift of fundamental tensor g_{ij} on $\widetilde{TM}$ is defined by

$$\mathbb{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j, \tag{5.1}$$

with δy^i from (3.8').

The almost complex structure determined by the canonical nonlinear connection N is expressed by:

$$\mathbb{F} = \delta_i \otimes \delta y^i - \dot{\partial}_i \otimes dx^i. \tag{5.2}$$

Thus, one has:

Theorem 5.1. *We have:*

1° $\mathbb{G}$ *is a pseudo-Riemannian structure on* $\widetilde{TM}$ *and* $\mathbb{F}$ *is an almost complex structure on* $\widetilde{TM}$. *They depend only on* Σ_L.

2° *The pair* $(\mathbb{G}, \mathbb{F})$ *is an almost Hermitian structure.*

3° *The associated 2-form* θ *of* $(\mathbb{G}, \mathbb{F})$ *is given by*

$$\theta = g_{ij} \delta y^i \wedge dx^j. \tag{5.3}$$

4° θ *is an almost symplectic structure on* $\widetilde{TM}$.

5° *The following equality holds*

$$\theta = \overset{\circ}{\theta} - \frac{1}{4} F_{ij} dx^i \wedge dx^j. \tag{5.3'}$$

Since $\overset{\circ}{\theta} = g_{ij} \overset{\circ}{\delta} y^i \wedge dx^j$ is the symplectic structure of Lagrange space L^n, [16, 17], we have $d\overset{\circ}{\theta} = 0$. So the exterior differential of θ can be expressed in the form

$$d\theta = -\frac{1}{4} dF_{ij} \wedge dx^i \wedge dx^j,$$

or in the following equivalent form

(5.4)

$$d\theta = -\frac{1}{12}(F_{ij|k} + F_{jk|i} + F_{ki|j})dx^k \wedge dx^i \wedge dx^j - \frac{1}{4}\dot{\partial}_k F_{ij}\delta y^k \wedge dx^i \wedge dx^j.$$

Corollary 5.1. 1° *The helicoidal tensor F_{ij} vanishes if and only if* $\theta = \overset{\circ}{\theta}$.

2° θ *is a symplectic structure on the phases space $\widetilde{TM}$ if and only if the following equations hold:*

$$F_{ij|k} + F_{jk|i} + F_{ki|j} = 0, \quad \dot{\partial}_k F_{ij} = 0.$$

Now we observe that a good application of this theory is the case of the Lagrangian mechanical systems $\Sigma_L = (M, L(x,y), F_e(x,y))$, where $L(x,y)$ is the Lagrangian from electrodynamics given by (2.3), and $F_e = A^i_{|j}(x)y^j\dot{\partial}_i$, where $A^i_{|j} = \partial_j A^i + A^s\gamma^i_{sj}$ and $A^i(x) = \gamma^{ij}(x)A_j(x)$.

§6. Finslerian Mechanical systems

An important particular case of previous theory is obtained by the Finslerian mechanical systems

$$\Sigma_F = (M, F(x,y), F_e(x,y)), \tag{6.1}$$

where $F^n = (M, F(x,y))$ is a Finsler space, [4, 5, 10, 15].

The evolution curves of Σ_F are given by the equations

$$\frac{d}{dt}(\dot{\partial}_i F^2) - \partial_i F^2 = F_i(x,y), \tag{6.2}$$

where $F^i(x,y)\dot{\partial}_i = F_e$ and $F^i = g^{ij}F_j$. Of course g_{ij} is the fundamental tensor of the Finsler space F^n.

The system (6.1) is equivalent to

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x,y)\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{1}{2}F^i(x,y), \; y^i = \frac{dx^i}{dt}. \tag{6.3}$$

If $F_e = 0$ then (6.3), in the canonical parametrization, give us the geodesics of the space F^n.

The canonical spray S of F^n is

$$\overset{\circ}{S} = y^i\partial_i - 2\overset{\circ}{G}{}^i\dot{\partial}_i, \; \overset{\circ}{G}{}^i = \frac{1}{2}\gamma^i_{jk}(x,y)y^jy^k, \tag{6.4}$$

and the Cartan nonlinear connection $\overset{\circ}{N}$ has the coefficients

$$\overset{\circ}{N}{}^i_j = \dot{\partial}_j\overset{\circ}{G}{}^i, \tag{6.5}$$

where $\gamma^i_{jk}(x,y)$ are the Christoffel symbols of the fundamental tensor $g_{ij}(x,y)$.

The most important result here is given by Miron - Frigioiu's Theorem:

Theorem 6.1. *We have:*

1° S_F *given by*

$$S_F = y^i\partial_i - 2\left(\overset{\circ}{G}{}^i - \frac{1}{4}F^i\right)\dot{\partial}_i = \overset{\circ}{S} + \frac{1}{2}F^i\dot{\partial}_i, \tag{6.6}$$

is a semispray on $\widetilde{TM}$.

2° S_F *is a dynamical system on the phases spaces* $\widetilde{TM}$ *depending only on the Finslerian mechanical system* Σ_F.

3° *The integral curves of* S_F *are the evolution curves of* Σ_F *given by (6.2).*

The energy $\varepsilon_F = F^2 = g_{ij}y^iy^j$ satisfies the equality (3.6) on every evoluton curve.

Therefore, we can say that the geometry of Σ_F is the geometry on $\widetilde{TM}$ of the semispray S_F. All considerations made for Σ_L in the §3, §4, §5 can be particularized without difficulties.

Examples. 1° Let $A_i(x)$ be a covector field and $A_{i|j}$ the covariant derivation with respect to $C\Gamma(\overset{\circ}{N})$. Consider the external forces $F_e = A_{i|j}g^{ih}\dot{\partial}_h$ which give a first example of systems Σ_F.

2° $F_e = a(x,y)y^i\dot{\partial}_i$. Systems Σ_F with F_e of this form have the evolution equations (6.2) of Lorentz type.

It is interesting to remark some properties of the electromagnetic fields $\mathcal{F}_{ij}$ and f_{ij} of Σ_F.

First of all, we have $f_{ij} = 0$ and from Theorems 4.2, 4.3 we deduce

Theorem 6.2. *The h− electromagnetic tensor* $\mathcal{F}_{ij}(x,y)$ *of* Σ_F *and the helicoidal tensor* F_{ij} *of same Finslerian mechanical system are in the following relation*

$$\mathcal{F}_{ij} = \frac{1}{4}F_{ij}. \tag{6.7}$$

Theorem 6.3. *The generalized Maxwell equations of* Σ_F *are*

$$\begin{aligned} &F_{ij|k} + F_{jk|i} + F_{ki|j} = 2\underset{ijk}{\sigma}(y^sR_{sijk} - R_{ijk}),\\ &F_{ij}|_k + F_{jk}|_i + F_{ki}|_j = 0. \end{aligned} \tag{6.8}$$

§7. The Riemannian mechanical systems

The most natural application of the previous theory is the particular case of the Riemannian mechanical systems when the external forces $F_e(x,y)$ depend on the points $x = (x^i)$ and on their velocities $y = \left(\dfrac{dx^i}{dt}\right)$.

A Riemannian mechanical system is the set $\Sigma_{\mathcal{R}} = (M, g(x), F_e(x,y))$, where $\mathcal{R}^n = (M, g(x))$ is a Riemann (or pseudo-Riemann) space, where metric tensor $g_{ij}(x)$ and $F_e = F^i(x,y)\dot{\partial}_i$ are the external forces. If $\dot{\partial}_j F^i = 0$, then $\Sigma_{\mathcal{R}}$ is the classical Riemann mechanical system.

The previous theory from the sections 3–6 can be applied by considering the kinetic energy $\mathcal{E}$ of $\Sigma_{\mathcal{R}}$:

$$\mathcal{E} = g_{ij}(x)y^i y^j \tag{7.1}$$

and the evolution equations of Σ given by the Lagrange equations

$$\frac{d}{dt}\frac{\partial\mathcal{E}}{\partial y^i} - \frac{\partial\mathcal{E}}{\partial x^i} = F_i(x,y),\ y^i = \frac{dx^i}{dt},\ F_i(x,y) = g_{ij}(x)F^j(x,y). \tag{7.2}$$

This system is equivalent to

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{1}{2}F^i\left(x, \frac{dx}{dt}\right), \tag{7.2'}$$

where $\gamma^i_{jk}(x)$ are the Christoffel symbols of metric tensor $g_{ij}(x)$.

Theorem 6.1 leads to:

Theorem 7.1. *We have:*

1° $S_{\mathcal{R}}$ *given by*

$$S_{\mathcal{R}} = y^i\partial_i - 2\left(\overset{\circ}{G}{}^i - \frac{1}{4}F^i\right)\dot{\partial}_i, \tag{7.3}$$

where

$$\overset{\circ}{G}{}^i = \frac{1}{2}\gamma^i_{jk}(x)y^j y^k, \tag{7.3'}$$

is a semispray on TM.

2° $S_{\mathcal{R}}$ *is a dynamical system on the phases spaces TM depending only on the Riemannian mechanical system* $\Sigma_{\mathcal{R}}$.

3° *The integral curves of* $S_{\mathcal{R}}$ *are the evolution curves of* $\Sigma_{\mathcal{R}}$.

It follows that on the evolution curves (7.3) the variation of kinetic energy ε is given by

$$\frac{d\mathcal{E}}{dt} = F_i\left(x, \frac{dx}{dt}\right)\frac{dx^i}{dt}. \tag{7.4}$$

Therefore we can say that *the geometry of $\Sigma_{\mathcal{R}}$ is the geometry on TM of the semispray $S_{\mathcal{R}}$.*

Example. For a Riemannian mechanical system $\Sigma_{\mathcal{R}}$ having the external forces of the covariant components

$$F_i(x, y) = A_{i|j}y^j, \; A_{i|j} = \partial_j A_i - A_s\gamma^s_{ij},$$

$A_i(x)$ being a given covector field (the electromagnetic potentials), the evolution equations (7.2) are the Lorentz equations from the electromagnetism.

The canonical nonlinear connection N of $\Sigma_{\mathcal{R}}$ has the coefficients

$$N^i_j = \gamma^i_{jk}(x)y^k - \frac{1}{4}\dot{\partial}_j F^i, \tag{7.5}$$

and the canonical metrical connection $C\gamma(N)$ has the coefficients

$$L^i_{jk}(x, y) = \gamma^i_{jk}(x), \; C^i_{jk}(x, y) = 0. \tag{7.6}$$

The $h-$ electromagnetic tensor $\mathcal{F}_{ij}$ and the helicoidal tensor F_{ij} satisfy the equations

$$\mathcal{F}_{ij} = \frac{1}{4}F_{ij},$$

and the Maxwell equations (4.6) are verified.

In the case of classical Riemannian mechanical systems $\Sigma_{\mathcal{R}}$, for which $\dfrac{\partial F_i}{\partial y^j} = 0$, the previous theory can be applied without difficulties.

Obviously, we can use the vector field $S_{\mathcal{R}}$ on the phase space TM for studying the qualitative problems as stability concerning the evolution curves of $\Sigma_{\mathcal{R}}$.

Part II. The dynamical systems of the Hamiltonian mechanical systems

The theory of dynamical systems of the Hamiltonian mechanical systems can be constructed step by step following the theory of Lagrangian mechanical systems. But the legitimacy of this theory is proved by means of $\mathcal{L}-$duality (Legendre duality) between the Lagrange spaces and Hamilton spaces. In this part of our paper we develop this theory using our papers [23, 24], and the book of R. Miron, D. Hrimiuc, H. Shimada and S. Sabău [19].

The content of this part is new.

§8. Preliminaries for the geometry of momenta space

Let M be a $C^\infty-$ real $n-$ dimensional manifold (called configurations space) and (T^*M, π^*, M) be the cotangent bundle of M. T^*M is called the momenta space. A point $u^* = (x,p) \in T^*M$, $\pi^*(u^*) = x$ has the local coordinates (x^i, p_i), (x^i) are the coordinates of the points $x \in M$ and (p_i) are the momenta, as local coordinates of the momenta p at point x.

A change of local coordinate $(x,p) \to (\tilde{x}, \tilde{p})$ at the point u^* is given by

$$\text{(8.1)} \qquad \begin{cases} \tilde{x}^i = \tilde{x}^i(x^j), \quad \det\left(\dfrac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0, \\ \\ \tilde{p}_i = \dfrac{\partial x^j}{\partial \tilde{x}^i} p_j. \end{cases}$$

The tangent space $T_{u^*}T^*M$ has the natural basis $\left(\dfrac{\partial}{\partial x^i} = \partial_i, \dfrac{\partial}{\partial p_i} = \dot{\partial}^i\right)$. With respect to (8.1) this basis transforms as follows:

$$\text{(8.2)} \qquad \begin{cases} \dfrac{\partial}{\partial x^i} = \dfrac{\partial \tilde{x}^j}{\partial x^i}\dfrac{\partial}{\partial \tilde{x}^j} + \dfrac{\partial \tilde{p}_j}{\partial x^i}\tilde{\dot{\partial}}^j, \\ \\ \dot{\partial}^i = \dfrac{\partial x^i}{\partial \tilde{x}^j}\tilde{\dot{\partial}}^j. \end{cases}$$

On the manifold T^*M there are globally defined Liouville 1-forms

$$\text{(8.3)} \qquad \tilde{p} = p_i dx^i,$$

and the natural symplectic structure

$$\text{(8.4)} \qquad \overset{\circ}{\theta} = dp_i \wedge dx^i.$$

Let $V = \ker d\pi^*$ be the vertical subbundle on T^*M. It defines a distribution V locally generated by vector fields $(\dot{\partial}^i)$. A supplementary distribution N to V is named a nonlinear connection on T^*M. We have

$$(8.5) \qquad T_{u^*}T^*M = N_{u^*} \oplus V_{u^*}, \ \forall u^* \in T^*M.$$

If the base manifold M is paracompact on T^*M there exists nonlinear connections N.

An adapted basis to N and V is $(\delta_i, \dot{\partial}^i)$, where

$$(8.6) \qquad \delta_i = \partial_i + N_{ji}\dot{\partial}^j.$$

The functions $N_{ji}(x,p)$ are the coefficients of the nonlinear connection N.

Under a change of coordinates (8.1) on T^*M we have $\tilde{\delta}_i = \dfrac{\partial x^j}{\partial \tilde{x}^i}\delta_j$ and

$$(8.7) \qquad \tilde{N}_{ij} = \frac{\partial x^s}{\partial \tilde{x}^i}\frac{\partial x^r}{\partial \tilde{x}^j}N_{rs} + p_r\frac{\partial^2 x^r}{\partial \tilde{x}^i \partial \tilde{x}^j}.$$

Then

$$(8.8) \qquad t_{ij} = N_{ij} - N_{ji},$$

is a $d-$ tensor field.

If $t_{ij} = 0$, the functions N_{ij} are called a symmetric nonlinear connection.

The dual adapted basis of $(\delta_i, \dot{\partial}^i)$ is $(dx^i, \delta p_i)$, where

$$(8.9) \qquad \delta p_i = dp_i - N_{ij}dx^j.$$

Here δp_i are 1-forms on T^*M.

If N is symmetric then the symplectic structure $\overset{\circ}{\theta}$ can be written as

$$(1.4') \qquad \overset{\circ}{\theta} = \delta p_i \wedge dx^i.$$

The integrability tensor of N is given by

$$(8.10) \qquad R_{kij} = \delta_i N_{kj} - \delta_j N_{ki},$$

and $R_{kij} = 0$ gives us necessary and sufficient conditions for integrability of the distribution N.

The notion of $N-$ linear connection on T^*M can be found in the books [19, 21].

§9. Hamilton spaces. Variational problem

The notion of Hamilton space has been introduced by the author in [19]. It is defined as a pair $H^n = (M, H(x,p))$, where H is a scalar function on the momenta space T^*M, of class C^∞ on the manifold $\widetilde{T^*M} = T^*M \setminus \{0\}$ and continuous on the null section of π^*, and the d–tensor

$$g^{ij}(x,p) = \frac{1}{2}\dot{\partial}^i \dot{\partial}^j H, \tag{9.1}$$

has the rank $n = \dim M$ and constant signature on $\widetilde{T^*M}$.

We have ([19]):

Theorem 9.1. 1° *In a Hamilton space $H^n(M, H(x,p))$ with paracompact configurations space M, then there exist nonlinear connections determined only by H^n.*

2° *One of them, say $\overset{\circ}{N}$, has the coefficients:*

$$\overset{\circ}{N}_{ij} = -\frac{1}{2}g_{jh}\left[\frac{1}{4}g_{ik}\dot{\partial}^k\{H, \dot{\partial}^h H\} + \dot{\partial}^h \partial_i H\right]. \tag{9.2}$$

3° *$\overset{\circ}{N}$ is symmetric.*

In the formula (9.2), $\{,\}$ is the Poisson bracket.

$\overset{\circ}{N}$ is called the canonical nonlinear connection of H^n.

Now, the variational problem for a $C^\infty-$ Hamiltonian $H(x,p)$ can be formulated.

Consider a smooth curve on a domain of a local chart $\pi^{*-1}(\mathcal{U})$ in T^*M, $c : t \in [0,1] \to (x^i(t), p_i(t)) \in \pi^{*-1}(\mathcal{U})$ and the functional

$$I(c) = \int_0^1 \left[p_i(t)\frac{dx^i}{dt} - \frac{1}{2}H(x(t), p(t))\right] dt. \tag{9.3}$$

Obviously, $I(c)$ is invariant with respect to (8.1).

A variation $\bar{c}$ of c is

$$\bar{x}^i(t) = x^i(t) + \varepsilon_1 v^i(t), \bar{p}_i(t) = p_i(t) + \varepsilon_2 \eta_i(t), \tag{9.4}$$

where $v^i(t), \eta_i(t)$ is a vector field and $\eta_i(t)$ a covector field along the curve c for which:

$$\begin{aligned} &v^i(0) = v^i(1) = 0, \frac{dv^i}{dt}(0) = \frac{dv^i}{dt}(1) = 0 \\ &\eta_i(0) = \eta_i(1) = 0. \end{aligned} \tag{9.5}$$

Here, $\varepsilon_1, \varepsilon_2$ are real numbers, sufficiently small in absolute value such that $\mathrm{Im}\bar{c} \subset \pi^{*-1}(\mathcal{U})$.

$I(\bar{c})$ is given by

$$(9.3') \qquad I(\bar{c})(\varepsilon_1, \varepsilon_2) = \int_0^1 \left\{ [p_i(t) + \varepsilon_2 \eta_i(t)] \left(\frac{dx^i}{dt} + \varepsilon_1 \frac{dv^i}{dt} \right) - \frac{1}{2} H(x + \varepsilon_1 v, p + \varepsilon_2 \eta) \right\} dt.$$

The necessary conditions as $I(c)$ be an extremal value of $I(\bar{c})$ are

$$(9.6) \qquad \left. \frac{\partial I(\bar{c})}{\partial \varepsilon_1} \right|_{\varepsilon_1 = \varepsilon_2 = 0} = 0, \quad \left. \frac{\partial I(\bar{c})}{\partial \varepsilon_2} \right|_{\varepsilon_1 = \varepsilon_2 = 0} = 0.$$

Using the classical method of variational calculus, by means of (9.3') we obtain

$$\int_0^1 \left(p_i \frac{dv^i}{dt} - \frac{1}{2} \frac{\partial H}{\partial x^i} v^i \right) dt = 0, \quad \int_0^1 \left(\frac{dx^i}{dt} - \frac{1}{2} \frac{\partial H}{\partial p_i} \right) \eta_i dt = 0,$$

which lead to

$$(9.7) \qquad \int_0^1 \left(\frac{dx^i}{dt} - \frac{1}{2} \frac{\partial H}{\partial p_i} \right) \eta_i dt = 0, \quad \int_0^1 \left(\frac{dp^i}{dt} + \frac{1}{2} \frac{\partial H}{\partial x_i} \right) v^i dt = 0.$$

But $v^i(t)$ and $\eta_i(t)$ being arbitrary, we obtain

Theorem 9.2. *The necessary conditions of extrem (9.7) imply that the curve $c(t) = (x^i(t), p_i(t))$ is a solution of the following Hamilton - Jacobi equations*

$$(9.8) \qquad \frac{dx^i}{dt} - \frac{1}{2} \frac{\partial H}{\partial p_i} = 0, \quad \frac{dp_i}{dt} + \frac{1}{2} \frac{\partial H}{\partial x^i} = 0.$$

Let N be a symmetric nonlinear connection, then the equations (9.7) are equivalent to

$$(9.7') \qquad \frac{dx^i}{dt} - \frac{1}{2} \frac{\partial H}{\partial p_i} = 0, \quad \frac{\delta p_i}{dt} + \frac{1}{2} \frac{\delta H}{\delta x^i} = 0.$$

The equations (9.7') show the geometrical meaning of the Hamilton - Jacobi equations with respect to the change of coordinates (8.1).

The curves $c(t)$, which verify (9.8) are called extremal curves for Hamiltonian $H(x,p)$. If $H(x,p)$ is the fundamental function of a Hamilton space H^n, then the extremal curves $c(t)$ are called geodesics of H^n.

Theorem 9.3. *The Hamiltonian $H(x,p)$ is constant along every extremal curve $c(t)$.*

Now, consider a Hamilton space $H^n = (M, H(x,p))$. Thus we can proof without difficulties:

Theorem 9.4. 1° *For a Hamilton space H^n there exists a vector field $\overset{\circ}{\xi} \in \chi(\widetilde{T^*M})$ with the property*

$$i_{\overset{\circ}{\xi}} \overset{\circ}{\theta} = -dH, \tag{9.9}$$

where $i_{\overset{\circ}{\xi}} \overset{\circ}{\theta}$ is the interior product of $\overset{\circ}{\xi}$ and $\overset{\circ}{\theta}$.

2° *The vector field $\overset{\circ}{\xi}$ is given by*

$$\overset{\circ}{\xi} = \frac{1}{2}(\dot{\partial}^i H \partial_i - \partial_i H \dot{\partial}^i). \tag{9.10}$$

3° *The integral curves of $\overset{\circ}{\xi}$ are given by the Hamilton - Jacobi equations (9.7). $\overset{\circ}{\xi}$ is called the Hamiltonian vector of the space H^n.*

If $\overset{\circ}{N}$ is the canonical nonlinear connection of H^n, then in adapted basis $(\overset{\circ}{\delta}_i, \dot{\partial}^i)$, the vector field $\overset{\circ}{\xi}$ takes the invariant form:

$$\overset{\circ}{\xi} = \frac{1}{2}\left(\dot{\partial}^i H \, \overset{\circ}{\delta}_i - \overset{\circ}{\delta}_i \, H \dot{\partial}^i\right). \tag{9.9'}$$

Therefore, we can say that $\overset{\circ}{\xi}$ is a dynamical system of the Hamilton space H^n having the Hamilton - Jacobi equations as evolution equations.

§10. The Hamilton mechanical systems

Following the ideas from the first part of this paper we can introduce the next definition.

Definition 10.1. *A Hamiltonian mechanical system is a triple:*

$$\Sigma_H = (M, H(x,p), F_e(x,p)), \tag{10.1}$$

where $H^n = (M, H(x,p))$ is a Hamilton space and

$$F_e(x,p) = F_i(x,p)\dot{\partial}^i \tag{10.2}$$

is a given vertical vector field on the momenta space T^*M.
F_e is called the external forces field.

The evolution equations of Σ_H can be defined by means of equations (9.6) from the variational problem.

Postulate 2. *The evolution equations of the Hamiltonian mechanical system* Σ_H *are the following* **Hamilton equations***:*

$$\frac{dx^i}{dt} - \frac{1}{2}\dot{\partial}^i H = 0, \ \frac{dp_i}{dt} + \frac{1}{2}\partial_i H = \frac{1}{2}F_i(x,p). \tag{10.3}$$

Obviously, for $F_e = 0$, the equation (10.3) give us the geodesics of the Hamilton space H^n.

Using the canonical nonlinear connection $\overset{\circ}{N}$ we can write the Hamilton equations in an invariant form, which allow to prove the geometrical meaning of these equations.

Examples. 1° Consider $H^n = (M, H(x,p))$ the Hamilton spaces of electrodynamics, [19]:

$$H(x,p) = \frac{1}{mc}\gamma^{ij}(x)p_i p_j - \frac{2e}{mc^2}A^i(x)p_i + \frac{e^3}{mc^3}A_i(x)A^i(x),$$

and $F_e = p_i\dot{\partial}^i$. Then Σ_H is a Hamiltonian mechanical system determined only by H^n.

2° $H^n = (M, K^2(x,p))$ is a Cartan space and $F_e = p_i\dot{\partial}^i$.

3° $H^n = (M, \varepsilon(x,p))$ with $\varepsilon(x,p) = \gamma^{ij}(x)p_i p_j$ and $F_e = a(x)p_i\dot{\partial}^i$.

Returning to the general theory, we can prove:

Theorem 10.1. *The following properties hold:*

1° ξ *given by*

$$\xi = \frac{1}{2}[\dot{\partial}^i H\partial_i - (\partial_i H - F_i)\dot{\partial}^i] \tag{10.4}$$

is a vector field on $\widetilde{T^*M}$.

2° ξ *is determined only by the Hamiltonian mechanical system* Σ_H.

3° *The integral curves of* ξ *are given by the Hamilton equation (10.3).*

The previous Theorem is not difficult to prove if we remark the following expression of ξ:

$$\xi = \xi_0 + \frac{1}{2}F_e. \tag{10.5}$$

Also we have:

Proposition 10.1. *The variation of $H(x,p)$ along the evolution curves of Σ_H is given by:*

$$\frac{dH}{dt} = F_i \frac{dx^i}{dt}. \tag{10.6}$$

The vector field ξ on $\widetilde{T^*M}$ is called the canonical dynamical system of the Hamilton mechanical system Σ_H.

Therefore we can say that the geometry of Σ_H is the geometry of pair (H^n, ξ).

§11. Geometrical properties of Σ_H

The fundamental tensor $g^{ij}(x,p)$ of the space H^n is the fundamental or the metric tensor of Σ_H. However, other fundamental geometric notions, as the canonical nonlinear connection of Σ_H cannot be introduced in a straightforward manner. They will be defined by means of $\mathcal{L}$– duality between the Lagrangian and the Hamiltonian mechanical systems Σ_L and Σ_H.

Let $\Sigma_L = (M, L(x,y), \underset{1}{F_e}(x,y))$, $\underset{1}{F_e} = \underset{1}{F^i}(x,y)\dot{\partial}_i$ be a Lagrangian mechanical system. The mapping

$$\varphi : (x,y) \in TM \to (x,p) \in T^*M,\ p_i = \frac{1}{2}\dot{\partial}_i L$$

is a local diffeomorphism called the Legendre transformation.

Let ψ be the inverse of φ and

$$H(x,p) = 2p_i y^i - L(x,y),\ y = \psi(x,p). \tag{11.1}$$

One can prove that $H^n = (M, H(x,p))$ is a Hamilton space, [19], called the $\mathcal{L}$– dual of Lagrange space $L^n = (M, L(x,y))$.

One proves that φ transforms, [19]:

1° The canonical semispray $\overset{\circ}{S}$ of L^n in the Hamilton vector $\overset{\circ}{\xi}$ of H^n.

2° The canonical nonlinear connection $\overset{\circ}{N}_L$ of L^n into the canonical nonlinear connection $\overset{\circ}{N}_H$ of H^n.

3° The external forces $\underset{1}{F_e}$ of Σ_L into external forces F_e of Σ_H, with $F_i(x,p) = g_{ij}(x,p)F_1^j(x,\psi(x,p))$.

4° The canonical nonlinear connection N_L of Σ_L with coefficients (3.7) into the canonical nonlinear connection N_H of Σ_H with the coefficients

$$N_{ij}(x,p) = \overset{\circ}{N}_{ij}(x,p) + \frac{1}{4} g_{ih} \dot{\partial}^h F_j, \tag{11.2}$$

where $\overset{\circ}{N}_{ij}$ are given by (9.2).

Therefore we can introduce:

Postulate 3. *The canonical nonlinear connection N of the Hamiltonian mechanical system Σ_H is given by the coefficients N_{ij}, (11.2).*

Of course we can prove directly that N_{ij}, given in (11.2), are the coefficients of a nonlinear connection. It is canonical for Σ_H, since N depend only on the system Σ_H.

The torsion of N is

$$t_{ij} = \frac{1}{4}(g_{ih}\dot{\partial}^h F_j - g_{jh}\dot{\partial}^h F_i). \tag{11.3}$$

Obviously, $\dot{\partial}^i F_j = 0$, implies that N is symmetric.

Let $(\delta_i, \dot{\partial}^i)$ be the adapted basis to N and V and $(dx^i, \delta p_i)$ its cobasis:

$$\delta_i = \partial_i + N_{ji}\dot{\partial}^j, \quad \delta p_i = dp_i - N_{ij}dx^j. \tag{11.4}$$

The tensor of integrability of the nonlinear connection N is

$$R_{kij} = \delta_i N_{kj} - \delta_j N_{ki}. \tag{11.5}$$

The condition $R_{kij} = 0$ characterize the integrability of the distribution N.

The canonical $N-$metrical connection $C\Gamma(N) = (H^i_{jk}, C_i^{jk})$ of the Hamiltonian mechanical system Σ_H is given by the following theorem:

Theorem 11.1. *The following properties hold:*

1) There exists only one $N-$linear connection $C\Gamma = (N_{ij}, H^i_{jk}, C_i^{jk})$ which depend on the Hamiltonian system Σ_H and satisfies the axioms:

1° *N_{ij} from (11.2), (9.2) is the canonical nonlinear connection.*

2° *$C\Gamma$ is $h-$ metric:*

$$g^{ij}{}_{|k} = 0. \tag{11.6}$$

3° *$C\Gamma$ is $v-$metric:*

$$g^{ij}|^k = 0. \tag{4.6'}$$

4° *$C\Gamma$ is $h-$torsion free:*

$$T^i_{jk} = H^i_{jk} - H^i_{kj} = 0. \tag{11.7}$$

5° *$C\Gamma$ is $v-$torsion free*

$$S_i^{jk} = C_i^{jk} - C_i^{kj} = 0. \tag{11.7'}$$

2) The coefficients of $C\Gamma$ are given by the generalized Christoffel symbols:

$$(11.8)\qquad \begin{aligned} H^i_{jk} &= \frac{1}{2}g^{is}(\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}),\\ C_i^{jk} &= -\frac{1}{2}g_{is}(\dot{\partial}^j g^{sk} + \dot{\partial}^k g^{js} - \dot{\partial}^s g^{jk}). \end{aligned}$$

Now, we have all data needed to construct the geometry of Hamilton mechanical system Σ_H. Therefore we can now investigate the electromagnetic and gravitational fields of Σ_H.

§12. The Cartan mechanical systems

An important class of systems Σ_H is obtained when the Hamiltonian $H(x,p)$ is 2-homogeneous with respect to momenta p_i, [2, 3, 19].

Definition 12.1. *A Cartan mechanical system is a set*

$$(12.1)\qquad \Sigma_C = (M, K(x,p), F_e(x,p)),$$

where $\mathcal{C} = (M, K(x,p))$ is a Cartan space and

$$(12.2)\qquad F_e = F_i(x,p)\dot{\partial}^i,$$

are the external forces.

The fact that $\mathcal{C}^n$ is a Cartan spaces implies:
1° $K(x,p)$ is a positive scalar function on T^*M.
2° $K(x,p)$ is a positive 1-homogeneous with respect to momenta p_i.
3° The pair $H^n = (M, K^2(x,p))$ is a Hamilton space.
Therefore, we have
a. The fundamental tensor $g^{ij}(x,p)$ is given by

$$(12.3)\qquad g^{ij} = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j K^2.$$

b. We have

$$(12.4)\qquad K^2 = g^{ij}p_i p_j.$$

c. The Cartan tensor is given by

$$C^{ijk} = \frac{1}{4}\dot{\partial}^i\dot{\partial}^j\dot{\partial}^k K^2,\ p_i C^{ijk} = 0.$$

d. $\mathcal{C}^n = (M, K(x,p))$ is the $\mathcal{L}-$ dual of the Finsler space $F^n = (M, F(x,y))$.

e. The canonical nonlinear connection $\overset{\circ}{N}$, introduced by the author, has the coefficients

$$\overset{\circ}{N}_{ij} = \gamma^h_{ij} p_h - \frac{1}{2}(\gamma^h_{sr} p_h p^r)\dot{\partial}^s g_{ij}, \tag{12.5}$$

$\gamma^i_{jk}(x,p)$ being the Christoffel symbols of $g_{ij}(x,p)$.

Obviously, the geometry of Σ_C is obtained from the geometry of Σ_H taking $H = K^2(x,p)$.

Hence, we have:

Postulate 2'. *The evolution equations of the Cartan mechanical system Σ_C are the Hamilton equations:*

$$\frac{dx^i}{dt} - \frac{1}{2}\dot{\partial}^i K^2 = 0, \frac{dp_i}{dt} + \frac{1}{2}\partial_i K^2 = \frac{1}{2}F_i(x,p). \tag{12.6}$$

A first result is given by

Proposition 12.1. 1° *The energy of the Hamiltonian K^2 is given by $\varepsilon_{K^2} = p_i \dot{\partial}^i K^2 - K^2 = K^2$.*

2° *The variation of energy $\varepsilon_{K^2} = K^2$ along to every evolution curve (12.6) is*

$$\frac{dK^2}{dt} = F_i \frac{dx^i}{dt}. \tag{12.7}$$

Example. The mechanical system Σ_C, with $K(x,p) = \{\gamma^{ij}(x)p_i p_j\}^{1/2}$, $(M, \gamma_{ij}(x))$ being a Riemann spaces and $F_e = a(x,p)p_i\dot{\partial}^i$.

Theorem 10.1 of Part II can be particularized as follows.

Theorem 12.1. *The following properties hold good:*

1° *ξ given by*

$$\xi = \frac{1}{2}[\dot{\partial}^i K^2 \partial_i - (\partial_i K^2 - F_i)\dot{\partial}^i], \tag{12.8}$$

is a vector field on $\widetilde{T^*M}$.

2° *ξ is determined only by the Cartan mechanical system Σ_C.*

3° *The integral curves of ξ are given by the evolution equations (12.6) of Σ_C.*

The vector ξ is **the canonical dynamical system** of the Cartan mechanical system Σ_C.

Therefore we can say that the geometry of Σ_C is the geometry of the pair $(\mathcal{C}^n, \xi)$.

The fundamental object fields of this geometry are $\mathcal{C}^n, \xi, F_e$, and the canonical nonlinear connection N with the coefficients

$$N_{ij} = \overset{\circ}{N}_{ij} + \frac{1}{4} g_{ih} \dot{\partial}^h F_j, \tag{12.9}$$

with $\overset{\circ}{N}_{ij}$ from (12.5). Taking into account that the vector fields $\delta_i = \partial_i - N_{ji}\dot{\partial}^j$ determine an adapted basis to N, we get the canonical $N-$ metrical connection $C\Gamma(N)$ of $\Sigma_{\mathcal{C}}$.

Theorem 12.2. *The canonical $N-$ metrical connection $C\Gamma(N)$ of the Cartan mechanical system $\Sigma_{\mathcal{C}}$ has the coefficients*

$$H^i_{jk} = \frac{1}{2} g^{is}(\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}), \ C_i^{jk} = g_{is} C^{sjk}. \tag{12.10}$$

Using the canonical connections N and $C\Gamma(N)$ one can study the electromagnetic and gravitational fields on the momenta space T^*M of the Cartan mechanical systems $\Sigma_{\mathcal{C}}$, as well as the dynamical system ξ of $\Sigma_{\mathcal{C}}$.

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